Exercise 3.18

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January 5, 2015

Exercise. *Find a 2- form which is not the product of 1- forms.*

Solve. We first prove that any 2-form on $T_p\mathbf{R}^2$ can be expressed as the product of two 1-forms.Any 2- form on $T_p\mathbf{R}^2$ can be expressed as $a dx \wedge dy$,where a is a real number.

$$(a_1 dx + b_1 dy) \wedge (a_2 dx + b_2 dy) = (a_1 b_2 - b_1 a_2) dx \wedge dy.$$

So let $a_1 = a, b_2 = 1, b_1 = 0$ then we are done.

Next we prove that any 2- form on $T_p\mathbf{R}^3$ can be expressed as the product of two 1-forms.Any 2-form on $T_p\mathbf{R}^3$ can be expressed as $a dx \wedge dy + b dy \wedge dz + c dz \wedge dx$,where a, b, c are real numbers.

$$\begin{aligned}&(a_1 dx + b_1 dy + c_1 dz) \wedge (a_2 dx + b_2 dy + c_2 dz)\\ &= (a_1 b_2 - a_2 b_1) dx \wedge dy + (b_1 c_2 - c_1 b_2) dy \wedge dz + (c_1 a_2 - a_1 c_2) dz \wedge dx.\end{aligned}$$

Let

$$\begin{cases} a = a_1 b_2 - a_2 b_1, \\ b = b_1 c_2 - b_2 c_1, \\ c = c_1 a_2 - c_2 a_1 \end{cases}$$

then

$$(b, c, a) = (a_1, b_1, c_1) \times (a_2, b_2, c_2).$$

According to the geometrical interpretion of the vector product,we know that the solution $(a_1, b_1, c_1), (a_2, b_2, c_2)$ exists.

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Next we prove that there exist 2- forms on $T_p\mathbf{R}^4$ such that it can not be expressed as the product of two 1- forms.

$$\begin{aligned}
&(a_1 dx + b_1 dy + c_1 dz + d_1 dw) \wedge (a_2 dx + b_2 dy + c_2 dz + d_2 dw) \\
&= (a_1 b_2 - a_2 b_1) dx \wedge dy + (a_1 c_2 - a_2 c_1) dx \wedge dz + (a_1 d_2 - a_2 d_1) dx \wedge dw \\
&+ (b_1 c_2 - b_2 c_1) dy \wedge dz + (b_1 d_2 - d_1 b_2) dy \wedge dw + (c_1 d_2 - c_2 d_1) dz \wedge dw,
\end{aligned}$$

let $a_1 b_2 - a_2 b_1 = 0$,then

$$a_1 c_2 - a_2 c_1 : b_1 c_2 - b_2 c_1 = a_1 d_2 - a_2 d_1 : b_1 d_2 - d_1 b_2,$$

So the two form $dx \wedge dz + dx \wedge dw + dy \wedge dz + 2dy \wedge dw + dz \wedge dw$ in $T_p\mathbf{R}^4$ can not be expressed as the product of two 1- forms. $\square$