陶哲轩实分析习题17.1.4

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设$T:\mathbf{R}^n\to\mathbf{R}^m$是线性变换，证明，存在数$M>0$,使得 对于一切$x\in\mathbf{R}^n$,$||T(x)||\leq M||x||$.

证明. 设T对应的矩阵为

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \tag{1}$$

$\forall x=(a_1,\cdots,a_n)\in\mathbf{R}^n$,

$$T(x)=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \tag{2}$$

则

$$||T(x)||^2=\sum_{k=1}^{m}(a_{k1}a_1+\cdots+a_{kn}a_n)^2 \tag{3}$$

根据柯西不等式,

$$(a_{k1}a_1+\cdots+a_{kn}a_n)^2\leq(a_{k1}^2+\cdots+a_{kn}^2)(a_1^2+\cdots+a_n^2) \tag{4}$$

因此

$$\sum_{k=1}^{m}(a_{k1}a_1+\cdots+a_{kn}a_n)^2\leq(a_1^2+\cdots+a_n^2)(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^2) \tag{5}$$

可见，令

$$M=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^2} \tag{6}$$

即可.

□