Mathematics for Data Science Tutorial 1 (week 2)

Semester 2, 2019

1. How much rain could be collected from the rooftops of houses in Adelaide in one year? How does this compare to the amount of water the population of Adelaide uses in a year?

   Discuss this question with your colleagues! Try to do it without using any calculators or Googles.

   Solution: This question is adapted from *Guesstimation 2.0 : Solving Today's Problems on the Back of a Napkin*; you can download it or read it online from the Library website.

   The proxy we'll need to use for rooftops here are families; in the piano tuner problem from lectures we estimated around 500,000 households in Adelaide. A typical one-storey house is say 6m $\times$ 20m, $\approx$ 100m^2 of roof. A quick Google search suggests around 500mm of rain fall on these rooftops per year, giving

   $$500,000 \times 100 \times 0.5 = 25 \times 10^6 \mathrm{m}^3 = 25 \times 10^9 L$$

   of water collected.

   Now think about what the largest household uses of water are (and forget the rest). These are most likely:

   - toilet flushes ($\approx$ 10L),
   - showers ($\approx$ 10L/min),
   - washing/dishwasher loads (say $\approx$ 50L each),
   - watering lawns ($\approx$ 100L per instance?)

   Add up 5 flushes + 2 5-minute showers + 1 load per day for roughly 200L, over 7 days is about 1400L, add on one lawn watering, to get 1500L per household per week. In one year this would be around 8×10^4L.

   So in a year the city of Adelaide will use $(5 \times 10^5) \times (8 \times 10^4) \approx 4 \times 10^{10}$L of water. I've rounded everything up a little bit. So we estimate that the roofs collect around 5-10% of the water required for all the population.

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2. Consider the function

   $$f(x) = \frac{6 - 10x}{8x + 7}.$$

(a) What is the domain of f?

Solution: Thinking about what can 'break' this function, we have that the denominator cannot be equal to zero. Therefore the value of x for which $8x + 7 = 0$ can't be in the domain, i.e., $x = -7/8$. So the domain is $\mathcal{D} = \{x \in \mathbb{R} | x \neq -7/8\}$.

(b) Find the inverse function $f^{-1}(x)$.

Solution: Set $y = \frac{6-10x}{8x+7}$. Then solving for x we have

$$\begin{aligned}(8x+7)y &= 6 - 10x \\ (8y+10)x &= 6 - 7y \\ x &= \frac{6-7y}{8y+10} = f^{-1}(y)\end{aligned}$$

so

$$f^{-1}(x) = \frac{6-7x}{8x+10}.$$

(c) Verify for this function that $f(f^{-1}(x)) = x$.

Solution:

$$\begin{aligned}f(f^{-1}(x)) &= \frac{6 - 10\frac{6-7x}{8x+10}}{8\frac{6-7x}{8x+10} + 7} \\ &= \frac{6(8x+10) - 10(6-7x)}{8(6-7x) + 7(8x+10)} \\ &= \frac{118x}{118} \\ &= x.\end{aligned}$$

(d) What is the range of f? (Use the fact that the range of f^{-1} is equal to the domain of f.)

Solution: Similarly to before, the value which 'breaks' $f^{-1}(x)$ by making the denominator equal to zero is $x = -10/8 = -5/4$. So the domain of $f^{-1}(x)$, and therefore the range of $f(x)$, is, $\mathcal{D} = \{x \in \mathbb{R} | x \neq -5/4\}$.

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3. Consider the function

$$f(x) = \begin{cases} |2x - x^2| & \text{if } -1 \le x < 2 \\ (x-3)H(x-3) & \text{if } x \ge 2 \end{cases}$$

on the domain $\mathcal{D} = [-1, 4]$.

Recall that $H(x)$ is the Heaviside function,

$$H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}$$

The domain can be broken into four sub-intervals such that on each sub-interval, $f(x)$ can be expressed as a polynomial. Find the sub-intervals and the corresponding polynomial functions. Hence sketch the graph of $f(x)$ and find the range, $\mathcal{R}$.

Solution:

The function $g(x) = 2x - x^2$ is a quadratic function, so its graph is a parabola. Because of the -1 multiplying the x^2, the parabola opens downwards. Since $2x - x^2 = x(2 - x)$, the curve $y = g(x)$ crosses the x-axis at $x = 0$ and $x = 2$. Since the it is a parabola, its vertex will be midway between $x = 0$ and $x = 2$, so when $x = 1$. Since $g(1) = 1$, the vertex of the parabola will be at the point $(1, 1)$.

For $-1 \le x < 0$, $g(x) < 0$, but the absolute value function will mean than the function $|g(x)|$ takes the positive values $-g(x)$ (so this part of the parabola is reflected in the x-axis). Hence $f(x) = x^2 - 2x$ on $[-1, 0)$.

For $0 < x < 2$ $g(x)$ is positive, so $f(x) = |g(x)| = g(x) = 2x - x^2$.

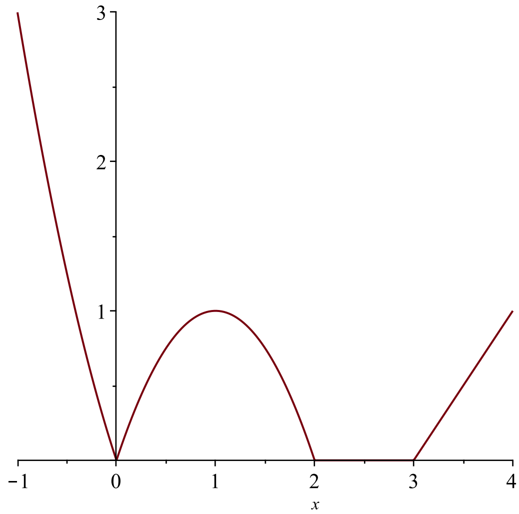
Over the interval $[2, 3)$, $x - 3 < 0$. Hence, the Heaviside function $H(x - 3) = 0$, so $f(x) = 0$ on this interval.

On the interval $[3, 4]$, $x - 3 \ge 0$, so $H(x - 3) = 1$. Hence $f(x) = x - 3$ on this interval.

Putting all this information together, we have

$$f(x) = \begin{cases} x^2 - 2x & \text{if } x \in [-1, 0) \\ 2x - x^2 & \text{if } x \in [0, 2) \\ 0 & \text{if } x \in [2, 3) \\ x - 3 & \text{if } x \in [3, 4] \end{cases}$$

We can now sketch the graph of f:



By looking at the graph, we can see $\mathcal{R} = [0, 3]$

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4. If $f(x) = \sinh x$, $x \in \mathbb{R}$, find the inverse function, $f^{-1}(x)$, giving your answer in terms of the natural logarithm function.
   Hint: use the definition to write $\sinh x$ *in terms of the exponential function, and obtain a quadratic equation satisfied by* e^x.

   Solution: We note that $f(x)$ is a one-to-one function *e.g.*, by checking its graph. Hence, the inverse function, f^{-1} exists. To find it, we follow the procedure outlined in the lecture notes. We begin by writing $y = \sinh x = \frac{e^x - e^{-x}}{2}$. Noting that $(e^x)^2 = e^{2x}$ we follow the hint by re-writing this equation as

$$2ye^x = e^{2x} - 1 \quad \Rightarrow \quad e^{2x} - 2ye^x - 1 = 0,$$

   which is a quadratic for e^x. Using the quadratic formula gives,

$$e^x = y \pm \sqrt{y^2 + 1}.$$

   However, we note that $y < \sqrt{y^2 + 1}$, whilst $e^x > 0$ for any value of x, so we must take the positive square root. Hence $e^x = y + \sqrt{y^2 + 1}$. Taking the natural logarithm then gives

$$x = \ln\left(y + \sqrt{y^2 + 1}\right) = f^{-1}(y).$$

   As the final step, we re-write the RHS in terms of x to get

$$f^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right).$$

5. Find all solutions in $[0, \pi]$ of the equation

$$4\cos^2 x + 3\sin^2 x - 2\sin x = 2,$$

giving your answers in exact form in terms of the arcsin function.
Hint: Use one of the properties of the sin *and* cos *functions to convert the equation into a quadratic in* $\sin x$.

Solution:

We know that, for any x, $\cos^2 x = 1 - \sin^2 x$. Hence, the equation can be written

$$4(1 - \sin^2 x) + 3\sin^2 x - 2\sin x = 4 - \sin^2 x - 2\sin x = 2,$$

or equivalently

$$\sin^2 x + 2\sin x - 2 = 0.$$

We note that this is a quadratic in $\sin x$.

Using the quadratic formula (or other suitable method), we find

$$\sin x = -1 \pm \sqrt{3}.$$

Now, $-1-\sqrt{3} \approx -2.73 < -1$ and since the values of the sin function lie in the interval $[-1, 1]$, there is no real number x with $\sin x = -1 - \sqrt{3}$. On the other hand, $-1+\sqrt{3} \approx 0.73 < 1$, and is therefore an acceptable value for $\sin x$.

Thinking about the graph of the sin function, there are clearly 2 numbers $x \in [0, \pi]$ whose sin is $\sqrt{3} - 1$, namely $\arcsin(\sqrt{3} - 1)$ and the other one which is "on the other side of $\pi/2$",
namely $\pi - \arcsin(\sqrt{3} - 1)$.

Hence, $x = \arcsin(\sqrt{3} - 1)$, $x = \pi - \arcsin(\sqrt{3} - 1)$ are the solutions.