Mathematics for Data Science Tutorial 5 (week 10)

Semester 2, 2019

1. Semaphore is a telegraphy system widely used in the maritime world during the 19th century in which a signal person holds two flags in different arrangements (one in each hand) to form different characters. Each flag can be held in one of eight different positions (say N,NE,E,SE,S,SW,W,NW). A character is is defined by the position of the two flags (noting it does not matter which hand is holding which).

   (a) How many different characters could you potentially make? (Note: two flags can be held in the same position.)

   (b) What if we were to disallow two flags being in the same position?

   Solution:

   (a) There are $n = 8$ possible choices for each flag and we want to take $r = 2$ at a time, with repetition, thus the total number of characters is $\binom{r+(n-1)}{r} = \binom{9}{2} = 36$.
   (Note that if the hand holding each flag happened to be important this becomes an ordered permutation with repetition for which there are $n^r = 8^2 = 64$ outcomes, but it would be too hard to see which hand is holding which flag over long distances at sea.)

   (b) There are 8 different characters in which both flags are in the same position (one for each possible position). We can directly subtract these from the 36 combinations obtained before to obtain $36 - 8 = 28$ possible characters.
   Alternatively, we observe that this is now an unordered combination *without* repetition, and thus there are $\binom{n}{r} = \binom{8}{2} = 28$ possible characters.
   Note: Most common semaphore uses 30 characters, the 28 above, plus the character with both flags in the S position, and lastly the error/attention signal which involves waving the flags up and down on either side.

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2. An investor has $\$16,000$ to invest in five possible companies where each investment must be a multiple of $\$1,000$.

   (a) If all of the money is invested, how many possible investment strategies are there?

   (b) What if at least $\$1,000$ must be invested in each company, and all of the money must be invested?

(c) If, in addition to investing at least \$1,000 in each company, at least \$10,000 must be invested in total, how many investment strategies are there?

Solution:

(a) We have $n = 5$ objects (the companies) taken $r = 16$ at a time (each choice is where \$1,000 goes) with repetition, and the order of the choice is unimportant. It follows that there are $\binom{n+(r-1)}{r} = \binom{20}{16} = \binom{20}{4} = 4845$ different investment strategies.

(b) In this case, \$5,000 is already distributed evenly amongst the 5 companies and it only remains to determine what to do with the remaining \$11,000. Thus it is equivalent to the first problem but with $r = 11$, and therefore there are $\binom{n+(r-1)}{r} = \binom{15}{11} = \binom{15}{4} = 1365$ investment strategies.
Another way to think about this is line up the 16 stacks of \$1,000, and we then want to choose 4 of the gaps between the stacks in which we will place dividers to that the \$16,000 is divided into 5 portions. There are $n = 15$ gaps between the stacks, and we want to choose $r = 4$ of them in any order without repetition, and thus there are $\binom{n}{r} = \binom{15}{4} = 1365$ possible outcomes.

(c) Here we just need to add up the number of investment options if \$16,000, ..., \$11,000 or \$10,000 is invested. That is

$$\begin{aligned}
&\binom{15}{4} + \binom{14}{4} + \binom{13}{4} + \binom{12}{4} + \binom{11}{4} + \binom{10}{4} + \binom{9}{4} \\
=&1365 + 1001 + 715 + 495 + 330 + 210 + 126 \\
=&4242\,.
\end{aligned}$$

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3. Suppose someone has forgotten their 4 digit PIN to unlock their phone.

(a) They make a random guess, what is the probability of it is correct?

(b) What is the probability of a random guess containing repeat digits?

(c) Suppose they know that their pin has no repeat digits, what is the probability of guessing correctly with this knowledge?

(d) If their PIN has no repeat digits, and they remember one digit but not necessarily which of the 4 it is, what is the probability of guessing correctly?

Solution:

(a) There are 10^4 possible PINs, so a random guess has $1/10^4$ chance of being correct.

(b) First, consider how many PINs have no repeat digits. Such a PIN can be generated as follows. The first digit can be chosen as any of the 10 digits, the second can only be one of the remaining 9 digits, the third can only be one of the remaining 8 digits and the last can only be one of the 7 unused digits. Thus there are $10 \times 9 \times 8 \times 7 = 5040$ PINs with no repeat digits which means that $10^4 - 5040 = 4960$ have repeat digits. Thus the probability of making a guess that has repeat digits is $4960/10000 = 0.496$.

(c) Since there are $10 \times 9 \times 8 \times 7 = 5040$ PINs with no repeat digits, with this knowledge a random guess has $1/5040$ chance (approximately double).

(d) If they know one digit, there are 4 possible places it could be in, and there are then $9 \times 8 \times 7 = 504$ possibilities for the remaining three digits. Thus there are $4 \times 504 = 2016$ possible outcomes to guess from meaning the chance of a correct guess is $1/2016$.

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4. A fast food chain is doing a study on the most popular items on their menu. Let B be the event a customer buys a burger, S be the event they buy a soft drink and C be the event they buy chips. From a large survey they determine that

- $\Pr(B) = 0.65$
- $\Pr(C) = 0.5$
- $\Pr(S) = 0.55$
- $\Pr(B \cap C) = 0.25$
- $\Pr(B \cap S) = 0.35$
- $\Pr(C \cap S) = 0.3$
- $\Pr(B \cap S \cap C) = 0.15$

Based on this, determine each of the following

(a) the probability a customer buys chips but not soft drink;

(b) the probability a customer buys a burger and/or chips;

(c) the probability a customer buys at least one of burger, soft drink or chips;

(d) the probability a customer does not buy a burger, soft drink nor chips;

(e) the probability a customer bought a burger, but no chips and no soft drink;

Solution:

(a) The corresponding event is $C \cap S^c$ for which we have

$$\Pr(C \cap S^c) = \Pr(C) - \Pr(C \cap S) = 0.5 - 0.3 = 0.2\,.$$

(b) The corresponding event is $B \cup C$ and using the inclusion/exclusion principle

$$\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C) = 0.65 + 0.5 - 0.25 = 0.9\,.$$

(c) The corresponding event is $B \cup S \cup C$, using the inclusion/exclusion principle we have

$$\begin{aligned}\Pr(B \cup S \cup C) &= \Pr(B) + \Pr(S) + \Pr(C) \\ &\quad - \Pr(B \cap C) - \Pr(B \cap S) - \Pr(C \cap S) \\ &\quad + \Pr(B \cap S \cap C) \\ &= 0.65 + 0.5 + 0.55 - 0.25 - 0.35 - 0.3 + 0.15 \\ &= 0.95\,.\end{aligned}$$

(d) This event is $B^c \cap S^c \cap C^c$, or equivalently $(B \cup S \cup C)^c$ for which

$$\Pr((B \cup S \cup C)^c) = 1 - \Pr(B \cup S \cup C) = 1 - 0.95 = 0.05\,.$$

(e) This event is $B \cap (C \cup S)^c$, or equivalently $B \cap C^c \cap S^c$. Observe that

$$\Pr(B \cap (C \cup S)^c) = \Pr(B) - \Pr(B \cap (C \cup S))\,.$$

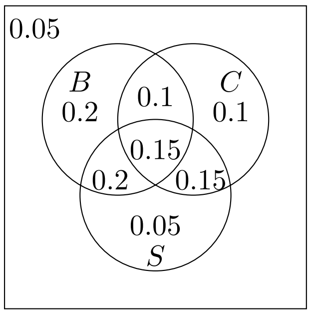
Using the inclusion/exclusion principle (twice) we have

$$\begin{aligned}\Pr(B \cap (C \cup S) &= \Pr(B) + \Pr(C \cup S) - \Pr(B \cup (C \cup S)) \\ &= \Pr(B) + (\Pr(C) + \Pr(S) - \Pr(C \cap S)) - \Pr(B \cup C \cup S) \\ &= 0.65 + 0.5 + 0.55 - 0.3 - 0.95 \\ &= 0.45\,.\end{aligned}$$

Thus we have

$$\Pr(B \cap (C \cup S)^c) = \Pr(B) - \Pr(B \cap (C \cup S)) = 0.65 - 0.45 = 0.2\,.$$

Alternatively, we could answer each of these by first constructing a Venn diagram with the appropriate probabilities determined for each piece, i.e.



One then need only identify the correct region for each question and add up the appropriate probabilities.