A GEOMETRIC PROOF OF DARBOUX'S THEOREM

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ABSTRACT. By using Lagrange's mean value theorem and the intermediate value theorem,we provide a geometric proof of Darboux's theorem.

In this note we give a geometric proof of Darboux's theorem,with the aid of Lagrange's mean value theorem and the intermediate value theorem.After I completed the proof,I found that the proof is essentially not new,Lars Olsen had already provided a very similar proof in [1].However,I decided to publish my proof because its geometric feature,and after all,there are slight differences between our approaches.

Darboux's theorem,which shows the intermediate property of derivatives,can be stated as below.

Theorem 1 (Darboux's theorem)**.** *Let I be an open interval.$a, b \in I$ and $a < b$.Let $f : I \to \mathbf{R}$ be a differentiable function,and $f'(a) < f'(b)$. Then $\forall t \in (f'(a), f'(b))$, there exists $\xi \in (a, b)$,such that $f'(\xi) = t$.*

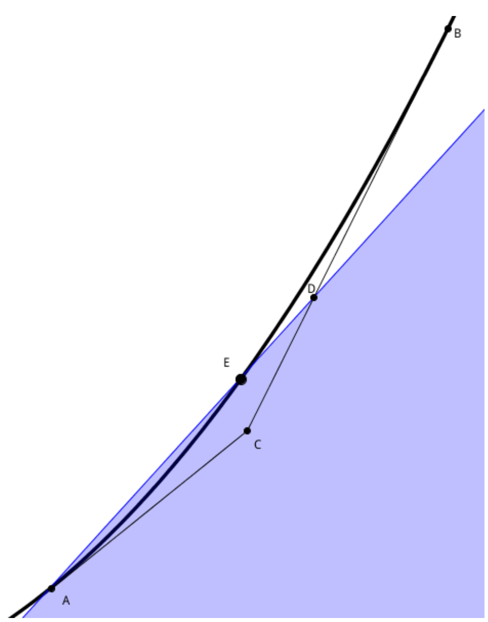

FIGURE 1

Proof. As shown in figure (1),there are five points $A(a, f(a))$,$B(b, f(b))$,$C(x_c, y_c)$, $D(x_d, y_d)$,$E(x_e, y_e)$ in the figure.The curve represents the graph of the function f over (a, b).Line AC is the tangent line of the curve at the point A.Line BC is the tangent line of the curve at the point B. For an arbitrary point D on the segment BC(excluding points B, C),line AD must intersect with the curve at a point E over the interval (a, b).This is because,the slope of the line AD is larger than the slope of the line AC(which is $f'(a)$),so the curve passing through the point A must intersect with the shaded region(a closed half plane whose boundary is line AD) over the interval (a, b),but this continuous curve also passes through the point B which is not in the shaded region.So according to the intermediate value theorem,the curve must intersect with the line AD at a point E over the interval (a, b).According to Lagrange's mean value theorem,there exists $\xi_1 \in (a, x_e)$ such that

$$f'(\xi_1) = \text{the slope of the line } AD. \tag{1}$$

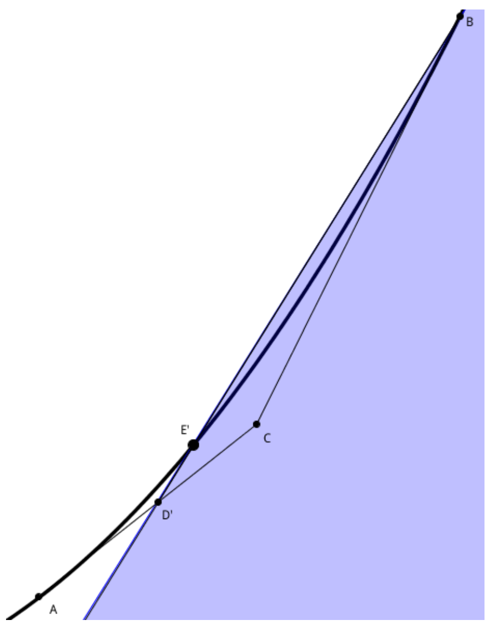

FIGURE 2

Similary,as shown in figure (2),for an arbitrary point D' on the segment AC(excluding points A, C),line BD' must intersect with the curve at a point $E'(x'_e, y'_e)$ over the interval (a, b).This is because,the slope of the line BD' is smaller than the slope of the line BC(which is $f'(b)$),so the curve passing through the point B must intersect with the shaded plane(a closed half plane whose boundary is line BD')over the interval (a, b),but this continous curve also passes through the point A which is not in the shaded plane.So according to the intermediate value theorem,the curve must intersect with the line BD' at a point E' over the interval (a, b).According to Lagrange's mean value theorem,there exists $\xi_2 \in (x'_e, b)$ such that

$$f'(\xi_2) = \text{the slope of the line } BD'. \tag{2}$$

Also,according to Lagrange's mean value theorem,there exists $\xi_3 \in (a,b)$ such that

$$f'(\xi_3) = \text{the slope of the line } AB. \tag{3}$$

It can be easily verified from elementary geometry that when D and D' move continuously over the segment BC and AC respectively,with the extreme points of both segments excluded,we have

$$\{\text{the slope of the line } AD\} \cup \{\text{the slope of the line } BD'\} \cup \{\text{the slope of the line } AB\}$$
$$= (\text{the slope of the line } AC, \text{the slope of the line } BC) = (f'(a), f'(b)).$$

So Darboux's theorem is finally proved by combining this relationship with equation (1), (2) and (3). □

References

[1] Lars Olsen , A New Proof of Darboux's Theorem ,*Amer. Math. Monthly*,**111** (2004)713–715.

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