PROPERTIES OF COMPLEX DIFFERENTIATION

LUQING YE

- The uniqueness of derivative:If f is differentiable at z_0,then $f'(z_0)$ is well defined,i.e,$f'(z_0)$ is a unique complex number.

  Proof. Suppose
  $$f(x) = f(z_0) + c_1(x - z_0) + o_1(x - z_0),$$
  $$f(x) = f(z_0) + c_2(x - z_0) + o_2(x - z_0),$$
  where $c_1 \neq c_2$,and $\lim_{x\to z_0; x\neq z_0} \frac{o(x-z_0)}{x-z_0} = 0$,then
  $$0 = (c_1 - c_2)(x - z_0) + o_3(x - z_0) \iff c_1 - c_2 = 0.$$
  So $f'(z_0)$ is unique. □

- If f is differentiable at z_0 then f is continuous at z_0.

  Proof. f is differentiable at z_0 means that there exists a complex number $f'(z_0)$ such that
  $$f(x) = f(z_0) + f'(z_0)(x - z_0) + o(x - z_0),$$
  so $\lim_{x\to z_0} f(x) = f(z_0)$. □

- If f and g are differentiable at z_0,then $f + g$ and fg also are,and
  $$(f + g)'(z_0) = f'(z_0) + g'(z_0).$$

  Proof. f is differentiable at z_0,this means that
  $$f(x) = f(z_0) + f'(z_0)(x - z_0) + o_1(x - z_0),$$
  $$g(x) = g(z_0) + g'(z_0)(x - z_0) + o_2(x - z_0).$$
  So
  $$f(x) + g(x) = [f(z_0) + g(z_0)] + [f'(z_0) + g'(z_0)](x - z_0) + o_3(x - z_0).$$
  $$f(x)g(x) = f(z_0)g(z_0) + [f(z_0)g'(z_0) + g(z_0)f'(z_0)](x - z_0) + o_4(x - z_0).$$
  □

- If $g(z_0) \neq 0$,and g is differentiable at z_0,then $1/g$ is differentiable at z_0,and
  $$\left(\frac{1}{g}\right)'(z_0) = \frac{-g'(z_0)}{g(z_0)^2}.$$

Proof.

$$\begin{aligned}\frac{1}{g(x)}-\frac{1}{g(z_0)} &= \frac{g(z_0)-g(x)}{g(x)g(z_0)}\\ &= \frac{-g'(z_0)(x-z_0)-o(x-z_0)}{g(x)g(z_0)}.\end{aligned}$$

Done. ☐

- If f is differentiable at z_0 and g is differentiable at $f(z_0)$,then the composite function $g\circ f$ is differentiable at z_0 and

$$(g\circ f)'(z_0)=g'(f(z_0))f'(z_0).$$

Proof. g is differentiable at $f(z_0)$,this means that

$$g(f(x))=g(f(z_0))+g'(f(z_0))(f(x)-f(z_0))+o(f(x)-f(z_0)).$$

f is differentiable at z_0,so

$$f(x)-f(z_0)=f'(z_0)(x-z_0)+o(x-z_0),$$

So

$$\begin{aligned}g(f(x)) &= g(f(z_0))+g'(f(z_0))(f'(z_0)(x-z_0)+o(x-z_0))+o(f(x)-f(z_0))\\ &= g(f(z_0))+g'(f(z_0))f'(z_0)(x-z_0)+o(x-z_0)g'(f(z_0))+o(f(x)-f(z_0)).\end{aligned}$$

Done. ☐

College of Science, Hangzhou Normal University,Hangzhou City,Zhejiang Province,China

E-mail address: yeluqingmathematics@gmail.com