Theorem7.1

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Theorem (7.1). *Let $f(x,y)$ be continuous in the domain D,then any solution of*

$$y' = f(x,y), y(x_0) = y_0$$

is also a solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt$$

and conversely.

Proof. $\Leftarrow$:If $g(x)$ is a solution of

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt,$$

then we have

$$g(x) = y_0 + \int_{x_0}^{x} f(t,g(t))dt. \tag{1}$$

So $g(x_0) = y_0$. f is continuous,so g is differentiable[1].Differentiate both sides of (1),we get[2]

$$g'(x) = f(x,g(x)). \tag{2}$$

$\Rightarrow$:We have

$$g'(x) = f(x,g(x)), g(x_0) = y_0,$$

[1]Note the if f is not continuous,but Riemann integrable,then g is not necessarily differentiable.

[2]Note that if f is not continuous,then $g'(x)$ does not necessarily equal to $f(x,g(x))$.

So g' is continuous.Integrate both sides,we get

$$g(x) = \int f(x, g(x))dx + C.$$

Let $H(x) = \int f(x, g(x))$,we have

$$g(x) = H(x) + C.$$

We know that

$$g(x_0) = y_0,$$

so

$$y_0 = H(x_0) + C, \Rightarrow C = y_0 - H(x_0).$$

So

$$g(x) = \int_{x_0}^{x} f(t, g(t))dt + y_0.$$

$\square$