Practice Questions (week 7)

Semester 2, 2019

These questions are all about linear algebra – linear (in)dependence, eigenvalues, and eigenvectors.

1. Determine, with reasons, whether the following statements are (A) always true, (B) always false, or (C) might be true or false.

   (a) If $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^5$ and $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is linearly dependent.

   (b) If $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathbb{R}^2$ lie on the same straight line through the origin, then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

   (c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

   (d) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (e) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$.

   (f) There is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ of linearly independent vectors in $\mathbb{R}^4$.

   Solution:

   (a) (A) Always true, since one vector is a linear combination of the others, or $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 + 0\mathbf{v}_4 + 0\mathbf{v}_5 = \mathbf{0}$.

   (b) (A) Always true, since one vector is a multiple of the other.

   (c) (A) Always true, since if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ for some x_1, x_2, x_3 not all zero, then $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$ without all coefficients being forced to be 0.

   (c) (C) Neither e.g. result is false for $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (0, 1), \mathbf{v}_3 = (1, 1)$, but true for $\mathbf{v}_1 = (1, 0), \mathbf{v}_2 = (2, 0), \mathbf{v}_3 = (1, 1)$.

   (d) (A) Always true. Suppose $a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_1 - \mathbf{v}_2) = 0$. Then $(a + b)\mathbf{v}_1 + (a - b)\mathbf{v}_2 = 0$. Since $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, this gives $a + b = 0, a - b = 0$ giving $a = b = 0$ only.

   (e) (B) Always false. The maximum number of linearly independent vectors in $\mathbb{R}^4$ is four.

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2. Consider the vectors $\mathbf{u} = (2, 1, 4, 3)$, $\mathbf{v} = (3, 4, 1, 2)$, $\mathbf{w} = (1, 2, -1, 0)$ in $\mathbb{R}^4$.

   (a) Are the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ linearly independent?

   (b) Are the vectors $\mathbf{v}$, $\mathbf{w}$ linearly independent?

(c) Are the vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{0}$ linearly independent?

(d) Are the vectors $\mathbf{u}$, $\mathbf{w}$, $5\mathbf{u} - 3\mathbf{w}$ linearly independent?

Solution:

(a) No, as $2\mathbf{u} + (-3)\,\mathbf{v}\, + 5\mathbf{w} = \mathbf{0}$.

(b) Yes, as $\mathbf{v}$ is not a multiple of $\mathbf{w}$.

(c) No, as $\mathbf{0}$ is in the list.

(d) No, as $5.\mathbf{u} + (-3).\mathbf{w} + (-1).(5\mathbf{u} - 3\mathbf{w}) = \mathbf{0}$.

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3. Which of the following sets of vectors are linearly independent?

(a) $\{(3,4),\ (4,3)\}$

(b) $\{(2,1,-3,6),\ (5,3,7,8),\ (1,1,13,-4)\}$

(c) $\{(2,1,3),\ (2,-2,-5),\ (7,3,9)\}$

(d) $\{\mathbf{e}_1,\ \mathbf{e}_1 + 2\mathbf{e}_2,\ \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3,\ \ldots,\ \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + \cdots + n\mathbf{e}_n\}$ where $\mathbf{e}_i \in \mathbb{R}^n$ is the vector which has 1 in the i-th place and 0 everywhere else.

Solution:

(a) Linearly independent: the 2 vectors are not multiples of each other.

(b) Linearly dependent: as $\begin{bmatrix} 2 & 5 & 1 \\ 1 & 3 & 1 \\ -3 & 7 & 13 \\ 6 & 8 & -4 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $2\mathbf{u} - \mathbf{v} + \mathbf{w} = \mathbf{0}$.

(c) Linearly independent: as $\begin{bmatrix} 2 & 2 & 7 \\ 1 & -2 & 3 \\ 3 & -5 & 9 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) Linearly independent: as $A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 2 & 2 & \ldots & 2 \\ 0 & 0 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & n \end{bmatrix}$ row reduces to the identity $n \times n$ matrix.

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4. For which value(s) of d are the following sets of vectors linearly dependent? Justify your answers.

   (a) $\{(1,-1,4)),(3,-5,5),(-1,5,d)\}$

   (b) $\{(3,7,-2),(-6,d,4),(9,1,-4)\}$

   Solution:

   (a) Linearly dependent if and only if $d = 10$ as $\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 5 & d \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -10+d \end{bmatrix}$.

   (b) Linearly dependent if and only if $d = -14$ as $\begin{bmatrix} 3 & -6 & 9 \\ 7 & d & 1 \\ -2 & 4 & -4 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & -2 & 0 \\ 0 & d+14 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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5. * Let $A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 9 & 5 \\ 1 & 3 & 1 \end{bmatrix}$ and let W be the set of all linear combinations of the columns of A.

   (a) Show that $(4,10,2)$ is in W (you should be able to do this by inspection, that is, without any row operations).

   (b) Solve the homogeneous system with augmented matrix $[A|\mathbf{0}]$.

   (c) Given that $(1,2,-1)$ is a solution of the linear system $[A|\mathbf{b}]$, what is the vector $\mathbf{b}$?

   (d) Write down the general solution of $[A|\mathbf{b}]$ (with the vector $\mathbf{b}$ from part (c)), without directly solving this system.

   Solution: (a) By inspection we see that $\begin{bmatrix} 4 \\ 10 \\ 2 \end{bmatrix}$ is given by the following linear combination of the columns: $0\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 9 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

(b) Row reducing gives:

$$\begin{array}{ccc|c}
1 & 5 & 2 & 0 \\
-1 & 9 & 5 & 0 \\
1 & 3 & 1 & 0 \\
\hline
1 & 5 & 2 & 0 \\
0 & 14 & 7 & 0 \\
0 & -2 & -1 & 0 \\
\hline
1 & 5 & 2 & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
\hline
1 & 0 & -\frac{1}{2} & 0 \\
0 & 1 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{array}$$

Let $x_3 = t$, then $x_1 = \frac{1}{2}t$ and $x_2 = -\frac{1}{2}t$, so $(x_1, x_2, x_3) = t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, for any $t \in \mathbb{R}$.

(c) The first coordinate of $\mathbf{b}$ is $1 \times 1 + 5 \times 2 + 2 \times (-1) = 9$. The second coordinate of $\mathbf{b}$ is $-1 \times 1 + 9 \times 2 + 5 \times (-1) = 12$. The third coordinate of $\mathbf{b}$ is $1 \times 1 + 3 \times 2 + 1 \times (-1) = 6$. So $\mathbf{b} = \begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix}$.

(d) The general solution of $[A|\mathbf{b}]$ is given by adding together the general solution of $[A|\mathbf{0}]$ and a particular solution of $[A|\mathbf{b}]$. In this case we have $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

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6. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Prove that $\mathbf{w}$ is a linear combination of $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$ and $\mathbf{u}$. (Another way of saying this is that $\mathbf{w}$ is in the *span* of $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$ and $\mathbf{u}$, $\mathbf{w} \in \operatorname{span}\{\mathbf{u}, \mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}\}$.)

Solution: One to do this is by trial and error. A more systematic way is as follows. We want to find scalars x, y and z such that

$$x\mathbf{u} + y(\mathbf{u} - \mathbf{v}) + z(\mathbf{v} - \mathbf{w}) = \mathbf{w}.$$

In other words, we want the vector equation

$$(x + y)\mathbf{u} + (z - y)\mathbf{v} - z\mathbf{w} = \mathbf{w}$$

to be satisfied. This is clearly satisfied if we choose x, y and z so that $x + y = 0$, $z - y = 0$ and $-z = 1$. Therefore take $z = -1$, $y = -1$ and $x = 1$. As a check we see that indeed

$$\mathbf{u} - (\mathbf{u} - \mathbf{v}) - (\mathbf{v} - \mathbf{w}) = \mathbf{w}.$$

7. Let A be an $n \times n$ matrix.

   (a) Define what it means for $\mathbf{x} \in \mathbb{R}^n$ to be an eigenvector of A with eigenvalue λ.

   (b) Define the characteristic polynomial of A.

Solution:

   (a) An eigenvector of A with eigenvalue λ is a vector $\mathbf{x} \neq 0$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$.

   (b) The characteristic polynomial of A is $|\lambda I - A|$.

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8. Find all eigenvalues and eigenvectors for the following matrices.

   (a) $A_1 = \begin{bmatrix} 0 & 3 \\ 6 & -3 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

   (c) $A_3 = \begin{bmatrix} 1 & 5 & 5 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

Solution:

   (a) First find the characteristic polynomial of A_1:

$$\begin{aligned} |\lambda I - A_1| = & \begin{vmatrix} \lambda & -3 \\ -6 & \lambda + 3 \end{vmatrix} \\ = & \lambda(\lambda + 3) - 18 \\ = & \lambda^2 + 3\lambda - 18 \\ = & (\lambda + 6)(\lambda - 3) \end{aligned}$$

   So the two eigenvalues of A_2 are $\lambda = -6, 3$. To compute the eigenspaces, solve $(\lambda I - A_1)\mathbf{x} = \mathbf{0}$.

$$\begin{array}{rr|r}
\multicolumn{3}{c}{\lambda = -6} \\
\boxed{-6} & -3 & 0 \\
-6 & -3 & 0 \\
\hline
1 & 1/2 & 0 \\
0 & 0 & 0
\end{array}
\qquad
\begin{array}{rr|r}
\multicolumn{3}{c}{\lambda = 3} \\
\boxed{3} & -3 & 0 \\
-6 & 6 & 0 \\
\hline
1 & -1 & 0 \\
0 & 0 & 0
\end{array}$$

So the eigenspaces are: $\mathbb{E}_{-6} = \left\{ s \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$ and $\mathbb{E}_3 = \left\{ s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$,

(b) The characteristic polynomial is:

$$\begin{aligned}
|\lambda I - A_2| &= \begin{vmatrix} \lambda - 2 & -1 & 0 \\ -1 & \lambda - 3 & -1 \\ 0 & -1 & \lambda - 2 \end{vmatrix} \\
&= (\lambda - 2) \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} \\
&= (\lambda - 2)\left[(\lambda - 3)(\lambda - 2) - 1\right] - (\lambda - 2) \\
&= (\lambda - 2)\left[(\lambda - 3)(\lambda - 2) - 2\right] \\
&= (\lambda - 2)\left[\lambda^2 - 5\lambda + 4\right] \\
&= (\lambda - 2)(\lambda - 1)(\lambda - 4)
\end{aligned}$$

So the three eigenvalues of A_2 are $\lambda = 2, 1, 4$.

- $\lambda = 2$:

$$\begin{array}{rrr|r}
0 & -1 & 0 & 0 \\
-1 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 \\
\hline
\boxed{-1} & -1 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 \\
0 & \boxed{-1} & 0 & 0 \\
0 & -1 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}$$

  So the eigenspace is $\mathbb{E}_2 = \left\{ s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$.

- $\lambda = 1$:

$$\begin{array}{rrr|r}
-1 & -1 & 0 & 0 \\
-1 & -2 & -1 & 0 \\
0 & -1 & -1 & 0 \\
\hline
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}$$

So the eigenspace is $\mathbb{E}_1 = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$.

- $\lambda = 4$:

$$\begin{array}{rrr|r} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

So the eigenspace is $\mathbb{E}_4 = \left\{ s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$.

(c) Using the first column to compute the determinant:

$$\begin{aligned} |\lambda I - A_3| &= \begin{vmatrix} \lambda - 1 & -5 & -5 \\ 0 & \lambda - 3 & -2 \\ 0 & -2 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} \lambda - 3 & -2 \\ -2 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1) \left[(\lambda - 3)(\lambda - 3) - 4 \right] \\ &= (\lambda - 1) \left[\lambda^2 - 6\lambda + 5 \right] \\ &= (\lambda - 1)(\lambda - 1)(\lambda - 5) \end{aligned}$$

So the three eigenvalues of A_3 are $\lambda = 1, 1, 5$.

- $\lambda = 1$:

$$\begin{array}{rrr|r} 0 & \boxed{\text{-5}} & -5 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Eigenspace is $\mathbb{E}_1 = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$.

- $\lambda = 5$:

$$\begin{array}{rrr|r} \boxed{4} & -5 & -5 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ \hline 1 & -5/4 & -5/4 & 0 \\ 0 & \boxed{2} & -2 & 0 \\ 0 & -2 & 2 & 0 \\ \hline 1 & 0 & -10/4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

So the eigenspace is $\mathbb{E}_5 = \left\{ s \begin{bmatrix} 10/4 \\ 1 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$.

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9. Show that if A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then
   (a) cA has eigenvalues $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ for any constant $c \in \mathbb{R}$.
   (b) If A^{-1} exists, then A^{-1} has eigenvalues $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$
   (c) A^m has eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ for $m = 1, 2, 3, \ldots$.

   Solution:

   (a) If $0 = |\lambda_i I - A|$ then
   $$0 = c^n|\lambda_i I - A| = |c(\lambda_i I - A)| = |(c\lambda_i)I - (cA)|.$$
   Hence cA has eigenvalues $c\lambda_i, i = 1, 2, \ldots, n$.

   (b) If $0 = |\lambda_i I - A|$ then $\lambda_i \neq 0$ (otherwise $\det A = 0$ and so A^{-1} doesn't exist). Hence
   $$0 = |\lambda_i I - A| = |A|.|\lambda_i.A^{-1} - I| = (\lambda_i)^n|A|.|A^{-1} - \frac{1}{\lambda_i}I|$$
   since $\lambda_i \neq 0$. Hence $0 = |\frac{1}{\lambda_i}I - A^{-1}|$ so $\frac{1}{\lambda_i}$ is an eigenvalue of A^{-1} for all $i = 1, 2, \ldots, n$.

   (c) $A\mathbf{x} = \lambda_i\mathbf{x}$ so $A^2\mathbf{x} = A(A\mathbf{x}) = A.\lambda_i\mathbf{x} = \lambda_i(A\mathbf{x}) = \lambda_i^2\mathbf{x}$. Then, if $A^k\mathbf{x} = \lambda_i^k\mathbf{x}$, we get
   $$A^{k+1}\mathbf{x} = A.A^k\mathbf{x} = A.\lambda_i^k\mathbf{x} = \lambda_i^k.(A\mathbf{x}) = \lambda_i^k.(\lambda_i\mathbf{x}) = \lambda_i^{k+1}\mathbf{x}.$$
   So by mathematical induction, if λ_i is an eigenvalue for A, then $A^m\mathbf{x} = \lambda_i^m\mathbf{x}$ i.e. A^m has eigenvalues λ_i^m for all i. Note they have the same eigenvectors.

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10. (a) Show that $\lambda = 0$ is an eigenvalue of A if and only if A is not invertible.
    (b) Without calculation find one eigenvalue and two linearly independent eigenvectors of
    $$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

    Solution:

(a) The determinant of A is equal to the product of eigenvalues, which is equal to zero if and only if one of the eigenvalues equals zero. The matrix is invertible if and only if the determinant is not equal to zero. Hence the result.

(b) The determinant of A is zero, so $\lambda = 0$ must be an eigenvalue. The vectors

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

when pre-multiplied by A, take combinations of columns which result in zero, and therefore give the zero vector, thus they are eigenvectors corresponding to the eigenvalue $\lambda = 0$.

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11. Suppose that for some 3×3 matrix A, we have

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) Give one eigenvalue of A.

(b) Explain why $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for this eigenvalue and hence find $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) If det(A)=12, then what is the multiplicity of the eigenvalue from (a)?

Solution:

(a) 2 is an eigenvalue of A.

(b) Since $[1\ 1\ 1]^t, [1\ 0\ 1]^t \in \mathbb{E}_2$ and $\mathbb{E}_2$ is a subspace then $[0\ 1\ 0]^t = [1\ 1\ 1]^t - [1\ 0\ 1]^t \in \mathbb{E}_2$.
Hence $A[0\ 1\ 0]^t = [0\ 2\ 0]^t$.

(c) The multiplicity m of an eigenvalue is greater than or equal to the dimension of the corresponding eigenspace. As the two eigenvectors are linearly independent, the dimension of $\mathbb{E}_2$ is at least 2, and so the multiplicity of the eigenvalue 2 is at least 2. As A is a 3×3 matrix, the multiplicity of the eigenvalue 2 can be at most 3. The product of the three eigenvalues is equal to the determinant, so $2 \times 2 \times \lambda_3 = 12$, and so $\lambda_3 = 3$, and the eigenvalue 2 has multiplicity 2.

12. Consider the matrices

$$A_1 = \begin{bmatrix} -12 & 7 \\ -7 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 3 & 9 \\ 0 & -7 & -18 \\ 0 & 2 & 5 \end{bmatrix}.$$

For each matrix A_i above

(a) Determine the eigenvalues and eigenvectors.

(b) For each eigenvalue, state its multiplicity and give the dimension of its associated eigenspace.

(c) Hence determine whether the matrix A_i is diagonalisable, stating the reason for your answer.

(d) For each diagonalisable matrix A_i, determine a matrix P such that $P^{-1}A_iP = D$, where D is a diagonal matrix. What is D? (For the purposes of this exercise, order your eigenvalues from smallest to largest i.e. $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$.)

(e) Determine if each matrix A_i is invertible and, where possible, use your results from (d) to write the inverse matrix A_i^{-1} in the form $P\Delta P^{-1}$, where Δ is a diagonal matrix. (You do not need to find A^{-1} if not possible by this method.)

Solution:

A_1: (a) A_1 has eigenvalue -5 with multiplicity 2. Eigenvectors are $\mathbf{x} = t(1, 1), t \neq 0$.

(b) $\mathbb{E}_1 = \text{span}\{(1, 1)\}$, $\dim(\mathbb{E}_1) = 1$.

(c) A_1 is not diagonalisable as it does not have two linearly independent eigenvectors.

(d) No answer required as A_1 is not diagonalisable.

(e) $\det(A) = 25 \neq 0$ so A is invertible though not diagonalisable.

A_2: (a) A_2 has eigenvalues 2 and 8 with multiplicities 2 and 1 respectively. For $\lambda = 2$ the eigenvectors are $\vec{x} = s(-1, 1, 0) + t(-1, 0, 1)$, s and t not both zero. For $\lambda = 8$ the eigenvectors are $\vec{x} = t(1, 1, 1)$, $t \neq 0$.

(b) $\mathbb{E}_2 = \text{span}\{(-1, 1, 0), (-1, 0, 1)\}$, $\dim(\mathbb{E}_2) = 2$. $\mathbb{E}_8 = \text{span}\{(1, 1, 1)\}$, $\dim(\mathbb{E}_8) = 1$.

(c) A_2 is diagonalisable as it has 3 linearly independent eigenvectors.

(d) $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

(e) $\det(A_2) = 32 \neq 0$ so A_2 is invertible. $\Delta = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$ with P as above.

A_3: (a) A_3 has eigenvalue -1 with multiplicity 3. The eigenvectors are $\vec{x} = t(1,0,0) + s(0,-3,1)$, s and t not both zero.

(b) $\mathbb{E}_3 = \operatorname{span}\{(1,0,0),(0,-3,1)\}$, $\dim(\mathbb{E}_3) = 2$.

(c) A_3 is not diagonalisable as it does not have three linearly independent eigenvectors.

(d) No answer required as A_3 is not diagonalisable.

(e) $\det(A_3) = -1 \neq 0$ so A_3 is invertible though not diagonalisable.

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13. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

(a) Verify that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue λ_1?

(b) Use the trace of A to find a second eigenvalue λ_2.

(c) Find the eigenspace for λ_2.

(d) Write down an invertible matrix P such that $P^{-1}AP = D$, where $D = \operatorname{diag}(\lambda_1, \lambda_2)$.

Solution:

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $(1,1)^t$ is an eigenvector of A corresponding to the eigenvalue 5.

(b) A is a 2×2 matrix so it has two eigenvalues. We know $\lambda_1 = 5$. But $\operatorname{tr}(A) = 2 + 1 = 3 = \lambda_1 + \lambda_2$ so $\lambda_2 = -2$.

(c) $-2I - A = \begin{bmatrix} -4 & -3 \\ -4 & -3 \end{bmatrix}$. Solving $(-2I - A)\vec{x} = \mathbf{0}$ gives $\vec{x} = s[-3/4\ 1]^t$. Thus $\mathbb{E}_{-2} = \operatorname{span}\{[-3/4\ 1]^t\}$.

(d) $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -3/4 \\ 1 & 1 \end{bmatrix}$.

14. Show that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is not diagonalizable.

Solution: The characteristic polynomial of A is

$$\begin{vmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 3)(\lambda - 2)^2$$

and so A has eigenvalues $\lambda = 3$ (multiplicity 1) and $\lambda = 2$ (multiplicity 2). By Theorem 21.1, A is diagonalizable if and only if it has 3 linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Since $\lambda = 3$ has multiplicity 1 the $\lambda = 3$ eigenspace has dimension 1 (since the dimension of an eigenspace is always $\leq$ to the multiplicity of the corresponding eiegen-value). So if the linearly independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ exist then only one of them could be an eigenvector for $\lambda = 3$. So the only way A can be diagonalizable is if the $\lambda = 2$ eigenspace has 2 linearly independent eigenvectors and hence is of dimension 2. To find the $\lambda = 2$ eigenspace $\mathbb{E}_2$ we solve $(\lambda I - A)\mathbf{x} = \mathbf{0}$ when $\lambda = 2$:

-1	0	0	0
0	0	0	0
0	-1	0	0
1	0	0	0
0	1	0	0
0	0	0	0

Hence $\mathbb{E}_2 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ has dimension 1. Therefore A is not diagonalizable.