HISTORIA MATHEMATICA (1978), 437-442

A HISTORY OF THE DIVERGENCE THEOREM

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SUMMARIES

This paper traces the development of the divergence theorem in three dimensions from 1813 to 1901, in its Cartesian coordinate form (1813-1875) by George Green, Carl F. Gauss and M. V. Ostrogradskii and then in its vector form (1880-1901) by Oliver Heaviside and Josiah W. Gibbs.

Cette étude trace le développement du théorème de divergence en trois dimensions de 1813 à 1901, dans sa frome cartésienne (1813-1875) par George Green, Carl F. Gauss et Michel Ostrogradskii, et ensuite dans sa forme vectorielle (1880-1901) par Olivier Heaviside et Josiah W. Gibbs.

THE CARTESIAN COORDINATE FORM

In its Cartesian coordinate form the divergence theorem states that if T is a region in 3-dimensional Euclidean space bounded by a closed surface S, and if P, Q and R are continuously differentiable functions, then

$$\iiint_T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma)\, dA, \tag{1}$$

where α, β and γ are the direction angles of the positively oriented normal to the surface S. The development of this result has been associated with the work of three men.

George Green (1793-1841) was a self-educated mathematician whose contributions to mathematical physics remained relatively unknown until some time after his death in 1841. His work was read and highly regarded by such men as Sir William Thomson (Lord Kelvin) and George Gabriel Stokes, two prominent mathematical physicists of the 19th century. In fact, Thomson is credited with "the re-discovery" of Green's work in 1846 [Boyer 1968, 583] and Stokes is quoted as saying that Green's works were very "remarkable both for the elegance and rigour of the analysis

0315-0860/78/0054-0437$02.00/0

and for the ease with which he arrives at most important results" [Stokes 1905, xii].

Every reference to Green concerning his role in the development of the divergence theorem leads to a paper which he published privately at Nottingham in 1828. That work was entitled "An Essay on the Application of Mathematical Analysis to the Theories of Elasticity and Magnetism" [Green 1970, 23]. A survey of Green's paper shows that he did prove a number of integral theorems, now referred to as Green's identities, relating volume integrals to surface integrals. Green's first identity states that

$$(2)\qquad \iiint_T \left[u\,\Delta v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right] dV = \iint_S u\,\frac{\partial v}{\partial n}\,dA,$$

where $\Delta v = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}$ is the Laplacian of v. In the surface integral, $\dfrac{\partial}{\partial n}$ refers to the directional derivative in the direction of n, the outer normal to S. If we assume that T is a region to which formula (1) applies, and that u and v together with the first derivatives of u and the first and second derivatives of v are continuous in T, then we may prove (2) as a consequence of (1). To see this, let

$$P = u\,\frac{\partial v}{\partial x}, \quad Q = u\,\frac{\partial v}{\partial y}, \text{ and } R = u\,\frac{\partial v}{\partial z}.$$

Then $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$ is the expression under the triple integral in (2), and

$$P\cos\alpha + Q\cos\beta + R\cos\gamma = u\,\frac{\partial v}{\partial n}.$$

Thus Green's first identity is a particular case of formula (1). However, Green neither stated nor proved (1), and his argument proving (2) was different from the type of argument used in contemporary proofs of the divergence theorem. This suggests that when he published his paper in 1828, Green was unaware of the form of the divergence theorem as it is known today. It would appear that if the theorem is to be credited to Green, more convincing evidence is required. Such evidence, however, is apparently non-existent.

Evidence establishing the contribution of Carl Friedrich Gauss to the early development of the divergence theorem is found in a paper written by him in 1813 [Gauss 1877, 5-7]. That paper was devoted to the study of the forces of attraction and repulsion between bodies in space. Early in that work, Gauss proved a number of results relating volume integrals to surface integrals. As in Green's case, Gauss did not state the divergence theorem as such. But if one reads a proof of the theorem as given by James Clerk Maxwell in 1873 [Maxwell 1904, 21], one sees immediately that this proof is clearly an extension of the ideas presented by Gauss in his work of 1813. Thus, although the statement of the Cartesian coordinate form of the divergence theorem did not actually appear in Gauss's work, the major ideas surrounding its proof did.

The third name associated with the divergence theorem is Michael V. Ostrogradskii (1801-1861). In 1820 Ostrogradskii completed his studies at the University of Kharkov with honors, but for political reasons he was deprived of a diploma. He moved to France and lived in Paris from 1822 to 1827 where he continued to advance his knowledge of the mathematical sciences by attending lectures at the Collège de France and the University of Paris. He established personal contact with many of the great French mathematicians such as Laplace, Legendre, Fourier, Poisson and Cauchy. He considered Cauchy to be one of his greatest teachers.

James Clerk Maxwell in his treatise on electricity and magnetism credits the divergence theorem to Ostrogradskii, referring the reader to [Ostrogradskii 1831a]. Examination of a copy of that paper shows that the divergence theorem was not in it. In the period 1826-1831 Ostrogradskii wrote a number of papers. One which did contain the divergence theorem was presented to the St. Petersburg Academy in November 1828, and was published in 1831. It was entitled "Note sur la théorie de la chaleur" [Ostrogradskii 1831b]. The statement and proof of formula (1) is presented in the first few pages of this paper.

For our purposes the above paper is significant for two reasons: first, it contains the Cartesian coordinate form (1) of the divergence theorem; second, the method of proof is essentially the same as that given by Maxwell in his work of 1873. This suggests that perhaps Maxwell first found the result in Ostrogradskii's work. However, the footnote in Maxwell's text which credits the theorem to Ostrogradskii was apparently omitted from the first edition [Crowe 1967, 146]. Since later editions were published posthumously, one cannot say with certainty that Maxwell learned of the theorem from Ostrogradskii. However, in light of the evidence presented thus far the following seems probable: the first person to state and prove the divergence theorem as it appears in formula (1) was Ostrogradskii. His method of proof was quite similar to an approach used by Gauss in his work of 1813.

Further evidence establishing Ostrogradskii as the author of formula (1) was furnished by A. P. Yushkevich [1965]. A summary of parts of Yushkevich's paper pertaining to the development of the divergence theorem is given below. Earlier, in 1957, V. I. Antropova published a paper which analyzed the extensive literature on the question of the authorship of (1). She gave credit to Gauss and Green, but quite justly stressed the priority of Ostrogradskii in the discovery itself. Antropova was also aware that the problem of identifying the authorship of (1) was made more difficult by the fact that it is found in several other forms. For example, the formula

$$\int X\,dm = \int (\gamma P_1 + \beta P_2 + \alpha P_3)\,ds,$$

where

$$X = \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} + \frac{\partial P_3}{\partial z},$$

is arrived at in a memoir read by Poisson to the Paris Academy of Sciences on April 14, 1828 and published in 1829. So, regardless of Ostrogradskii's important application and generalization of formula (1), it appears that it was presented by Poisson a few months before Ostrogradskii's paper [1831b].

In 1963 Yushkevich, working in Paris with A. T. Grigoryan, found in the archives of the Academy of Sciences of the French Institute several unpublished articles of Ostrogradskii which were written in his own handwriting (in French) and presented with his signature to the Academy in 1824-1827. Two of these memoirs include the divergence theorem. The first was entitled "Demonstration d'un théoreme du calcul intégral" and was presented to the Academy on February 13th, 1826. It contained the formulation and proof of the theorem essentially as in Ostrogradskii's later note [1831b]. The second, entitled "Mémoire sur la propagation de la chaleur dans l'intérieur des corps solides", was presented to the Academy in 1827 and gave another formulation of the theorem without proof.

The Paris works of Ostrogradskii, which were found and analyzed by Yushkevich, shed important light on the question of the discovery of formula (1). Yushkevich shows that the formula is presented and proved in the third and fourth pages of the memoir, "Demonstration d'un théoreme du calcul intégral", presented to the Academy on February 13th, 1826. In that paper Ostrogradskii analyzes a triple integral

$$\int \left(a\frac{\partial p}{\partial x} + b\frac{\partial q}{\partial y} + c\frac{\partial r}{\partial z}\right)\omega$$

in space bounded by a closed surface $f(x,y,z) = 0$. Here p, q and r are functions of x, y and z, a, b, and c are constants and ω represents the differential volume element. This integral is divided into three parts and each of the integrals

$$\int a \frac{\partial p}{\partial x} \omega, \quad \int b \frac{\partial q}{\partial y} \omega, \quad \int c \frac{\partial r}{\partial z} \omega$$

is transformed into a double integral on the surface. The proof, which does not differ from the one found in [Ostrogradskii 1831b], contains the formula

$$\int \left(a \frac{\partial p}{\partial x} + b \frac{\partial q}{\partial y} + c \frac{\partial r}{\partial z}\right) \omega =$$
$$\int (a\ P \cos \alpha + b\ Q \cos \beta + c\ R \cos \gamma)\ \varepsilon\ ,$$

where ε is the differential element of area on the surface and P, Q and R are the values of the functions p, q and r on the surface.

Thus Yushkevich shows that the integral formula of Ostrogradskii was proven in a work officially presented to the Academy of Sciences in Paris on February 13, 1826. That is more than two years before the memoir of Poisson which was presented on April 14, 1828. It was also two years before Green's work of 1828.

THE VECTOR FORM

In its vector form, formula (1) is replaced by

$$\text{(3)} \qquad \iiint_T \nabla \cdot F\ dV = \iint_S F \cdot da\ ,$$

where $\nabla \cdot F$ is the divergence of the vector field F and da is the vector representing the element of surface area.

The development of the vector calculus is due primarily to the independent efforts of two men, Oliver Heaviside (1850-1925) and Josiah W. Gibbs (1839-1903). In 1893 Heaviside published his famous work on electromagnetic theory [Heaviside 1950]. In it he devoted a chapter to vector algebra and analysis. This chapter has been described as the first "extensive published treatment of modern vector analysis" [Crowe 1967, 169]. It also seems to be the first explicit reference to the divergence theorem as such. The term divergence had been used earlier, but apparently never in direct reference to the theorem. The vector analytic proof of the divergence theorem as presented by Heaviside and also by Gibbs [Wilson 1901, 186] is quite different from the type of argument used in the Cartesian coordinate version of the same theorem. Their proofs appeal directly to physical intuition and the use of vectors to represent physical quantities.

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ACKNOWLEDGEMENT

The author would like to thank A. P. Yushkevich for his helpful comments and suggestions in the writing of this paper and for supplying a copy of his paper on the early unpublished works of Ostrogradskii. He is also grateful to Professor Ida Pizem of Manhattan College for her translation of parts of Professor Yushkevich's work.