Mathematics for Data Science I
Practice Questions (week 4-5)

Semester 2, 2019

These questions are all about linear algebra – systems of equations and matrices.

1. Which of the following equations are linear?

   (a) $x_1 - 5x_2 + x_3^2 = 0$

   (b) $x_1 - 2x_3 = 7 - x_2$

   (c) $x_3 = 0$

   (d) $x_1 - x_2x_3 = 1$

   (e) $0 = 8$

   Solution: (c) is linear: it fits the definition of a linear equation precisely. So does (e)! (b) is also linear, because it can easily be rewritten in the standard form $x_1 + x_2 - 2x_3 = 7$. (a) and (d) are not linear, because a linear equation must not contain products of the variables with each other or themselves.

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2. (a) Solve the following two linear systems.

   (i) $\begin{aligned} 2x + y &= 4 \\ -x + 3y &= 5 \end{aligned}$ (ii) $\begin{aligned} x + y &= -1 \\ 4x + y &= 2 \end{aligned}$

   (b) What does it mean to say that two linear systems (of the same size) are equivalent?

   (c) Are the two systems in (a) equivalent?

   (d) Can we transform one of the two systems into the other by elementary operations?

   Solution: (a) System (i) has the unique solution $(x, y) = (1, 2)$. System (ii) has the unique solution $(x, y) = (1, -2)$.

   (b) It means that the two systems have the same solution set, that is, they have exactly the same solutions.

   (c) No, because they do not have the same solutions.

   (d) No, because systems that can be transformed into one another by elementary operations are equivalent.

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3. Solve the following systems of linear equations.

(a) $$\begin{aligned} 2x_1 + 6x_2 &= 12 \\ 4x_1 - 3x_2 &= 19 \end{aligned}$$ (b) $$\begin{aligned} x_1 - 3x_2 + 2x_3 &= -10 \\ 2x_2 - x_3 &= 6 \\ x_1 + x_3 &= 0 \end{aligned}$$

Solution: These systems of equations may be solved by the matrix method, or just by substitution.
(a) The first equation can be rewritten as $x_1 = 6 - 3x_2$, substituting into the second then gives $24 - 15x_2 = 19$ which gives the solution $(x_1, x_2) = (5, 1/3)$.
(b) The second and third equations give $x_2 = (x_3+6)/2$ and $x_1 = -x_3$, so we can substitute into the first equation to get $-x_3 - \frac{3}{2}(x_3 + 6) + 2x_3 = -10$ which gives the solution $(x_1, x_2, x_3) = (-2, 4, 2)$.

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4. Solve the following systems of linear equations.

(a) $$\begin{aligned} 2x_1 + 6x_2 &= 3 \\ x_1 + 3x_2 &= 1 \end{aligned}$$ (b) $$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 - x_3 &= 2 \end{aligned}$$

Solution: (a) Multiplying the second equation by 2 gives $2x_1 + 6x_2 = 2$ which clearly contradicts the first equation so there are no solutions. Alternatively, in matrix form we have the augmented matrix $\begin{bmatrix} 2 & 6 & | & 12 \\ 1 & 3 & | & 1 \end{bmatrix}$ and halving the first row, and then subtracting it from the second row gives $\begin{bmatrix} 1 & 3 & | & 6 \\ 0 & 0 & | & -5 \end{bmatrix}$, so the last row says that $0 = -5$, so there are no solutions.
(b) This system has more variables than equations, so there will not be a unique solution. There will be either infinitely many solutions or no solutions. Let $x_3 = t$ then $x_2 = 2 + t$, and $x_1 = 4 - 2x_2 = 4 - 2(2 + t) = -2t$, so the solutions are $(x_1, x_2, x_3) = (-2t, 2 + t, t)$ for any $t \in \mathbb{R}$.

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5. Consider the following system of linear equations.

$$\begin{gathered} x_1 - 2x_2 = 1 \\ 3x_1 + ax_2 = 4 \end{gathered}$$

(a) If the system has the unique solution $(x_1, x_2) = (5, 2)$, what is a?
(b) If the system has no solutions, what is a?

Solution: (a) The second equation is $15 + 2a = 4$, so $a = -11/2$.
(b) The first equation says that $x_1 = 1 + 2x_2$. Substituting into the second we get $3 + 6x_2 + ax_2 = 4$ or $(6 + a)x_2 = 1$. This will give a solution for x_2 (and hence also for x_1) unless $6+a = 0$,that is, $a = -6$, in which case there are no solutions.

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6. How many 5×1 matrices are there in reduced row echelon form?

   Solution: A 5×1 matrix has only one column. If there is a nonzero entry in the column, then it must be a pivot at the top of the column, with the other entries equal to zero. Otherwise the whole column is zero. So the answer is: two.

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7. For each of the following matrices in reduced row echelon form, find the set of solutions to the corresponding system of equations (you can call the variables $x_1, x_2, \ldots$).

$$\text{(a)} \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & | & 4 \\ 0 & 0 & 1 & 2 & 0 & | & 3 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 1 & 4 & 8 & 3 & | & -1 \end{bmatrix}$$

   Solution: (a) $(x_1, x_2) = (0, 1)$
   (b) Let $x_3 = t$. Then $(x_1, x_2, x_3) = (0, 1, t) = (0, 1, 0) + t(0, 0, 1)$.
   (c) Let $x_2 = s$ and $x_4 = t$. Then we have $x_1 + 2x_2 + 3x_4 = 4$, so $x_1 = 4-2s-3t$. Also $x_3+2x_4 = 3$ so $x_3 = 3-2t$ and $x_5 = 2$. Putting these together we get
   $(x_1, x_2, x_3, x_4, x_5) = (4-2s-3t, s, 3-2t, t, 2) = (4, 0, 3, 0, 2)+s(-2, 1, 0, 0, 0)+t(-3, 0, -2, 1, 0)$.
   (d) Let $x_2 = r$, $x_3 = s$ and $x_4 = t$. Then
   $(x_1, x_2, x_3, x_4) = (-1-4r-8s-3t, r, s, t) = (-1, 0, 0, 0)+r(-4, 1, 0, 0)+s(-8, 0, 1, 0) + t(-3, 0, 0, 1)$.

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8. Suppose that you bring the matrix

$$\begin{bmatrix} 5 & 0 & -4 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

into reduced row echelon form. Which columns will contain a pivot? Do not calculate the reduced row echelon form. Just look at the matrix!

Solution: Columns 1, 3, and 4 will. First, we turn the $(1,1)$-entry 5 into a pivot by dividing the first row by 5. Then we use the $(2,3)$-entry as a pivot and use it to eliminate the entries -4 and -3 above and below. That leaves a nonzero $(3,4)$-entry (in fact 22), so we get a third pivot in the 4th column.

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9. (a) Are the following two matrices in reduced row echelon form?

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Are they row equivalent?

Solution: (a) Yes. They satisfy the four defining properties of the reduced row echelon form.

(b) No, because they are different. Remember that every matrix is row equivalent to a *unique* matrix in reduced row echelon form. Hence two different matrices in reduced row echelon form cannot be equivalent.

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10. Use row operations to put the following matrices into reduced row echelon form.

$$\text{(a)} \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ 1 & 0 & | & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 2 & 0 & | & 2 \\ 1 & 1 & 0 & | & 1 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 1 & 2 & 3 & | & 4 \\ \alpha & 0 & 1 & | & 2 \end{bmatrix} \quad \text{for some } \alpha \neq 0$$

Solution: (a) Swap R_1 and R_3, and then swap R_2 and R_3 to get $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$.

(b) Do $R_3 \to R_3 - R_1$ and $R_2 \to \frac{1}{2} R_2$; giving $\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$.

Next do $R_3 \to R_3 - R_2$ to get $\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & -1 & | & -1 \end{bmatrix}$ then $R_3 \to -R_3$:

$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$ and $R_1 \to R_1 - R_3$ gives the rref $\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$

(c) $R_2 \to R_2 - \alpha R_1$ gives $\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & -2\alpha & 1-3\alpha & | & 2-4\alpha \end{bmatrix}$ and then $R_2 \to \frac{-1}{2\alpha} R_2$ gives

$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & \frac{-1}{2\alpha} + \frac{3}{2} & | & \frac{-1}{\alpha} + 2 \end{bmatrix}$ and finally $R_1 \to R_1 - 2R_2$ gives

$\begin{bmatrix} 1 & 0 & \frac{1}{\alpha} & | & \frac{2}{\alpha} \\ 0 & 1 & \frac{-1}{2\alpha} + \frac{3}{2} & | & \frac{-1}{\alpha} + 2 \end{bmatrix}$

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11. Find the solution sets of the following systems of linear equations.

(a)
$$\begin{aligned} 4x + 2y + 5z &= 8 \\ 2x + 4y + 4z &= 1 \\ -2y - z &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 4x + 2y + 4z &= 8 \\ x + 4y + 4z &= 1 \\ 2x - 6y - 3z &= 6 \end{aligned}$$

Solution: (a) The corresponding augmented matrix is

$$\begin{bmatrix} 4 & 2 & 5 & | & 8 \\ 2 & 4 & 4 & | & 1 \\ 0 & -2 & -1 & | & 2 \end{bmatrix}$$

which row reduces to $\begin{bmatrix} 1 & 0 & 1 & | & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

Let $z = t$. Then $x = \frac{5}{2} - t$ and $y = -1 - \frac{1}{2}t$ so the solutions are $(x, y, z) = (\frac{5}{2} - t, -1 - \frac{1}{2}t, t) = (\frac{5}{2}, -1, 0) + t(-1, -\frac{1}{2}, 1)$ for $t \in \mathbb{R}$.

(b) The corresponding augmented matrix is $\begin{bmatrix} 4 & 2 & 4 & | & 8 \\ 1 & 4 & 4 & | & 1 \\ 2 & -6 & -3 & | & 6 \end{bmatrix}$ which row reduces to

$\begin{bmatrix} 1 & 0 & 0 & | & \frac{15}{7} \\ 0 & 1 & 0 & | & -\frac{2}{7} \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$. Thus there is a unique solution, $(x, y, z) = (\frac{15}{7}, -\frac{2}{7}, 0)$.

12. Suppose three foods are used to make a meal. The foods have the following units per gram of vitamin B and vitamin C respectively.

	Vitamin B	Vitamin C
Food 1	12	18
Food 2	8	10
Food 3	4	2

By setting up a system of linear equations, determine if it is possible to combine these foods into a meal of 500 grams with precisely 2000 units of Vitamin B and 1000 units of Vitamin C. If it is possible, in how many ways can it be done?

Solution: Let x_1 equal the number of grams of Food 1, x_2 equal the number of grams of Food 2 and x_3 equal the number of grams of Food 3 in the meal. Since the meal is of weight 500 grams, we must have $x_1 + x_2 + x_3 = 500$. On the other hand, from the table, the meal will contain $12x_1 + 8x_2 + 4x_3$ units of Vitamin B, therefore $12x_1+8x_2+4x_3 = 2000$. Similarly $18x_1+10x_2+2x_3 = 1000$. Therefore x_1, x_2 and x_3 satisfy the linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 500 \\ 12x_1 + 8x_2 + 4x_3 &= 2000 \\ 18x_1 + 10x_2 + 2x_3 &= 1000 \end{aligned}$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & | & 500 \\ 12 & 8 & 4 & | & 2000 \\ 18 & 10 & 2 & | & 1000 \end{bmatrix}$$

Do the row operations $R_2 \to R_2 - 12R_1$, $R_3 \to R_3 - 18R_1$ to obtain the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & | & 500 \\ 0 & -4 & -8 & | & -4000 \\ 0 & -8 & -16 & | & -8000 \end{bmatrix}$$

Now do the row operation $R_2 \to -1/4R_2$ to obtain

$$\begin{bmatrix} 1 & 1 & 1 & | & 500 \\ 0 & 1 & 2 & | & 1000 \\ 0 & -8 & -16 & | & -8000 \end{bmatrix}$$

The row operations $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 8R_2$ then put the augmented matrix into reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 & | & -500 \\ 0 & 1 & 2 & | & 1000 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

There are infinitely many solutions of this system: $x_1 = t - 500$, $x_2 = 1000 - 2t$, $x_3 = t$. However, we seek a solution with $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$ (you can't eat negative amounts of food). Therefore we must have $t - 500 \geq 0$ and $1000 - 2t \geq 0$. Hence $t = 500$ and so there is only one way to make the meal: just take a meal with 500 grams of Food 3.

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13. * Use linear equations to find a polynomial $p(x)$ with $p(0) = 1$, $p(1) = 0$, $p(2) = 5$, $p(3) = 22$.

Solution: We seek to prescribe the values of $p(x)$ at 4 points, so the lowest degree we can expect to work is 3. Thus we take $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Substituting $x = 0, 1, 2, 3$ into $p(x)$ gives the linear system

$$\begin{array}{rcrcrcrcl} & & & & & & a_0 & = & 1 \\ a_3 & + & a_2 & + & a_1 & + & a_0 & = & 0 \\ 8a_3 & + & 4a_2 & + & 2a_1 & + & a_0 & = & 5 \\ 27a_3 & + & 9a_2 & + & 3a_1 & + & a_0 & = & 22 \end{array}$$

Applying Gauss-Jordan elimination to the augmented matrix gives

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 5 \\ 27 & 9 & 3 & 1 & 22 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

so $p(x) = x^3 - 2x + 1$.

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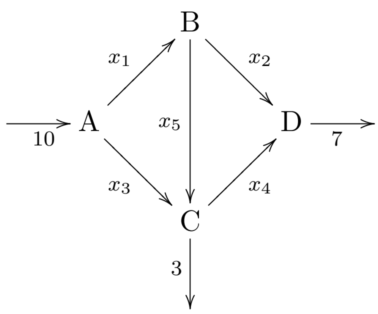
14. * Consider the following network of irrigation channels with flows measured in, say, thousands of litres per day.



(a) Set up and solve a linear system for the possible flows x_1, x_2, x_3, x_4, x_5, using x_4 and x_5 as free variables.

(b) Can we close both channels BC and CD?

(c) Find the maximum and minimum possible flow x_4 through channel CD.

Solution: (a) Each of the four nodes contributes a linear equation:

$$\begin{array}{rcrcrcrcrcl} x_1 & & & + & x_3 & & & & & = & 10 \\ & & x_2 & & & + & x_4 & & & = & 7 \\ x_1 & - & x_2 & & & & & - & x_5 & = & 0 \\ & & & & x_3 & - & x_4 & + & x_5 & = & 3 \end{array}$$

Applying Gauss-Jordan elimination to the augmented matrix of this linear system gives

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -1 & 7 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We express the basic variables x_1, x_2, x_3 in terms of the free variables x_4, x_5 and get

$$x_1 = 7 - x_4 + x_5, \quad x_2 = 7 - x_4, \quad x_3 = 3 + x_4 - x_5$$

(b) The only constraint on the variables is that they must be nonnegative. If we close BC and CD, that is, take $x_4 = x_5 = 0$, then $x_1 = 7$, $x_2 = 7$, and $x_3 = 3$, which is allowed, so the answer is: yes.

(c) All the variables must be nonnegative. In particular, $x_4 \geq 0$. Also, $7 - x_4 = x_2 \geq 0$ gives $x_4 \leq 7$. All values of x_4 between 0 and 7 are possible. For example, taking $x_4 = x_5$ between 0 and 7 makes $x_1 = 7$, x_2, and $x_3 = 3$ all nonnegative. So the maximum possible flow through CD is 7, and the minimum possible flow through CD is 0.

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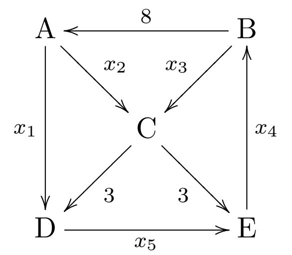
15. * Consider the following road network with traffic flows measured in, say, hundreds of cars per hour.



(a) Set up and solve a linear system for the possible flows x_1, x_2, x_3, x_4, x_5.

(b) Find the maximum and minimum possible traffic flow for each road AD, AC, BC, EB, and DE.

(c) Is it possible to close two of these five roads at once?

Solution: (a) Each of the five nodes contributes a linear equation:

$$\begin{array}{rcl}
x_1 + x_2 & = & 8 \\
x_3 - x_4 & = & -8 \\
x_2 + x_3 & = & 6 \\
x_1 - x_5 & = & -3 \\
- x_4 + x_5 & = & -3
\end{array}$$

Applying Gauss-Jordan elimination to the augmented matrix of this linear system gives

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & -1 & 0 & -8 \\ 0 & 1 & 1 & 0 & 0 & 6 \\ 1 & 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -1 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 0 & 1 & 11 \\ 0 & 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We express the basic variables x_1, x_2, x_3, x_4 in terms of the free variable x_5 and get

$$x_1 = x_5 - 3, \quad x_2 = -x_5 + 11, \quad x_3 = x_5 - 5, \quad x_4 = x_5 + 3$$

(b) The only constraint on the variables is that they must be non-negative. From $x_5 - 5 = x_3 \geq 0$ we deduce that $x_5 \geq 5$. From $-x_5 + 11 = x_2 \geq 0$ we deduce that $x_5 \leq 11$. From $5 \leq x_5 \leq 11$ we get:

$$2 \leq x_1 \leq 8, \quad 0 \leq x_2 \leq 6, \quad 0 \leq x_3 \leq 6, \quad 8 \leq x_4 \leq 14$$

(c) Of the five roads, the only two that could be closed are AC ($x_2 = 0$) and BC ($x_3 = 0$) – the variables x_1, x_4, and x_5 cannot take the value zero by (b). We have $x_2 = 0$ when $x_5 = 11$, and we have $x_3 = 0$ when $x_5 = 5$, so these cannot happen simultaneously. Thus the answer is: no.