FOUNDATIONS FOR A GENERAL THEORY OF FUNCTIONS OF A VARIABLE COMPLEX MAGNITUDE

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1

If we consider z to be a variable magnitude which can gradually assume all possible real values,then we call w a function of z,when each of its real values corresponds to a single value of undetermined magnitude such as w_0.If w also constantly change while z continuously goes through all the value lying between two fixed values,then we call this function within these intervals a constant or a continuous function.

Obviously,this definition does not set up any absolute law between the individual values of the function,because when we assign a determinate value to this function,the way in which it continues outside of this interval remains totally arbitrary.

We can express the slope function (dependence) of magnitude $w(z)$ by a mathematical law so that we can find the corresponding value of w for every value of z through determinate numerical operations(Groessen operationen).Previously,people have only considered a certain kind of function (functiones continuae according to Euler's usage) as having the ability of being able to determine all the value of z lying between a given interval by using that same slope function law;[1]however,in the meantime,new research has shown that there are analytic expressions that can represent each and every constant function for a given interval[2].This holds,regardless of whether the slope function of magnitude w (magnitude z) is conditionally defined as an arbitrary given numerical operation,or as an determinate numerical operation.As a result of the theorems mentioned above,both concepts are congruent.

But the situation is different when we do not limit the variability of magnitude z to real values,but instead allow complex values of the form $x+yi$(where $i=\sqrt{-1}$).

Assume that $x+yi$ and $x+yi+dx+dyi$ are two infinitesimally slightly different values for magnitude z,which correspond to the values $u+vi$ and $u+vi+du+dvi$ for magnitude w.So then,if the slope function of magnitude $w(z)$ is an arbitrarily given one,then generally speaking,the ratio $\frac{du+dvi}{dx+dyi}$ changes for the values for

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[1] Sentences which I do not understand are typeset in blue.

[2] Fourier theorem.

dx, dy,because when we have $dx + dyi = \varepsilon e^{\varphi i}$,then

$$\begin{aligned} \frac{du + dv\, i}{dx + dy\, i} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) i \\ &\quad + \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) i\right] \frac{dx - dy\, i}{dx + dy\, i} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) i \\ &\quad + \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) i\right] e^{-2\varphi i} \end{aligned}$$

[3] However,regardless of the manner in which we define w as a function of z through these simple numerical operation,the value of the differential quotient $\frac{dw}{dz}$ is always independent of the special values of differential dz.Obviously,not every arbitrary slope function of complex magnitude w (complex magnitude z) can be expressed in this manner.

[3]Here we give a modern derivation of the Cauchy-Riemann equations.Let $f(x,y) = (u,v)$,where $f : \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable.So there is a linear map which maps (dx, dy) to (du, dv),i.e,

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

We also want that

$$du + dvi = (c + di)(dx + dyi) + o(dx + dyi),$$

where $c + di$ is a complex number,and

$$\lim_{dx+dyi \to 0} \frac{o(dx + dyi)}{dx + dyi} = 0.$$

So

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

So

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = c, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = d.$$

Riemann's derivation gives us more information than the modern method we give above.If the Cauchy-Riemann equations are not satisfied by $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$,then the linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$ maps a circle centering at the $(0,0)$ to an ellipse,Riemann's derivation provide us error term $\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) i\right] e^{-2\varphi i}$.