# Linear algebra,Exercise 8

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**Exercise** (Tao). *For this question,the field of scalars will be the complex numbers instead of the reals(i.e,all matrices,etc.are allowed to have complex entries).Let $\theta$ be a real number,and let $A$ be the $2\times 2$ rotational matrix*

$$A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

- *Show that $A$ has eigenvalues $e^{i\theta},e^{-i\theta}$.What are the eigenvectors corresponding to $e^{i\theta}$ and $e^{-i\theta}$?*
- *Write $A=QDQ^{-1}$ for some invertible matrix $Q$ and diagonal matrix $D$.*

*Proof.*

- $$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\lambda\begin{pmatrix}x\\ y\end{pmatrix},$$

  so

  $$\begin{pmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

  where $(x,y)\neq(0,0)$.So

  $$(\cos\theta-\lambda)^2+\sin^2\theta=0\iff\lambda^2-2\lambda\cos\theta+1=0\iff\lambda=\cos\theta\pm i\sin\theta.$$

  When $\lambda=\cos\theta+i\sin\theta$,then

  $$\begin{pmatrix}-i\sin\theta & -\sin\theta\\ \sin\theta & -i\sin\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

  So when $\theta=\pi k,k\in\mathbf{Z}$,the eigenvector corresponding to $e^{i\theta}$ is arbitary,otherwise,the eigenvector is in the form of $(iy,y),y\in\mathbf{C}\backslash\{0\}$.

  When $\lambda=\cos\theta-i\sin\theta$,then $ix\sin\theta-y\sin\theta=0$.When $\theta=\pi k,k\in\mathbf{Z}$,then the eigenvector corresponding to $e^{i\theta}$ is arbitrary.Otherwise,the eigenvector is in the form of $(x,ix),x\in\mathbf{C}\backslash\{0\}$.

- Denote the linear transformation corresponding to the matrix $A$ by $L_A$. The eigenvector of $A$,$(i,1)$ and $(1,i)$,are linearly independent in $\mathbf{C}$.Let $\alpha=((i,1),(1,i))=(w_1,w_2)$ be an ordered basis,let $\beta=((1,0),(0,1))=(v_1,v_2)$ be another ordered basis.Then

  $$[L_A]_\alpha^\alpha=\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}.$$

  And

  $$\begin{cases}w_1=iv_1+v_2,\\ w_2=v_1+iv_2.\end{cases}$$

  So

  $$[I]_\alpha^\beta=\begin{pmatrix}i & 1\\ 1 & i\end{pmatrix},[I]_\beta^\alpha=\begin{pmatrix}\frac{-i}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-i}{2}\end{pmatrix}.$$

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So

$$A = [L_A]_\beta^\beta = [I]_\alpha^\beta [L_A]_\alpha^\alpha [I]_\beta^\alpha.$$

Done.

$\square$