Practice Questions (week 8)

Semester 2, 2019

The first two questions are about principal component analysis (PCA). The remainder are about probability – permutations and combinations, discrete random variables, and discrete probability distributions.

1. Consider a principal component analysis (PCA) of a set of data $X \in \mathbb{R}^{n\times p}$ (with $\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} X_{ij}$, $X' = X - \mathbf{1}_{n\times 1}\bar{\mathbf{x}}$ and $C = \frac{1}{n-1}(X')^T X'$ as usual) Let $a \in \mathbb{R}$ be non-zero.
   (a) How do the principal components of aX compare to those of X?
   (b) Let $A = a\mathbf{1}_{n\times p}$, how do the principal components of $A + X$ compare to those of X?

2. Determine the principal components of the following data

$$X = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 4 & 3 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$$

3. The university assigns IDs to students and staff using 7 digit numbers.
   (a) How many IDs are there assuming that the first digit cannot be zero?
   (b) How many IDs are there in which no digit is repeated twice (allowing a zero as the first digit)?
   (c) How many IDs are there in which no digit is repeated twice if the first digit cannot be zero?

4. Suppose a tutorial of 30 students is broken up into 5 working groups of 6 students to work together on problems.
   (a) Taking a single working group of 6 students, how many possible outcomes are there?
   (b) Suppose the 30 students consist of 15 distinct pairs of friends who always work together, how many possible outcomes are there now for a single working group of 6 students?

5. You've just sat down to tackle these practice questions but only have time to do five of them.
   (a) How many possible combinations of the practice questions could you answer?

(b) How many possible combinations of the practice questions could you answer assuming you are completing this one?

(c) Suppose you answer another four questions tomorrow, how many combinations are there of the five questions you do today and four you do tomorrow?
(Note: this question does not have to be one of the nine, and we implicitly distinguish the questions done on each day)

6. How many different letter arrangements can be made from the letters in the words

   (a) MODEL;
   (b) DATA;
   (c) ADELAIDE;
   (d) KARRAWIRRA?

7. In Australian rules football (AFL) there are eighteen teams. Eight of these teams will make it into the finals (playoffs). Of these eight, only two will participate in the grand final. Determine the number of possible ways the final eight and final two can be formed. That is, how many ways are there to form a list of 10 teams from 18 where the first 8 are those that make it to the finals and the last 2 are those that make it to the grand final (and thus appear twice in the list).

8. Prove that
$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2 .$$
(Hint: Use the formula $\displaystyle\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$).

9. Prove each of the following (hint: use the binomial theorem)

   (a)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n ,$$
   (b)
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 .$$
   (c)
$$\sum_{k=0}^{n} (-1)^k 2^k \binom{n}{k} = 3^n .$$

10. We will consider how many ways a committee of any size can be formed from n people with one of the committee members selected to be chair person.

(a) Describe why there are $k\binom{n}{k}$ ways to pick a committee of size k from n people with one of the k members selected as chair.

(b) Use the definition of $\binom{n}{k}$ to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(c) Use the above two steps to show that the number of ways to form a committee of any size from n people with one of the committee members selected as chair is $n \times 2^{n-1}$.
(Hint: you'll also need the result $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ from the previous question.)

11. 10 items are to be distributed amongst 3 bins. How many combinations are there if

    (a) both the items and bins are distinguishable;

    (b) the items are identical and only the bins are distinguishable;

12. We will use induction (on n) to prove the identity

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

    (a) First show the base case $n = k$ is true.

    (b) Now assuming the identity holds for n, show that it also holds for $n + 1$.

13. Expand $(2x + y^3)^5$.

14. A tiler has n white tiles and m black tiles and wants to lay them in a line to form a path. There are more black tiles than white tiles (i.e. $m > n$). The tiler wants to ensure no two white tiles are next to each other. Come up with a combinatorical formula for how many different patterns the tiler could potentially make?

15. Let n be a positive integer. How many solutions are there to

$$x_1 + x_2 + \cdots + x_r = n$$

with the condition that

    (a) each x_i is a (strictly) positive integer?
    (Hint: consider laying out the n objects and choosing how to divide them up into r sets having at least one object in each.)

    (b) each x_i is a non-negative integer?

    (c) x_1 is (strictly) positive, but the remaining x_i are non-negative integers?