Tutorial 1: Basics

1 Solutions

Exercise 1:

1. Show that $P(n) : \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all integer $n \geq 1$.

   Proof Base case: $P(n)$ holds for $n = 1$ as we have $\sum_{i=1}^{n} i^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6} = 1$.
   Induction hypothesis (I.H.): Assume that the statement $P(k)$ is true for k, i.e.

   $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

   holds.
   Induction step: Show that the statement $P(k+1)$ holds for k+1.
   We have

   $$\begin{aligned}
   \sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \sum_{i=1}^{k} i^2 \\
   &=^{I.H.} (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \\
   &= \frac{6 \cdot (k+1)^2 + k(k+1)(2k+1)}{6} \\
   &= \frac{(k+1) \cdot (6(k+1) + k(2k+1)}{6} \\
   &= \frac{(k+1) \cdot (2k^2 + 7k + 6)}{6} \\
   &= \frac{(k+1) \cdot (k+2) \cdot (2k+3)}{6} \\
   &= \frac{(k+1) \cdot (k+2) \cdot (2(k+1)+1)}{6}
   \end{aligned}$$

   which is the right hand side of the statement.

2. Show $fib(n) \leq 2^n$ for all non-negative integers n.

Proof Base case: We have $fib(0) = 0 \leq 2^0 = 1$ and $f(1) = 1 \leq 2^1 = 2$ and therefore the statement is correct for $n = 0$ and $n = 1$.

Induction hypothesis (I.H.): Assume that the statement is true for all i, $0 \leq i \leq k$ and $k \geq 1$, i.e. $fib(i) \leq 2^i$, $0 \leq i \leq k$, holds.

Induction step: Show that the statement is true for all i, $0 \leq i \leq k+1$:
We have $fib(k+1) = fib(k) + fib(k-1) \leq^{I.H.} 2^k + 2^{k-1} \leq 2 \cdot 2^k = 2^{k+1}$.

3. Show $\sum_{i=0}^{n} a \cdot r^i = \frac{a(1-r^{(n+1)})}{1-r}$ for all integer $n \geq 0$.

Proof Base case: For n=0, we have $\sum_{i=0}^{n} a \cdot r^i = a \cdot r^0 = a = \frac{a(1-r^1)}{1-r} = a$.
Induction hypothesis (I.H.): Assume that the statement is true for all $0 \leq i \leq k$, i.e. $\sum_{i=0}^{i} a \cdot r^i = \frac{a(1-r^{(i+1)})}{1-r}$ holds.
Induction step: Show that the statement is true for all i, $0 \leq i \leq k+1$:
We have

$$\begin{aligned}
\sum_{i=0}^{k+1} a \cdot r^i &= a \cdot r^{k+1} + \sum_{i=0}^{k} a \cdot r^i \\
&=^{I.H.} a \cdot r^{k+1} + \frac{a(1-r^{k+1})}{1-r} \\
&= \frac{(1-r)a \cdot r^{k+1} + a(1-r^{k+1})}{1-r} \\
&= \frac{a((1-r)r^{k+1} + 1 - r^{k+1})}{1-r} \\
&= \frac{a(r^{k+1} - r^{k+2} + 1 - r^{(k+1)})}{1-r} \\
&= \frac{a(1-r^{k+2})}{1-r}
\end{aligned}$$

Exercise 2: If $g(n) = O(f(n))$ then there is a constant c and an n_0 such that for all $n \geq n_0$, we have $g(n) \leq cf(n)$. This implies that $g(n) + f(n) \leq cf(n) + f(n) = (c+1)f(n)$ for the same choice of c and n_0. Hence, $g(n) + f(n) = O(f(n))$.

Exercise 3: Show $n^k = o(c^n)$ for any fixed integer k and any $c > 1$.

Proof We have to show that for all $c_1 > 0$ the exists an n_0 such that for all $n \geq n_0$ $n^k \leq c_1 \cdot c^n$ holds. Let $c_1 > 0$, $c > 1$, and k be fixed constants. We have

$$\begin{aligned}
& n^k \leq c_1 \cdot c^n \\
\iff\ & \log_c(n^k) \leq \log_c(c_1 \cdot c^n) \\
\iff\ & k \cdot \log_c(n) \leq \log_c(c_1) + n \\
\iff\ & k \cdot \log_c(n) - \log_c(c_1) \leq n
\end{aligned}$$

k and $\log_c(c_1)$ are constants. Choosing n_0 such that $n_0 \geq k \cdot \log_c(n_0) - \log_c(c_1)$ holds fullfills the condition and we have $n^k \leq c_1 \cdot c^n$ for all $n \geq n_0$.