Algorithm and Data Structure Analysis (ADSA)

Hashing (1)

Motivation for Data Structures

- Worst case analysis of data structures:

Name	**Insert(x)**	**Remove(x)**	**Find(x)**
Linked Lists	O(1)	O(1)	Θ(n)
AVL Trees	O(log n)	O(log n)	O(log n)

- Can we have constant time insertion and removal, yet have a better find?

Associative Arrays

Idea: consider a different use of arrays.

- Don't change array size on insert or remove.
- On remove, simply clear the element at the index.
- Assume we know the index of x.
  - insert(x) is O(1)
  - remove(x) is O(1)
  - find(x) is O(1)

Associative Arrays

- Associative array S stores elements
- Each element e in S has a unique key: *key(e)*. Clearly, each key has a unique element.
- Need an index in S for each possible key.

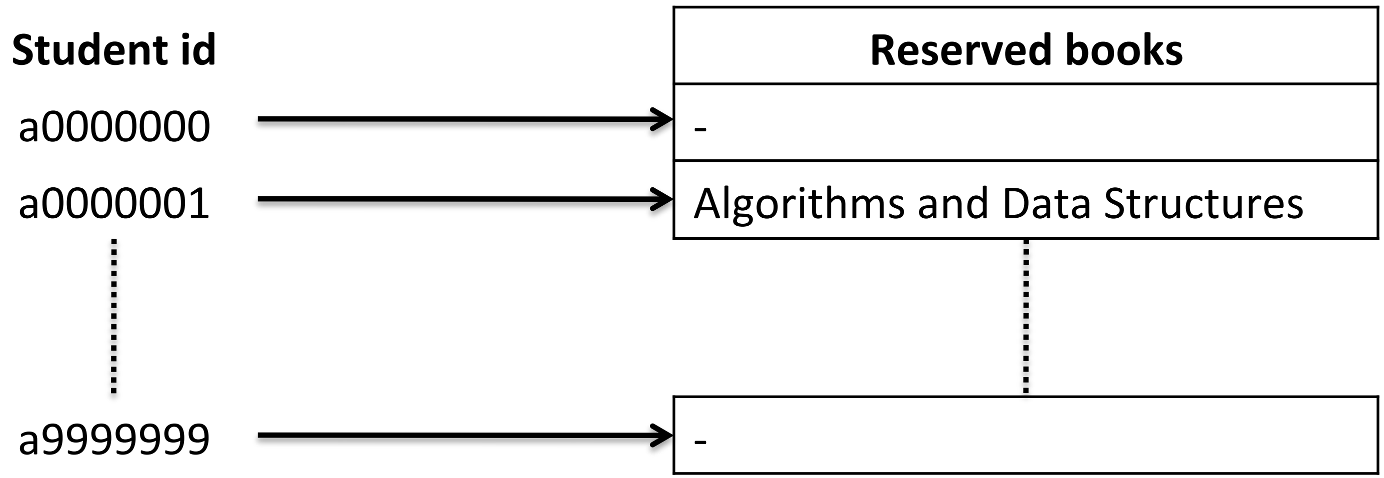
S.insert(e: Element): $S := S \cup \{e\}$

S.remove(k: Key): $S := S \setminus \{e\}$

S.find(k: Key): if e in S, return e. Else return null.

Associative Arrays

- Problem: number of possible keys is MASSIVE.
- Library example: how many students borrow books? How many student ids are there?



Associative Arrays

- Let N be the number of potential keys in S
- Let n be the number of elements in S

- Having an associative array S of size N elements is too costly in terms of space.
- Want to have S of size $O(n)$.

Hash Tables

- Idea: use hash function *h* to map potential keys to *m* values, where $m < N$.
- Let *t* be a hash table of size *m*.

- Store element *e* in index *h(key(e))* of *t*.

Hash Tables

- Example hash function: *key(e)* are student ids, *h(key(e))* are last two digits of student ids.
- *N* is 10^7, *m* is 10^2.

key(e)	**h(key(e))**	**Table t**
All ids ending in 00 →	00 →	Introduction to Software Eng.
All ids ending in 01 →	01 →	Algorithms and Data Structures
⋮	⋮	⋮
All ids ending in 99 →	99 →	Some Dinosaur Book

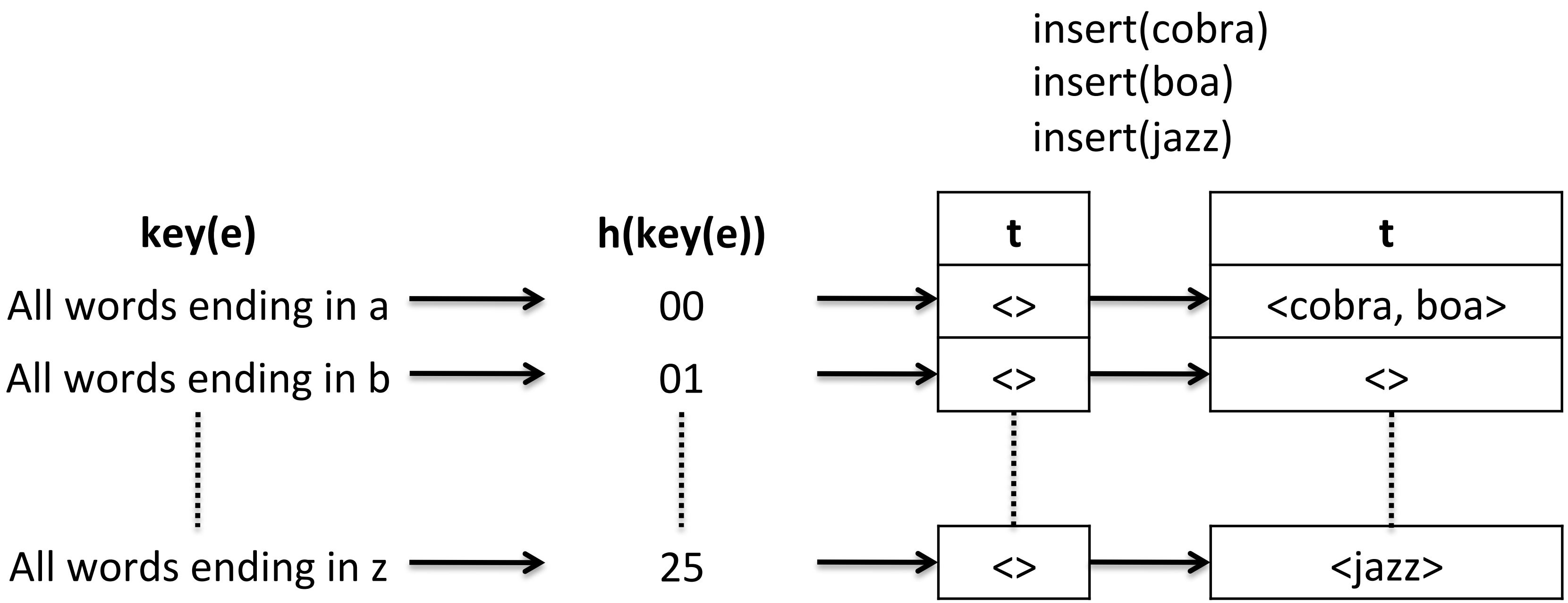
Hash Tables: Collisions

- Smaller table to store elements means some elements may get stored in the same index.
- Previous example, a0000000 and a1995400.
- If only one element per table entry, only one element can be stored.

- How do we handle collisions?
  - Think linked lists…

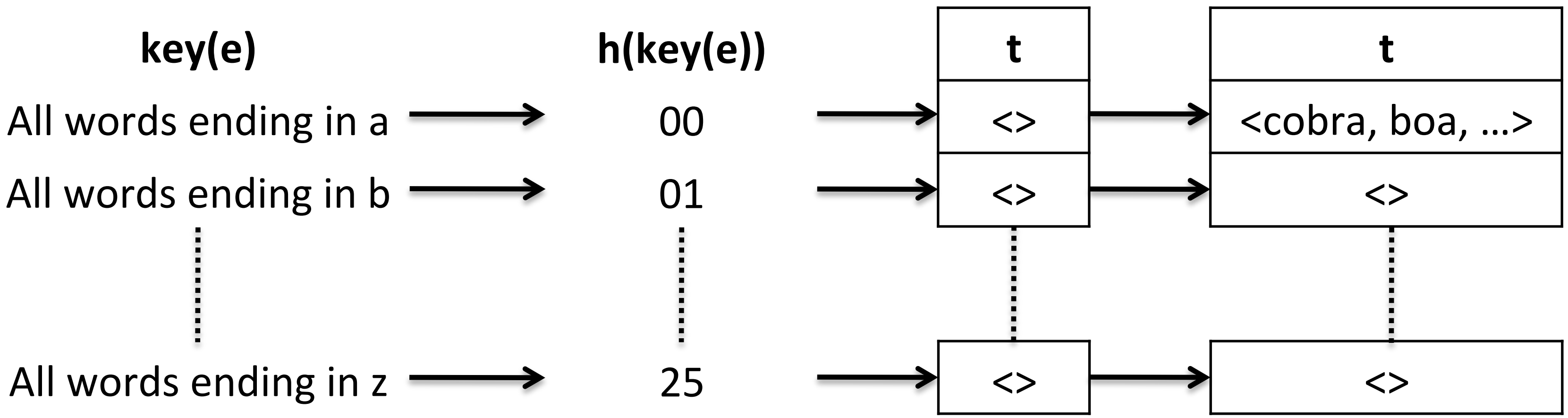
Hashing with Chaining

- Solution: let t be a table of linked lists.
- Example: Storing words.



Hashing with Chaining

- Worst case performance: hash function of elements returns the same value.
- In example, insert cobra, boa, ABBA, zebra



Chaining Limitations

- N = number of potential keys
- m = number of possible hash function values
- n = number of elements
- Thus hash functions will have sets of N/m keys mapped to the same index of t.
- As (usually) $n < N/m$, it is possible to have all n elements in one table entry.

Insert(e)

- insert(*e*: Element)
  - Get index *h(key(e))*
  - Add *e* to the end of the list at *t[h(key(e))]*

- What is the worst case complexity?

Insert(e)

- insert(*e*: Element)
  - Get index *h(key(e))*
  - Add *e* to the end of the list at *t[h(key(e))]*

- Hash function is O(1)
- Worst case insert of linked list is O(1)
- Thus insert(*e*: Element) is O(1).

Find(k)

- find(*k*: Key)
  - Get index *h(k)*
  - Search through list at *t[h(k)]*.
  - If element *e* with unique key *k* is in list, return *e*. Else return null.

- What is the worst case complexity?

Find(k)

- find(*k*: Key)
  - Get index *h(k)*
  - Search through list at *t[h(k)]*.
  - If element *e* with unique key *k* is in list, return *e*. Else return null.

- Hash function is O(1)
- Worst case find of linked list is $\Theta(n)$
- Thus find(*k*: Key) is $\Theta(n)$.

Remove(k)

- remove(*k*: Key)
  - Get index *h(k)*
  - Search through list at *t[h(k)]*.
  - If element *e* with unique key *k* is in list, remove *e*.

- What is the worst case complexity?

Remove(k)

- remove(*k*: Key)
  - Get index *h(k)*
  - Search through list at *t[h(k)]*.
  - If element *e* with unique key *k* is in list, remove *e*.

- Hash function is O(1)
- Worst case find of linked list is Θ(*n*)
- Worst case remove of linked list is O(1)
- Thus remove(*k*: Key) is Θ(*n*).

Average Case Analysis

Theorem 4.1: If *n* elements are stored in a hash table *t* with *m* entries and a random hash function is used, the expected execution time of remove or find is O(1+ *n/m*).

Note: a random hash function maps *e* to all *m* table entries with the same probability.

Average Case Analysis

Proof:

Execution time for remove and find is constant time plus the time scanning the list *t[h(k)]*.

Let the random variable X be the length of the list *t[h(k)],* and let *E[X]* be the expected length of the list.

Thus the *expected* execution time = O(1 + E[X]).

Average Case Analysis

Proof (continued):

Let S be the set of n elements contained in t.

For each e, let X_e be an indicator variable which indicates whether e hashes to the same value as k.

ie: **if** $h(key(e)) = h(k)$ **then** $X_e = 1$ **else** $X_e = 0$.

$$X = \sum_{e \in S} X_e$$

(ie how many e's are in table entry h(key(e)))

Average Case Analysis

Proof (continued):

$$E[X] = E\left[\sum_{e \in S} X_e\right]$$

$$= \sum_{e \in S} E[X_e]$$

$$= \sum_{e \in S} prob(X_e = 1)$$

Average Case Analysis

Proof (continued):

$$E[X] = \sum_{e \in S} prob(X_e = 1)$$ *(From last slide)*

$$= \sum_{e \in S} 1/m$$ *(As function maps e to all m with equal probability)*

$$= n/m$$ *(Because n elements in S)*

Average Case Analysis

Proof (continued):

Expected execution time = $O(1 + E[X])$,

$E[X] = n/m$

Thus the expected execution time for remove and find under hashing with chaining is $O(1 + n/m)$, and constant if $m = \Theta(n)$