1: If a 4 by 4 matrix has $\det A = \frac{1}{2}$,find $\det(2A)$,$\det(-A)$,$\det A^2$,and $\det(A^{-1})$.

Solution:

- $\det(2A) = 2^4 \det A = 8.$
- $\det(-A) = (-1)^4 \det A = \frac{1}{2}.$
- $\det A^2 = \det A \cdot \det A = \frac{1}{4}.$
- $\det(A^{-1}) = \frac{1}{\det A} = 2.$

2: If a 3 by 3 matrix has $\det A = -1$,find $\det(\frac{1}{2}A)$,$\det(-A)$,$\det(A^2)$,and $\det(A^{-1})$.

Solution:

- $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}.$
- $\det(-A) = (-1)^3 \det A = 1.$
- $\det(A^2) = \det A \cdot \det A = 1.$
- $\det(A^{-1}) = \frac{1}{\det A} = -1.$

3: *Row exchange*:Add row 1 to row 2,then subtract row 2 from row 1.Then add row 1 to row 2 and multiply row 1 by -1 to reach B.Which rules show the following?

$$\det B = \begin{vmatrix} c & d \\ a & b \end{vmatrix} \textbf{ equals } -\det A = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Those rules could replace Rule 2 in the definition of the determinant.

Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.Then,the operation flow mentioned in the exercise can be represented as below:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix} \to \begin{pmatrix} -c & -d \\ a+c & b+d \end{pmatrix} \to \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \to \begin{pmatrix} c & d \\ a & b \end{pmatrix} = B.$$

Rule 5:Subtracting a multiple of one row from another row leaves the same determinant.Rule 5 can replace Rule 2.

4: By applying row operations to produce an upper triangular U,compute

$$\det \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \textbf{ and } \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

Exchange rows 3 and 4 of the second matrix and recompute the pivots and determinant.

Solution:

- Let $A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$.Then we perform row operations to matrix A:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \xrightarrow{row2+row1\times(-2)} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \xrightarrow{row3+row1} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$\xrightarrow{row4+row2\times 2} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{row4+row3\times\frac{5}{2}} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} = U.$$

So $\det A = \det U = 20$.

- Let $B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$.Then we perform row operations to matrix B:

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \xrightarrow{row2+row1\times\frac{1}{2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \xrightarrow{row3+row2\times\frac{2}{3}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow{row4+row3\times\frac{3}{4}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & -\frac{11}{4} \end{bmatrix} = U.$$

So $\det B = \det U = -11$.
When row 3 and 4 of the second matrix are changed,the matrix B turned into the matrix $B' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$.Perform row operations to the matrix B',we get matrix $U' = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -\frac{11}{3} \end{bmatrix}$.So $\det B' = \det U' = 11$.

5: Count row exchanges to find these determinants:

$$\det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \pm 1 \textbf{ and } \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = -1.$$

Solution: Exchange row 1 and 4,then row 2 and 3 of the first matrix,we get an identity matrix.So the determinant of the first matrix is 1.
Exchange row 1 and 4,then row 3 and 4,then row 2 and row 3 of the second matrix,we get an identity matrix.So the determinant of the second matrix is -1.

6: For each n,how many exchanges will put (row n,row $n-1,\cdots$,row 1) into the normal order (row $1,\cdots$,row $n-1$,row n)?Find $\det P$ for the n by n permutation with 1s on the reverse diagonal.Problem 5 had $n=4$.

Solution: $(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}$.Putting (row n,row $n-1,\cdots$,row 1) into the normal order (row $1,\cdots$,row $n-1$,row n) requires $\frac{n(n-1)}{2}$ row exchanges.$\det P=(-1)^{\frac{n(n-1)}{2}}$.

7: Find the determinants of

- **a rank one matrix**

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}.$$

- **the upper triangular matrix**

$$U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

- **the lower triangular matrix U^T.**
- **the inverse matrix U^{-1}.**
- **the "reverse-triangular" matrix that results from row exchanges,**

$$M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}.$$

Solution:

- $\det A = 0$.
- $\det U = 16$.
- $\det U^T = 16$.
- $\det U^{-1} = \frac{1}{16}$.
- $\det M = (-1)^2 \det U = \det U = 16$.

8: Show how rule 6($\det = 0$ if a row is zero) comes directly from rules 2 and 3.

Solution: $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} c-c & d-d \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} + \begin{vmatrix} -c & -d \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} - \begin{vmatrix} c & d \\ c & d \end{vmatrix} = 0.$

9: Suppose you do two row operations at once,going from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \textbf{ to } \begin{bmatrix} a-mc & b-md \\ c-la & d-lb \end{bmatrix}.$$

Find the determinant of the new matrix,by rule 3 or by direct calculation.

Solution:

$$\begin{aligned}
\begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix} &= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} - \begin{vmatrix} mc & md \\ c-la & d-lb \end{vmatrix} \\
&= \begin{vmatrix} c-la & d-lb \\ mc & md \end{vmatrix} - \begin{vmatrix} c-la & d-lb \\ a & b \end{vmatrix} \\
&= \left(\begin{vmatrix} c & d \\ mc & md \end{vmatrix} - \begin{vmatrix} la & lb \\ mc & md \end{vmatrix} \right) - \left(\begin{vmatrix} c & d \\ a & b \end{vmatrix} - \begin{vmatrix} la & lb \\ a & b \end{vmatrix} \right) \\
&= - \begin{vmatrix} la & lb \\ mc & md \end{vmatrix} - \begin{vmatrix} c & d \\ a & b \end{vmatrix} \\
&= -ml \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} \right) + \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\
&= (1-ml)(ad-bc).
\end{aligned}$$

10: If Q is an orthogonal matrix,so that $Q^TQ = I$,prove that $\det Q$ equals $+1$ or -1.What kind of box is formed from the rows(or columns)of Q?

Solution: $\det Q^TQ = \det Q \det Q = \det I = 1$,so $\det Q = \pm 1$.The rows or columns of Q form a n dimensional unit cube,in which n is the rank of Q.

11: Prove again that $\det Q = 1$ **or** -1 **using only the Product rule.If** $|\det Q| > 1$ **then** $\det Q^n$ **blows up.How do you know this can't happen to** Q^n**?**

Solution: Because Q^n is still an orthogonal matrix.Its determinant can't blow up because the rows of Q^n form an n dimensional unit cube.

12: Use row operations to verify that the 3 **by** 3 **"Vandermonde determinant" is**

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

Solution: When either two of a, b, c are equal,both the determinant and $(b-a)(c-a)(c-b)$ are zero.So in this case the identity holds.When neither two of a, b, c are equal,

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c^2-a^2) - \frac{b^2-a^2}{b-a}(c-a) \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}.$$

So $\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$

13:

- **A skew-symmetric matrix satisfies** $K^T = -K$**,as in**

$$K = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

  In the 3 **by** 3 **case,why is** $\det(-K) = (-1)^3 \det K$**?On the other hand** $\det K^T = \det K$**(always).Deduce that the determinant must be zero.**

- **Write down a** 4 **by** 4 **skew-symmetric matrix with** $\det K$ **not zero.**

Solution:

- $\det(-K) = (-1)^3 \det K = -\det K$.$\det(-K) = \det K^T = \det K$.So $\det K = -\det K$,so $\det K = 0$.

- $\det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \neq 0.$

14: True or false,with reason if true and counterexample if false:

- **If A and B are identical except that $b_{11} = 2a_{11}$,then $\det B = 2\det A$.**
- **The determinant is the product of the pivots.**
- **If A is invertible and B is singular,then $A + B$ is invertible.**
- **If A is invertible and B is singular,then AB is singular.**
- **The determinant of $AB - BA$ is zero.**

Solution:

- False.Counterexample:$\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0, \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1.$
- True.
- False.Counterexample:$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$
- True.
- False.
$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix},$$
then $\det(AB - BA) = \det \begin{bmatrix} 3 & -7 \\ -2 & -3 \end{bmatrix} \neq 0.$

15: If every row of A adds to zero,prove that $\det A = 0$.If every row adds to 1,prove that $\det(A - I) = 0$.Show by example that this does not imply $\det A = I$.

Solution: When every row of A adds to zero,then A is singular,so $\det A = 0$.
When every row of A adds to one,then every row of $A - I$ adds to zero,so $\det(A - I) = 0$.
$\det(A - I) = 0$ does not imply that $\det A = 1$.For example,$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$.

16: Find these 4 by 4 determinants by Gaussian elimination:

$$\det\begin{bmatrix}11&12&13&14\\21&22&23&24\\31&32&33&34\\41&42&43&44\end{bmatrix} \text{ and } \det\begin{bmatrix}1&t&t^2&t^3\\t&1&t&t^2\\t^2&t&1&t\\t^3&t^2&t&1\end{bmatrix}.$$

Solution:

$$\det\begin{bmatrix}11&12&13&14\\21&22&23&24\\31&32&33&34\\41&42&43&44\end{bmatrix} = \det\begin{bmatrix}11&12&13&14\\-1&-2&-3&-4\\-2&-4&-6&-8\\-3&-6&-9&-12\end{bmatrix} = 0.$$

$$\begin{aligned}\det\begin{bmatrix}1&t&t^2&t^3\\t&1&t&t^2\\t^2&t&1&t\\t^3&t^2&t&1\end{bmatrix} &= \det\begin{bmatrix}1&t&t^2&t^3\\0&1-t^2&t-t^3&t^2-t^4\\0&t-t^3&1-t^4&t-t^5\\0&t^2-t^4&t-t^5&1-t^6\end{bmatrix}\\ &= \det\begin{bmatrix}1&t&t^2&t^3\\0&1-t^2&1-t^3&t^2-t^4\\0&0&1-t^2&t-t^3\\0&0&t-t^3&1-t^4\end{bmatrix}\\ &= \det\begin{bmatrix}1&t&t^2&t^3\\0&1-t^2&1-t^3&t^2-t^4\\0&0&1-t^2&t-t^3\\0&0&0&1-t^2\end{bmatrix}\\ &= (1-t^2)^3.\end{aligned}$$

17: Find the determinant of

$$A=\begin{bmatrix}4&2\\1&3\end{bmatrix}, A^{-1}=\frac{1}{10}\begin{bmatrix}3&-2\\-1&4\end{bmatrix}, A-\lambda I=\begin{bmatrix}4-\lambda&2\\1&3-\lambda\end{bmatrix}.$$

For which value of λ is $A-\lambda I$ a singular matrix?

Solution: $\det A = \det\begin{bmatrix}4&2\\1&3\end{bmatrix} = \det\begin{bmatrix}4&2\\0&\frac{5}{2}\end{bmatrix} = 10.\det A^{-1} = \frac{1}{\det A} = \frac{1}{10}.$When $\det(A-\lambda I)=0$,

$\det(A-\lambda I) = \det\begin{bmatrix}4-\lambda&2\\1&3-\lambda\end{bmatrix} = -\det\begin{bmatrix}1&3-\lambda\\4-\lambda&2\end{bmatrix} = -\det\begin{bmatrix}1&3-\lambda\\0&2-(3-\lambda)(4-\lambda)\end{bmatrix} = (3-\lambda)(4-\lambda)-2=0,$

so $\lambda = 2$or 5.

18: Evaluate $\det A$ by reducing the matrix to triangular form (rules 5 and 7).

$$A=\begin{bmatrix}1&1&3\\0&4&6\\1&5&8\end{bmatrix}, B=\begin{bmatrix}1&1&3\\0&4&6\\0&0&1\end{bmatrix}, C=\begin{bmatrix}1&1&3\\0&4&6\\1&5&9\end{bmatrix}.$$

What are the determinants of $B, C, AB, A^T A$,and C^T?

Solution:

$$\det A = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} = 4.$$

$\det B = 4.\det C^T = \det C = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 6 \end{bmatrix} = 0.$

19: Suppose that $CD = -DC$,and find the flaw in the following argument:Taking determinant gives $(\det C)(\det D) = -(\det D)(\det C)$,so either $\det C = 0$ or $\det D = 0$.Thus $CD = -DC$ is only possible if C or D is singular.

Solution: $\det -DC = -(\det D)(\det C)$ is generally false.

20: Do these matrices have determinant $0, 1, 2$,or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: $\det A = 1.\det B = 2.\det C = 0.$

21: The inverse of a 2 by 2 matrix seems to have determinant $= 1$:

$$\det A^{-1} = \det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1.$$

What is wrong with this calculation?What is the correct $\det A^{-1}$?

Solution: $\det A^{-1} = \frac{1}{ad-bc}$.

22: Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Solution:

- $\det A = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$
- $\det A = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = 3.$

23: By applying row operations to produce an upper triangular U,compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \textbf{ and } \det \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Solution:

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 36.$$

$$\det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{bmatrix} = (1-t^2)^2.$$

25: Elimination reduces A to U.Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinant of $L, U, A, U^{-1}L^{-1}$,and $U^{-1}L^{-1}A$.

Solution: $\det L = 1$,$\det U = -6$,$\det A = \det L \det U = -6$.$\det U^{-1}L^{-1} = \det U^{-1} \det L^{-1} = -\frac{1}{6}$.$\det U^{-1}L^{-1}A = \det I = 1$.

26: If a_{ij} is i times j,show that $\det A = 0$.(Exception when $A = [1]$.)

Solution: Every two rows of A are linearly dependent,so $\det A = 0$.

27: If a_{ij} is $i + j$,show that $\det A = 0$.(Exception when $n = 1$ or 2).

Solution: Let the second row of A subtract the first row of A,we get a matrix A' in which all the element of the second row are 2.Let the third row of A' subtract the second row of A',we get a matrix A'' in which all the element of the third row are 3.The second and the third row of A'' are linearly dependent,so $\det A = \det A' = \det A'' = 0$.

28: Compute the determinant of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}, \textbf{ and } C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

Solution: $\det A = abc.\det B = -abcd.$

$$\det C = \det \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix} = \det \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = a(b-a)(c-b).$$

29: What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^TA)^{-1}A^T \textbf{ so } |P| = |A|\frac{1}{|A^T||A|}|A^T| = 1.$$

Solution: A^TA is not necessarily nonsingular.

30: Show that the partial derivatives of $\ln(\det A)$ give A^{-1}:

$$f(a,b,c,d) = \ln(ad-bc) \textbf{ leads to } \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = A^{-1}.$$

Solution:

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = A^{-1}.$$

31-33: Omitted.

34: If you know that $\det A = 6$,what is the determinant of B?

$$\det A = \begin{vmatrix} \textbf{row1} \\ \textbf{row2} \\ \textbf{row3} \end{vmatrix} = 6, \det B = \begin{vmatrix} \textbf{row1} + \textbf{row2} \\ \textbf{row2} + \textbf{row3} \\ \textbf{row3} + \textbf{row1} \end{vmatrix} =$$

Solution: $\det B = 0$.

35: Suppose the 4 by 4 matrix M has four equal rows all containing a, b, c, d. We know that $\det(M) = 0$. The problem is to find $\det(I + M)$ by any method:

$$\det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

Solution:

$$\begin{aligned}
\det(I+M) &= \det \begin{bmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} a & b+1 & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \\
&= -\det \begin{bmatrix} -1 & 1 & 0 & 0 \\ a & b+1 & c & d \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & a+b+1 & c & d \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & a+b+1 & c & d \\ 0 & -1 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & a+b+c+1 & d \\ 0 & 0 & -1 & 1 \end{bmatrix} \\
&= -\det \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & a+b+c+1 & d \end{bmatrix} = -\det \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & a+b+c+d+1 \end{bmatrix} \\
&= a+b+c+d+1.
\end{aligned}$$