Corollary7.6

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Corollary. *Let $u(x), p(x), q(x)$ be nonnegative continuous functions in the interval $|x-x_0| \le a$ and*

$$u(x) \le p(x) + \left| \int_{x_0}^{x} q(t)u(t)dt \right| for|x-x_0| \le a. \tag{1}$$

And $p(x) = c_0 + c_1|x-x_0|$,$q(x) = c_2$,where c_0, c_1, c_2 are non-negative constants, Then the following inequality holds:

$$u(x) \le \left(c_0 + \frac{c_1}{c_2}\right) \exp(c_2|x-x_0|) - \frac{c_1}{c_2}. \tag{2}$$

Proof. By symmetry,assume that $x \ge x_0$.And from Theorem7.3,we know that

$$u(x) \le p(x) + \int_{x_0}^{x} p(t)q(t) \exp\left(\int_t^x q(s)ds\right) dt \tag{3}$$

We know that

$$\int_{x_0}^{x} e^{c_2(x-t)}dt = -\frac{1}{c_2}e^0 + \frac{1}{c_2}e^{c_2(x-x_0)}.$$

Now we calculate

$$\int te^{c_2(x-t)}dt = \int t\frac{e^{c_2x}}{e^{c_2t}}dt = e^{c_2x}\int te^{-c_2t}dt = e^{c_2x}(-\frac{1}{c_2^2}e^{-c_2t}(c_2t+1)) + Ce^{c_2x}$$

So

$$\begin{aligned}\int_{x_0}^{x} te^{c_2(x-t)}dt &= e^{c_2x}(-\frac{1}{c_2^2}e^{-c_2x}(c_2x+1)) - e^{c_2x}(-\frac{1}{c_2^2}e^{-c_2x_0}(c_2x_0+1)) \\ &= \frac{-1}{c_2^2}(c_2x+1) + e^{c_2x}(\frac{1}{c_2^2}e^{-c_2x_0}(c_2x_0+1)).\end{aligned}$$

So (3) is equivalent to

$$
\begin{aligned}
u(x) &\leq c_0 + c_1(x - x_0) + \int_{x_0}^{x} (c_0 + c_1(t - x_0)) c_2 \exp\left(\int_t^x c_2 ds\right) dt \\
&= c_0 + c_1(x - x_0) + \int_{x_0}^{x} c_2(c_0 + c_1(t - x_0)) e^{(x-t)c_2} dt \\
&= c_0 + c_1(x - x_0) + c_0 c_2 \int_{x_0}^{x} e^{c_2(x-t)} dt + c_1 c_2 \int_{x_0}^{x} (t - x_0) e^{c_2(x-t)} dt \\
&= c_0 + c_1(x - x_0) + c_0 c_2 (-\frac{1}{c_2} e^0 + \frac{1}{c_2} e^{c_2(x-x_0)}) - c_1 c_2 x_0 (-\frac{1}{c_2} + \frac{1}{c_2} e^{c_2(x-x_0)}) \\
&+ c_1 c_2 (\frac{-1}{c_2^2}(c_2 x + 1) + e^{c_2 x}(\frac{1}{c_2^2} e^{-c_2 x_0}(c_2 x_0 + 1))) \\
&= e^{c_2(x-x_0)}(c_0 + \frac{c_1}{c_2}) - \frac{c_1}{c_2}.
\end{aligned}
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Done. $\square$