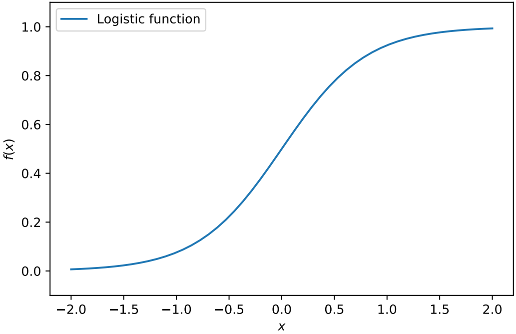
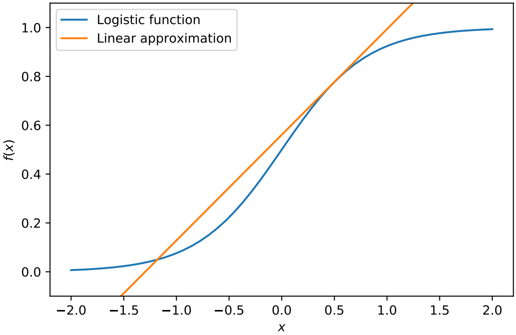
Course outline

1. Fundamentals
   - Notation
   - Functions
   - Approximation
2. Series
   - Summation
   - Taylor series
3. Linear algebra
   - Representing big, complex, data
   - Systems of equations
   - Dimension reduction
4. Probability
   - Discrete random variables
   - Continuous random variables & integration
5. Optimisation

Taylor series approximation



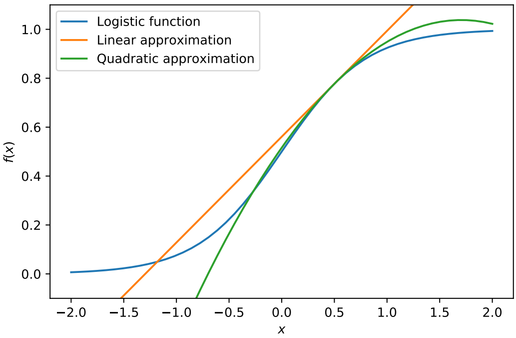
Linear approximation



Orange line is the tangent line at $a = 0.5$:

$$P_1(x) = f(a) + f'(x_0)(x - a)$$

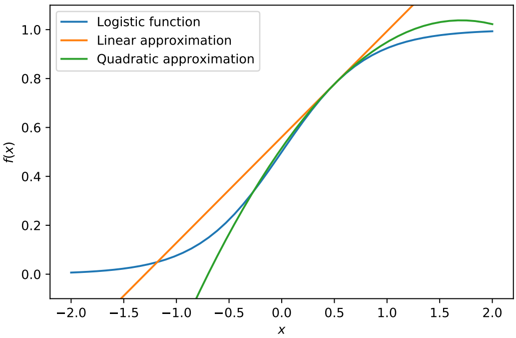
Quadratic approximation



Green line is a quadratic function:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Taylor polynomial approximation



In general:

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Linear approximation

How do we get to Taylor polynomials?

Let's find a degree 1 polynomial P_1 such that the value of $P_1(a)$ and $P_1'(a)$ agree with f at $x = a$.

Linear approximation

Example

Find the first order Taylor polynomial for $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(1.1)$.

Quadratic approximation

Find a degree 2 polynomial P_2 such that $P_2(a)$, $P_2'(a)$, and $P_2''(a)$ agree with f at a.

Quadratic approximation

Example

Approximate $\ln 1.1$ using the quadratic $P_2(x)$.

General polynomial approximation

nth degree polynomial approximation:

$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

Demand $P_n(a)$, $P_n'(a)$, ..., $P_n^{(n)}(a)$ agree with f at a.

Taylor and Maclaurin polynomials

Definition (Taylor Polynomial of degree n for $f(x)$ at $x = a$)

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor and Maclaurin polynomials

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Definition (Maclaurin Polynomial of degree n for $f(x)$)

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Taylor and Maclaurin polynomials

Example

Find the Taylor polynomial of degree n for $\ln x$ with $a = 1$. Use $P_4(x)$ to estimate $\ln 1.1$.

Theorem (Taylor's Theorem)

Suppose the function f has $(n+1)$ derivatives on some interval containing a and x. Then if

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the nth Taylor polyomial of f at a, $f(x) = P_n(x) + R_n(x)$ where the remainder

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

for some number z between a and x.

Proof by reference: Stewart, pg. 787-9

Taylor's Theorem

The point of this is:

$$\begin{aligned}\text{error} = |f(x) - P_n(x)| &= |R_n(x)| \\ &= \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \right|.\end{aligned}$$

If we can find C such that

$$|f^{(n+1)}(z)| \leq C, \quad \forall z \in [a, x]$$

then

$$|f(x) - P_n(x)| \leq \frac{C}{(n+1)!}|x-a|^{n+1}.$$

Taylor's Theorem

Example

Determine the accuracy of the use of $P_4(x)$ to estimate $\ln(1.1)$.

Taylor's Theorem

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Example

If we use the Maclaurin polynomial $P_n(x)$ for e^x, find the smallest value of n which gives $e = e^1$ to within the accuracy of 0.000005 (i.e 5-dp's).

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Example

Use a Maclaurin polynomial of degree 3 for $(1 + x)^{1/2}$ to approximate $\sqrt{5}$. Estimate the error.

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Example

Use a Maclaurin polynomial of degree 3 for $(1 + x)^{1/2}$ to approximate $\sqrt{5}$. Estimate the error.

Example

Find the Maclaurin polynomial of degree $n = 2k$ for $f(x) = \cos x$.

Taylor series

Definition (Taylor series of f at a)

If the function f has derivatives of all orders, then

$$f(x) = \lim_{n \to \infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

If $a = 0$ this is the *Maclaurin series*.

Some important Taylor series

$$e^x : \quad 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots \quad = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x : \quad x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x : \quad 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} : \quad 1 + x + x^2 + x^3 + \cdots \quad = \sum_{n=0}^{\infty} x^n$$

$$(1+x)^k \quad = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n$$

Taylor series

Theorem (Convergence of Power Series)

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

Then either

(1) *the series converges for all values of x, or*

(2) *the series converges only for $x = 0$, or*

(3) *there exists a number $R > 0$ such that $\sum_{n=0}^{\infty} a_n x^n$ converges for all x with $|x| < R$ and diverges for all x with $|x| > R$.*

The number R is called the radius of convergence. In case (1) we often write "$R = \infty$" and in case (2), $R = 0$.

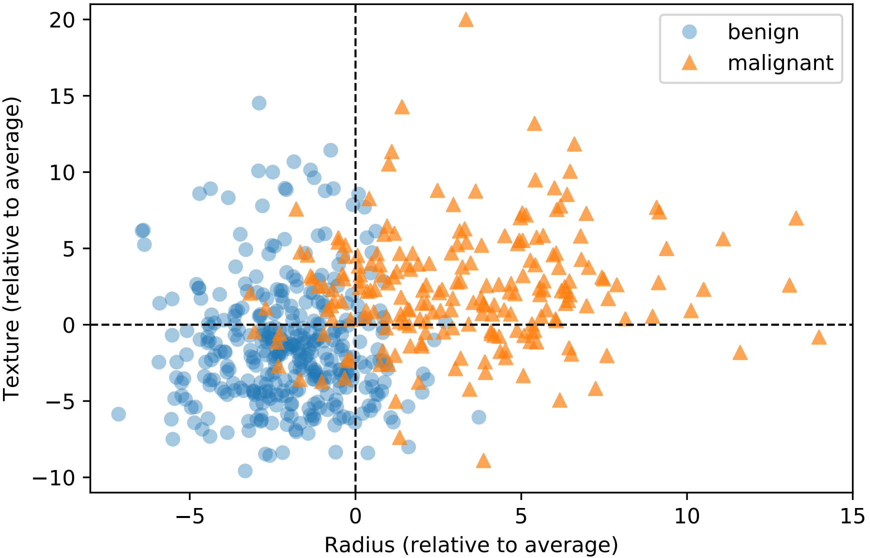
Taylor series

Example

Find the intervals of convergence for

- $\cos(x)$
- $1/(1-x)$

Who cares about Taylor series?



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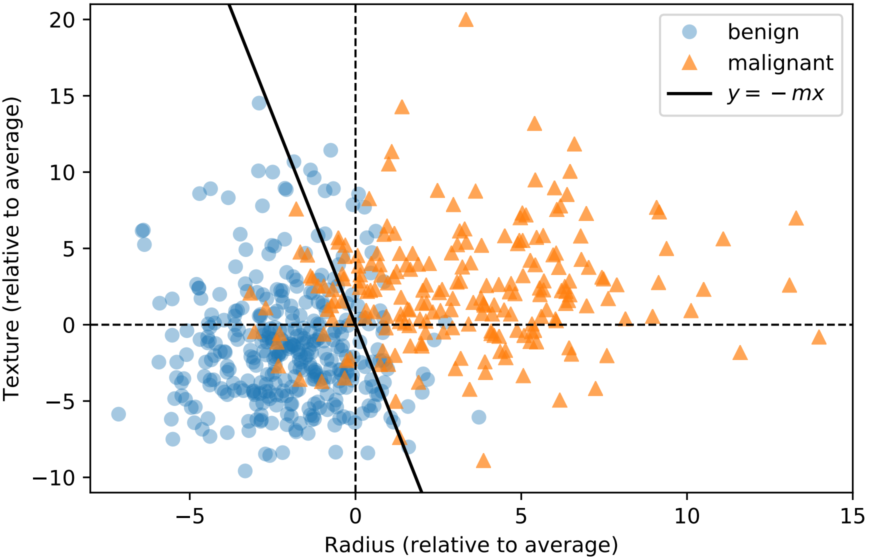
Who cares about Taylor series?



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Who cares about Taylor series?

Define a *loss function* $f(m)$:

$$f(m) = \sum_{i \in \{\text{misclassifications}\}} d\left((x_i, y_i) \text{ from line } y = -mx\right)$$
$$= \sum_{i \in \{\text{misclassifications}\}} \frac{|mx_i + y_i|}{\sqrt{m^2 + 1}}$$

This function $f(m)$ looks gross! No fun at all to differentiate.

Who cares about Taylor series?

Find the (approximate) minimum using *gradient descent*:

1. Guess a solution m
2. Change m by some amount h, $m \to m + h$, such that $f(m + h) < f(m)$
3. If h is very small, STOP. Otherwise GOTO 2.

But how do we choose h?

Who cares about Taylor series?

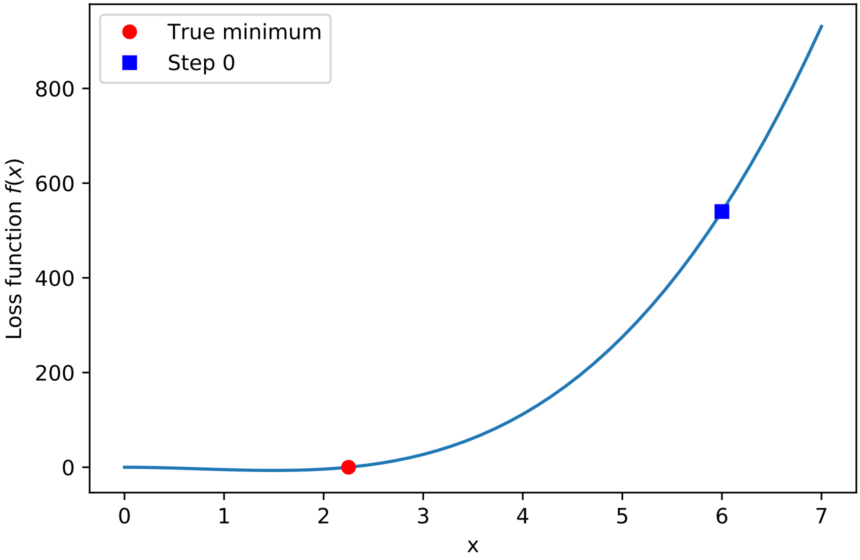
Let $x = m + h$, $a = m$. Taylor polynomial ($n = 1$):

$$f(m + h) \approx f(m) + f'(m)h$$

Choose $h = -\eta f'(m)$, so

$$f(m + h) = f(m) - \eta \left(f'(m)\right)^2 < f(m)$$

Who cares about Taylor series?



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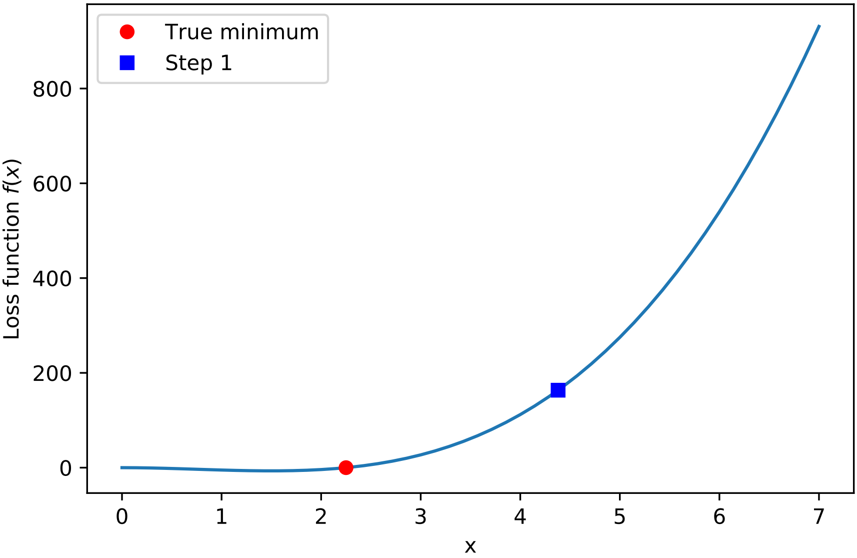

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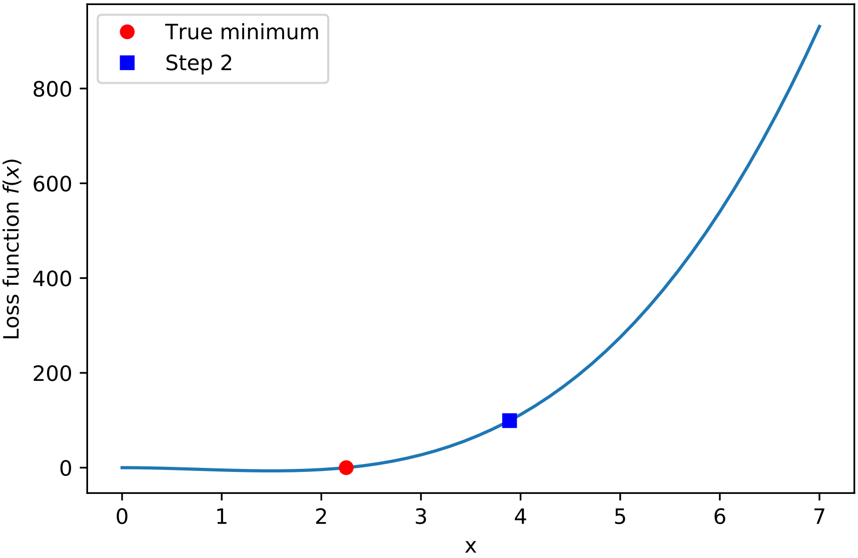

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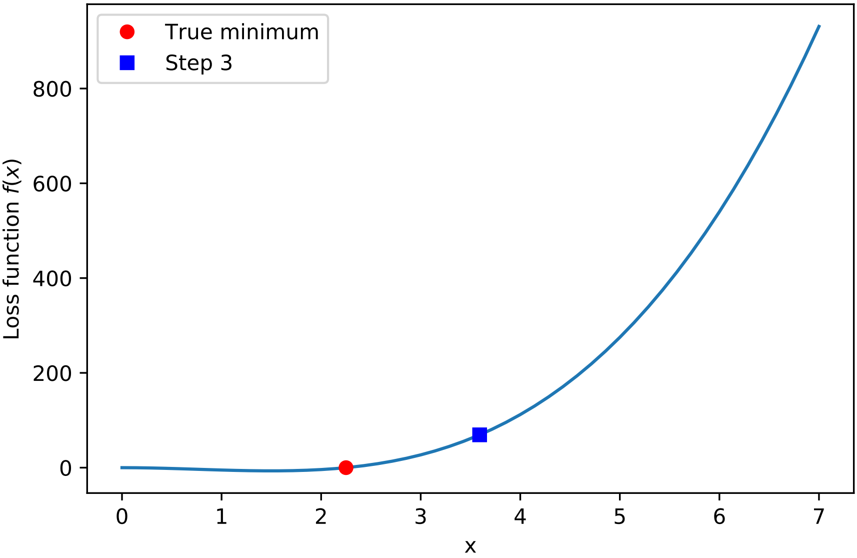

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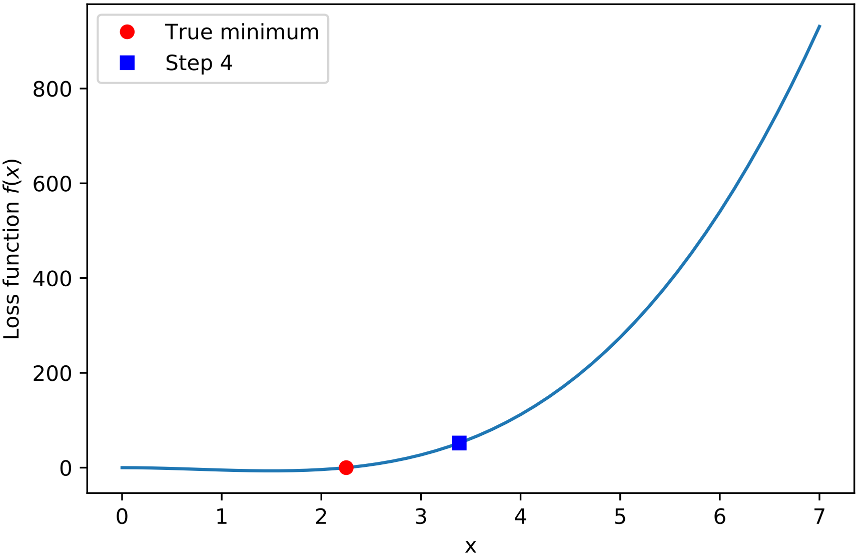

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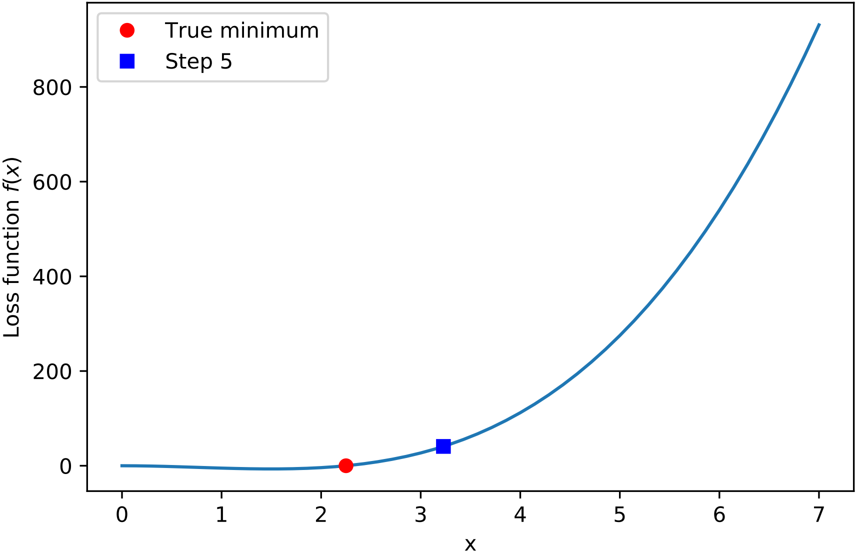

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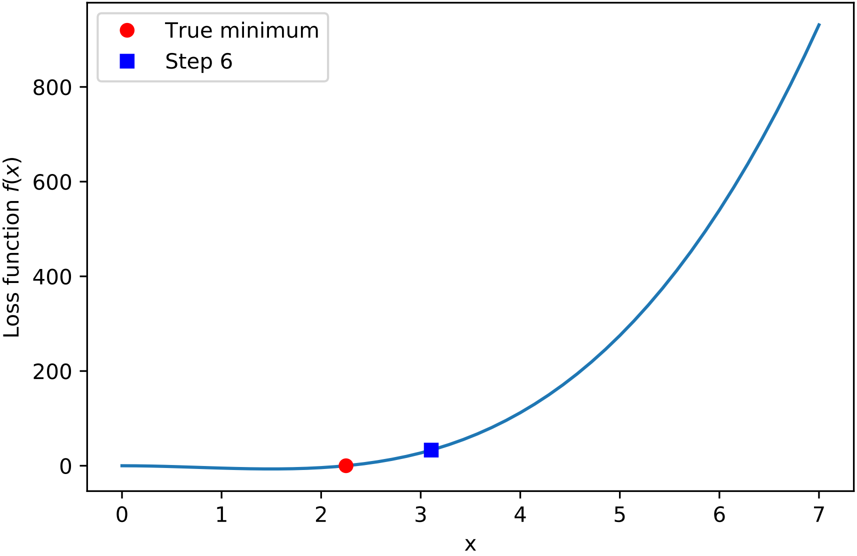

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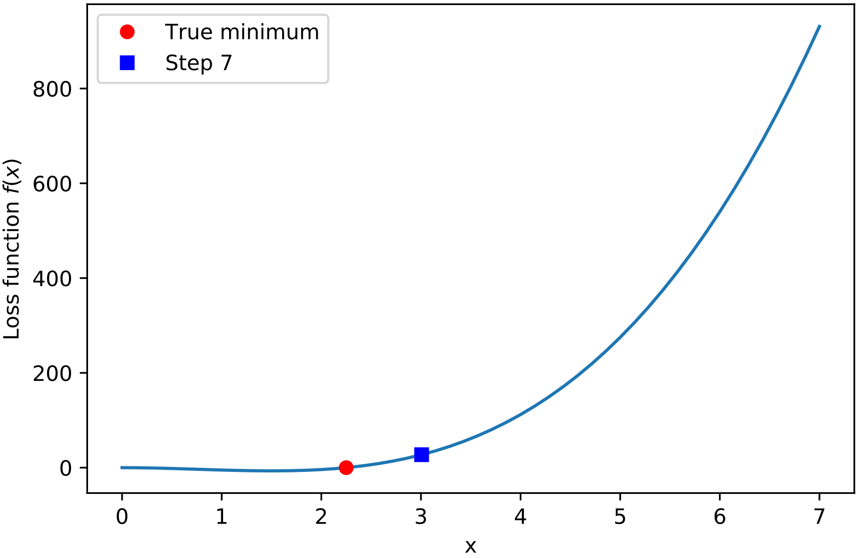

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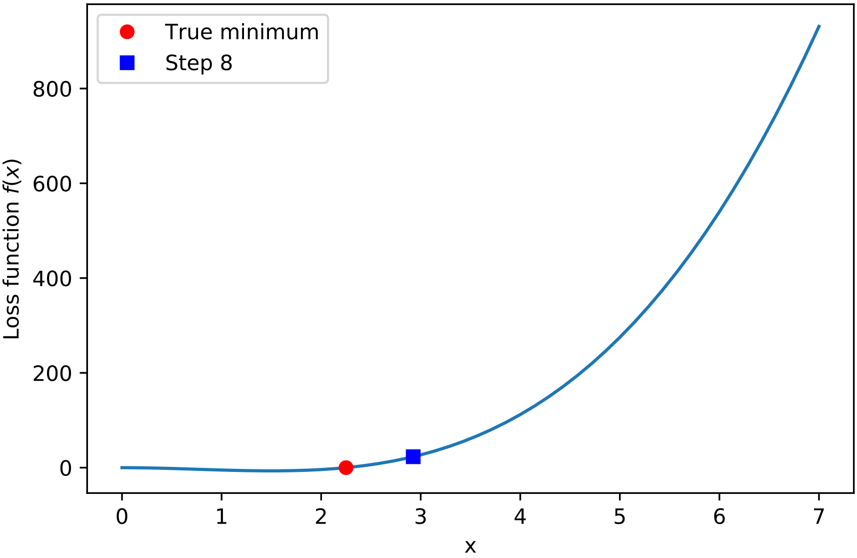

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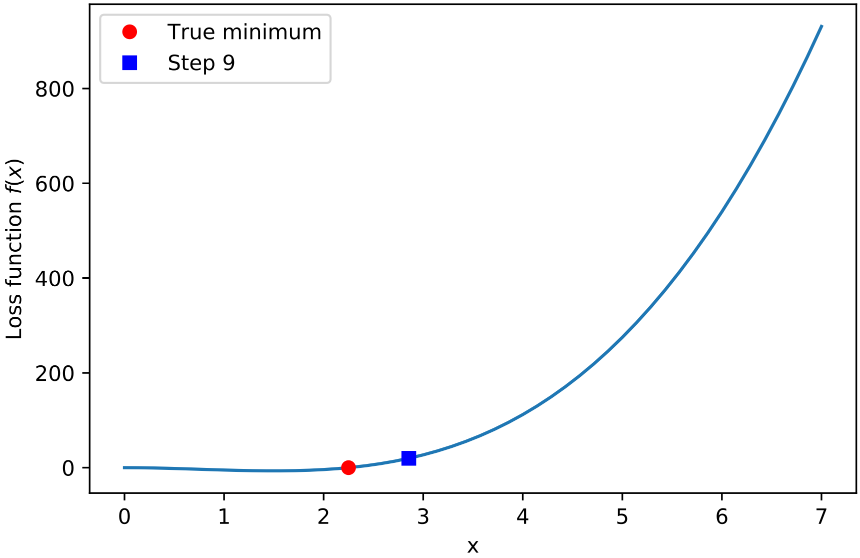

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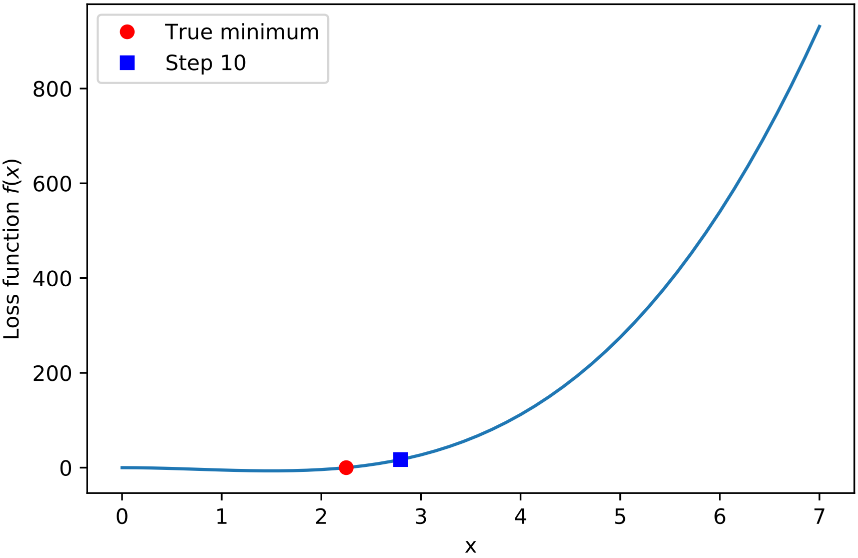

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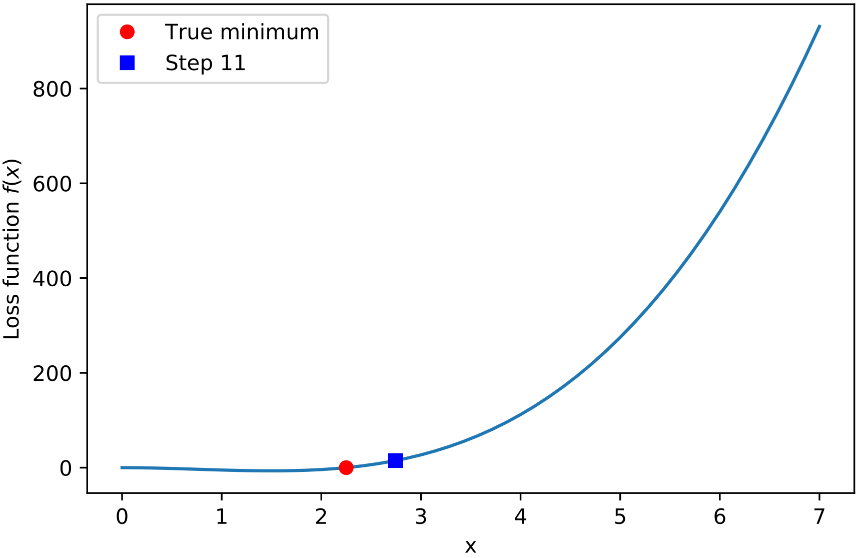

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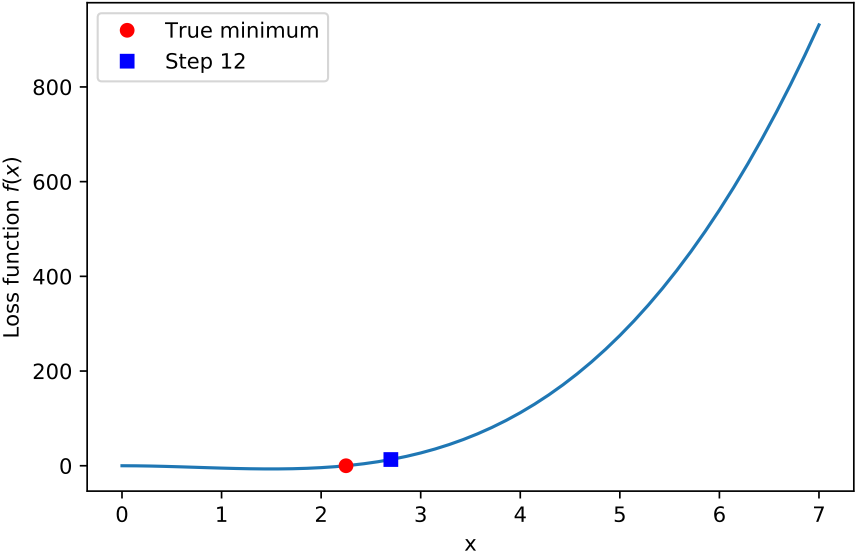

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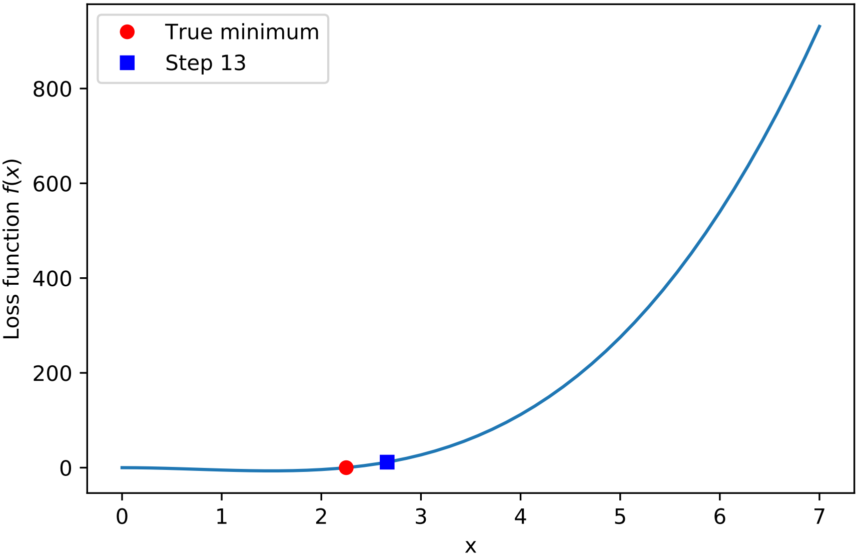

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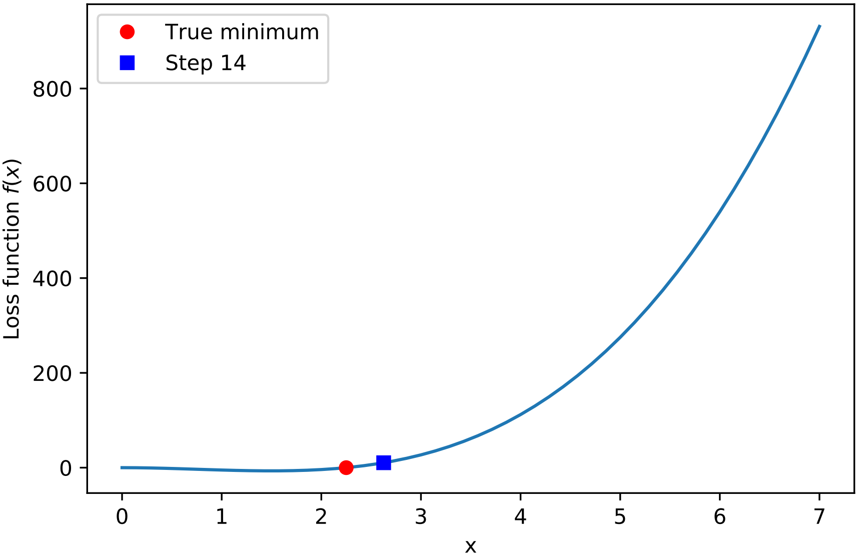

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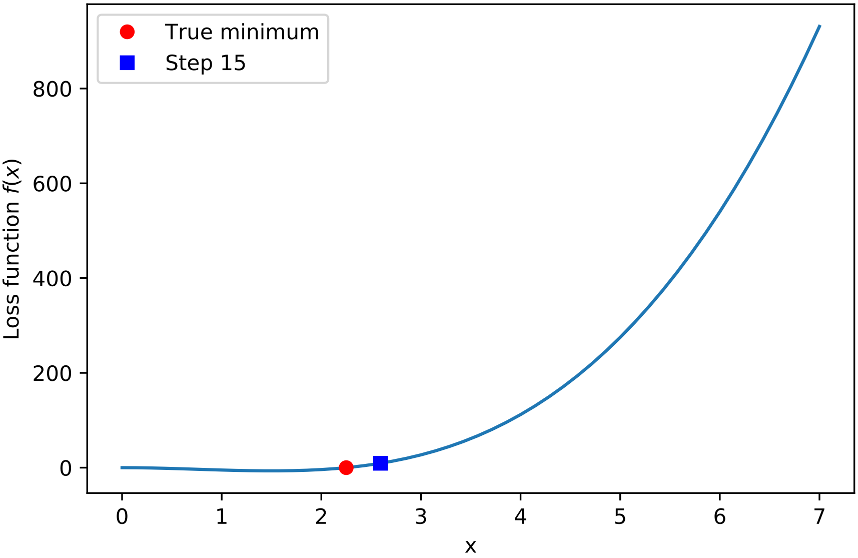

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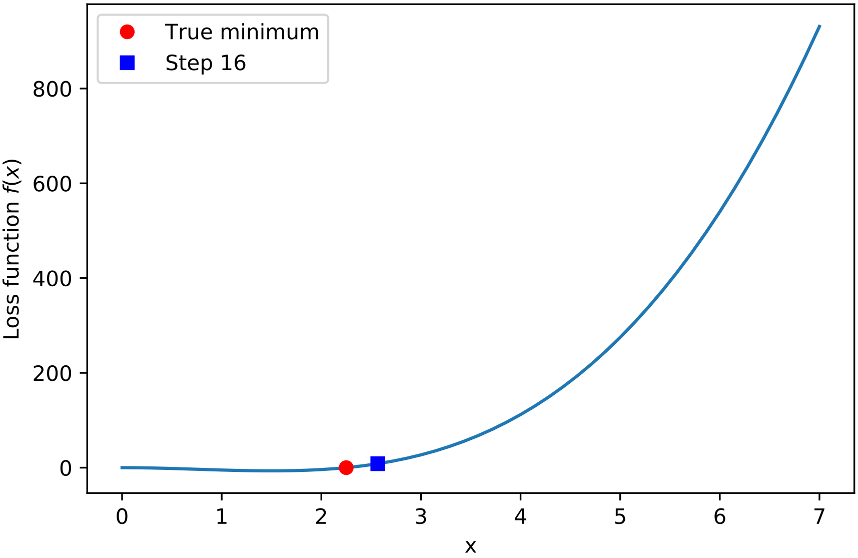

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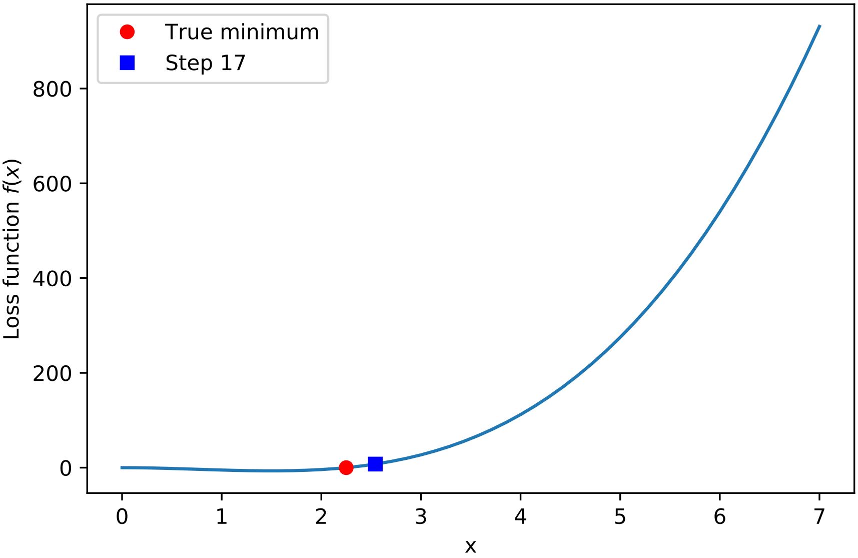

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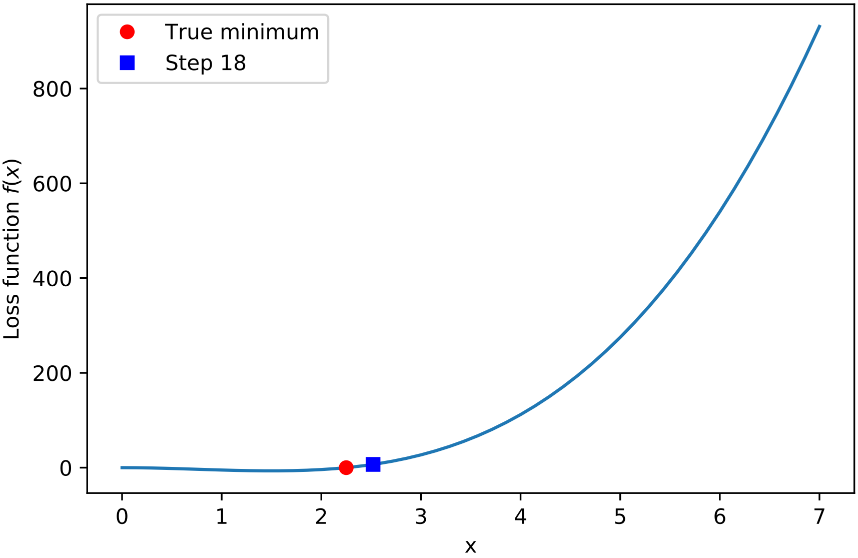

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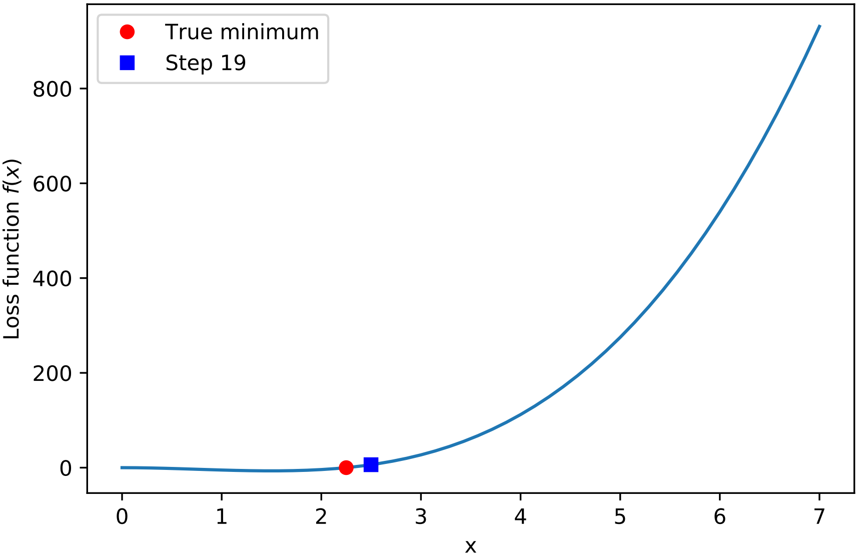

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