Bayesian Network

3007/7059 Artificial Intelligence

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Bayesian Network

- One kind of Probabilistic Graphic Models
- Represent probability model with a graph

Example Application

The potential customer is trying to insure his Mercedes Benz. It is his 3rd car. He was involved in 2 previous minor accidents. The car has airbags and anti-lock braking system. He is 38 years old, married with 2 kids and makes $120,000 annually. Is he a risky driver?

"I know you haven't had an accident in 15 years. That's why we're raising your rates - because you're about due one."

Concept

In the last lecture we saw how to do some simple inference in a set of three variables. Here we introduce two important ideas, and then show how they can be encoded in a *graphical model* or bayesian network

- Bayes rule
- Independence (and conditional independence)

Bayes' Rules

- It is convenient to build statistic model by using causal relationship: *P(effect|cause)*
- Real world requirement *P(cause|effect)*

Bayes’ rules: make connection between them!

Bayes' Rules

From product rule we can write

$$P(X,Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

Rearranging yields Bayes' rule

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{P(Y|X)P(X)}{\sum_X P(Y|X)P(X)} = \alpha P(Y|X)P(X)$$

Again $\alpha = \frac{1}{P(Y)}$ can be treated as a normalising constant.

The names of the various components are

$$\underbrace{P(X|Y)}_{posterior} = \frac{\overbrace{P(Y|X)}^{likelihood}\overbrace{P(X)}^{prior}}{\underbrace{P(Y)}_{evidence}}$$

Bayes' Rules

The difference between *P(effect|cause)* and *P(cause|effect)*?

Example

A doctor knows that meningitis causes the patient to have a stiff neck 50% of the time.

She also knows some facts: At any given time the probability that someone has meningitis is 1/50000, and the probability that a patient has stiff neck is 1/20.

A patient visits her with a stiff neck (indicated by event s). He is concerned that he might have meningitis (event m).

Performing statistical inference on the meningitis proposition using Bayes' rule yields

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.5 \times 1/50000}{1/20} = 0.0002$$

which is very small! This is because the $P(s) \gg P(m)$.

Observe the possibility of large discrepancies between the causal $P(Effect|Cause)$ and diagnostic $P(Cause|Effect)$ probabilities.

Independence

Another concept central to probability and statistics is independence.

Formally, random variables A and B are statistically independent if and only if

$$P(A|B) = P(A), \text{ or } P(B|A) = P(B), \text{ or } P(A, B) = P(A)P(B)$$

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With Independence, we can simplify the joint distribution.

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Example:

The variables $Toothache$, $Catch$ and $Cavity$ are independent from the variable $Weather$, i.e.,

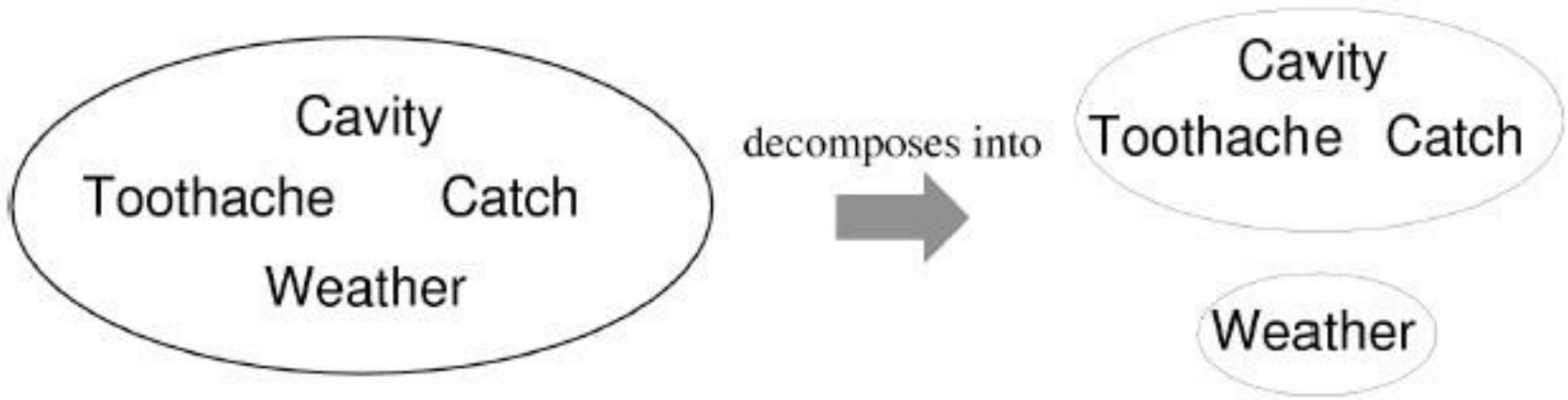
$$P(Toothache, Catch, Cavity, Weather)$$
$$= P(Toothache, Catch, Cavity)P(Weather)$$

2*2*2*4-1 = 31 independent entries

2*2*2-1+ 4-1 = 10 independent entries

Independence

This can be graphically represented as



This notion of independence is sometimes called absolute independence (we shall see different type of independence later).

It is worth noting that absolute independence is powerful (useful for simplifying statistical inference) but rare, e.g., dentistry is a large field with hundreds of variables, none of which are independent.

Conditional Independence

Formally, two random variables X and Y are conditionally independent given a third variable Z if and only if

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

Equivalently we can write

$$P(X|Y, Z) = P(X|Z) \quad \text{and} \quad P(Y|X, Z) = P(Y|Z)$$

Conditional Independence

- Absolute independence -> conditional independence?
- Conditional independence -> absolute independence?

Conditional Independence

- Absolute independence -> conditional independence? No
- Conditional independence -> absolute independence? No

Example

Now, variables $Catch$ and $Toothache$ are not independent: If the probe catches in the tooth, it probably has cavity and that probably causes toothache. $P(Toothache \mid Catch) \neq P(Toothache)$

The 2 variables are independent, however, given the presence or absence of cavity. Each is directly caused by the cavity, but neither affects the other: toothache depends on the state of the nerves in the tooth, whereas the probe's accuracy depends on the dentist' skill, to which the toothache is irrelevant.

$$P(Toothache \mid Catch, Cavity) = P(Toothache \mid Cavity)$$

$$P(Catch \mid Toothache, Cavity) = P(Catch \mid Cavity)$$

Simplification due to conditional independence

The joint probability table of $P(Toothache, Cavity, Catch)$ has 8 entries (see previous lecture notes). However, only 7 of these are independent since the entries must sum to 1.

If we write out the full joint distribution using chain rule and then apply conditional independence:

$$
\begin{aligned}
& P(Catch, Toothache, Cavity) \\
= \; & P(Catch|Toothache, Cavity)P(Toothache, Cavity) \\
= \; & P(Catch|Toothache, Cavity)P(Toothache|Cavity)P(Cavity) && \text{Chain rule} \\
= \; & P(Catch|Cavity)P(Toothache|Cavity)P(Cavity) && \text{Conditional independence}
\end{aligned}
$$

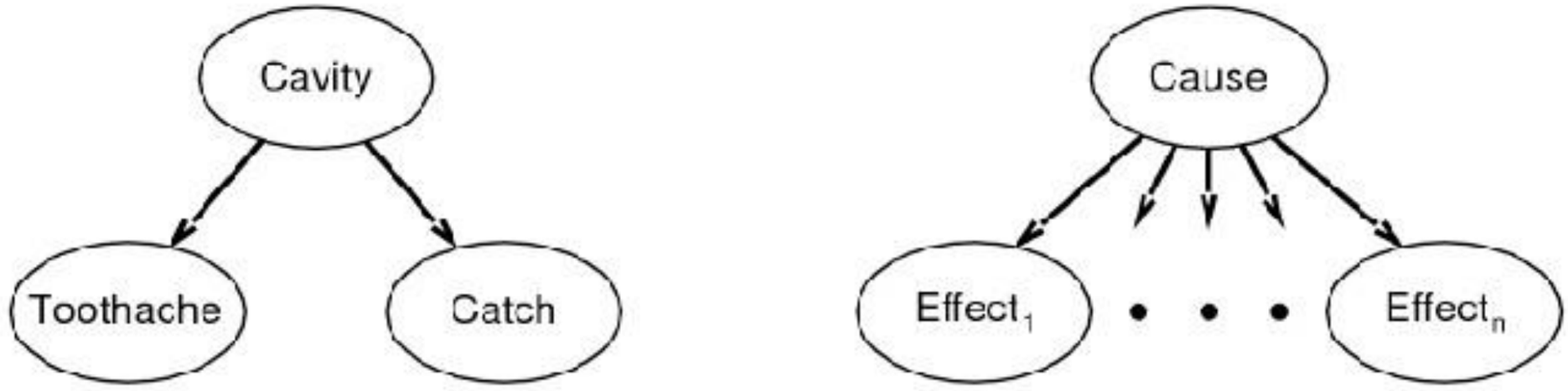
Assuming conditional independence on 2 of the variables allows us to reduce the number of independent entries from 7 to 5:

- 1 for $P(Cavity)$
- 2 for $P(Toothache|Cavity)$
- 2 for $P(Catch|Cavity)$

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

Naive Bayes

This is an example of a naive Bayes model:

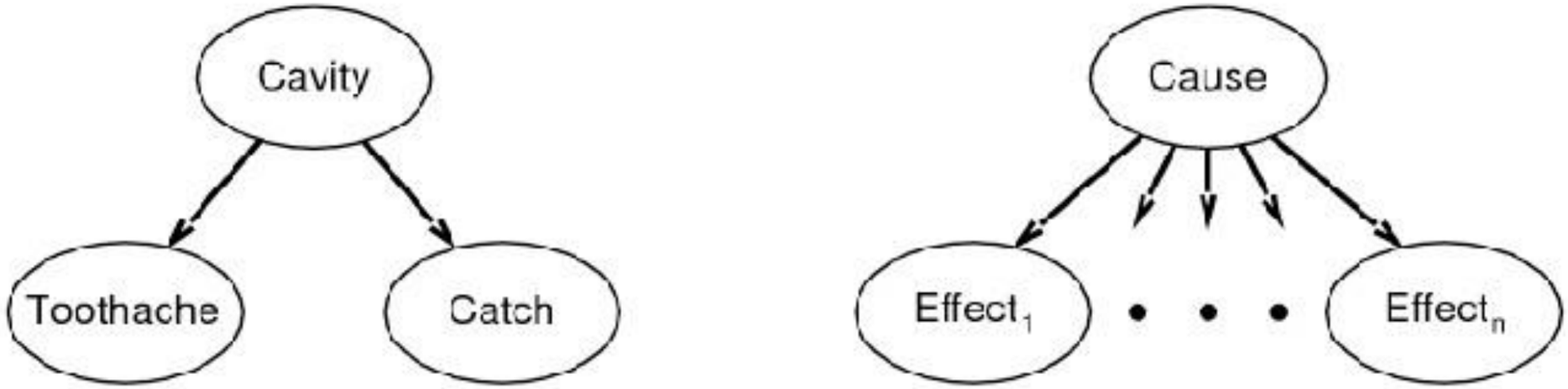


"Naive" because it is often used as a simplifying assumption in cases where the effect variables are not conditionally independent given the cause variable.

However, naive Bayes models can work well in cases where the conditional dependencies between effect variables are weak (this occurs in a surprisingly large number of real-life applications).

Naive Bayes

This is an example of a naive Bayes model:



The full joint probability:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i \mid Cause)$$

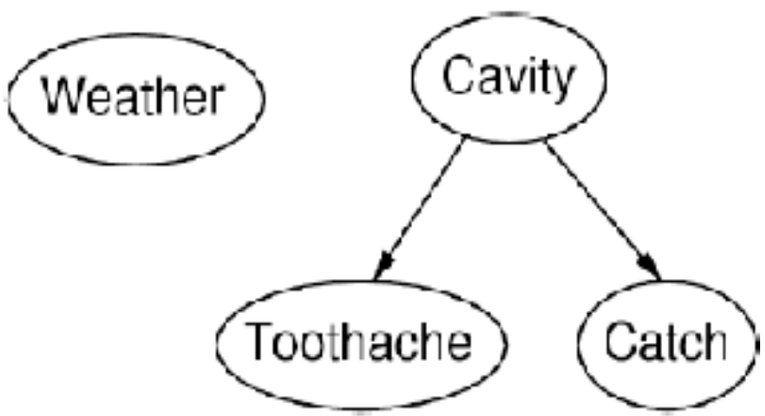
Bayesian Network

A Bayesian network comprises of the following:

- A set of nodes, one per variable.
- A directed, acyclic graph. This means if you start from a node and follow the arrows there is no way of getting back to the original node.

causal knowledge!

Example:



Bayesian Network

A Bayesian Network reflects a simple conditional independence statement. Namely that each variable is independent of its **nondescendents** in the graph given the state of its parents.

Each node is associated with a conditional probability

$$P(X_i|\{X_j\}) = P(X_i|Parents(X_i))$$

Example: Burglar problem

An inference problem

I'm at work, neighbour John called to say my alarm is ringing, but neighbour Mary didn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

Variables

$Burglar$, $Earthquake$, $Alarm$, $JohnCalls$, $MaryCalls$

Network topology reflects "causal" knowledge:

– A burglar can set the alarm off
– An earthquake can set the alarm off
– The alarm can cause Mary to call
– The alarm can cause John to call

Example: Burglar problem

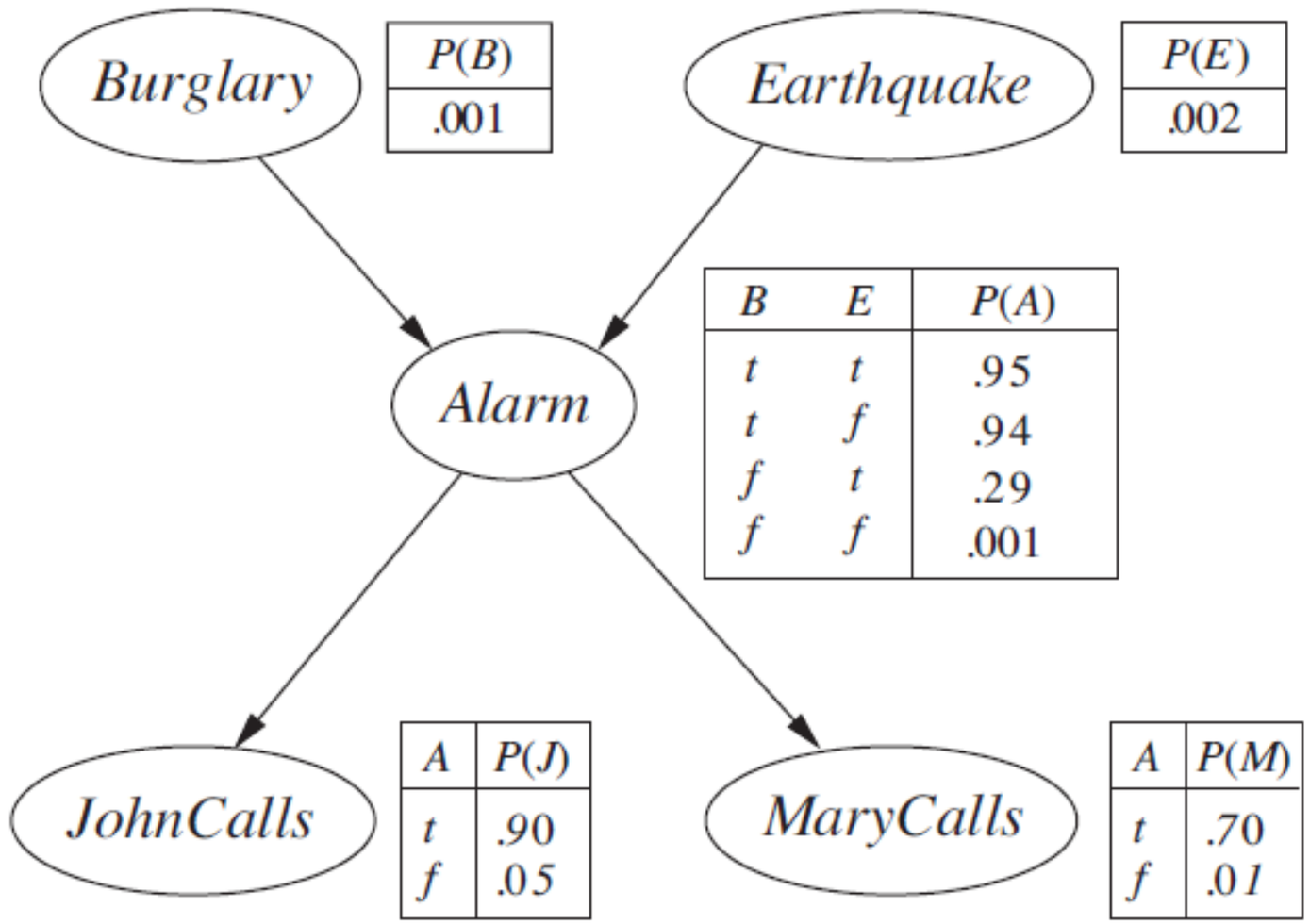
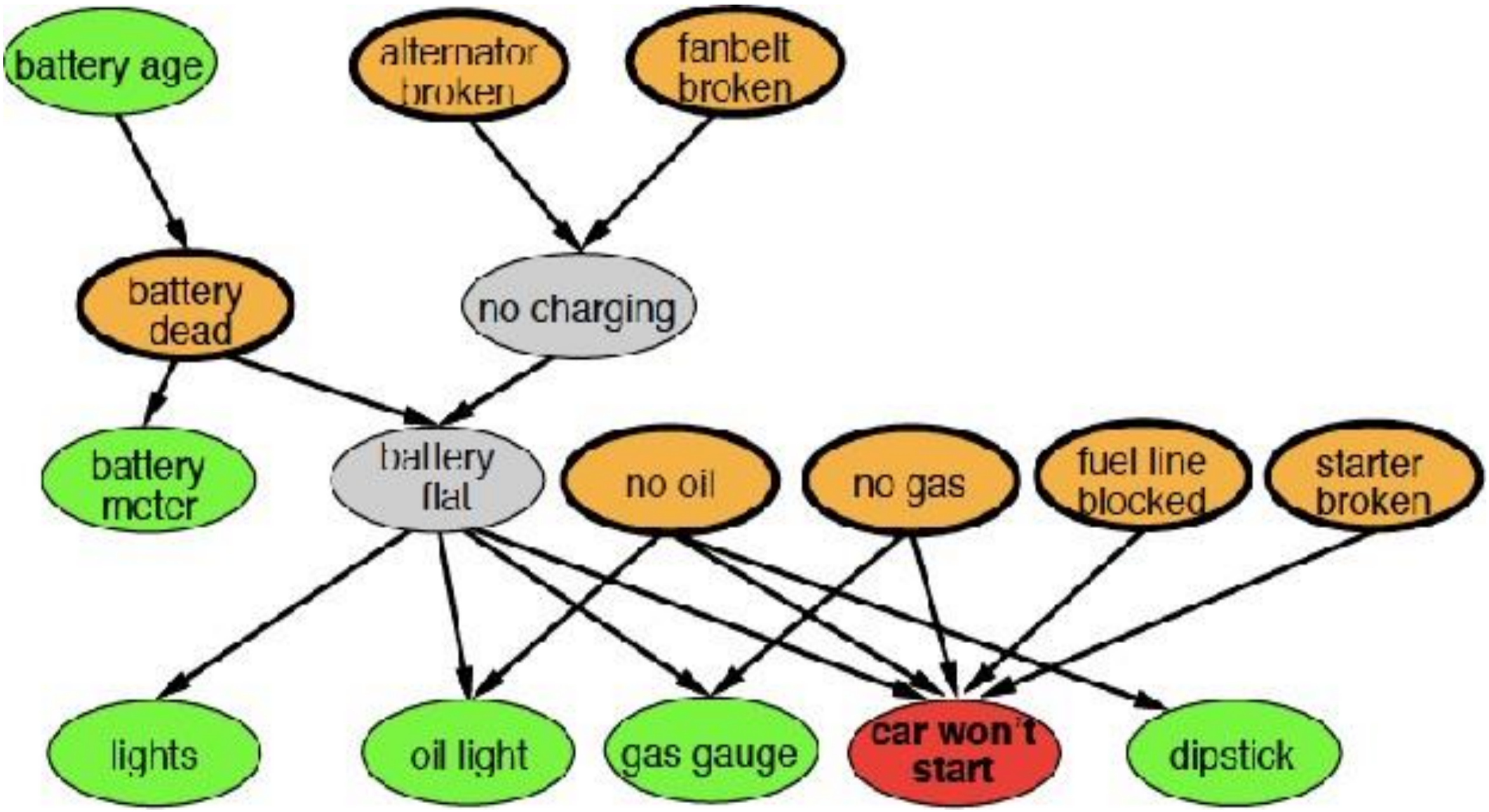
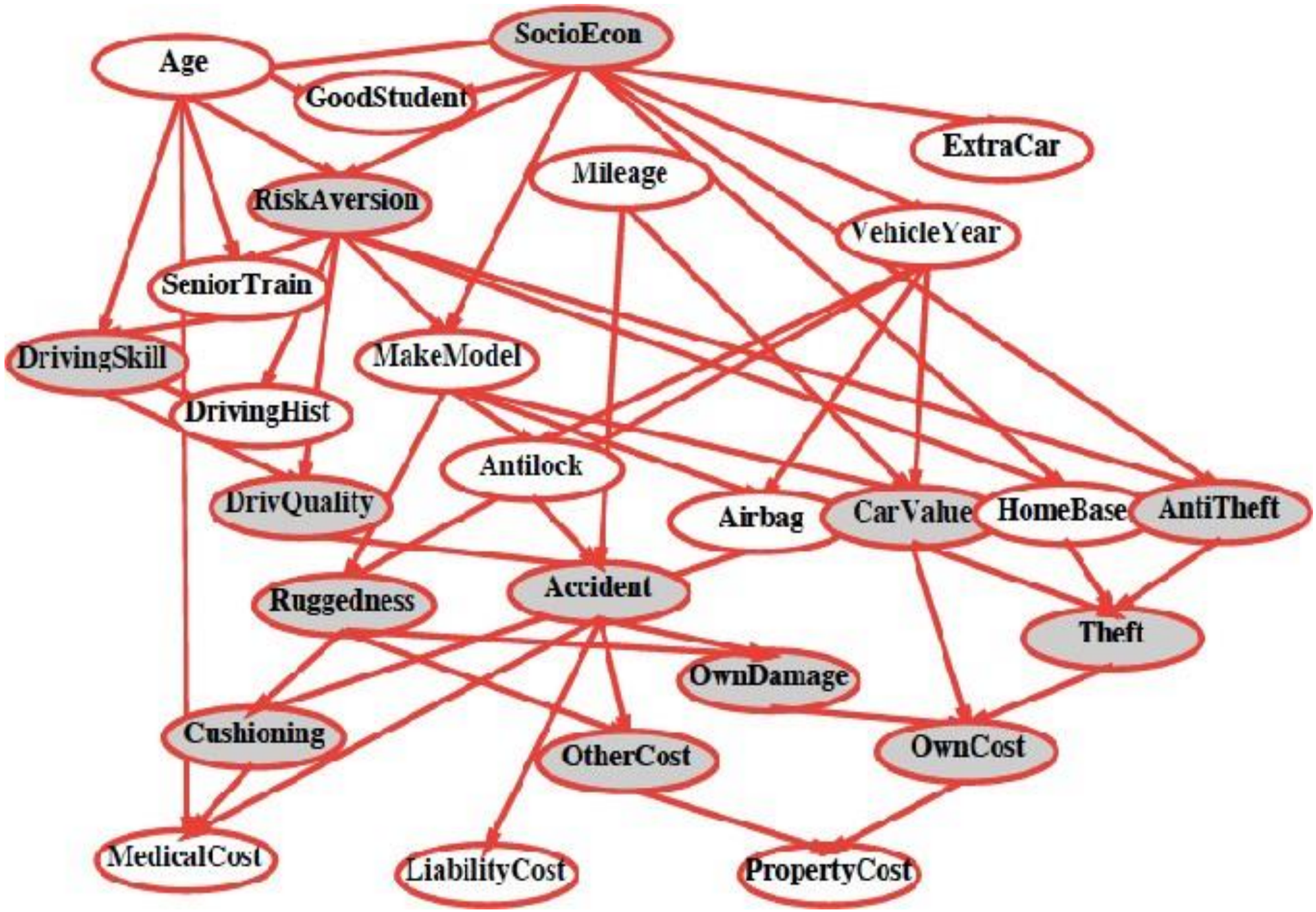


Figure 14.2 A typical Bayesian network, showing both the topology and the conditional probability tables (CPTs). In the CPTs, the letters B, E, A, J, and M stand for *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, and *MaryCalls*, respectively.

Example: Car start problem



Example: car insurance risk assessment



Global Semantics

The global semantics of a network define a joint distribution of all variables as the product of local conditional distributions.

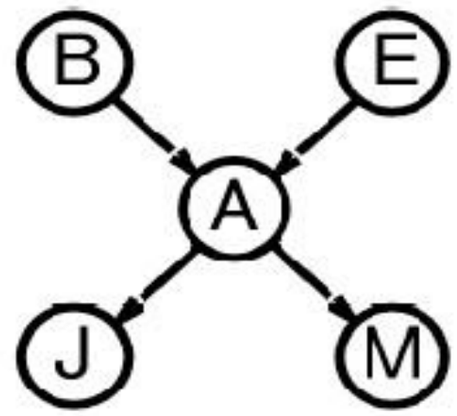
The joint distribution defined by a Bayesian Network with variables $X_1, \ldots, X_n$ is:

$$\begin{aligned} P(X_1, \ldots, X_n) &= P(X_1|Parents(X_1)) \times P(X_2|Parents(X_2)) \\ &\quad \times \cdots \times P(X_n|Parents(X_n)) \\ &= \prod_{i=1}^{n} P(X_i|Parents(X_i)) \end{aligned}$$

where $Parents(X_i)$ are parents of X_i as specified by the particular Bayesian Network.

Example

For the Burglar Alarm network,



the joint probability distribution of all variables as specified by the network is

$$P(J, M, A, B, E) = P(J|A)P(M|A)P(A|B, E)P(B)P(E)$$

Given evidence (i.e. observed values) for all the variables, we use the global semantics to obtain the joint probability of obtaining the evidences.

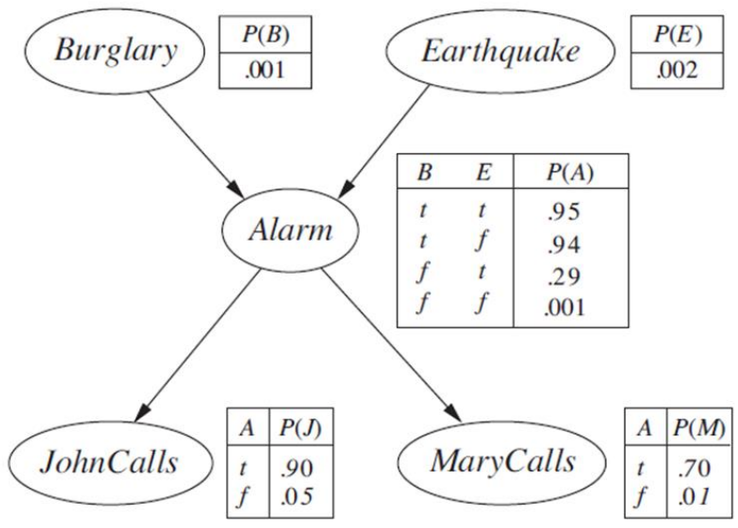
Example

Lets say the observed values are John and Mary called, the alarm is ringing, there is no burglary and no earthquake.

The joint probability of this is

$$\begin{aligned} P(j, m, a, \neg b, \neg e) &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

(for each of the component on the right-hand-side, simply read off the correspoding CPTs)

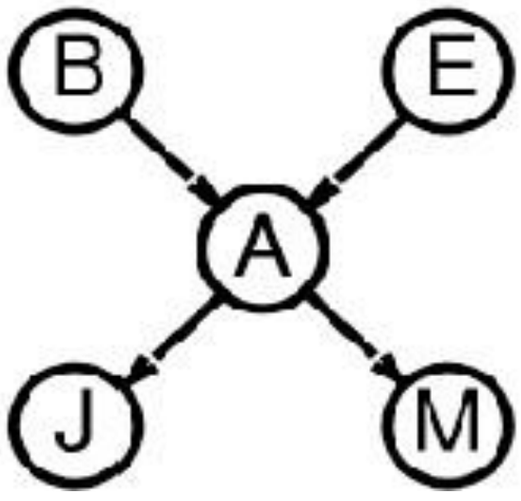


Compactness

The conditional independence assumptions encoded in a Bayesian Network defines a simplified joint distribution of the variables.

For the Burglar Alarm problem where there are 5 Boolean variables, without conditional independence assumptions we need to specify $2^5 - 1 = 31$ independent numbers to define the joint distribution.

Utilising the corresponding Bayesian Network, we require only $1 + 1 + 4 + 2 + 2 = 10$ independent numbers.



Compactness

More generally, a CPT for a Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values.

Each row requires one number p for $X_i = True$ (the number for $X_i = False$ is just $1 - p$).

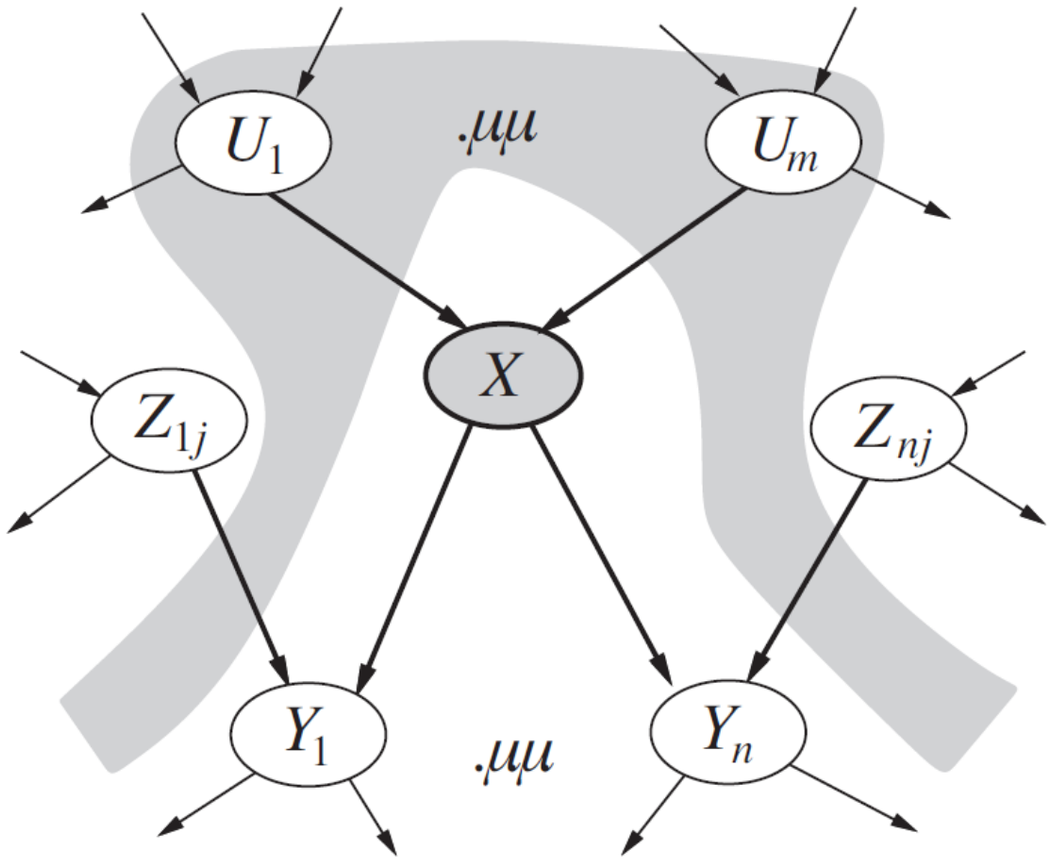
If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ independent numbers, where n is the total number of variables.

This implies that the required numbers grow linearly with n, versus $O(2^n)$ for the full joint distribution.

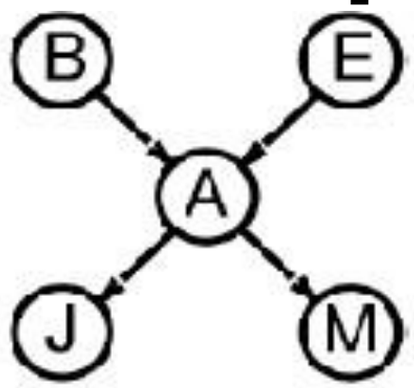
Local Semantics

Conditional independence assumptions can simply be “read off” the network topology.

Local semantics: each node is conditionally independent of its non-descendants given its parents.



Example



Variable J is not independent of variable M, i.e.,

$$P(J, M) \neq P(J)P(M)$$

Intuitively, if John calls, Mary will probably call as well since both would have heard the alarm. The reverse is true.

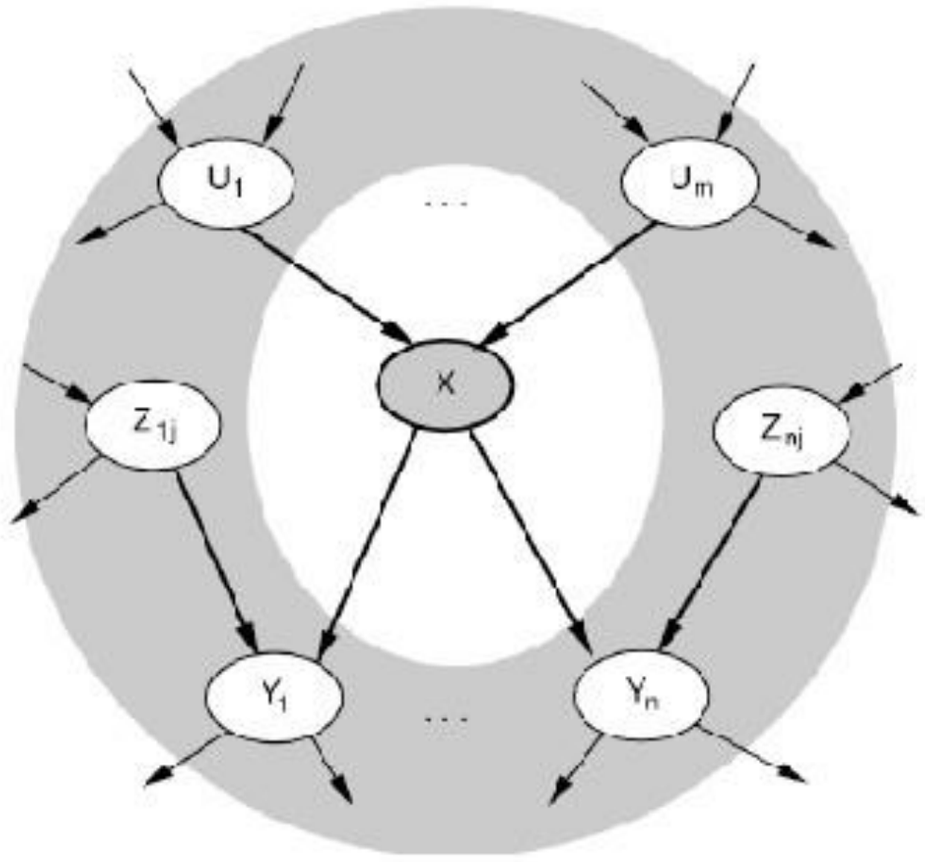
However J is conditionally independent of M given A, since A is the only parent of J and M is a non-descendent of J. So

$$P(J, M|A) = P(J|A)P(M|A)$$

If we know the alarm did ring, the fact that John calls has no bearing on the probability that Mary calls.

Markov blanket

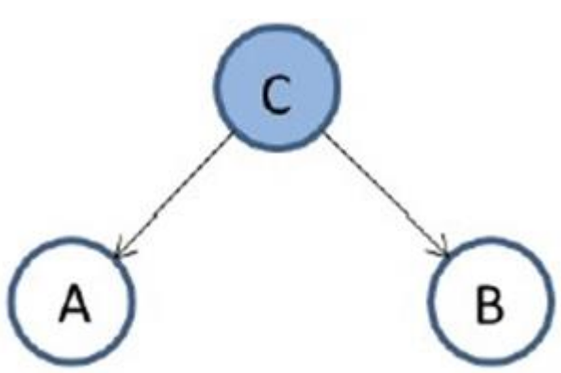
A more specific way to state the local semantics: A node is conditionally independent of all others given its parents, children, and children's parents— i.e., given the Markov blanket of the node.



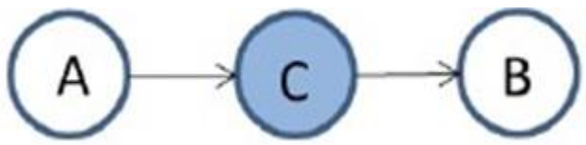
Why do we need to consider the children's parents??

Local Semantics

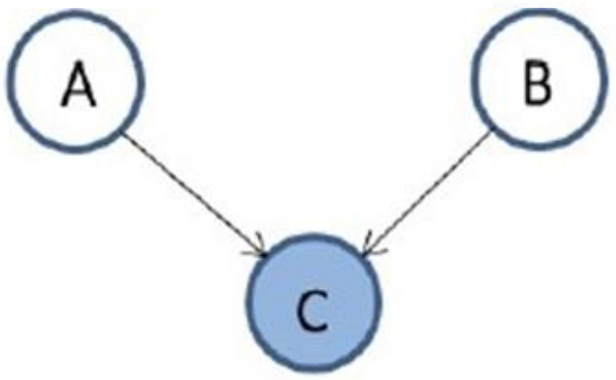
Consider the possible arrangement of a triplet of nodes in a directed acyclic graph



Fork

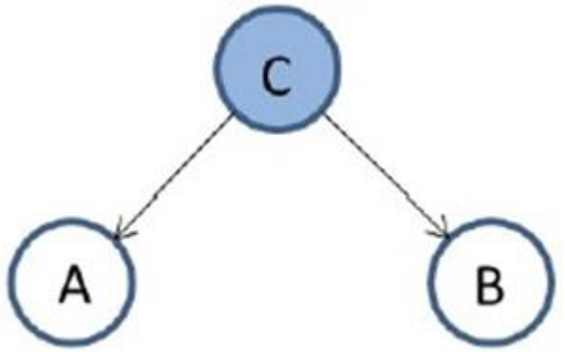


Chain



Inverted fork

Case 1, Fork (Tail-to-Tail)



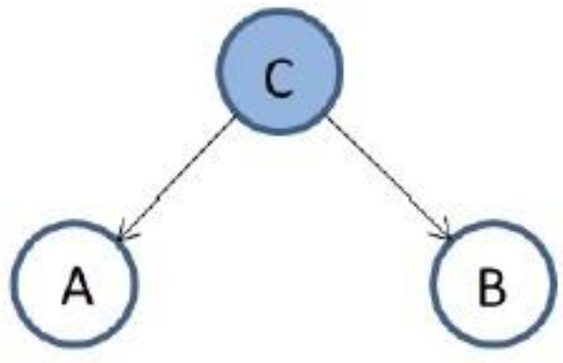
$$P(A, B, C) = P(A|C)P(B|C)P(C)$$

$$P(A, B) = \sum_C P(A|C)P(B|C)P(C)$$ Marginalization

In general, this does not factorize into the product $P(A)P(B)$, so

$$A \not\perp\!\!\!\perp B$$

Case 1, Fork (Tail-to-Tail)



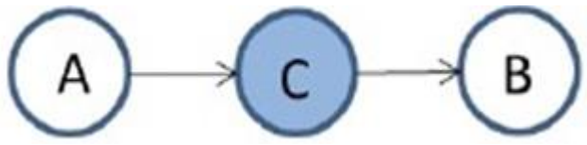
Product rule

$$\begin{aligned} P(A, B|C) &= P(A, B, C)/P(C) \\ &= P(A|C)P(B|C)P(C)/P(C) \\ &= P(A|C)P(B|C) \end{aligned}$$

A and B are not (unconditionally) independent, but A and B are *conditionally independent given* C

$$A \perp\!\!\!\perp B|C$$

Case 2, Chain (Head-to-Tail)



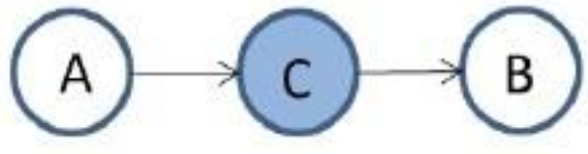
$$P(A, B, C) = P(A)P(C|A)P(B|C)$$

$$P(A, B) = P(A) \sum_C P(C|A)P(B|C)$$

In general, this does not factorize into the product $P(A)P(B)$, so

$$A \not\perp\!\!\!\perp B$$

Case 2, Chain (Head-to-Tail)

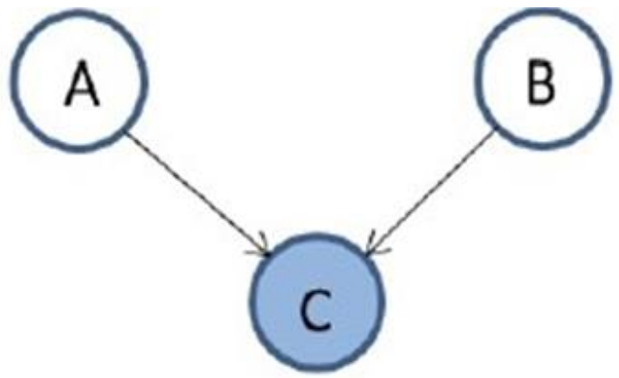


$$\begin{aligned} P(A, B|C) &= P(A, B, C)/P(C) \\ &= P(B|C)P(C|A)P(A)/P(C) \\ &= P(B|C)P(A|C) \end{aligned}$$

As for case 1, A and B are not (unconditionally) independent, but:

$$A \perp\!\!\!\perp B|C$$

Case 3, Inverted Fork (Head-to-Head)

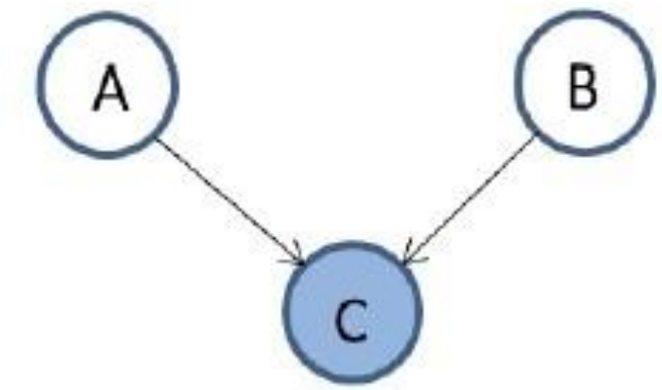


$$P(A, B, C) = P(A)P(B)P(C|A, B)$$

$$P(A, B) = \sum_C P(A)P(B)P(C|A, B) = P(A)P(B)$$

so $A \perp\!\!\!\perp B$

Case 3, Inverted Fork (Head-to-Head)

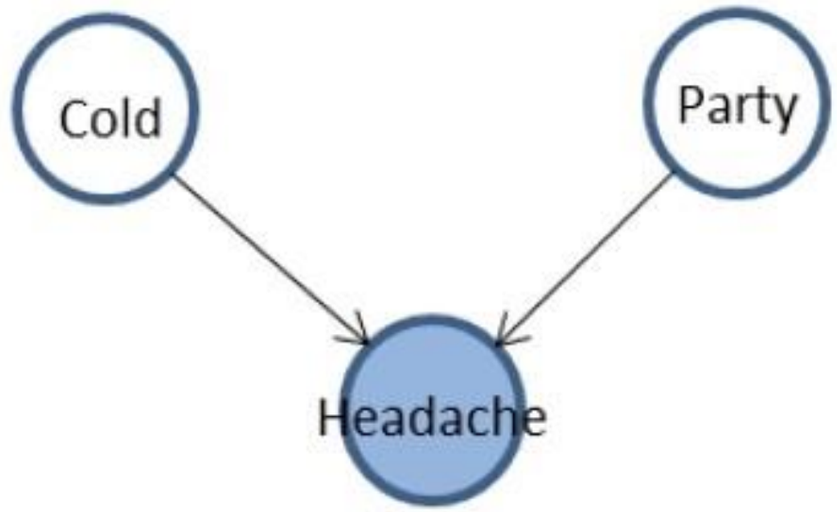


$$P(A, B|C) = P(A, B, C)/P(C)$$
$$= P(C|A, B)P(A)P(B)/P(C)$$

so $A \not\perp\!\!\!\perp B|C$

Case 3 The Explaining-Away

Head-to-head, multiple possible causes, same effect



$$P(\text{Party}) = \langle 0.1, 0.9 \rangle$$
$$P(\text{Cold}) = \langle 0.2, 0.8 \rangle$$

$$P(\text{headache}|\text{party}, \text{cold}) = 0.95$$
$$P(\text{headache}|\neg\text{party}, \text{cold}) = 0.7$$
$$P(\text{headache}|\neg\text{party}, \neg\text{cold}) = 0.1$$
$$P(\text{headache}|\text{party}, \neg\text{cold}) = 0.8$$

Case 3 The Explaining-Away

This is sufficient information for us to write down the full joint probability because $P(H, P, C) = P(H|P, C)P(P)P(C)$:

	party		¬ party	
	cold	¬ cold	cold	¬ cold
headache	0.019	0.056	0.144	0.072
¬ headache	0.001	0.024	0.036	0.648

Before any observations $P(\text{cold}) = 0.2$. Now suppose we observe that the person has a headache. What is the probability of having a cold now?

Recall the general rule we used in last lecture

$$P(Cold|headache) = \alpha \sum_{Party} P(Cold, headache, Party)$$

$$= \alpha\langle 0.019 + 0.144, 0.056 + 0.072\rangle = \langle 0.56, 0.44\rangle$$

$P(cold|headache)$ $P(\neg cold|headache)$

Case 3 The Explaining-Away

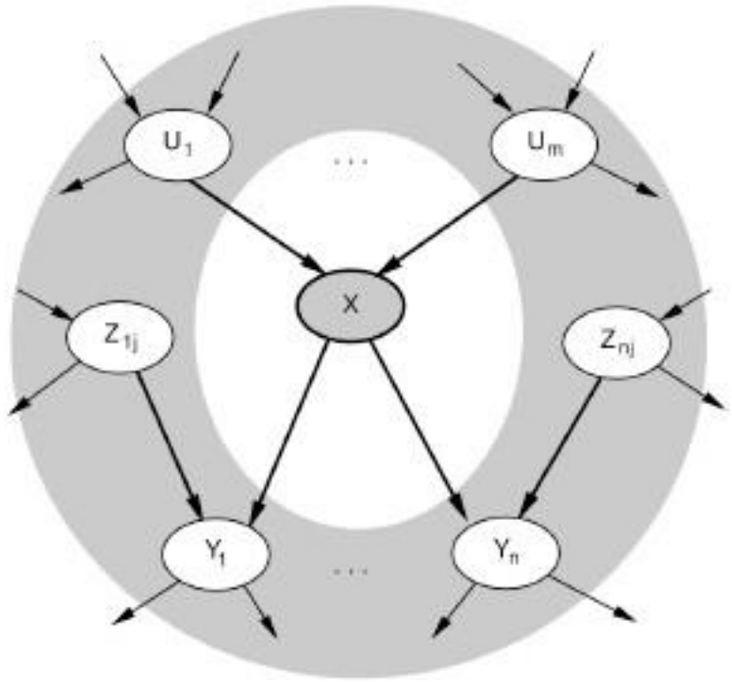
so the probability of having a cold has increased (as we would expect intuitively). Now suppose further that we observe that the person went to a party last night.

$$\begin{aligned} P(cold|headache, party) &= \frac{P(cold, headache, party)}{\sum_{Cold} P(Cold, headache, party)} \\ &= 0.019/(0.019 + 0.056) = 0.25 \end{aligned}$$

which is significantly less than $P(cold|headache) = 0.56$. The observation that she went to a party last night (so may well have a hangover) *explains away* the cold as a cause.

Markov blanket

A node is conditionally independent of all others given its parents, children, and children's parents— i.e., given the Markov blanket of the node. Why do we need to consider the children's parents?? Because of the explainng away effect.



Back to inference problem

So far we have learnt how to obtain the joint probability according to a Bayesian Network given the value of all variables.

However, the sort of problems we wish to solve are statistical inference problems, i.e. we have a query variable, some evidence variables, and some unobserved variables, i.e. we want to compute

$$P(X|e) = \alpha \sum_{\forall Y} P(X, e, Y)$$

Example: "I'm at work, neighbour John called to say my alarm is ringing, but neighbour Mary didn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?" $P(burglar|jCall, \neg mCall)$

How to accomplish this using Bayesian Networks? We shall study this in the next lecture.