Mathematics for Data Science Tutorial 2 (week 4)

Semester 2, 2019

1. Find $\displaystyle\sum_{i=2}^{11}(i+1)(i+2)$ using the results from lectures.

   Solution:

$$
\begin{aligned}
\sum_{i=2}^{11}(i+1)(i+2) &= \sum_{i=2}^{11} i^2+3i+2 = \sum_{i=2}^{11} i^2 + 3\sum_{i=2}^{11} i + 2\sum_{i=2}^{11} 1 \\
&= \left(\sum_{i=1}^{11} i^2 - 1\right) + 3\left(\sum_{i=1}^{11} i - 1\right) + 2(10) \\
&= \sum_{i=1}^{11} i^2 + 3\sum_{i=1}^{11} i + 16 = \frac{11(12)(23)}{6} + 3\frac{11(12)}{2} + 16 \\
&= 506 + 198 + 16 = 720
\end{aligned}
$$

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2. For the two series

$$
\text{(a)} \sum_{k=2}^{\infty} \frac{2^k}{k!} \qquad \text{and} \qquad \text{(b)} \sum_{n=1}^{\infty} 3^{n+1}4^{-n}\,,
$$

   what method would be appropriate for deciding whether each series in convergent? Apply them.

   Solution: For (a), the powers and factorials seem to suggest the usage of the ratio test. Here, $a_k = 2^k/k!$ and $a_{k+1} = 2^{k+1}/(k+1)!$, and so

$$
\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{2^{k+1}}{(k+1)!}\,\frac{k!}{2^k}\right| \\
&= \lim_{k\to\infty}\left|\frac{2^k 2}{k!(k+1)}\,\frac{k!}{2^k}\right| \\
&= 2\lim_{k\to\infty}\left|\frac{1}{k+1}\right| = 0 < 1\,.
\end{aligned}
$$

   Since this limit is less than 1, the series is absolutely convergent by the ratio test. (Note that the fact that the summation begins at $k = 2$ has no effect on the issue of convergence of the infinite series.)

For (b), the terms are almost of the form $(a/b)^n = x^n$, which is like a geometric series, but not quite. We should try to rearrange the summation to look like a geometric series

$$\begin{aligned}\sum_{n=1}^{\infty} 3^{n+1}4^{-n} &= 3\sum_{n=1}^{\infty} 3^n 4^{-n} \\ &= 3\sum_{n=1}^{\infty} \frac{3^n}{4^n} \\ &= 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\end{aligned}$$

which is a geometric series with $x = 3/4 < 1$. Therefore, this series is convergent, and we can even write out the sum:

$$\sum_{n=1}^{\infty} 3^{n+1}4^{-n} = \frac{1}{1-3/4} = 4.$$

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3. A smooth function $f(x)$ is such that its derivatives at $x = 1$ alternate in sign between ± 2, i.e., $f(1) = 2$, $f'(1) = -2$, $f''(1) = 2$, $f^{(3)}(1) = -2$, $f^{(4)}(1) = 2$, $f^{(5)}(1) = -2$, etc.
   (a) Write down the Taylor series of f.
   (b) Find the exact value of $f(1/2)$.
   (c) * What is the function $f(x)$?

Solution:

(a) At a general n value in $\{0, 1, 2, 3, \ldots\}$, $f^{(n)}(1) = (-2)^n$. Thus, using the formula for Taylor series centred around a general point c (in this case, $c = 1$), we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}(x-1)^n \ .$$

(b) Substituting $x = 1/2$ in the above expression, we get

$$f(1/2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}(1/2-1)^n = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}\left(\frac{1}{-2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{(-2)^n}{(-2)^n} = \sum_{n=0}^{\infty} \frac{1}{n!} \ .$$

We should recognise this final answer to be e, because we know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} \ .$$

(c) From (a),

$$f(x) = \sum_{n=0}^{\infty} \frac{(-2x+2)^n}{n!} = e^{2-2x} = e^2 e^{-2x}\,.$$

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4. (a) Calculate the degree 3 MacLaurin Polynomial, $P_3(x)$, for $\cosh x$. (Note that for $f(x) = \cosh x$, $f'(x) = \sinh x$, and $f''(x) = f(x) = \cosh x$.)

   (b) Evaluate $P(1)$.

   (c) Use the remainder term to estimate the error in using $P(1)$ to estimate $\cosh(1)$ (to put an upper bound on the error you should use the fact that $2 < e < 3$ to find a bound on $\cosh z$ for $0 < z < 1$).

Solution: Let $f(x) = \cosh x$. Then $f'(x) = \sinh(x)$, $f''(x) = \cosh(x)$ and $f'''(x) = \sinh(x)$, so $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$ and $f'''(0) = 0$, so $P_3(x) = 1 + \frac{x^2}{2}$, and hence $P(1) = 3/2$.
The remainder term is $R_3(x) = \frac{f^{(iv)}(z)x^4}{4!}$ for some z between 0 and x.
Now, $f^{(iv)}(z) = \cosh(z) = \frac{e^z+e^{-z}}{2}$.
For $x = 1$, we have that cosh is an increasing function on $[0,1]$, so $|f^{(iv)}(z)| \le \cosh(1) = \frac{e+e^{-1}}{2}$. Using $2 < e < 3$, then $e < 3$ and $1/e < 1/2$ so $\cosh(1) < \frac{3+\frac{1}{2}}{2} = \frac{7}{4}$, so $|R_3(1)| < \frac{7}{4.4!} = \frac{7}{96}$.
(Note that while P_3 and P_2 are the same polynomial, R_3 gives a better error estimate than R_2.)
