例20.2.3

叶卢庆
杭州师范大学理学院,学 号:1002011005
Email:h5411167@gmail.com
2013. 12. 7

例 (20.2.3)**.** 计算三重积分

$$I=\iiint_V (x^2+y^2+z^2)dxdydz,$$

V 是椭球面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

的内部区域.

解. 不妨设 $a,b,c>0$.还有注意,为书写简便 期间,以下积分中,任意常数 C 都不写.我们来求

$$\int_{-c}^{c}\int_{-a\sqrt{1-\frac{z^2}{c^2}}}^{a\sqrt{1-\frac{z^2}{c^2}}}\int_{-b\sqrt{1-\frac{z^2}{c^2}-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{z^2}{c^2}-\frac{x^2}{a^2}}}(x^2+y^2+z^2)dxdydz.$$

把这个求出来,我们的目的就达到了.令 $x=au,y=bv,z=cw$,则 Jacobi 行列 式为

$$\begin{vmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc.$$

我们只用求

$$abc\int_{-1}^{1}\int_{-\sqrt{1-w^2}}^{\sqrt{1-w^2}}\int_{-\sqrt{1-w^2-v^2}}^{\sqrt{1+w^2+v^2}}(a^2u^2+b^2v^2+c^2w^2)dudvdw.$$

进行球坐标变换,令$u = \rho\sin\phi\cos\theta$,$v = \rho\sin\phi\sin\theta$,$w = \rho\cos\phi$.则得到 Jacobi 行列式

$$\begin{vmatrix} \frac{\partial u}{\partial \rho} & \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial \rho} & \frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \theta} \\ \frac{\partial w}{\partial \rho} & \frac{\partial w}{\partial \phi} & \frac{\partial w}{\partial \theta} \end{vmatrix} = \rho^2\sin\phi.$$

因此化为

$$abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^2\rho^2\sin^2\phi\cos^2\theta + b^2\rho^2\sin^2\phi\sin^2\theta + c^2\rho^2\cos^2\phi)\rho^2\sin\phi d\rho d\phi d\theta.$$

我们知道,

$$\int_0^1 (a^2\rho^2\sin^2\phi\cos^2\theta + b^2\rho^2\sin^2\phi\sin^2\theta + c^2\rho^2\cos^2\phi)\rho^2\sin\phi d\rho$$
$$= \frac{1}{5}(a^2\sin^2\phi\cos^2\theta + b^2\sin^2\phi\sin^2\theta + c^2\cos^2\phi)\sin\phi$$

我们知道,

$$\sin^3\phi = \frac{1}{4}(3\sin\phi - \sin 3\phi)$$
$$\cos^2\phi\sin\phi = \frac{1}{4}(\sin\phi + \sin 3\phi).$$

因此,

$$\frac{1}{5}(a^2\sin^2\phi\cos^2\theta + b^2\sin^2\phi\sin^2\theta + c^2\cos^2\phi)\sin\phi$$
$$= \frac{1}{20}a^2\cos^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}b^2\sin^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}c^2(\sin\phi + \sin 3\phi).$$

因此

$$\int \frac{1}{20}a^2\cos^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}b^2\sin^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}c^2(\sin\phi + \sin 3\phi)d\phi$$
$$= \frac{1}{20}a^2\cos^2\theta(-3\cos\phi + \frac{1}{3}\cos 3\phi) + \frac{1}{20}b^2\sin^2\theta(-3\cos\phi + \frac{1}{3}\cos 3\phi)$$
$$+ \frac{1}{20}(-\cos\phi - \frac{1}{3}\cos 3\phi)c^2.$$

因此

$$\int_0^{\pi} \frac{1}{20}a^2\cos^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}b^2\sin^2\theta(3\sin\phi - \sin 3\phi) + \frac{1}{20}c^2(\sin\phi + \sin 3\phi)d\phi$$
$$= \frac{4}{15}a^2\cos^2\theta + \frac{4}{15}b^2\sin^2\theta + \frac{2}{15}c^2.$$

我们知道,

$$\int \frac{4}{15}a^2\cos^2\theta + \frac{4}{15}b^2\sin^2\theta + \frac{2}{15}c^2 d\theta = \frac{2}{15}a^2(\theta + \sin\theta\cos\theta) + \frac{2}{15}b^2(\theta - \sin\theta\cos\theta) + \frac{2}{15}c^2\theta.$$

因此

$$\int_0^{2\pi} \frac{4}{15}a^2\cos^2\theta + \frac{4}{15}b^2\sin^2\theta + \frac{2}{15}c^2 d\theta = \frac{4\pi}{15}a^2 + \frac{4\pi}{15}b^2 + \frac{4\pi}{15}c^2.$$

因此最后的答案为

$$abc(\frac{4\pi}{15}a^2 + \frac{4\pi}{15}b^2 + \frac{4\pi}{15}c^2).$$

□