Theorem7.3

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Theorem (7.3)**.** *Let $u(x), p(x), q(x)$ be nonnegative continuous functions in the interval $|x - x_0| \leq a$ and*

$$u(x) \leq p(x) + \left| \int_{x_0}^{x} q(t)u(t)dt \right| for |x - x_0| \leq a. \tag{1}$$

Then the following inequality holds:

$$u(x) \leq p(x) + \left| \int_{x_0}^{x} p(t)q(t) \exp\left(\left| \int_{t}^{x} q(s)ds \right| \right) dt \right| for |x - x_0| \leq a. \tag{2}$$

Proof. According to symmetry,let us assume that $x \geq x_0$.Then (1) turned to

$$u(x) \leq p(x) + \int_{x_0}^{x} q(t)u(t)dt \; for \; 0 \leq x - x_0 \leq a. \tag{3}$$

So there exists a non-negative continuous function $g(x)$,such that

$$u(x) + g(x) - p(x) = \int_{x_0}^{x} q(t)u(t)dt \; for \; 0 \leq x - x_0 \leq a. \tag{4}$$

Let $H(x) = u(x) + g(x) - p(x)$,then we have $H(x_0) = 0$ and

$$H'(x) = q(x)u(x) \; for \; 0 \leq x - x_0 \leq a. \tag{5}$$

(5) is equivalent to

$$H'(x) = q(x)(H(x) + p(x) - g(x)). \tag{6}$$

(6) is equivalent to

$$H'(x) - q(x)H(x) + (-q(x)p(x) + q(x)g(x)) = 0. \tag{7}$$

(7) is an ordinary differential equation for x and $H(x)$,solving it gives

$$H(x)e^{\int -q(x)dx} = C + \int \left((q(x)p(x) - q(x)g(x))e^{\int -q(x)dx}\right) dx.$$

Let

$$T(x) = \int -q(x)dx, K(x) = \int \left((q(x)p(x) - q(x)g(x))e^{T(x)}\right) dx$$

Then we have

$$H(x)e^{T(x)} = C + K(x).$$

Because $H(x_0) = 0$,we have

$$C + K(x_0) = H(x_0)e^{T(x_0)} = 0,$$

which means that $C = -K(x_0)$.So

$$H(t)e^{T(t)} = K(t) - K(x_0) = \int_{x_0}^{t} \left((q(x)p(x) - q(x)g(x))e^{T(x)}\right) dx$$

So

$$(u(t) - (p(t) - g(t)))e^{T(t)} = \int_{x_0}^{t} \left(q(x)(p(x) - g(x))e^{T(x)}\right) dx$$

So

$$\begin{aligned} u(t) - (p(t) - g(t)) &= \int_{x_0}^{t} q(x)(p(x) - g(x))e^{\int_t^x -q(s)ds} dx \\ &= \int_{x_0}^{t} q(x)(p(x) - g(x))e^{\int_x^t q(s)ds} dx \\ &= \int_{x_0}^{t} q(x)p(x)e^{\int_x^t q(s)ds} dx - \int_{x_0}^{t} q(x)g(x)e^{\int_x^t q(s)ds} dx. \end{aligned}$$

So

$$u(t) - p(t) \leq \int_{x_0}^{t} q(x)p(x)e^{\int_x^t q(s)ds}.$$

□