习题2.5.2

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习题 (2.5.2). 证明方程

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1}$$

有形如 $u = u(\phi(x,y))$ 的积分因子的充要条件是

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial \phi}{\partial x} - P\frac{\partial \phi}{\partial y}} = f(\phi(x,y)),$$

并写出这个积分因子.

证明. 由题目条件可知,

$$u(\phi(x,y))P(x,y)dx + u(\phi(x,y))Q(x,y)dy = 0 \tag{2}$$

是一个恰当微分方程,当且仅当

$$\frac{\partial[u(\phi(x,y))P(x,y)]}{\partial y} = \frac{\partial[u(\phi(x,y))Q(x,y)]}{\partial x}.$$

即

$$\frac{\partial u(\phi(x,y))}{\partial y}P(x,y) + u(\phi(x,y))\frac{\partial P}{\partial y} = \frac{\partial u(\phi(x,y))}{\partial x}Q(x,y) + u(\phi(x,y))\frac{\partial Q}{\partial x}.$$

根据链法则,即

$$\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y}P(x,y) + u(\phi(x,y))\frac{\partial P}{\partial y} = \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x}Q(x,y) + u(\phi(x,y))\frac{\partial Q}{\partial x}.$$

整理一下,可得

$$u(\phi(x,y))\frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{Q(x,y)\frac{\partial \phi}{\partial x}-P(x,y)\frac{\partial \phi}{\partial y}}=\frac{\partial u}{\partial \phi}.$$

$\square$