FINDING THE FERMAT POINT VIA ANALYSIS

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Let P_1, P_2, P_3 be three given points in $\mathbf{R}^2$,and P be an arbitrary point in $\mathbf{R}^2$.The famous Fermat's problem to Torricelli asks for the location of P,such that

$$|P_0P_1| + |P_0P_2| + |P_0P_3|$$

is a minimum.Then P is called the Fermat point of the triangle $P_1P_2P_3$(Triangle $P_1P_2P_3$ is nondegenerate).There exist several elegant geometrical solutions in the literature.In this note,we consider finding the Fermat point by using methods in advanced calculus.The main tools we use are the extreme value theorem,Fermat's theorem,and the intermediate value theorem,which are listed below.

Theorem 1 (The extreme value theorem)**.** *Let $f : D \to \mathbf{R}$ be a continuous function,where D is a nonempty bounded closed set in $\mathbf{R}^2$.Then f must attain a minimum on D.That is,there exist point ξ in D such that $f(\xi) \leq f(x)$ for all $x \in D$.*

Theorem 2 (Fermat's theorem)**.** *Let $f : C \to \mathbf{R}$ be a differentiable function,where C is a nonempty open set in $\mathbf{R}^2$.Suppose $x_0 \in D$ is a local extreme of f,then $f'(x_0) = \mathbf{0}$.*

Theorem 3 (The intermediate value theorem)**.** *Let $f : I \to \mathbf{R}$ be a continuous function,where $I = [a, b]$ is a closed interval of $\mathbf{R}$.For any real number u between $f(a)$ and $f(b)$,there exist $\xi \in [a, b]$ such that $f(\xi) = u$.*

And we need two lemmas.

Lemma 4. *Let $M_1M_2M_3$ be a nondegenerate triangle,let M be an arbitrary point in the interior of the triangle region.As is shown in Figure* (1)*. Then*

$$rad\angle M_1MM_3 > rad\angle M_1M_2M_3$$

,where $rad\angle M_1MM_3$ is the radian measure of the angle $\angle M_1MM_3$.

Proof. Extend the segment M_3M to N,where N is a point on the segment M_1M_2.Then $\mathrm{rad}\angle M_1MM_3 > \mathrm{rad}\angle M_1NM_3$,and $\mathrm{rad}\angle M_1NM_3 > \mathrm{rad}\angle M_1M_2M_3$.So $\mathrm{rad}\angle M_1MM_3 > \mathrm{rad}\angle M_1M_2M_3$. □

Lemma 5. *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuous in the neighborhood U of $P \in \mathbf{R}^2$,and continuously differentiable in the deleted neighborhood $U\backslash\{P\}$ of P[1]. On $U\backslash\{P\}$,when*

[1]The deleted neighborhood of a point is the neighborhood of a point which exclude the point itself.

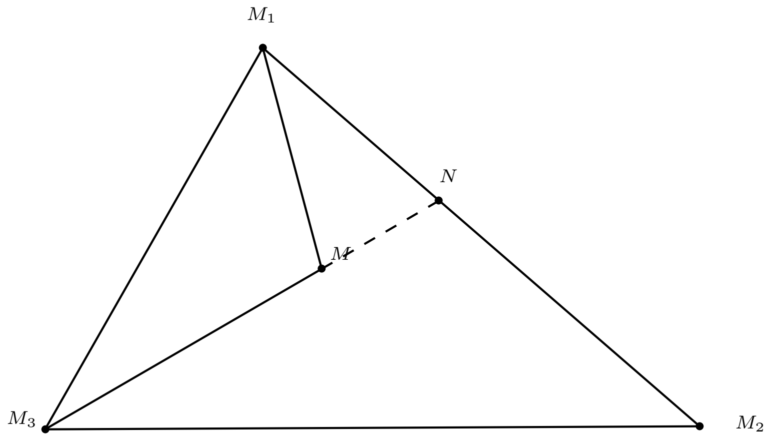

FIGURE 1

the diameter ξ of U is small enough,the derivative of f approximates a nonzero linear map[2] *from* $\mathbf{R}^2$ *to* $\mathbf{R}$,*i.e,*

$$\lim_{\xi\to 0;\mathbf{x}\in U\setminus\{P\}} f'(\mathbf{x}) = T,$$

where T is a nonzero linear map from $\mathbf{R^2}$ *to* $\mathbf{R}$. *Then P is not a local extreme of f.*

First we prove the existence of the Fermat point of the triangle $P_1P_2P_3$.Zuo Quanru and Lin Bo already used a sophisticated version of this method in [1].

Let $P = (x, y)$,$P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$.Let

$$f(x, y) = |PP_1| + |PP_2| + |PP_3| = \sum_{i=1}^{3} \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

[2] A zero linear map maps any vectors into the zero vector.

Theorem 6 (Existence of the Fermat point)**.** *Any triangle $P_1P_2P_3$ has a Fermat point.*

Proof. Draw a circle O_1 centered at P_1,whose radius r is large enough.Then

$$D_1 = \{|P - P_1| \le r : P \in \mathbf{R}^2\}$$

is a bounded closed disk.According to the extreme value theorem,f must attain a minimum on D_1.When r is large,the minimum point of f on D_1 is the minimum point of f on the whole plane $\mathbf{R}^2$.Thus the existence of the Fermat point of the triangle $P_1P_2P_3$ is guaranteed. □

Now we prove the uniqueness of the Fermat point of the triangle $P_1P_2P_3$,in the mean time,we find the exact location of the Fermat point.

If $P_0 = (x_0, y_0)$ is a minimum point of f,and $P_0 \ne P_1, P_2, P_3$,then according to Fermat's theorem,we have

$$(1) \qquad \begin{cases} \dfrac{\partial f}{\partial x}(x_0, y_0) = \displaystyle\sum_{i=1}^{3} \dfrac{x_0 - x_i}{\sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}} = 0, \\ \dfrac{\partial f}{\partial y}(x_0, y_0) = \displaystyle\sum_{i=1}^{3} \dfrac{y_0 - y_i}{\sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}} = 0. \end{cases}$$

Let vectors

$$\mathbf{L_0} = (x_0 - x_1, y_0 - y_1), \mathbf{M_0} = (x_0 - x_2, y_0 - y_2), \mathbf{N_0} = (x_0 - x_3, y_0 - y_3).$$

Then the simultaneous equations (1) is equivalent to

$$(2) \qquad \frac{\mathbf{L_0}}{|\mathbf{L_0}|} + \frac{\mathbf{M_0}}{|\mathbf{M_0}|} + \frac{\mathbf{N_0}}{|\mathbf{N_0}|} = \mathbf{0}.$$

Notice that $\frac{\mathbf{L_0}}{|\mathbf{L_0}|}, \frac{\mathbf{M_0}}{|\mathbf{M_0}|}, \frac{\mathbf{N_0}}{|\mathbf{N_0}|}$ are unit vectors.It can be easily verified that equation (2) holds if and only if the point P_0 is in the interior of the triangle region,and the radian measure of the angle between any two of the unit vectors is $\frac{2\pi}{3}$.So the location of the Fermat point of the triangle $P_1P_2P_3$ has two possibilities.

(1) When there exist a point P_0 which satisfies the equation (2),the Fermat point is in the interior of the triangle region,or on the vertex of the triangle $P_1P_2P_3$ whose corresponding interior angle is the largest among the three interior angles.Be aware at present we haven't proved the uniqueness of the Fermat point,we will do it later in theorem 8.
(2) When there is no point P_0 which satisfies the equation (2),the Fermat point must be on the vertex of the triangle whose corresponding interior angle is the largest among the three interior angles.In this case,the Fermat point is unique.Later,in theorem 8,we will show this is only possible when the radian measure of all the interior triangle are equal or more than $\frac{2\pi}{3}$.

Combine the item (2) with lemma 4 and theorem 5,we have the following corollary:

Corollary 7. *When the radian measure of all the interior angle of the triangle $P_1P_2P_3$ is equal or larger than $\frac{2\pi}{3}$,then the Fermat point must be on the vertex of the triangle whose corresponding interior angle is the largest among the three interior angles.*

Now we prove

Theorem 8. *When the radian measure of all the interior angles of the triangle $P_1P_2P_3$ are less than $\frac{2\pi}{3}$,we can find a unique point in the interior of the triangle region which satisfies equation* (2)*.*

Proof. As is shown in figure (2),draw a family of circles in which all of the circles pass through P_2 and P_3.Suppose that an interior point of the triangle region F is on a given circle,then $\text{rad}\angle P_3FP_2$ is a constant.Now we let this given circle move while keep the property that this circle passes through P_2, P_3.When the center of this circle moves downward to infinity,$\text{rad}\angle P_3FP_2$ tends to π.When the center of the circle moves from infinity to a location such that the circle passes through P_1, P_2 and P_3,then $\text{rad}\angle P_3FP_2$ becomes $\text{rad}\angle P_3P_1P_2$,which is less than $\frac{2\pi}{3}$.So according to the intermediate value theorem,there exist a location such that when the center of the circle moves to this location, $\text{rad}\angle P_3FP_2$ becomes $\frac{2\pi}{3}$,denote this circle by O'.Now let F moves on this circle.When F tends to the line P_1P_3,$\text{rad}\angle P_1FP_3$ tends to π while $\text{rad}\angle P_1FP_2$ tends to $2\pi - \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. .When F tends to the line P_1P_2, $\text{rad}\angle P_1FP_2$ tends to π while $\text{rad}\angle P_1FP_3$ tends to $2\pi - \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.So according to the intermediate value theorem,there exists a point F' on the circle O' such that $\text{rad}\angle P_1F'P_3 = \text{rad}\angle P_1F'P_2$,i.e,both of them are equal to $\frac{2\pi - \frac{2\pi}{3}}{2} = \frac{2\pi}{3}$.So F' satisfies equation (2).

And the uniqueness of the point which satisfies the equation (2) is obvious by lemma 4. □

Next we prove

Theorem 9. *When the radian measure of all the interior angles of the triangle $P_1P_2P_3$ are less than $\frac{2\pi}{3}$,then item* (2) *does not hold,which means the Fermat point is in the interior of the triangle and is unique.*

Proof. Let $P = (x, y)$ be an arbitrary point in the small deleted neighborhood of P_1.Let vectors

$$\mathbf{L} = (x - x_1, y - y_1), \mathbf{M} = (x - x_2, y - y_2), \mathbf{N} = (x - x_3, y - y_3).$$

When the diameter of the deleted neighborhood ξ is small enough,the radian measure of the angle between the unit vectors

$$\frac{\mathbf{M}}{|\mathbf{M}|}, \frac{\mathbf{N}}{|\mathbf{N}|}$$

approximates $\text{rad}\angle P_3P_1P_2$.To be more precise,

$$\lim_{\xi \to 0} \text{rad}\angle P_3PP_2 = \text{rad}\angle P_3P_1P_2 < \frac{2\pi}{3},$$

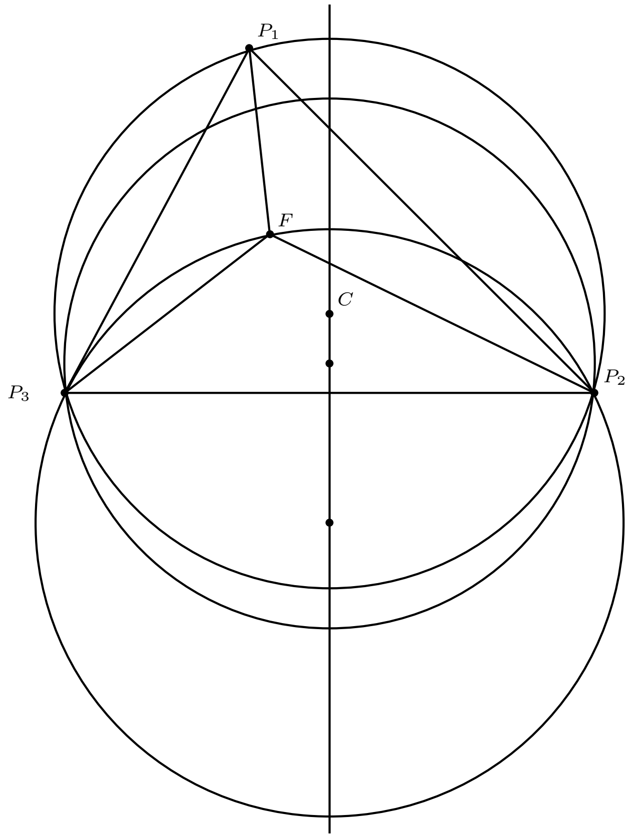

FIGURE 2

which implies that

$$\lim_{\xi \to 0} \left| \frac{\mathbf{M}}{|\mathbf{M}|} + \frac{\mathbf{N}}{|\mathbf{N}|} \right| > 1,$$

So

$$\lim_{\xi \to 0} \left| \frac{\mathbf{L}}{|\mathbf{L}|} + \frac{\mathbf{M}}{|\mathbf{M}|} + \frac{\mathbf{N}}{|\mathbf{N}|} \right| > c,$$

where $c \in \mathbf{R}^+$ is a constant.So according to lemma 5,P_1 is not a minimum point of f.Similarly,P_2, P_3 are not minimum points of f.So the theorem holds. □

REFERENCES

[1] Zuo Quanru,Lin Bo.Fermat Points of finite Point Sets in Metric Spaces.Journal of Math.(PRC),1997.V.17,No.3,pp359-364

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