Linear algebra,Exercise 2

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Exercise. [a] *Let* $T : V \to W$ *be a linear transformation from one vector space* V *to another vector space* W*.Let* $v_1, \cdots, v_n$ *be vectors in* V*.Assume the span of* $v_1, \cdots, v_n$ *contains the null space* $N(T)$ *of* T*,and assume that the vectors* $T(v_1), \cdots, T(v_n)$ *span* W*.Prove that the vectors* $v_1, \cdots, v_n$ *span* V*.*

[a] This exercise is from www.math.ucla.edu/ tao/resource/general/115a.3.02f/practice.pdf

Proof. Prove by contradiction.Otherwise,there exists $v_{n+1} \in V$,such that $v_{n+1} \notin Span(v_1, \cdots, v_n)$.From $Span(T(v_1), \cdots, T(v_n)) = W$,we have

$$T(v_{n+1}) = a_1 T(v_1) + \cdots + a_n T(v_n),$$

so

$$T(v_{n+1} - a_1 v_1 - \cdots - a_n v_n) = 0_w,$$

which means that $v_{n+1} - a_1 v_1 - \cdots - a_n v_n \in N(T)$.From $N(T) \subset Span(v_1, \cdots, v_n)$,we have

$$v_{n+1} - a_1 v_1 - \cdots - a_n v_n = b_1 v_1 + \cdots + b_n v_n,$$

so

$$v_{n+1} = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n.$$

This contradicts $v_{n+1} \notin Span(v_1, \cdots, v_n)$. □

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