EXERCISE 2.6.2

LUQING YE

Exercise 1 (2.6.2)**.** *Prove that the function $f(z) = \sqrt{|xy|}$ is not differentiable at the origin,even though it satisfies the Cauchy-Riemann equations here.*

Proof. For a complex number $z = x + yi$,

$$f(x+yi) = \sqrt{|xy|}.$$

Define $g(x,y) = (\sqrt{|xy|}, 0)$.When $xy > 0$,$g(x,y) = (\sqrt{xy}, 0)$,when $xy < 0$,$g(x,y) = (\sqrt{-xy}, 0)$,when $xy = 0$,$g(x,y) = (0,0)$.We now prove that g is not differentiable at the origin.Suppose that g is differentiable at the origin,then $g'(\mathbf{0}) = \mathbf{0}$,this is because when $x = 0$ or $y = 0$,$g(x,y) = 0$.But,when $y = kx$,where $k > 0$ is a constant,then $g(x,y) = g(x,kx) = (\sqrt{k}|x|, 0)$,so when $x \geq 0$,$g(x,kx) = (\sqrt{k}x, 0)$,so at this time,

$$\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \sqrt{k} & \frac{1}{\sqrt{k}} \\ 0 & 0 \end{pmatrix}.$$

Which is a contradiction.So g is not differentiable at the origin.So f is not differentiable. □

College of Science, Hangzhou Normal University,Hangzhou City,Zhejiang Province,China

E-mail address: yeluqingmathematics@gmail.com