FINDING THE FERMAT POINT VIA ANALYSIS

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Let P_1, P_2, P_3 be three given points in $\mathbf{R}^2$,and P be an arbitrary point in $\mathbf{R}^2$.The classical Fermat's problem to Torricelli asks for the location of P,such that

$$|PP_1| + |PP_2| + |PP_3|$$

is a minimum.Then P is called the Fermat point of the triangle $P_1P_2P_3$(Triangle $P_1P_2P_3$ is nondegenerate).There exist several elegant geometrical solutions in the literature.In this note,we consider finding the Fermat point by using methods in advanced calculus.The main tools we use are the extreme value theorem,Fermat's theorem,and the intermediate value theorem,which are listed below.

Theorem 1 (The extreme value theorem)**.** *Let $f : D \to \mathbf{R}$ be a continuous function,where D is a nonempty bounded closed set in $\mathbf{R}^2$.Then f must attain a minimum on D.That is,there exists a point ξ in D such that $f(\xi) \leq f(x)$ for all $x \in D$.*

Theorem 2 (Fermat's theorem)**.** *Let $f : C \to \mathbf{R}$ be a differentiable function,where C is a nonempty open set in $\mathbf{R}^2$.Suppose $x_0 \in C$ is a local extreme point of f,then $f'(x_0)$ is a zero linear map*[1] *from $\mathbf{R}^2$ to $\mathbf{R}$.*

Theorem 3 (The intermediate value theorem)**.** *Let $f : I \to \mathbf{R}$ be a continuous function,where $I = [a, b]$ is a closed interval of $\mathbf{R}$.For any real number u between $f(a)$ and $f(b)$,there exists a $\xi \in [a, b]$ such that $f(\xi) = u$.*

And we need four lemmas.

Lemma 4. *Let $M_1M_2M_3$ be a nondegenerate triangle,let M be an arbitrary point in the interior of the triangle region.As shown in figure* (1)*. Then*

$$rad\angle M_1MM_3 > rad\angle M_1M_2M_3$$

,where $rad\angle M_1MM_3$ is the radian measure of the angle $\angle M_1MM_3$.

Proof. Extend the segment M_3M to N,where N is a point on the segment M_1M_2.Then $\text{rad}\angle M_1MM_3 > \text{rad}\angle M_1NM_3$,and $\text{rad}\angle M_1NM_3 > \text{rad}\angle M_1M_2M_3$.So $\text{rad}\angle M_1MM_3 > \text{rad}\angle M_1M_2M_3$. □

Lemma 5. *For unit vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ on the Euclidean plane $\mathbf{R}^2$,*

$$\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3} = \mathbf{0}$$

if and only if the radian measure of the angle between any two of the unit vectors is $\frac{2\pi}{3}$.

[1] A zero linear map maps any vectors to the zero vector.

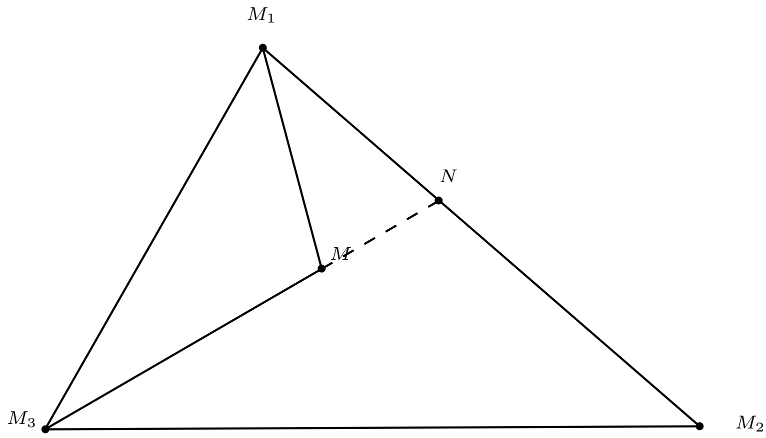

FIGURE 1

Proof. The proof is easy via plane geometry,so is left to the reader. ☐

Lemma 6. *When the radian measure of all the interior angles of the triangle $P_1P_2P_3$ are less than $\frac{2\pi}{3}$,then we can find a unique point F' in the interior of the triangle region satisfying*

$$rad\angle P_1F'P_2 = rad\angle P_2F'P_3 = rad\angle P_3F'P_1 = \frac{2\pi}{3}.$$

Proof. As shown in figure (2),suppose that an interior point of the triangle region F is on a circle which passes through P_2, P_3,then rad$\angle P_3FP_2$ is a constant,i.e,rad$\angle P_3FP_2$ remains unchanged when F moves on the circle.Now let this circle move while keeping the property that the circle passes through P_2, P_3.When the center of this circle moves downward to infinity,rad$\angle P_3FP_2$ tends to π.When the center of this circle moves from infinity to a location such that the circle passes through P_1, P_2 and P_3,then rad$\angle P_3FP_2$ becomes rad$\angle P_3P_1P_2$,which is less than $\frac{2\pi}{3}$.So according to the intermediate value theorem,there exists a location G such that when the center of this circle moves to G,then $\text{rad}\angle P_3FP_2$ becomes $\frac{2\pi}{3}$.Denote the circle centering at G by O'.Now let F move on O'.When F tends to the line P_1P_3,$\text{rad}\angle P_1FP_3$ tends to π while $\text{rad}\angle P_1FP_2$ tends to $2\pi - \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. When F tends to the line P_1P_2, $\text{rad}\angle P_1FP_2$ tends to π while $\text{rad}\angle P_1FP_3$ tends to $2\pi - \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.So according to the intermediate value theorem,there exists a point F' on the circle O' such that $\text{rad}\angle P_1F'P_3 = \text{rad}\angle P_1F'P_2$,i.e,both of them are equal to $\frac{2\pi - \frac{2\pi}{3}}{2} = \frac{2\pi}{3}$.So

$$\text{rad}\angle P_1F'P_2 = \text{rad}\angle P_2F'P_3 = \text{rad}\angle P_3F'P_1 = \frac{2\pi}{3}.$$

And the uniqueness of the point F' is obvious by lemma 4. □

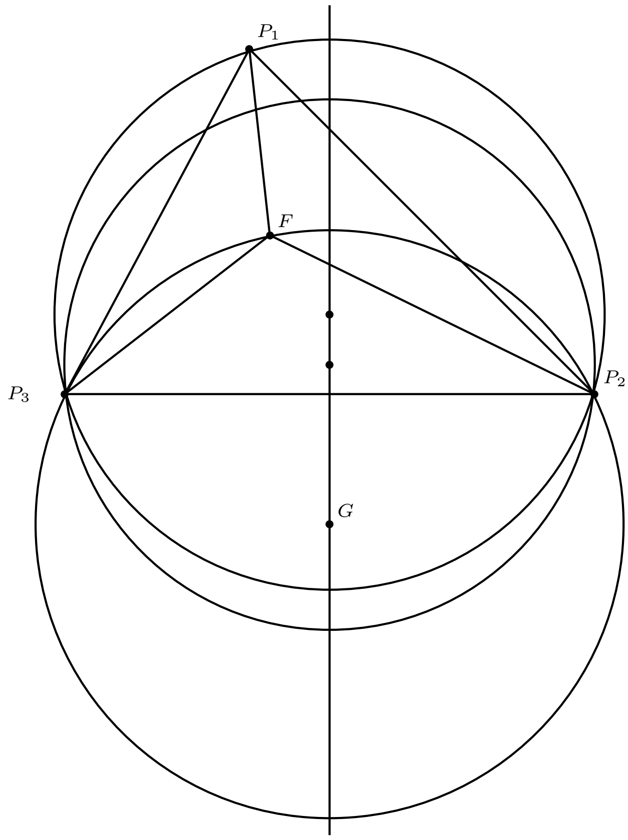

FIGURE 2

Lemma 7. *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous,and f be differentiable in the deleted neighborhood*[2] *of $P \in \mathbf{R}^2$.A line l in $\mathbf{R}^2$ passes through P.For any sequence of points $Q_1, Q_2, \cdots, Q_n, \cdots$ on l such that $\lim_{n\to\infty} |Q_nP| = 0$(P is not in the sequence),if $\lim_{n\to\infty} f'(Q_n)$ exists and is not a zero linear map from $\mathbf{R}^2$ to $\mathbf{R}$,then P is not a local extreme point of f.*

[2]The deleted neighborhood of a point is the neighborhood of the point excluding the point itself.

Proof. The proof is left to the reader. □

First we prove the existence of the Fermat point of the triangle $P_1P_2P_3$.Zuo Quanru and Lin Bo already used a sophisticated version of this method in [1].

Let $P=(x_P,y_P)$,$P_1=(x_{P_1},y_{P_1})$, $P_2=(x_{P_2},y_{P_2})$, $P_3=(x_{P_3},y_{P_3})$.Let

$$f(x,y)=|PP_1|+|PP_2|+|PP_3|=\sum_{i=1}^{3}\sqrt{(x_P-x_{P_i})^2+(y_P-y_{P_i})^2}.$$

Theorem 8 (Existence of the Fermat point)**.** *Any triangle $P_1P_2P_3$ has a Fermat point.*

Proof. Draw a circle O_1 centering at P_1,whose radius r is large enough.Then

$$D_1=\{|P-P_1|\leq r : P\in\mathbf{R}^2\}$$

is a bounded closed disk.According to the extreme value theorem,f must attain a minimum on D_1.When r is large,the minimum point of f on D_1 is the minimum point of f on the whole plane $\mathbf{R}^2$.Thus the existence of the Fermat point of the triangle $P_1P_2P_3$ is guaranteed. □

Now we prove the uniqueness of the Fermat point of the triangle $P_1P_2P_3$,in the mean time,we find the exact location of the Fermat point.Theorem (9) and Theorem (10) are our main theorems.

If $P_0=(x_{P_0},y_{P_0})$ is a minimum point of f,and $P_0\notin\{P_1,P_2,P_3\}$,then according to Fermat's theorem,we have

$$\begin{cases}\dfrac{\partial f}{\partial x}(x_{P_0},y_{P_0})=\displaystyle\sum_{i=1}^{3}\frac{x_{P_0}-x_{P_i}}{\sqrt{(x_{P_0}-x_{P_i})^2+(y_{P_0}-y_{P_i})^2}}=0,\\ \dfrac{\partial f}{\partial y}(x_{P_0},y_{P_0})=\displaystyle\sum_{i=1}^{3}\frac{y_{P_0}-y_{P_i}}{\sqrt{(x_{P_0}-x_{P_i})^2+(y_{P_0}-y_{P_i})^2}}=0.\end{cases}\tag{1}$$

Let vectors

$\mathbf{L_0}=(x_{p_0}-x_{p_1},y_{p_0}-y_{p_1}),\mathbf{M_0}=(x_{p_0}-x_{p_2},y_{p_0}-y_{p_2}),\mathbf{N_0}=(x_{p_0}-x_{p_3},y_{p_0}-y_{p_3}).$

Then the simultaneous equations (1) is equivalent to

$$\frac{\mathbf{L_0}}{|\mathbf{L_0}|}+\frac{\mathbf{M_0}}{|\mathbf{M_0}|}+\frac{\mathbf{N_0}}{|\mathbf{N_0}|}=\mathbf{0}.\tag{2}$$

When $P_0\in\{P_1,P_2,P_3\}$,equation (2) is not defined,because in this case,one of $|\mathbf{L_0}|$, $|\mathbf{M_0}|$, $|\mathbf{N_0}|$ is 0.

Notice that $\frac{\mathbf{L_0}}{|\mathbf{L_0}|}$, $\frac{\mathbf{M_0}}{|\mathbf{M_0}|}$, $\frac{\mathbf{N_0}}{|\mathbf{N_0}|}$ are unit vectors.According to lemma 5,it is easy to verify that equation (2) holds if and only if the point P_0 is in the interior of the triangle region,and the radian measure of the angle between any two of the unit vectors is $\frac{2\pi}{3}$.

If there is no point satisfying equation (2),then there is no minimum point of f on $\mathbf{R}^2\backslash\{P_1, P_2, P_3\}$,which means that there is no Fermat point of the triangle except points P_1, P_2, P_3.But according to the existence of the Fermat point(Theorem (8)),we know that the Fermat point of the triangle do exist,so the Fermat point of the triangle must be on the vertex of the triangle $P_1P_2P_3$ whose corresponding interior angle is the largest among the three interior angles.Combine the analysis in this paragraph and in last paragraph with lemma (4),we have

Theorem 9. *When the radian measure of an interior angle of the triangle $P_1P_2P_3$ is equal or larger than $\frac{2\pi}{3}$,then the Fermat point must be on the vertex of the triangle whose corresponding interior angle is the largest among the three interior angles,and the Fermat point is unique.*

Next we prove

Theorem 10. *When the radian measure of all the interior angles of the triangle $P_1P_2P_3$ are less than $\frac{2\pi}{3}$,then the Fermat point must be in the interior of the triangle and is unique,denoted by P_0.And $rad\angle P_1P_0P_2 = rad\angle P_2P_0P_3 = rad\angle P_3P_0P_1 = \frac{2\pi}{3}$.*

Proof. According to lemma (6),there exists a unique point P_0 in the interior of the triangle satisfying the condition $\mathrm{rad}\angle P_1P_0P_2 = \mathrm{rad}\angle P_2P_0P_3 = \mathrm{rad}\angle P_3P_0P_1 = \frac{2\pi}{3}$.

And the point P_0 is the unique point satisfying equation (2),which means that P_0 is the only possible minimum point of f except P_1, P_2, P_3.So if we managed to prove that none of the points P_1, P_2, P_3 is the minimum point of f,then we managed to prove that P_0 is the unique Fermat point of f.Now we do this job.

Draw a line l passing through the point P_1.Now we prove that for any sequence of points $Q_1 = (x_1, y_1), Q_2 = (x_2, y_2), \cdots, Q_n = (x_n, y_n), \cdots$ on l such that $\lim_{n\to\infty} |Q_nP_1| = 0$($P_1$ is not in the sequence),we have $\lim_{n\to\infty} f'(Q_n)$ exists and is a nonzero linear map from $\mathbf{R}^2$ to $\mathbf{R}$,then according to lemma 7,we can prove that P_1 is not a minimum point of f.

Let $\mathbf{L_n} = (x_n - x_{P_1}, y_n - y_{P_1}), \mathbf{M_n} = (x_n - x_{P_2}, y_n - y_{P_2}), \mathbf{N_n} = (x_n - x_{P_3}, y_n - y_{P_3})$.Then

$$\frac{\mathbf{L_n}}{|\mathbf{L_n}|} + \frac{\mathbf{M_n}}{|\mathbf{M_n}|} + \frac{\mathbf{N_n}}{|\mathbf{N_n}|} = (\frac{\partial f}{\partial x}(x_n, y_n), \frac{\partial f}{\partial y}(x_n, y_n)).$$

And we have

$$\lim_{n\to\infty} \left| \frac{\mathbf{M_n}}{|\mathbf{M_n}|} + \frac{\mathbf{N_n}}{|\mathbf{N_n}|} \right| > 1,$$

this is because $\mathrm{rad}\angle P_2P_1P_3 < \frac{2\pi}{3}$.So as n goes to infinity,$\frac{\mathbf{L_n}}{|\mathbf{L_n}|} + \frac{\mathbf{M_n}}{|\mathbf{M_n}|} + \frac{\mathbf{N_n}}{|\mathbf{N_n}|}$ tends to a nonzero vector.So as n goes to infinity,$f'(Q_n)$ tends to a nonzero linear map from $\mathbf{R}^2$ to $\mathbf{R}$.Done.

So P_1 is not a minimum point of f.Similarly,P_2, P_3 are not minimum points of f.So P_0 is the unique minimum point of f. $\square$

References

[1] Zuo Quanru,Lin Bo.Fermat Points of finite Point Sets in Metric Spaces[J].Journal of Mathematics.(PRC),1997-03

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