Theorem7.10:Ascoli-Arzela Theorem

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2013. 11. 25

Theorem (Ascoli-Arzela). *An infinite set S of functions uniformly bounded and equicontinuous in $[\alpha, \beta]$ contains a sequence which converges uniformly in $[\alpha, \beta]$.*

Proof. For a given $x_0 \in [\alpha, \beta]$,the set

$$F_0 = \{f_1(x_0), f_2(x_0), \cdots, f_n(x_0), \cdots\}$$

is bounded and infinite.And we make a completion of the set F_0 so as to turn F_0 to a complete set F_0',which means that any limiting point of F_0' is still in F_0'.According to the accumulating point theorem,F_0' has at least one limiting point which is still in F_0'.Let a_0 be a limiting point of F_0'.There are infinitely many curves of functions that passes through any given neighborhood of (x_0, a_0),i.e,for any given neighborhood P_0 of (x_0, a_0),the set

$$P_0 \cap \{x_0\} \times F_0$$

is an infinite set.Now we consider the interval series $[x_0, x_1)$,(x_2, x_3),(x_4, x_5),$\cdots$,(x_n, x_{n+1}),$\cdots$, where

$$[x_0, x_1) \cup (x_2, x_3) \cup (x_4, x_5) \cdots \cup (x_n, x_{n+1}) \cup \cdots = [x_0, \beta].$$

And $[x_0, x_1) \cap (x_2, x_3) \neq \emptyset$,$(x_4, x_5) \cap (x_6, x_7) \neq \emptyset$,$\cdots$,$(x_n, x_{n+1}) \cap (x_{n+2}, x_{n+3}) \neq \emptyset$,$\cdots$.For any given positive real number δ,there exist a positive real number ε,such that

$$x_1 - x_0 = x_3 - x_2 = x_5 - x_4 = \cdots = x_{n+1} - x_n = \cdots = \varepsilon,$$

and that $\forall x, y \in [x_0, x_1)$ or $\forall x, y \in (x_n, x_{n+1})$,and for all $n \in \mathbf{N}^+$,we have

$$|f_n(x) - f_n(y)| \leq \delta.$$

Done. □