Exercise2.6

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Exercise. *Let*

$$D(x,y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

If,for some point (x_0, y_0)*,you know* $D(x_0, y_0) > 0$*,then show that the signs of* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$ *and* $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$ *are the same.*

Proof. Let (a, b) be a vector,and t be a real number.Let $g(t) = f(x_0 + ta, y_0 + tb)$.According to Taylor's formula,

$$g(t) = g(0) + \frac{g'(0)}{1!}t + \frac{g''(0)}{2!}t^2 + \cdots$$

It is easy to verify that

$$g'(h) = a\frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb) + b\frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb),$$

and

$$\begin{aligned} g''(h) &= a\left[a\frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb) + b\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb)\right] \\ &+ b\left[a\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb) + b\frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb)\right] \\ &= a^2\frac{\partial^2 f}{\partial x^2}(x_0 + ha, y_0 + hb) + 2ab\frac{\partial^2 f}{\partial y \partial x}(x_0 + ha, y_0 + hb) + b^2\frac{\partial^2 f}{\partial y^2}(x_0 + ha, y_0 + hb). \end{aligned}$$

Let $b = 1$,then

$$g''(0) = a^2\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + 2a\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

Regard $g''(0)$ as a quadratic equation of a,then $\Delta < 0$.So $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) > 0$. $\square$

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