EXERCISE2.8.1

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Exercise 1. *Let the function f be holomorphic in the open disk D.Prove that each of the following conditions forces f to be constant.*

- *$f' = 0$ through out D.*

  Proof. Let $f(x+yi) = u + vi$.Then we have
  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$
  through out D.So both u and v will be constant,so f is constant. □

- *f is real valued in D.*

  Proof. Let $f(x+yi) = u + 0i$,then
  $$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0,$$
  through out D.so u is a constant. □

- *$|f|$ is a constant in D.*

  Proof. Let $f(x+yi) = u + vi$.$|f|$ is a constant,so
  $$g(x,y) = u(x,y)^2 + v(x,y)^2$$
  is a constant c.So
  $$\frac{\partial g}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0.$$
  $$\frac{\partial g}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0.$$
  $$\begin{vmatrix} u & -v \\ v & u \end{vmatrix} = c,$$
  when $c = 0$,then $u = v = 0$,so f is a constant.When $c \neq 0$,then
  $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$
  through out D,so u is a constant,so v is also a constant. □

- *$\arg f$ is constant in D.*

Proof. Let $f(x+yi) = u+vi$,we have $\lambda_1 v + \lambda_2 u = 0$,where $\lambda_1, \lambda_2 \in \mathbf{R}$ are constants,and $\lambda_1^2 + \lambda_2^2 > 0$.Differentiating both sides of the equation,we have

$$\lambda_1 \frac{\partial v}{\partial x} + \lambda_2 \frac{\partial u}{\partial x} = 0,$$

so

$$-\lambda_1 \frac{\partial u}{\partial y} + \lambda_2 \frac{\partial u}{\partial x} = 0.$$

Similarly,we have

$$\lambda_1 \frac{\partial u}{\partial x} + \lambda_2 \frac{\partial u}{\partial y} = 0.$$

So

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0,$$

so f is constant. □

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