Exercise. *Show that any 3- form on $T_p\mathbf{R}^4$ can be written as the product of three 1-forms.*

Proof. Any three form on $T_p\mathbf{R}^4$ can be expressed as

$$a_1 dx \wedge dy \wedge dz + a_2 dx \wedge dy \wedge dw + a_3 dx \wedge dz \wedge dw + a_4 dy \wedge dz \wedge dw. \tag{1}$$

(1) is equivalent to

$$dx \wedge (a_1 dy \wedge dz + a_2 dy \wedge dw + a_3 dz \wedge dw) + a_4 dy \wedge dz \wedge dw. \tag{2}$$

We know that there exists p_1, p_2, p_3 and q_1, q_2, q_3,such that

$$a_1 dy \wedge dz + a_2 dy \wedge dw + a_3 dz \wedge dw = (p_1 dy + p_2 dz + p_3 dw) \wedge (q_1 dy + q_2 dz + q_3 dw),$$

so

$$\begin{aligned} & dx \wedge (a_1 dy \wedge dz + a_2 dy \wedge dw + a_3 dz \wedge dw) + a_4 dy \wedge dz \wedge dw \\ & = dx \wedge (p_1 dy + p_2 dz + p_3 dw) \wedge (q_1 dy + q_2 dz + q_3 dw) + a_4 dy \wedge dz \wedge dw. \end{aligned}$$

For the intersection of any two three dimensional subspace of a four dimensional linear space exists and the intersection is a two dimensional subspace of the four dimensional space,so when (p_1, p_2, p_3) and (q_1, q_2, q_3) are linearly independent,then there exists n_1, n_2, n_3 such that

$$dy \wedge dz \wedge dw = (p_1 dy + p_2 dz + p_3 dw) \wedge (q_1 dy + q_2 dz + q_3 dw) \wedge (n_1 dy + n_2 dz + n_3 dw).$$

So

$$\begin{aligned} & dx \wedge (a_1 dy \wedge dz + a_2 dy \wedge dw + a_3 dz \wedge dw) + a_4 dy \wedge dz \wedge dw \\ & = dx \wedge (p_1 dy + p_2 dz + p_3 dw) \wedge (q_1 dy + q_2 dz + q_3 dw) + a_4 dy \wedge dz \wedge dw \\ & = (p_1 dy + p_2 dz + p_3 dw) \wedge (q_1 dy + q_2 dz + q_3 dw) \wedge (dx + a_4 n_1 dy + a_4 n_2 dz + a_4 n_3 dw). \end{aligned}$$

When (q_1, q_2, q_3) and (p_1, p_2, p_3) are linearly dependent,then

$$\begin{aligned} & dx \wedge (a_1 dy \wedge dz + a_2 dy \wedge dw + a_3 dz \wedge dw) + a_4 dy \wedge dz \wedge dw \\ & = a_4 dy \wedge dz \wedge dw. \end{aligned}$$

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