THE UNIVERSITY of ADELAIDE

Student ID:	
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Examination in the School of Mathematical Sciences

Practice exam

109685	**MATHS 1004**	**Mathematics for Data Science I**
109685	**MATHS 1004UAC**	**Mathematics for Data Science – University of Adelaide College**

Time for completing booklet: 120 mins (plus 10 mins reading time).

Question	Marks	
1	/12	
2	/14	
3	/9	
4	/11	
5	/10	
6	/14	
Total	/70	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators without remote communications or CAS capability are allowed.
- You may bring in one double-sided A4 formula sheet into the examination.

Do not commence writing until instructed to do so.

12 Total

Question 1.

Answer true or false to each of the following assertions. You must also provide a *very brief* (one to two lines) justification for each of your answers.

/2 marks

1(a) The matrix

$$\begin{bmatrix} 2 & 4 & -1 & 8 \\ -4 & 3 & 5 & 6 \\ 3 & 2 & 0 & 4 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

is invertible.

[1 mark] [1 mark] **Solution** False. ※ The second and fourth columns are linearly dependent. ※

/2 marks

1(b) A set of vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent if there is no non-trivial solution to $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \mathbf{0}$.

[1 mark] [1 mark] **Solution** True. ※ This is the definition. ※

/2 marks

1(c) An even function has an inverse.

[1 mark] [1 mark] **Solution** False. ※ It is not 1-1 (since $f(x) = f(-x)$). ※

/2 marks

1(d) Two die are rolled. The events described by 'both numbers are odd' and 'the sum greater than 10' are mutually exclusive.

[1 mark] [1 mark] **Solution** True. ※ The intersection of these events is empty, e.g. since only the outcomes $(5,6)$, $(6,5)$, $(6,6)$ have sums greater than 10 and both numbers odd in none of these. ※

/2 marks

1(e) A biased coin is tossed. The probability of flipping heads is 0.7 and the probability of tails is 0.4.

[1 mark] [1 mark] **Solution** False. ※ The probabilities do not sum to 1. ※

/2 marks

1(f) E, F are two events in a sample space S with probabilities $\Pr(E \cap F) = 0.8$, $\Pr(E) = 0.6$.

[1 mark] [1 mark] **Solution** False. ※ Since $E \cap F \subset E$ we must have $\Pr(E \cap F) \leq \Pr(E)$. ※

14 Total

Question 2.

2(a) Consider the following two matrices X and Y:

$$X = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}; \quad Y = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 3 \end{bmatrix}$$

If possible, calculate the following. If not possible, give a reason for your answer.

/1 mark

(i) $X + Y$

[1 mark] **Solution** Not defined (shapes/dimensions are different). ※

/1 mark

(ii) XY

Solution

$$XY = \begin{bmatrix} -1 & 4 & -3 \\ 7 & -10 & 12 \end{bmatrix}$$

[1 mark] ※

/1 mark

(iii) YX

[1 mark] **Solution** Not defined (Y is order 3×3 while X is order 2×3 and $3 \neq 2$).※

/1 mark

(iv) $Y^{\intercal}$

Solution

$$Y^{\intercal} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

[1 mark] ※

/1 mark

(v) the determinant $|X|$

[1 mark] **Solution** Not defined since X is not square. ※

/1 mark

(vi) the determinant $\det(Y)$

[1 mark] **Solution** $\det(Y) = 6$. ※

/1 mark

(vii) The eigenvalues of Y

[1 mark] **Solution** $\lambda = 1, 2, 3$ since the eigenvalue equation is $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$. ※

2(b) Consider the system of equations

$$\begin{aligned} 2x - z - 4 &= 0 \\ y - 2z &= 0 \\ -x + 3y + z &= 1 \end{aligned}$$

/1 mark

(i) Write down the matrix A and vector $\boldsymbol{b}$ such that the above system of equations can be written in matrix form $A\boldsymbol{x} = \boldsymbol{b}$.

Solution

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

[1 mark] ※

/5 marks

(ii) Determine the inverse of

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}$$

by wrtiting in the augmented form $[A|I]$ and performing the required row operations.

Solution

$$R_1 \to R_1/2 \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

[1 mark] ※

$$R_3 \to R_3 + R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 3 & 1/2 & 1/2 & 0 & 1 \end{array}\right]$$

[1 mark] ※

$$R_3 \to R_3 - 3R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 13/2 & 1/2 & -3 & 1 \end{array}\right]$$

[1 mark] ※

Please turn over for page 5.

$$R_3 \to \frac{2}{13} R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/13 & -6/13 & 2/13 \end{array}\right]$$

[1 mark] ※

$$R_1 \to R_1 + R_3/2, \quad R_2 \to R_2 + 2R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/13 & -3/13 & 1/13 \\ 0 & 1 & 0 & 2/13 & 1/13 & 4/13 \\ 0 & 0 & 1 & 1/13 & -6/13 & 2/13 \end{array}\right]$$

[1 mark] ※

Hence

$$A^{-1} = \begin{bmatrix} 7/13 & -3/13 & 1/13 \\ 2/13 & 1/13 & 4/13 \\ 1/13 & -6/13 & 2/13 \end{bmatrix}$$

/1 mark

(iii) Hence, or otherwise, find the solution $\boldsymbol{x}$.

Solution

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 29/13 \\ 12/13 \\ 6/13 \end{bmatrix}$$

[1 mark] ※

9 Total

Question 3.

3(a) Consider the matrix

$$A = \begin{bmatrix} -7 & 5 \\ -3 & 9 \end{bmatrix}.$$

/2 marks

(i) Solve the characteristic equation to show that the eigenvalues for this matrix are $\lambda = 8$ and $\lambda = -6$.

[1 mark]
[1 mark]

Solution Characteristic equation ※ gives after some simplification $\lambda^2 - 2\lambda - 48 = (\lambda - 8)(\lambda + 6) = 0$. ※ Solving this gives the result.

/4 marks

(ii) Hence, find the eigenvectors of A.

Solution For $\lambda = -6$:

$$\left[\begin{array}{cc|c} -1 & 5 & 0 \\ -3 & 15 & 0 \end{array}\right]$$

Row reduces to

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

[1 mark] ※

[1 mark] giving $\boldsymbol{v}_1 = (1, 1/5)t$. ※

For $\lambda = 8$:

$$\left[\begin{array}{cc|c} -15 & 5 & 0 \\ -3 & 1 & 0 \end{array}\right]$$

Row reduces to

$$\left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

[1 mark]
[1 mark]

※ giving $\boldsymbol{v}_2 = (1/3, 1)t$. ※

/3 marks

3(b) Consider a principal component analysis of some dataset X. This dataset has the covariance matrix

$$C = \begin{bmatrix} 1 & 16 & -32 \\ -4 & -41 & 88 \\ -2 & -20 & 43 \end{bmatrix}$$

The largest eigenvalue of C is $\lambda = 3$. Use this to determine a principle component of X.

Please turn over for page 7.

Solution Principal components solve the eigenvector equation $C\boldsymbol{v} = \lambda\boldsymbol{v}$. Therefore one of the principle components
[1 mark] satisfies $C\boldsymbol{v} = 3\boldsymbol{v}$, or equivalently $(C - 3I)\boldsymbol{v} = \mathbf{0}$. ※

We now row reduce [(C-3 I)—**0**] to obtain

$$\left[\begin{array}{ccc|c} -2 & 16 & -32 & 0 \\ -4 & -44 & 88 & 0 \\ -2 & -20 & 40 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} -2 & 16 & -32 & 0 \\ 0 & -76 & 152 & 0 \\ 0 & -36 & 72 & 0 \end{array}\right]$$

[1 mark] ※

$R_1 \to -R_1/2, \quad R_2 \to -R_2/76, \quad R_3 \to -R_3/36$

$$\left[\begin{array}{ccc|c} 1 & -8 & 16 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

[1 mark] ※

$R_1 \to R_1 + 8R_2, \quad R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the corresponding principle component is $(0, 2, 1)t$
[1 mark] for $t \in \mathbb{R}$.※

11 Total

Question 4.

4(a) A company employs 7 engineers, 4 data scientists and 2 mathematicians. The director randomly chooses 3 of these employees to work on a new research project.

/1 mark (i) How many different types of teams could be chosen (thinking only of the expertise of each member)?

/1 mark (ii) Now taking account of individuals in the company, how many different possible teams are there?

/1 mark (iii) To give some structure to the team, the director decides to appoint one member as research leader. How many compositions of teams are there now (thinking only of the expertise of each member)?

/1 mark (iv) Repeat the previous question but accounting for the individuals in the team.

/1 mark (v) The research project becomes much bigger than anticipated and so a fourth team member is added to the team. How many different team compositions are there now (thinking only of the expertise of each member)?

Solution

(i) Let E, D, M denote the profession of each member. The possible outcomes are

$$\{EEE, EED, EEM, EDD, EDM, \\ EMM, DDD, DDM, DMM\}$$

noting that the order of the members is not important, and a team of 3 mathematicians is not possible. Hence there are 9 different team compositions. ※ [1 mark]
(Alternatively, observe that if there were 3 mathematicians there would be $\binom{3+(3-1)}{3} = 10$ combinations, from which we can then subtract the one impossible outcome.)

(ii) This is simply a problem of choosing 3 people out of the 13 in an unordered manner. It follows that there are $\binom{13}{3} = 286$ possible teams. ※ [1 mark]

(iii) Suppose the leader is chosen first, it could be any one of E, D, M. If the leader is E or D, then the remainder of the team can be any one of EE, ED, EM, DD, DM, MM, that is $\binom{2+(3-1)}{2} = 6$ outcomes. However, if the leader is M, then all but MM are possible for the remainder of the team. Therefore, there are $2 \times 6 + 5 = 17$ possible compositions (aternatively formulate the result as $3 \times \left(\binom{2+(3-1)}{2} - 1\right)$. ※ [1 mark]

(iv) Again, suppose the leader is chosen first, there are 13 possible choices. Making up the remainder of the team then has $\binom{12}{2} = 66$ possible outcomes. Thus the total number of teams with a team leader is $13 \times 66 = 858$. ※ [1 mark]

(Alternatively, for every possible team from part ii there are 3 possible choices for leader, hence $286 \times 3 = 858$ outcomes.)

(v) If there were 4 mathematicians available there would be $\binom{4+(3-1)}{4} = 15$ possible outcomes. However, one of these contains 4 mathematicians (the team $MMMM$) and two contain three mathematicians (the teams $EMMM$ and $DMMM$). Therefore only 12 of these are valid combinations. ※ [1 mark]

Alternatively one could list all 12, i.e.

$$\{EEEE, EEED, EEEM, EEDD, EEDM, EEMM, EDDD, EDDM, EDMM, DDDD, DDDM, DDMM\}.$$

/2 marks

4(b) (i) Suppose the probability of passing the final exam having completed the practice exam is 0.9 and the probability of passing the final exam without completing the practice exam is 0.45. If 80 students in the class complete the practice exam and 40 do not, what is the probability that a student chosen at random will pass the final exam.

Solution Using the law of total probability, let F denote passing the final exam, and P denote completing the practice exam.

$$\begin{aligned}\Pr(F) &= \Pr(F|P)\Pr(P) + \Pr(F|P^c)P(P^c) \\ &= 0.9\frac{80}{80+40} + 0.45\frac{40}{80+40} \\ &= 0.75\end{aligned}$$

※ [2 marks]

/1 mark

(ii) Suppose that 80% of students who completed all assignments also completed the practice exam. Is this enough information to determine the probability of a student passing the final exam given they have completed all of the assignments.

Solution No, we have only been given $\Pr(P|A) = 0.8$ which is insufficient to determine $\Pr(F|A) = \Pr(F \cap A)/\Pr(A)$, where A denotes completing all assignments.

[1 mark] ※

/3 marks

4(c) A researcher studying the genes of a particular plant species. They have found that 40% of their plant samples are significantly taller than the other. In 80% of the tall plants they have identified a particular gene they believe is a marker for the plant being tall. However, 30% of the short plants are also found to have this gene marker. Suppose a new seedling has been analysed and is found to have the gene marker, what is the probability it will grow tall?

Solution Let T denote the event a plant grows tall, and M be the event that it has the gene marker. We want to know $\Pr(T|M)$ based on the available data. We have been given $\Pr(T) = 0.4$ (and thus $\Pr(T^c) = 0.6$), $\Pr(M|T) = 0.8$ and $\Pr(M|T^c) = 0.3$. [1 mark] ※ Using Bayes' theorem we have

$$\begin{aligned}
\Pr(T|M) &= \frac{\Pr(M \cap T)}{\Pr(M)} \\
&= \frac{\Pr(M|T)\Pr(T)}{\Pr(M|T)\Pr(T) + \Pr(M|T^c)\Pr(T^c)} \\
&= \frac{0.8 \times 0.4}{0.8 \times 0.4 + 0.3 \times 0.6} \\
&= \frac{0.32}{0.32 + 0.18} = 0.64\,.
\end{aligned}$$

[2 marks] ※

10 Total

Question 5.

5(a) Consider the probability density function defined by

$$f(x) = \begin{cases} a - cx & \text{if } 0 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

/2 marks

(i) What restrictions must a and c satisfy? Give a reason for your answer.

Solution Since the area under the curve must be one then $\frac{4}{2}(a + (a - 4c)) = 4a - 8c = 1$. ※ Additionally, we must have $f(x) = a - cx \ge 0$ for $x \in [0, 4)$ and so $a \ge 0$ and also $a - 4c \ge 0$. ※

[1 mark]

[1 mark]

/1 mark

(ii) Find the expected value of x (for any a, c).

Solution

$$\begin{aligned}\mathbb{E}[X] &= \int_0^4 x f(x) dx \\ &= \int_0^4 ax - cx^2 dx \\ &= \left[\frac{1}{2}ax^2 - \frac{1}{3}cx^3\right]_0^4 = 8a - \frac{64}{3}c\,.\end{aligned}$$

[1 mark] ※

/2 marks

(iii) Is there a valid a, b for which $\mathbb{E}[X] = 3$?

Solution This would mean we have both $4a - 8c = 1$ and $8a - \frac{64}{3}c = 3$. ※ Subtracting twice the former from the latter we obtain $-\frac{16}{3}c = 1$, or $c = -\frac{3}{16}$. It follows that $4a = 1 + 8c = -\frac{1}{2}$, but then $a = -\frac{1}{8} < 0$ which violates the condition that $f(0) \ge 0$. Hence $\mathbb{E}[X] = 3$ is not possible. ※

[1 mark]

[1 mark]

5(b) Consider a function $f(x)$ for which

- $f(x) = x^{-4}$ for $x \ge 1$
- $f(x) \ge 0$ for $x \in [-2, 1]$
- $f(x) < 0$ for $x < -2$

/1 mark

(i) Write down, using correct set notation, the largest domain over which this function could be a valid probability density function.

Please turn over for page 12.

[1 mark] **Solution** $\mathcal{D} = [-2, \infty) = \{x \in \mathbb{R} | x \geq -2\}$. ※ (Either
[1 mark] way is fine.) Because the PDF needs to be positive. ※

For the remainder, suppose that the valid domain $\mathcal{D}$ of f which makes X a probability density function contains $[1, \infty)$.

/2 marks

(ii) What is the probability that X is between 2 and 4?

Solution

$$\begin{aligned} P(2 \leq X \leq 4) &= \int_2^4 x^{-4} dx \\ &= \left[-\frac{1}{3} x^{-3}\right]_2^4 \\ &= \frac{1}{3}\left(\frac{1}{8} - \frac{1}{64}\right) = \frac{7}{192} \end{aligned}$$

[2 marks] ※

/2 marks

(iii) What is $\Pr(X \in \mathcal{D} \cap [-2, 1))$?

Solution The total area must be 1 and so $\Pr(X \in$
[1 mark] $\mathcal{D} \cap [-2, 1)) = 1 - \Pr(X \geq 1)$. ※ Since

$$\Pr(X \geq 1) = \int_1^\infty x^{-3}\, dx = \left[-\frac{1}{3} x^{-3}\right]_1^\infty = \frac{1}{3},$$

[1 mark] then $\Pr(-X \in \mathcal{D} \cap [-2, 1)) = \frac{2}{3}$.※

14 Total

Question 6.

/3 marks

6(a) Use the priniciple of mathematical induction to prove that

$$\frac{\partial^n}{\partial x^n} x^2 e^x = (n(n-1) + 2nx + x^2)e^x$$

[1 mark]

Solution Base case $n = 0$ gives $x^2e^x = (0(-1)+0x+x^2)e^x$. ※

(It is also okay to take $n = 1$ as a base case to find $\frac{\partial}{\partial x} x^2 e^x = (2x + x^2)e^x$)

Now assume it is true for the case n. Then for the case $n+1$ we have

$$\begin{aligned}
\frac{\partial^{n+1}}{\partial x^{n+1}} x^2 e^x &= \frac{\partial}{\partial x}\left(\frac{\partial^n}{\partial x^n} x^2 e^x\right) \\
&= \frac{\partial}{\partial x}(n(n-1) + 2nx + x^2)e^x \\
&= (2n + 2x)e^x + (n(n-1) + 2nx + x^2)e^x \\
&= ((n+1)n + 2(n+1)x + x^2)e^x\,,
\end{aligned}$$

[2 marks]

as required. ※

/3 marks

6(b) Using the result above and mathematical induction, show that

$$\sum_{k=0}^{n} \frac{\partial^k}{\partial x^k} x^2 e^x = \frac{1}{3}(n+1)(n(n-1) + 3nx + 3x^2)e^x\,.$$

[1 mark]

Solution For the base case $n = 0$ we have $x^2e^x = \frac{1}{3}(1)(1(0)+0x + 3x^2)e^x$. ※

Now, assume it is true for the case n, then for the case $n+1$

we have

$$\begin{aligned}\sum_{k=0}^{n+1}\frac{\partial^k}{\partial x^k}x^2e^x &= \frac{\partial^{n+1}}{\partial x^{n+1}}x^2e^x + \sum_{k=0}^{n}\frac{\partial^k}{\partial x^k}x^2e^x \\ &= ((n+1)n + 2(n+1)x + x^2)e^x \\ &\quad + \frac{1}{3}(n+1)(n(n-1)+3nx+3x^2)e^x \\ &= \frac{1}{3}\Big(3(n+1)n + 6(n+1)x + 3x^2 \\ &\qquad + (n+1)n(n-1) + 3(n+1)nx + 3(n+1)x^2\Big)e^x \\ &= \frac{1}{3}\Big((n+1)n(n+2) + 3(n+1)(n+2)x + 3(n+2)x^2\Big)e^x \\ &= \frac{1}{3}(n+2)\Big((n+1)n + 3(n+1)x + 3x^2\Big)e^x\,,\end{aligned}$$

[2 marks] as required. ※

/3 marks

6(c) Using the result above, or otherwise, calculate the limit

$$\lim_{n\to\infty}\frac{e^{-x}}{n^3}\sum_{k=0}^{n}\frac{\partial^k}{\partial x^k}x^2e^x = \frac{1}{3}\,.$$

Solution First we observe that our previous results show that

$$\frac{e^{-x}}{n^3}\sum_{k=0}^{n}\frac{\partial^k}{\partial x^k}x^2e^x = \frac{(n+1)(n(n-1)+3nx+3x^2)}{3n^3}\,.$$

[1 mark] ※ which we can further expand to

$$\begin{aligned}\frac{(n+1)(n(n-1)+3nx+3x^2)}{3n^3} &= \frac{n^3 - n + 3n^2x + 3nx + 3nx^2 + 3x^2}{3n^3} \\ &= \frac{1}{3} + \frac{x}{n} + \frac{-1+3x+3x^2}{3n^2} + \frac{x^2}{n^3}\,.\end{aligned}$$

[1 mark] ※ Observe that for any given x, all terms of the form x^i/n^j with $j>0$ go to zero as $n\to\infty$. It follows that

$$\lim_{n\to\infty}\frac{e^{-x}}{n^3}\sum_{k=0}^{n}\frac{\partial^k}{\partial x^k}x^2e^x = \frac{1}{3}\,.$$

[1 mark] ※

/2 marks

6(d) Recall the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Using this series, write down the Maclaurin series for $\frac{\partial^3}{\partial x^3}x^2e^x$.

Please turn over for page 15.

Solution From above we know that $\frac{\partial^3}{\partial x^3} x^2 e^x = (6 + 6x + x^2)e^x$. It follows that the McLaurin series is then

$$
\begin{aligned}
(6+6x+x^2)\sum_{n=0}^{\infty}\frac{x^n}{n!} &= \sum_{n=0}^{\infty} 6\frac{x^n}{n!} + 6\frac{x^{n+1}}{n!} + \frac{x^{n+2}}{n!} \\
&= 6 + (6+6)x + \sum_{n=2}^{\infty}\left(\frac{6}{n!} + \frac{6}{(n-1)!} + \frac{1}{(n-2)!}\right)x^n \\
&= 6 + 12x + \sum_{n=2}^{\infty}\frac{6+6n+n(n-1)}{n!}x^k \\
&= \sum_{n=0}^{\infty}\frac{6+5n+n^2}{n!}x^k
\end{aligned}
$$

[2 marks] ※ (Alternatively, the coefficients can be obtained via $\frac{1}{n!}\frac{\partial^{3+n}}{\partial x^{3+n}} x^2 e^x$ evaluated at $x = 0$.)

/3 marks

6(e) Using Taylor's theorem, which says that the remainder in an nth-order Maclaurin series polynomial is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

for some z between 0 and x, along with the fact that the nth derivative e^x is bounded between $e^0 = 1$ and $e^1 = e$ for $x \in [0, 1]$, determine the number of terms n such that the error in a Maclaurin polynomial approximation of $\frac{\partial^3}{\partial x^3} x^2 e^x$ at $x = 1$ is no greater than 0.005.

Solution We need that

$$|R_n(1)| = \frac{|f^{(n+1)}(z)|}{(n+1)!} 1^{n+1} \leq 0.005\,.$$

[1 mark] ※ We have

$$
\begin{aligned}
|f^{(n+1)}(z)| &= \left|\frac{\partial^{n+4}}{\partial z^{n+4}} z^2 e^z\right| \\
&= |(n+4)(n+3) + 2(n+4)z + z^2|\,|e^z| \\
&\leq (n^2 + 9n + 21)e
\end{aligned}
$$

[1 mark] ※ Therefore we need

$$\frac{(n^2+9n+21)e}{(n+1)!} \leq 0.005$$

[1 mark] Trial and error gives $\frac{(8^2+9\times 8+21)e}{(8+1)!} \approx 0.00118$, so $n = 8$ is good enough. ※

End of examination questions.