

# A geometric proof of Darboux's theorem

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**Abstract.** By using Lagrange's mean value theorem and the intermediate value theorem, we provide a geometric proof of Darboux's theorem.

In this note we give a geometric proof of Darboux's theorem, with the aid of Lagrange's mean value theorem and the intermediate value theorem. After I completed the proof, I found that the proof is essentially not new, Lars Olsen had already provided an extremely similar proof in [1]. However, I decide to publish my proof because its geometric feature, and after all, there are slight differences between our approach.

Darboux's theorem, which shows the intermediate property of derivatives, can be stated as below.

**Theorem (Darboux's theorem).** *Let  $I$  be an open interval,  $a, b \in I$  and  $a < b$ . Let  $f : I \rightarrow \mathbf{R}$  be a differentiable function, and  $f'(a) < f'(b)$ . Then  $\forall t \in (f'(a), f'(b))$ , there exists  $\xi \in (a, b)$ , such that  $f'(\xi) = t$ .*

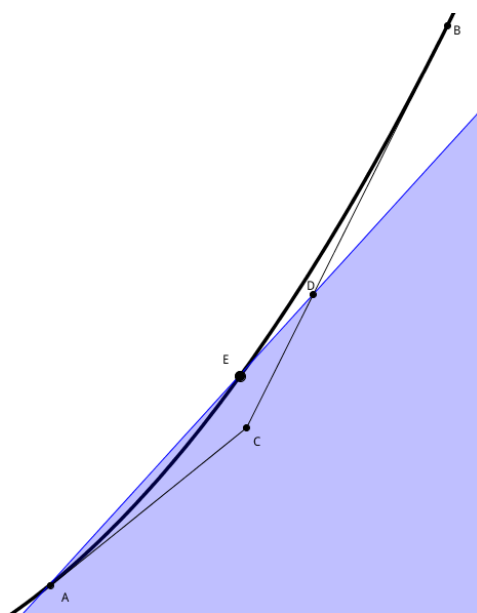


Figure 1.

*Proof.* As shown in figure (1), there are five points  $A(a, f(a))$ ,  $B(b, f(b))$ ,  $C(x_c, y_c)$ ,  $D(x_d, y_d)$ ,  $E(x_e, y_e)$  in the figure. The curve represents the graph of the function  $f$  over  $(a, b)$ . Line  $AC$  is the tangent line of the curve at the point  $A$ . Line  $BC$  is the tangent line of the curve at the point  $B$ . For an arbitrary point  $D$  on the segment  $BC$  (excluding points  $B, C$ ), line  $AD$  must intersect with the curve at a point  $E$  over

the interval  $(a, b)$ . This is because, the slope of the line  $AD$  is larger than the slope of the line  $AC$  (which is  $f'(a)$ ), so the curve passing through the point  $A$  must intersect with the shaded region (a closed half plane whose boundary is line  $AD$ ) over the interval  $(a, b)$ , but this continuous curve also passes through the point  $B$  which is not in the shaded region. So according to the intermediate value theorem, the curve must intersect with the line  $AD$  at a point  $E$  over the interval  $(a, b)$ . According to Lagrange's mean value theorem, there exists  $\xi_1 \in (a, x_e)$  such that

$$f'(\xi_1) = \text{the slope of the line } AD. \quad (1)$$

Similarly, as shown in figure (2), for an arbitrary point  $D'$  on the segment  $AC$  (excluding points  $A, C$ ), line  $BD'$  must intersect with the curve at a point  $E'(x'_e, y'_e)$  over the interval  $(a, b)$ . This is because, the slope of the line  $BD'$  is smaller than the slope of the line  $BC$  (which is  $f'(b)$ ), so the curve passing through the point  $B$  must intersect with the shaded plane (a closed half plane whose boundary is line  $BD'$ ) over the interval  $(a, b)$ , but this continuous curve also passes through the point  $A$  which is not in the shaded plane. So according to the intermediate value theorem, the curve must intersect with the line  $BD'$  at a point  $E'$  over the interval  $(a, b)$ . According to Lagrange's mean value theorem, there exists  $\xi_2 \in (x'_e, b)$  such that

$$f'(\xi_2) = \text{the slope of the line } BD'. \quad (2)$$

Also, according to Lagrange's mean value theorem, there exists  $\xi_3 \in (a, b)$  such that

$$f'(\xi_3) = \text{the slope of the line } AB. \quad (3)$$

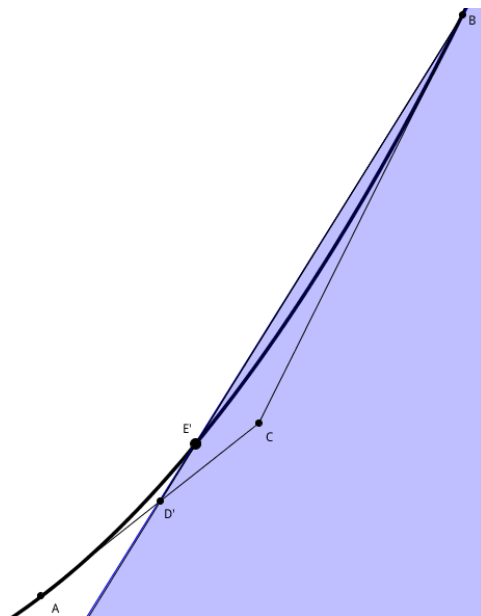


Figure 2.

It can be easily verified from elementary geometry that when  $D$  and  $D'$  move continuously over the segment  $BC$  and  $AC$  respectively, with the extreme points of both

segments excluded, we have

$$\begin{aligned} & \{\text{the slope of the line } AD\} \cup \{\text{the slope of the line } BD'\} \cup \{\text{the slope of the line } AB\} \\ &= (\text{the slope of the line } AC, \text{the slope of the line } BC) = (f'(a), f'(b)). \end{aligned}$$

So Darboux's theorem is finally proved by combining this relationship with equation (1), (2) and (3). ■

#### REFERENCES

1. Lars Olsen, A New Proof of Darboux's Theorem, *Amer. Math. Monthly*, **111** (2004) 713–715.

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