Practice Questions (week 10)

Semester 2, 2019

These questions are about conditional probability, Bayes' formula, law of total probability, generalised Bayes' rule, naive Bayes classifiers.

- Ross, *A first course in Probability* (6th Ed.), Chapter 3.

1. (a) Suppose there is a 40% chance it rains today, and a 25% chance that it rains both today and tomorrow. If it happens to rain later today, what is the chance of rain tomorrow?

   (b) Suppose there is a 10% chance of storm today, and a 30% chance of storm tomorrow if there is a storm today. What is the chance of a storm on both days?

Solution:

(a) We have

$$\begin{aligned}\Pr(\text{rain tomorrow}|\text{rain today}) &= \frac{\Pr(\text{rain today and tomorrow})}{\Pr(\text{rain today})}\\ &= \frac{0.25}{0.40} = 0.675\,.\end{aligned}$$

(b) We have

$$\begin{aligned}\Pr(\text{storm today and tomorrow}) &= \Pr(\text{storm tomorrow}|\text{storm today})\,\Pr(\text{storm today})\\ &= 0.3 \times 0.1 = 0.03\,.\end{aligned}$$

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2. Two fair dice are rolled. What is the (conditional) probability of:

   (a) obtaining a sum of at least 10 given at least one 6?

   (b) rolling at least one 6 given the sum is at least 10?

   (c) obtaining a sum of at least 10 given the first dice rolled is a 6?

   (d) obtaining a double digit product given a sum of at most 7?

   (e) obtaining a sum of at most 7 given a double digit product?

Solution:

(a) There are 11 outcomes in which at least one die is a 6, specifically

$$(1,6),(2,6),\ldots,(5,6),(6,6),(6,5),\ldots,(6,1)\,.$$

The sum is greater than or equal to 10 in only 5 of these, namely

$$(4,6),(5,6),(6,4),(6,5),(6,6)\,.$$

Thus

$$\begin{aligned}\Pr(\text{sum} \geq 10|\text{at least one } 6) &= \frac{n(\text{sum} \geq 10 \text{ and at least one } 6)}{n(\text{at least one } 6)}\\ &= \frac{5}{11} \approx 0.4545\,.\end{aligned}$$

(b) We already know that $n(\text{sum}> 10 \text{ and at least one } 6) = 5$. The number of ways in which the sum is at least 10 is 6, namely

$$(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\,.$$

Therefore

$$\begin{aligned}\Pr(\text{at least one } 6|\text{sum} \geq 10) &= \frac{n(\text{sum} \geq 10 \text{ and at least one } 6)}{n(\text{sum} \geq 10)}\\ &= \frac{5}{6} \approx 0.8333\,.\end{aligned}$$

(c) There are six outcomes in which the first die is a 6 and three of these have a sum of at least 10, therefore

$$\begin{aligned}\Pr(\text{sum} \geq 10|\text{first die is } 6) &= \frac{n(\text{sum} \geq 10 \text{ and first die is } 6)}{n(\text{first die is } 6)}\\ &= \frac{3}{6} = 0.5\,.\end{aligned}$$

Alternatively, if the first dice rolled is given to be a 6 then we only need to roll a 4 or higher on the second roll to obtain a sum of at least 10, clearly this event has probability 1/2.

(d) The outcomes with sum of at most 7 are

$$\begin{aligned}(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(2,5),(3,1),\ldots,(3,4),\\ (4,1),(4,2),(4,3),(5,1),(5,2),(6,1)\end{aligned}$$

and of these only $(2,5),(3,4),(4,3),(5,2)$ have a product with double digits (product ≥ 10). Thus

$$\begin{aligned}\Pr(\text{prod} \geq 10|\text{sum} \geq 7) &= \frac{n(\text{prod} \geq 10 \text{ and sum} \geq 7)}{n(\text{sum} \geq 7)}\\ &= \frac{4}{21} \approx 0.1905\,.\end{aligned}$$

(e) We already know that $n(\text{prod} \geq 10 \text{ and sum} \geq 7) = 4$. The outcomes with a double digit product are

$$(2,5),(2,6),(3,4),(3,5),(3,6),(4,3),(4,4),(4,5),(4,6),$$
$$(5,2),(5,3),(5,4),(5,5),(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\,,$$

which is a total of 19. Therefore

$$\begin{aligned}\Pr(\text{sum} \geq 7|\text{prod} \geq 10) &= \frac{n(\text{prod} \geq 10 \text{ and sum} \geq 7)}{n(\text{prod} \geq 10)} \\ &= \frac{4}{19} \approx 0.2105\,.\end{aligned}$$

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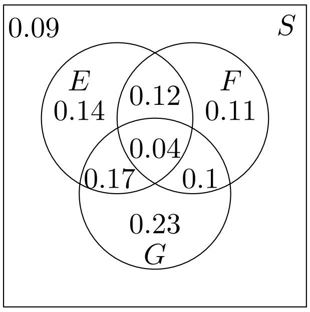
3. Consider events A, B, C in a sample space S with the probability of the various intersections as indicated in the following Venn diagram



Determine each of the following:

(a) $\Pr(E|F)$

(b) $\Pr(E \cup F|G^c)$

(c) $\Pr(E \cap G|E \cup F)$

(d) $\Pr(F^c|G)$

Solution: The probability of various events (and their intersections) are easily obtained by adding up the appropriate portions in the Venn diagram.

(a)

$$\begin{aligned}\Pr(E|F) &= \frac{\Pr(E \cap F)}{\Pr(F)} \\ &= \frac{0.12 + 0.04}{0.11 + 0.12 + 0.04 + 0.1} \\ &= \frac{0.16}{0.37} \approx 0.432\,.\end{aligned}$$

(b)

$$\begin{aligned}\Pr(E \cup F | G^c) &= \frac{\Pr((E \cup F) \cap G^c)}{\Pr(G^c)} \\ &= \frac{0.14 + 0.12 + 0.11}{0.09 + 0.14 + 0.12 + 0.11} \\ &= \frac{0.37}{0.46} \approx 0.804\,.\end{aligned}$$

(c)

$$\begin{aligned}\Pr(E \cap G | E \cup F) &= \frac{\Pr((E \cap G) \cap (E \cup F))}{\Pr(E \cup F)} \\ &= \frac{0.17 + 0.04}{0.14 + 0.12 + 0.11 + 0.17 + 0.04 + 0.1} \\ &= \frac{0.21}{0.68} \approx 0.309\,.\end{aligned}$$

(Notice that $(E \cap G) \cap (E \cup F) = E \cap G$)

(d)

$$\begin{aligned}\Pr(F^c | G) &= \frac{\Pr(F^c \cap G)}{\Pr(G)} \\ &= \frac{0.17 + 0.23}{0.23 + 0.1 + 0.04 + 0.17} \\ &= \frac{0.40}{0.54} \approx 0.741\,.\end{aligned}$$

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4. A magician shuffles a standard deck of 52 playing cards and you pick one at random. They get to ask one question before trying to guess your card. Which of the following questions should they ask to have the best chance of guessing your card correctly?

- Is it a black card?
- Is it a hearts card?
- Is it a queen?
- Is it the 2 of spades?

Solution: Let B denote the event the card you picked is black, H denote the event it is a hearts card, Q denote the event it is a queen card and $2S$ denote the event it is the 2 of spades. Clearly $\Pr(B) = 1/2$, $\Pr(H) = 1/4$, $\Pr(Q) = 1/13$ and $\Pr(S2) = 1/52$. Let $C1, C2, C3, C4$ denote the event that the magician guesses your card correctly after asking question 1, 2, 3 or 4 respectively. If the magician first asks if your card is a black card, then if the answer is yes they only need to guess the correct card from the 26 black cards, that is $\Pr(C1|B) = 1/26$. Similarly, if the answer is no they need only guess the correct card from the 26 red cards, hence $\Pr(C1|B^c) = 1/26$. It follows from the law of total probability that

$$\Pr(C1) = \Pr(C1|B)\Pr(B) + \Pr(C1|B^c)\Pr(B^c) = \frac{1}{26}\frac{1}{2} + \frac{1}{26}\frac{1}{2} = \frac{1}{26}.$$

Now, if the magician asks if your card is a hearts card, if the answer is yes they need only guess the correct card out of the 13 hearts ($\Pr(C2|H) = 1/13$), but if it is not they need only guess the correct card from the remaining 39 ($\Pr(C2|H^c) = 1/39$). That is,

$$\Pr(C2) = \Pr(C2|H)\Pr(H) + \Pr(C2|H^c)\Pr(H^c) = \frac{1}{13}\frac{1}{4} + \frac{1}{39}\frac{3}{4} = \frac{1}{26}.$$

Now, if the magician asks if your card is a queen card, if the answer is yes they need only guess the correct card out of the 4 queens, but if it is not they need only guess the correct card from the remaining 48 cards. That is,

$$\Pr(C3) = \Pr(C3|Q)\Pr(Q) + \Pr(C3|Q^c)\Pr(Q^c) = \frac{1}{4}\frac{1}{13} + \frac{1}{48}\frac{12}{13} = \frac{1}{26}.$$

Lastly, if the magician asks if your card is the 2 of spades, if the answer is yes they are guaranteed to guess the correct card, but if it is not they need only guess the correct card from the remaining 51. That is,

$$\Pr(C4) = \Pr(C4|2S)\Pr(2S) + \Pr(C4|2S^c)\Pr(2S^c) = 1\frac{1}{52} + \frac{1}{51}\frac{51}{52} = \frac{1}{26}.$$

Thus, it does not matter which question they ask, the probability of guessing the correct card is 1/26 in each case (which is double the chance of a 'blind' guess).

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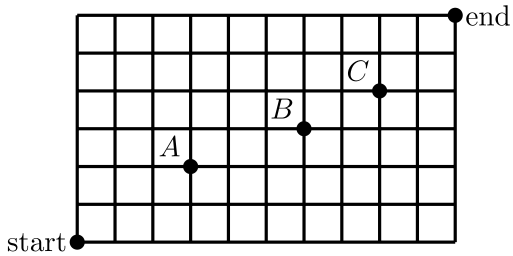
5. Consider an ant that must traverse the grid shown below from start to end only moving up or right along each edge. Suppose that every valid path is equally likely. (See also the similar question from last weeks practice questions.)



(a) What is the probability the ant passes through the point B given it passes through the point A?

(b) What is the probability the ant passes through the point A given it passes through the point C?

(c) What is the probability the ant passes through the point C given it passes through the point B?

(d) What is the probability the ant passes through the point C given it passes through both the points A and B?

(e) What is the probability the ant passes through the point B given it passes through both the points A and C?

Solution:

(a) Let A, B, C denote the events of passing through the respective points.

Going from start to end requires 6 up and 10 right movements. Thinking about the problem in terms of placing the up movements in containers surrounding the right movements then the total number of possible paths is $\binom{6+(11-1)}{6} = 8008$. Similarly, by breaking the path into traversals of two smaller grids, the number of paths passing through point A is

$$\binom{2+(4-1)}{2}\binom{4+(8-1)}{4} = 10 \times 330 = 3300\,.$$

Thus $\Pr(A) = \frac{3300}{8008} \approx 0.412$. The number of paths passing through both A and B can be obtained by considering the number of paths through the three smaller grids between start, A, B and end. We then obtain

$$\binom{2+(4-1)}{2}\binom{1+(4-1)}{1}\binom{3+(5-1)}{3} = 10\times 4\times 35 = 1400\,,$$

such paths and therefore $\Pr(A\cap B) = \frac{1400}{8008} \approx 0.175$. Consequently $\Pr(B|A) = \frac{\Pr(A\cap B)}{\Pr(A)} = \frac{1400}{3300} \approx 0.424$.

Alternatively, the fact that the ant passes through A can be seen as reducing the problem to considering what portion of valid paths from A to end also pass through B. There are $\binom{4+(8-1)}{4} =$ 330 paths of which $\binom{1+(4-1)}{1}\binom{3+(5-1)}{3} = 140$ pass through B, thus yielding $140/330 \approx 0.424$.

(b) Going through this quicker this time

$$
\begin{aligned}
\Pr(A|C) &= \frac{\Pr(A\cap C)}{\Pr(C)} \\
&= \frac{\binom{2+(4-1)}{2}\binom{2+(6-1)}{2}\binom{2+(3-1)}{2}/8008}{\binom{4+(9-1)}{4}\binom{2+(3-1)}{2}/8008} \\
&= \frac{10\times 21\times 6/8008}{495\times 6/8008} = \frac{210}{495} \approx 0.424\,.
\end{aligned}
$$

(c) Similarly

$$
\begin{aligned}
\Pr(C|B) &= \frac{\Pr(B\cap C)}{\Pr(B)} \\
&= \frac{\binom{3+(7-1)}{3}\binom{1+(3-1)}{1}\binom{2+(3-1)}{2}/8008}{\binom{3+(7-1)}{3}\binom{3+(5-1)}{3}/8008} \\
&= \frac{84\times 3\times 6/8008}{84\times 35/8008} = \frac{18}{35} \approx 0.514\,.
\end{aligned}
$$

(d) We have $\Pr(C|A\cap B) = \frac{\Pr(A\cap B\cap C)}{\Pr(A\cap B)}$. Recall that from the first part we have $\Pr(A\cap B) = \frac{1400}{8008}$. The number of paths passing through A, B and C is

$$
\binom{2+(4-1)}{2}\binom{1+(4-1)}{1}\binom{1+(3-1)}{1}\binom{2+(3-1)}{2} = 10\times 4\times 3\times 6 = 720\,,
$$

and therefore $\Pr(A\cap B\cap C) = \frac{720}{8008}$. Thus $\Pr(C|A\cap B) = \frac{720}{1400} \approx 0.514$.

(Alternatively, observe that given the path passes through B, then whether or not the path will pass through C is not effected by what the path does before B, that is we expect $\Pr(C|A\cap B) = \Pr(C|B)$. However, a similar argument does not work with $\Pr(B|A\cap C)$, why not?)

(e) From previous parts we already have $\Pr(A\cap B\cap C) = \frac{720}{8008}$ and $\Pr(A\cap C) = \frac{1260}{8008}$ and therefore

$$
\Pr(B|A\cap C) = \frac{\Pr(A\cap B\cap C)}{\Pr(A\cap C)} = \frac{720}{1260} \approx 0.571\,.
$$

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6. A box contains fair and biased coins. Each biased coin has a probability of flipping heads 60% of the time. It is known that 30% of the coins in the box are biased. A coin is randomly selected from the box and flipped.

(a) What is the probability it is biased if the result is heads?

(b) Suppose we flip the same coin a second time and it again return heads, what is the probability it is biased.

(c) How many times in a row would we need to flip heads to be at least 50% certain the chosen coin is a biased one?

Solution:

(a) Let B be the event that a biased coin was chosen, and H be the event that heads was flipped. We know that $\Pr(H|B) = 0.6$ and $\Pr(B) = 0.3$. Additionally, by the law of total probability

$$\Pr(H) = \Pr(H|B)\Pr(B) + \Pr(H|B^c)\Pr(B^c) = 0.6\times 0.3 + 0.5\times 0.7 = 0.53\,.$$

(Note $\Pr(H|B^c)$ is the probability of flipping heads for the fair coin.)
Using Bayes' rule we have

$$\Pr(B|H) = \frac{\Pr(H|B)\Pr(B)}{\Pr(H)} = \frac{0.6\times 0.3}{0.53} \approx 0.340\,.$$

(b) Let HH be the event of two heads. Observe that $\Pr(HH|B) = 0.6^2 = 0.36$ and $\Pr(HH|B^c) = 0.5^2 = 0.25$. Consequently $\Pr(HH) = 0.36\times 0.3 + 0.25\times 0.7 = 0.283$ and using Bayes' rule we obtain

$$\Pr(B|HH) = \frac{\Pr(HH|B)\Pr(B)}{\Pr(HH)} = \frac{0.36\times 0.3}{0.283} \approx 0.382\,.$$

(c) Let H^n denote the event of flipping n consecutive heads. One has $\Pr(H^n|B) = 0.6^n$ and $\Pr(H^n|B^c) = 0.5^n$ and therefore $\Pr(H^n) = 0.6^n\times 0.3 + 0.5^n\times 0.7$ and from Bayes' rule

$$\Pr(B|H^n) = \frac{\Pr(H^n|B)\Pr(B)}{\Pr(H^n)} = \frac{0.6^n\times 0.3}{0.6^n\times 0.3 + 0.5^n\times 0.7}\,,$$

and by trial and error we find $\Pr(B|H^5) \approx 0.516$ is the first time this exceeds 50%, so 5 consecutive heads would be needed.

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7. A movie recommendation website has $70,000$ users that enjoy horror movies and $130,000$ that do not. After the release of a new horror movie all of the users were polled. $55,000$ of the users that enjoy horror movies said they enjoyed the new movie, whereas $90,000$ users that do not enjoy horror also did not enjoy the new movie. Given a randomly chosen user that enjoyed the new movie, what is the probability they enjoy horror movies generally?

Solution: Let H be the event of the user enjoying horror and E be the event they enjoyed the new movie. We want to determine $\Pr(H|E)$. From the given data we know there are $200,000$ users. Additionally, $95,000$ users enjoyed the movie (noting $130,000 - 90,000 = 40,000$ are those that don't generally enjoy horror movies). Since the number that enjoyed the movie and generally enjoy horror is $55,000$ then we have

$$\Pr(H|E) = \frac{\Pr(H \cap E)}{\Pr(E)} = \frac{\frac{55000}{200000}}{\frac{95000}{200000}} = \frac{11}{19} \approx 0.579\,.$$

Alternatively, we could answer this question using Bayes' rule. From the data we know that $\Pr(H) = 7/20 = 0.35$ and $\Pr(H^c) = 13/20 = 0.65$. Additionally, $\Pr(E|H) = 55/70 = 11/14$ and $\Pr(E^c|H^c) = 9/13$. Observe that $\Pr(E|H^c) = 1 - \Pr(E^c|H^c) = 4/13$. From the law total probability we have

$$\begin{aligned}\Pr(E) &= \Pr(E|H)\Pr(H) + \Pr(E|H^c)\Pr(H^c) \\ &= \frac{11}{14}\frac{7}{20} + \frac{4}{13}\frac{13}{20} \\ &= \frac{19}{40}\,.\end{aligned}$$

Now, using Bayes' rule we have

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E)} = \frac{\frac{11}{14}\frac{7}{20}}{\frac{19}{40}} = \frac{11}{19} \approx 0.579\,.$$

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8. Let A, B be two events in a probability space S. What can you say about $\Pr(A|B)$ in each of the following cases? (You may assume $\Pr(B) > 0$.)

   (a) $A \subset B$

   (b) $B \subset A$

   (c) $A \subset B^c$

   (d) $B \subset A^c$

   (e) $\Pr(A) = 0.6$, $\Pr(B) = 0.4$ and A, B are independent events

   (f) $\Pr(B|A) = 0.8$, $\Pr(B) = 0.7$ and $\Pr(A) = 0.3$

Solution:

(a) With $A \subset B$ then $A \cap B = A$ and thus

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A)}{\Pr(A)} = 1\,.$$

(b) With $B \subset A$ then $A \cap B = A$ and thus $\Pr(A|B) = \frac{\Pr(B)}{\Pr(A)}$.
(No further simplifications can be made)

(c) With $A \subset B^c$ then $A \cap B = \emptyset$ (i.e. the events are mutually exclusive) and thus $\Pr(A \cap B) = 0$ which means $\Pr(A|B) = 0$.

(d) With $B \subset A^c$ we again have $A \cap B = \emptyset$ and so $\Pr(A|B) = 0$.

(e) Since A, B are independent $\Pr(A|B) = \Pr(A) = 0.6$.

(f) Via Bayes' rule $\Pr(A|B) = 0.8 \times 0.3/0.7 \approx 0.343$.

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9. Differential privacy is a technique that can be used to help protect the privacy of individuals participating in surveys (for more on this, see this SIAM News article.)
Consider a survey asking 'Have you ever smoked weed?'. Participants may hesitate to respond 'yes' truthfully due to fear of potential consequences if they were identified. To obscure the response of the individual, the survey can be designed as follows. If a participant answers 'yes', then record a 'yes', but if a participant answers 'no', then instead randomly record 'yes' or 'no' with $50:50$ chance. Let Y be the event of a 'yes' being recorded and let S be the event of the individual having smoked weed before. Suppose $\Pr(S) = 0.2$ and that 90% of participants are anticipated answer truthfully given this survey design (that is $\Pr(Y|S) = 0.9$, i.e. there is still some distrust).

(a) What is $\Pr(Y)$?

(b) What is $\Pr(S|Y)$? Explain how this increases the privacy of participants.

(c) If $\Pr(S)$ is unknown, e.g. let $\Pr(S) = p$, and the survey produces a result of $\Pr(Y) = 0.7$, then what is $\Pr(S)$?

(d) Why would it not be sensible to instead modify 'yes' responses in the design rather than 'no' responses?

Solution:

(a) It is reasonable to assume that those who have never smoked weed will always truthfully answer 'no', but then their responses are changed to 'yes' 50% of the time due to the survey design, that is $\Pr(Y|S^c) = 0.5$. Therefore, using the law of total probability

$$\Pr(Y) = \Pr(Y|S)\Pr(S) + \Pr(Y|S^c)Pr(S^c) = 0.9 \times 0.2 + 0.5 \times 0.8 = 0.58\,.$$

(b) Using Bayes' formula

$$\Pr(S|Y) = \frac{\Pr(Y|S)\Pr(S)}{\Pr(Y)} = \frac{0.9 \times 0.2}{0.58} \approx 0.310\,.$$

Privacy is provided because there is only a 31% chance that a recorded 'yes' implies that a participant has smoked weed before.

(c) In this case we have

$$0.7 = \Pr(Y) = \Pr(Y|S)\Pr(S) + \Pr(Y|S^c)Pr(S^c) = 0.9\times p + 0.5\times(1-p) = 0.5 + 0.4p\,,$$

and therefore $0.4p = 0.2$, that is $p = 0.5$ which means 50% of survey participants have smoked weed.

(d) Because all 'yes' responses in the survey would be solely from participants that have smoked weed, that is $\Pr(S|Y) = 1$.

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10. You have lost your TV remote. You presume it is equally likely to be in one of three rooms in your house. Let R_1, R_2, R_3 be the event that the remote is in each of the three rooms. You intend to start with a quick look in your bedroom (R_1 say), but it is quite a mess and you are only 60% confident you will find the remote if it is in there. Let F be the event that you find the remote during your quick search of the first room. If your quick search is unsuccessful, what is the probability the remote is in each of the three rooms?

Solution: We want to determine $\Pr(R_i|F^c)$ for each $i = 1, 2, 3$. Our assumption is that $\Pr(R_i) = \frac{1}{3}$ for $i = 1, 2, 3$ and that $\Pr(F|R_1) = 0.6$. Consequently one has $\Pr(F^c|R_1) = 0.4$. Additionally, if the remote happens to be in rooms 2 or 3 then the quick search is guaranteed to fail, that is $\Pr(F|R_2) = \Pr(F|R_3) = 0$. If follows that $\Pr(F^c|R_2) = \Pr(F^c|R_3) = 1$. Using Bayes' formula we have

$$\begin{aligned}
\Pr(R_1|F^c) &= \frac{\Pr(F^c|R_1)\Pr(R_1)}{\Pr(F^c|R_1)\Pr(R_1) + \Pr(F^c|R_2)\Pr(R_2) + \Pr(F^c|R_3)\Pr(R_3)} \\
&= \frac{0.4\frac{1}{3}}{0.4\frac{1}{3} + 1\frac{1}{3} + 1\frac{1}{3}} \\
&= \frac{2/15}{12/15} = \frac{1}{6} \approx 0.167\,.
\end{aligned}$$

Similarly for the second room we have

$$\begin{aligned}
\Pr(R_2|F^c) &= \frac{\Pr(F^c|R_2)\Pr(R_2)}{\Pr(F^c|R_1)\Pr(R_1) + \Pr(F^c|R_2)\Pr(R_2) + \Pr(F^c|R_3)\Pr(R_3)} \\
&= \frac{1\frac{1}{3}}{0.4\frac{1}{3} + 1\frac{1}{3} + 1\frac{1}{3}} \\
&= \frac{1/3}{12/15} = \frac{5}{12} \approx 0.417\,.
\end{aligned}$$

Lastly, it is straightforward to see that the result is the same for the third room, i.e. $\Pr(R_3|F^c) = \frac{5}{12}$.

11. Consider rolling two fair die. Let A be the event the first dice roll is even, B be the event the second dice roll is even, and C be the event that their sum is odd.

    (a) Show that the three events are pairwise independent (i.e. A, B are independent, A, C are independent and B, C are independent), but not independent (i.e. all together).

    (b) Consider now the event D that the multiple of the two dice rolls is odd. What can we say about the independence of A, B, C, D or any pair/triple of these?

Solution:

(a) A, B are clearly independent events (that is $\Pr(A \cap B) = \Pr(A)\Pr(B)$) and also $\Pr(A) = \Pr(B) = 0.5$. Further, C is equivalent to the event $(A \cap B^c) \cup (A^c \cap B)$. Since $A \cap B^c, A^c \cap B$ are mutually exclusive and the pairs A, B^c and A^c, B are also independent (since A, B are independent) one has

$$\begin{aligned}
\Pr(C) &= \Pr((A \cap B^c) \cup (A^c \cap B)) \\
&= \Pr(A \cap B^c) + \Pr(A^c \cap B) \\
&= \Pr(A)\Pr(B^c) + \Pr(A^c)\Pr(B) \\
&= 0.5 \times 0.5 + 0.5 \times 0.5 = 0.5\,.
\end{aligned}$$

Additionally, the event $C \cap A$ is equivalent to $B^c \cap A$ (i.e. the second roll must be odd if the first is even in order to get an odd sum) and therefore

$$\Pr(C \cap A) = \Pr(B^c \cap A) = \Pr(A)\Pr(B^c) = 0.25 = \Pr(A)\Pr(C)\,.$$

Thus A, C are independent. The proof is identical for B, C. However, observe that $A \cap B \cap C = \emptyset$ and therefore $\Pr(A \cap B \cap C) = 0$ whereas $\Pr(A)\Pr(B)\Pr(C) = 0.125$. Therefore A, B, C are not independent.

(b) Note that the event D is equivalent to the event $A^c \cap B^c$, i.e. since both dice rolls must be odd to obtain an odd multiple. It follows that $\Pr(D) = 0.25$ and also $D \cap A = D \cap B = \emptyset$, that is $\Pr(D \cap A) = \Pr(D \cap B) = 0$. Since $\Pr(D)\Pr(A) = \Pr(D)\Pr(B) = 0.125 \neq 0$ then neither of the pairs A, D nor B, D are independent. It follows that neither of the combinations A, C, D nor B, C, D nor A, B, C, D are independent. It remains to check the pair C, D. Notice that D is equivalent to $A^c \cap B^c$ which is in turn equivalent to C^c (i.e. odd multiple implies two odd rolls which implies an even sum). Therefore $\Pr(C \cap D) =$

$\Pr(\emptyset) = 0 \neq 0.125 = \Pr(C)\Pr(D)$. In summary, no subset of events which includes D is independent (apart from the subset $\{D\}$).

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12. As hotel guests enter the lobby during the day they greet the receptionist with 'g'day', 'hello' or 'bonjour' if they are from Australia, the USA or France respectively. The receptionist writes each of these down as they occur. The proportion of guests currently staying at the hotel are 30% Australian, 50% American and the remainder are French (and assume each comes and goes equally often). Suppose you pick a random letter from the last greeting written down by the receptionist and find that it is a consonant. What is the probability the last guest to greet the receptionist was an Australian?

    Solution: Let A, U, F denote the events that the last guest was from Australia, the USA or France respectively. Let C denote the event that the letter guessed was a consonant. Observe that $\Pr(C|A) = 3/4$, $\Pr(C|U) = 3/5$ and $\Pr(C|F) = 4/7$. We want to determine $\Pr(A|C)$ and using Bayes' formula we have

$$\begin{aligned}\Pr(A|C) &= \frac{\Pr(C|A)\Pr(A)}{\Pr(C|A)\Pr(A) + \Pr(C|U)\Pr(U) + \Pr(C|F)\Pr(F)} \\ &= \frac{0.75 \times 0.3}{0.75 \times 0.3 + 0.6 \times 0.5 + \frac{4}{7} \times 0.2} \\ &\approx \frac{0.225}{0.225 + 0.3 + 0.1143} \approx 0.352\,.\end{aligned}$$

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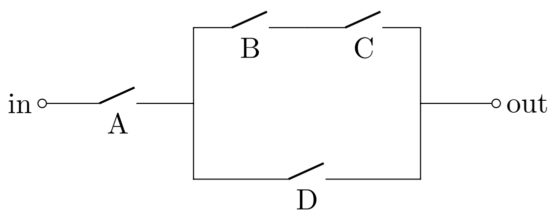
13. (a) Consider the following circuit with four switches that operate independently.



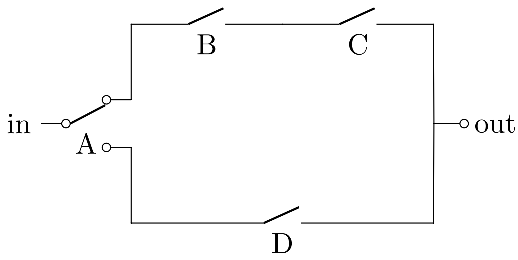
Let A, B, C, D be the events that the respective switches are closed. For convenience let $a = \Pr(A)$, and similar for switches B, C, D.

i. What is the probability that current flows through the circuit (i.e. from 'in' to 'out')?

ii. What is the probability that current flows through the circuit given switch B is closed?

iii. What is the probability that switch D is closed given current flows through the circuit?

(b) Consider the following modification to the previous circuit with the four switches again operating independently.



In this case let A be the event shown where the switch is in the 'up' position. A^c is then the event when the switch is in the 'down' position (i.e. there is no in-between).

i. What is the probability that current flows through the circuit (i.e. from 'in' to 'out')?

ii. What is the probability that current flows through the circuit given switch B is closed?

iii. What is the probability that switch D is closed given current flows through the circuit?

Solution:

(a) i. Let I be the event that current flows through the circuit. There are two possible paths for the current. These correspond to the events $A \cap D$ and $A \cap B \cap C$. Therefore we can write $I = (A \cap D) \cup (A \cap B \cap C)$. Using the inclusion/exclusion principle we have

$$\Pr(I) = \Pr(A \cap D) + \Pr(A \cap B \cap C) - \Pr(A \cap B \cap C \cap D)\,.$$

Given the switches operate independently then

$$\Pr(A \cap D) = \Pr(A)\Pr(D) = ad\,,$$

and similarly $\Pr(A \cap B \cap C) = abc$ and $\Pr(A \cap B \cap C \cap D) = abcd$. Thus the probability of current through the circuit is

$$\Pr(I) = ad + abc - abcd = a(bc + d - bcd)\,.$$

ii. The inclusion/exclusion principle still applies to $\Pr(I|B)$ so that

$$\Pr(I) = \Pr(A\cap D|B)+\Pr(A\cap B\cap C|B)-\Pr(A\cap B\cap C\cap D|B)\,.$$

The probability of current through the path $A \cap D$ is unaffected by switch B, that is $\Pr(A \cap D|B) = \Pr(A \cap D) = ad$. In the other two cases it is straightforward to show

$$\Pr(A \cap B \cap C|B) = \frac{\Pr(A)\Pr(B)\Pr(C)}{\Pr(B)} = ac\,,$$

and similarly $\Pr(A\cap B\cap C\cap D|B) = acd$. Therefore $\Pr(I|B) = a(c + d - cd)$.

iii. We want to determine $\Pr(D|I)$. Using Bayes' formula we have

$$\Pr(D|I) = \frac{\Pr(I|D)\Pr(D)}{\Pr(I)}\,.$$

For the event $I|D$ observe that if D is closed then current is guaranteed whenever A is closed. Therefore $\Pr(I|D) = \Pr(A)$ and we obtain

$$\Pr(D|I) = \frac{ad}{a(bc + d - bcd)} = \frac{d}{bc + d - bcd}\,.$$

(b) i. In this case the flow of current should be conditioned on the event A. The event $I|A$ is equivalent to $B \cap C$ (i.e. given switch A is up current can only flow it B, C are closed), and similarly $I|A^c$ is equivalent to D. Therefore

$$\Pr(I) = \Pr(I|A)\Pr(A)+\Pr(I|A^c)\Pr(A^c) = bc\times a+d\times(1-a) = abc+d-ad\,.$$

ii. We have

$$\Pr(I|B) = \Pr(I|A \cap B)\Pr(A|B) + \Pr(I|A^c \cap B)\Pr(A^c|B)\,.$$

Since A, B operate independently $\Pr(A|B) = A$ and $\Pr(A^c|B) = A^c$. Further, the event $I|A \cap B$ is equivalent to the event C, and the event $I|A^c \cap B$ is equivalent to the event D (noting B has no effect on the circuit when the A switch is in the 'down' position). Consequently

$$\Pr(I|B) = ca + d(1 - a) = ca + d - da\,.$$

iii. Using Bayes' formula we again have

$$\Pr(D|I) = \frac{\Pr(I|D)\Pr(D)}{\Pr(I)}\,.$$

The event $I|D$ requires some care. With switch D closed current is guaranteed to flow in the event A^c. However, current will also flow in the event $A \cap B \cap C$. That is $I|D$ is equivalent to $A^c \cup (A \cap B \cap C)$. Since $A^c, A \cap B \cap C$ are mutually exclusive then $\Pr(I|D) = (1-a) + abc$ and thus

$$\Pr(D|I) = \frac{(1-a+abc)d}{abc+d-ad}\,.$$

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14. Consider 3 biased coins, each with a different bias. Label the coins A, B, C and suppose each flips heads with probability $0.4, 0.5, 0.6$ respectively. Suppose the three coins are placed down in a row in a manner such that any ordering is equally likely. The coins are then flipped. Let L, M, R denote the events that the left, middle and right coin are heads.

    (a) Use a naive Bayes' classifier to determine what the most likely ordering of the coins is given the result $L \cap M^c \cap R$ (i.e. heads, tails and heads going from left to right).

    (b) Explain why this result fits with your intuition.

    (c) Explain why a naive Bayes' classifier is a particularly good approach for this problem.

    Solution:

    (a) There are six possible orderings, each with probability $1/6$ of occurring. Let C_1 denote the event that the order is A, B, C.

$$\begin{aligned}\Pr(C_1|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_1)\Pr(L|C_1)\Pr(M^c|C_1)\Pr(R|C_1)\\ &= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.4 \times 0.5 \times 0.6\\ &= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.12}{6}\,.\end{aligned}$$

    Let C_2 denote the event that the order is A, C, B.

$$\begin{aligned}\Pr(C_2|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_2)\Pr(L|C_2)\Pr(M^c|C_2)\Pr(R|C_2)\\ &= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.4 \times 0.4 \times 0.5\\ &= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.08}{6}\,.\end{aligned}$$

Let C_3 denote the event that the order is B, A, C.

$$\begin{aligned}
\Pr(C_3|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_3) \Pr(L|C_3) \Pr(M^c|C_3) \Pr(R|C_3) \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.5 \times 0.6 \times 0.6 \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.18}{6}.
\end{aligned}$$

Let C_4 denote the event that the order is B, C, A.

$$\begin{aligned}
\Pr(C_4|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_4) \Pr(L|C_4) \Pr(M^c|C_4) \Pr(R|C_4) \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.5 \times 0.4 \times 0.4 \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.08}{6}.
\end{aligned}$$

Let C_5 denote the event that the order is C, A, B.

$$\begin{aligned}
\Pr(C_5|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_5) \Pr(L|C_5) \Pr(M^c|C_5) \Pr(R|C_5) \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.6 \times 0.6 \times 0.5 \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.18}{6}.
\end{aligned}$$

Let C_6 denote the event that the order is C, B, A.

$$\begin{aligned}
\Pr(C_6|L \cap M^c \cap R) &= \frac{1}{\Pr(L \cap M^c \cap R)} \Pr(C_6) \Pr(L|C_6) \Pr(M^c|C_6) \Pr(R|C_6) \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{1}{6} 0.6 \times 0.5 \times 0.4 \\
&= \frac{1}{\Pr(L \cap M^c \cap R)} \frac{0.12}{6}.
\end{aligned}$$

Hence the classes C_3 and C_5 are equally most likely, that is the orderings B, A, C and C, A, B.
(Note it is not necessary to determine $\Pr(L \cap M^c \cap R)$, although it can be shown to be ≈ 0.1267.)

(b) It makes sense that the two most likely orderings are those in which the coins most likely to be heads are in those positions. (Or similarly, that the coin most likely to be tails is in that position.)

(c) Because each coin flip (or trait/property) is truly independent (i.e. this need not be 'assumed' of the 'data').

15. A medical researcher has gathered data on people with and without diabetes. Let D be the event that someone in their database has diabetes. Let W be the event that person hast weight at least 80kg. Let H be the event that person has height at least 160cm. Let A be the event that person has age at least 40 years. Let F be the event that person is female. Suppose that the researcher tabulates some summary statistics from the data which show that

- $\Pr(W|D) = 0.65$
- $\Pr(H|D) = 0.35$
- $\Pr(A|D) = 0.70$
- $\Pr(F|D) = 0.40$
- $\Pr(W|D^c) = 0.55$
- $\Pr(H|D^c) = 0.50$
- $\Pr(A|D^c) = 0.45$
- $\Pr(F|D^c) = 0.60$

They decide to construct a naive Bayes' classifier from this information to decide if their own patients have diabetes. They also use the estimate that 10% of the population has diabetes.

(a) Given a male patient who weighs 77kg, is 183cm tall, and is 27 years old, are they more likely to have, or not have diabetes based on this classifier?

(b) Why is a naive Bayes' classifier from this summary data likely to be a poor model/fit?

Solution:

(a) This particular patient corresponds to the event $W^c \cap H \cap A^c \cap F^c$. Under the standard assumption for naive Bayes' classifiers, i.e. that all of the traits/features are independent, one has

$$\begin{aligned}\Pr(D|W^c \cap H \cap A^c \cap F^c) &= \frac{\Pr(D)\Pr(W^c|D)\Pr(H|D)\Pr(A^c|D)\Pr(F^c|D)}{\Pr(W^c \cap H \cap A^c \cap F^c)} \\ &= \frac{0.1 \times (1-0.65) \times 0.35 \times (1-0.7) \times (1-0.4)}{\Pr(W^c \cap H \cap A^c \cap F^c)} \\ &= \frac{0.002205}{\Pr(W^c \cap H \cap A^c \cap F^c)},\end{aligned}$$

and similarly

$$\begin{aligned}\Pr(D^c|W^c \cap H \cap A^c \cap F^c) &= \frac{\Pr(D^c)\Pr(W^c|D^c)\Pr(H|D^c)\Pr(A^c|D^c)}{\Pr(W^c \cap H \cap A^c \cap F^c)}\Pr(F^c|D^c) \\ &= \frac{(1-0.1) \times (1-0.55) \times 0.5 \times (1-0.45) \times (1-0.6)}{\Pr(W^c \cap H \cap A^c \cap F^c)} \\ &= \frac{0.04455}{\Pr(W^c \cap H \cap A^c \cap F^c)}.\end{aligned}$$

Therefore it is much more likely that this patient does not have diabetes (by a factor of roughly 20).

(b) Because the traits/features W, H, A, F are expected to be correlated in general, in particular we expect W, H, F to be strongly correlated (i.e. weight, height and gender) amongst others. In other words

$$\Pr(W^c \cap H \cap A^c \cap F^c)|D) \neq \Pr(W^c|D)\,\Pr(H|D)\,\Pr(A^c|D)\,\Pr(F^c|D)\,.$$