Mathematics for Data Science Tutorial 3 (week 6)

Semester 2, 2019

1. State whether or not the following matrix operations are possible. Perform the operations where possible.

   (a)
$$\begin{bmatrix} 2 & 1 & 8 & 6 \\ 3 & 5 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -11 & -1 & 9 \\ 2 & 9.4 & -2 \end{bmatrix}$$

   Solution: Can't do it, because we can't add a 2×4 to a 2×3 matrix.

   (b)
$$\begin{bmatrix} 2 & 2 \\ 1 & -5.5 \\ 2.5 & 10 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & -7 \\ -1 & 3 \end{bmatrix}$$

   Solution: Can do this (matrices are same order), and the result is
$$\begin{bmatrix} 0 & 3 \\ 0 & -8.5 \\ 2.5 & -17 \\ 3 & -4 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} -3 & -2 & 9 & 1 \\ 2 & 3 & -1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 4 & -8 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

   Solution:
$$\begin{aligned}
\begin{bmatrix} -3 & -2 & 9 & 1 \\ 2 & 3 & -1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 4 & -8 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T &= \begin{bmatrix} -3 & -2 & 9 & 1 \\ 2 & 3 & -1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 4 & 2 & 1 & 0 \\ -8 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -2 & 9 & 1 \\ 2 & 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 4 & 2 & 0 \\ -16 & 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 2 & 11 & 1 \\ -14 & 3 & -1 & 2 \end{bmatrix}
\end{aligned}$$

(d)

$$\begin{bmatrix} 2 & 5 \\ 0 & -3 \\ 7 & 16 \end{bmatrix} - \begin{bmatrix} 2 & -5 & 0 \\ 2 & 9 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution: Can't do this, because we can't subtract a 3×3 from a 3×2 matrix. It doesn't matter that the last column is all zeros! The operation is not defined.

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2. In lectures we declared that $A(B+C) = AB + AC$, but ran away from doing an example. Verify this result for the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} A(B+C) &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 6 & 5 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4+12+6 & 4+10+12 \\ 12+12+2 & 12+10+4 \end{bmatrix} \\ &= \begin{bmatrix} 22 & 26 \\ 26 & 26 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} AB + AC &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+6+3 & 2+8+9 \\ 3+6+1 & 6+8+3 \end{bmatrix} + \begin{bmatrix} 3+6+3 & 2+2+3 \\ 9+6+1 & 6+2+1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 19 \\ 10 & 17 \end{bmatrix} + \begin{bmatrix} 12 & 7 \\ 16 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 22 & 26 \\ 26 & 26 \end{bmatrix}. \end{aligned}$$

Therefore, $A(B+C) = AB + AC$.

3. In lectures we also declared that $(AB)^T = B^T A^T$, without giving you any proof. Verify that this result is true for 2×2 matrices A and B.

Solution: Define two general 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

Now,

$$\begin{aligned} (AB)^T &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)^T \\ &= \left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \right)^T \\ &= \begin{bmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} B^T A^T &= \begin{bmatrix} e & g \\ f & h \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \\ &= \begin{bmatrix} ea+gb & ec+gd \\ fa+hb & fc+hd \end{bmatrix}, \end{aligned}$$

therefore $(AB)^T = B^T A^T$.

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4. Solve the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 2 \\ -2x_1 - x_2 + x_3 &= -1 \\ x_2 - 3x_3 &= 3 \\ -x_1 - x_3 &= 1 \end{aligned}$$

by first bringing its augmented matrix into reduced row echelon form.

Start by looking at the system and try to guess how many solutions it has. Once you've solved it, see if your guess was correct.

Solution: Gauss-Jordan elimination applied to the augmented matrix of the system gives:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ -2 & -1 & 1 & -1 \\ 0 & 1 & -3 & 3 \\ -1 & 0 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The so-called *basic* variables are x_1 and x_2: they correspond to the pivot columns. (Not necessary to know this terminology.) The free variable is x_3; let's call it t. We obtain the solutions

$$x_1 = -x_3 - 1 = -t - 1, \qquad x_2 = 3x_3 + 3 = 3t + 3, \qquad x_3 = t$$

In vector form, the solutions are

$$(x_1, x_2, x_3) = (-t - 1, 3t + 3, t) = t(-1, 3, 1) + (-1, 3, 0)$$

Maybe you thought that a system of 4 equations in 3 unknowns wouldn't have any solutions: there are too many conditions on the unknowns. But in this case there was a lot of redundancy among the 4 equations. Gauss-Jordan elimination boiled them down to 2 equations.
