Practice Questions (week 6)

Semester 2, 2019

These questions are all about linear algebra – matrix inverses and determinants.

1. Let A, B and C be invertible $n \times n$ matrices. Find expressions in terms of A, B, C and their inverses for the inverses of the following matrices. (i) ABC (ii) $AB^{-1}A$ (iii) $3ABC^2$ (iv) $-BA^{-1}CA$

   Solution: (i) $C^{-1}B^{-1}A^{-1}$. (ii) $A^{-1}BA^{-1}$ (iii) $\frac{1}{3}(C^{-1})^2B^{-1}A^{-1}$ (iv) $-A^{-1}C^{-1}AB^{-1}$.

---

2. For each of the following matrices, find the inverse using elementary row operations. (a) $\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

   Solution: (Note: You may ignore the *elementary matrices* to the right of each step of the row reduction.)
   (a)

$$\begin{array}{cc|cc ccc}
1 & 3 & 1 & 0 & & & \\
4 & 3 & 0 & 1 & & & \\
\hline
1 & 3 & 1 & 0 & E_1 = & 1 & 0 \\
0 & -9 & -4 & 1 & & -4 & 1 \\
\hline
1 & 3 & 1 & 0 & E_2 = & 1 & 0 \\
0 & 1 & 4/9 & -1/9 & & 0 & -1/9 \\
\hline
1 & 0 & -1/3 & 1/3 & E_3 = & 1 & -3 \\
0 & 1 & 4/9 & -1/9 & & 0 & 1
\end{array}$$

So $A^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 4/9 & -1/9 \end{bmatrix}$.

(b)

$$\begin{array}{ccc|ccccccc}
1 & 0 & 0 & 1 & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & 1 & 0 & & & & \\
1 & 2 & 3 & 0 & 0 & 1 & & & & \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & E_1 = & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & & 0 & 1 & 0 \\
0 & 2 & 3 & -1 & 0 & 1 & & -1 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & E_2 = & 1 & 0 & 0 \\
0 & 2 & 3 & -1 & 0 & 1 & & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & & 0 & 1 & 0 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & E_3 = & 1 & 0 & 0 \\
0 & 1 & 3/2 & -1/2 & 0 & 1/2 & & 0 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & & 0 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & E_4 = & 1 & 0 & 0 \\
0 & 1 & 0 & -1/2 & -3/2 & 1/2 & & 0 & 1 & -3/2 \\
0 & 0 & 1 & 0 & 1 & 0 & & 0 & 0 & 1
\end{array}$$

so $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & -3/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

(c)

$$\begin{array}{ccc|ccccccc}
0 & 0 & 1 & 1 & 0 & 0 & & & & \\
1 & 1 & 1 & 0 & 1 & 0 & & & & \\
0 & 1 & 2 & 0 & 0 & 1 & & & & \\ \hline
1 & 1 & 1 & 0 & 1 & 0 & E_1 = & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & & 0 & 0 & 1 \\ \hline
1 & 1 & 1 & 0 & 1 & 0 & E_2 = & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & & 0 & 1 & 0 \\ \hline
1 & 0 & -1 & 0 & 1 & -1 & E_3 = & 1 & -1 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 1 & -1 & E_4 = & 1 & 0 & 1 \\
0 & 1 & 2 & 0 & 0 & 1 & & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 1 & -1 & E_5 = & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & 0 & 1 & & 0 & 1 & -2 \\
0 & 0 & 1 & 1 & 0 & 0 & & 0 & 0 & 1
\end{array}$$

so $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

---

3. Find all values of α for which the following matrix is *not* invertible.

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ -1 & 3 & \alpha \end{bmatrix}$$

Solution:

Row reducing A gives $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha - 3 \end{bmatrix}$. To further row reduce to the identity we need $\alpha - 3 \neq 0$, so $\alpha \neq 3$. Thus A is not invertible if $\alpha = 3$.

---

4. Let a, b, c be fixed non-zero numbers.

(a) Find the inverse of $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ in terms of a, b, c.

(b) Find the inverse of $\begin{bmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{bmatrix}$.

Solution: (a) $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$.

(b) We can write this as $a \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, so the inverse is $\frac{1}{a} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} =$ $\begin{bmatrix} 1/2a & -1/2a & 1/2a \\ 1/2a & 1/2a & -1/2a \\ -1/2a & 1/2a & 1/2a \end{bmatrix}$ (after using the usual method to find the inverse).

---

5. True or False? Examine each of the following statements carefully and decide whether they are true or false. Give a short reason for your decision in each case.

(a) Let A and B be square matrices such that $AB = O$. If A is invertible then $B = O$.

(b) Let A and B be invertible matrices of the same size. Then

$$(A + B)^{-1} = A^{-1} + B^{-1}$$

(c) The matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 7 & 0 \\ 0 & 1 & -2 & 0 & 1 & 5 \\ 2 & 0 & 4 & -3 & 1 & 8 \\ 1 & -1 & 2 & 3 & 7 & 0 \\ 4 & 8 & 11 & -21 & 0 & -7 \\ 3 & 5 & -6 & 2 & 1 & 4 \end{bmatrix}$$

is invertible.

Solution: (a) True. Multiply both sides of $AB = O$ by A^{-1} to get $B = A^{-1}AB = A^{-1}O = O$.

(b) False. For instance $A = I_2$ and $B = -I_2$ are both invertible, but $A + B = O$ is not even invertible.

(c) False. Since Row 1 = Row 4, the reduced row echelon form of A has a zero row. Therefore A is not row equivalent to I_6, and hence A is not invertible.

---

6. Let A be an $n \times n$ matrix with two identical columns. Explain why A is not invertible.

Solution: **Method 1**: A^t has two identical rows, this means that A^t is not invertible (since the rref must have a zero row), so A must not be invertible since if it was then A^t would have inverse $(A^{-1})^t$.

Method 2: Suppose A is invertible. Then there is a pivot in each column of rref(A), which occurs in the corresponding row. Suppose the ith and jth columns are identical. Then carrying out the same row operations on the entries in these columns must give the same results, however the ith column should row reduce to give a pivot in the ith row, and the jth column should give a pivot in the jth row, hence we have a contradiction and A cannot be invertible.

---

7. Suppose that A is an invertible $n \times n$ matrix satisfying $A^3 - 3A + 2I = 0$. Find an expression for A^{-1} in terms of A and I.

Solution: $A(A^2 - 3I) = -2I$ so $A\left(-1/2(A^2 - 3I)\right) = I$ thus $A^{-1} = -1/2(A^2 - 3I)$.

---

8. Calculate the following determinants. Your answers should be formulas in terms of a and b.

(a) $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ a & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & a & b \\ b & a & 0 \\ -a & -b & -a \end{vmatrix}$

Solution: (a) Expanding along the 3rd column we have $1\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1 - a^2$.

(b) Can either use the diagonal method, or expanding along the first row we have $0 - a\begin{vmatrix} b & 0 \\ -a & -a \end{vmatrix} + b\begin{vmatrix} b & a \\ -a & -b \end{vmatrix} = -a(-ab-0)+b(-b^2+a^2) = a^2b - b^3 + a^2b = 2a^2b - b^3 = b(2a^2 - b^2)$.

---

9. * Let J_n be the $n \times n$ matrix all of whose entries are equal to 1. For example,

$$J_1 = [1], \quad J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Prove that if $n > 1$, then the matrix $I_n - J_n$ is invertible with inverse

$$(I_n - J_n)^{-1} = I_n - \frac{1}{n-1}J_n.$$

Here I_n is the $n \times n$ identity matrix.

Solution: We have to check that $(I_n - J_n)(I_n - \frac{1}{n-1}J_n) = I_n = (I_n - \frac{1}{n-1}J_n)(I_n - J_n)$. We know that it is not necessary to verify both equations. It is enough to check one of them. So let us prove that

$$(I_n - J_n)(I_n - \frac{1}{n-1}J_n) = I_n.$$

We expand the left hand side:

$$\begin{aligned}(I_n - J_n)(I_n - \frac{1}{n-1}J_n) &= I_n - J_n - \frac{1}{n-1}J_n + \frac{1}{n-1}J_n^2 \\ &= I_n - \frac{n}{n-1}J_n + \frac{1}{n-1}J_n^2.\end{aligned}$$

We need to think about the matrix power J_n^2. The (i,j) entry of this matrix is

$$[1\ 1\ \cdots 1]\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = n$$

and so we see that $J_n^2 = nJ_n$. Therefore

$$\begin{aligned}(I_n - J_n)(I_n - \frac{1}{n-1}J_n) &= I_n - \frac{n}{n-1}J_n + \frac{1}{n-1}J_n^2\\ &= I_n - \frac{n}{n-1}J_n + \frac{n}{n-1}J_n = I_n.\end{aligned}$$

---

10. Given that $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 5$, find:

(i) $\begin{vmatrix}d & e & f\\ g & h & i\\ a & b & c\end{vmatrix}$ (ii) $\begin{vmatrix}d & e & f\\ 2a+d & 2b+e & 2c+f\\ g & h & i\end{vmatrix}$ (iii) $\begin{vmatrix}a-4g & d & g\\ b-4h & e & h\\ c-4i & f & i\end{vmatrix}$

Solution: Given that $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 5$,

(i) $\begin{vmatrix}d & e & f\\ g & h & i\\ a & b & c\end{vmatrix} = (-1)(-1)\times 5 = 5 \quad (R_1 \leftrightarrow R_2, R_2 \leftrightarrow R_3)$

(ii) $\begin{vmatrix}d & e & f\\ 2a+d & 2b+e & 2c+f\\ g & h & i\end{vmatrix} = (-1)(2)\times 5 = -10$

(iii) $\begin{vmatrix}a-4g & d & g\\ b-4h & e & h\\ c-4i & f & i\end{vmatrix} = 5$

where the row operations are:
(ii) $R_1 \leftrightarrow R_2$; $R_2 \to 2R_2$ and $R_2 \to R_2 + R_1$;
(iii) $R_1 \to R_1 - 4R_3$; then take the transpose which does not affect the determinant.

---

11. Find the values of c for which the matrix $A = \begin{bmatrix}c & c & 0\\ c^2 & 2 & c\\ 0 & c & c\end{bmatrix}$ is invertible.

Solution: We know that A is invertible if and only if $\det A \neq 0$. Using one row operation – subtracting c times the first row from the second row, which does not affect the determinant – we get

$$\det A = \begin{vmatrix} c & c & 0 \\ 0 & 2-c^2 & c \\ 0 & c & c \end{vmatrix} = c \begin{vmatrix} 2-c^2 & c \\ c & c \end{vmatrix} = c^2(2-c^2-c) = -c^2(c-1)(c+2)$$

so the answer is: A is invertible if and only if $c \neq -2, 0, 1$.

---

12. Suppose A and B are $n \times n$ matrices with $\det A = 4$ and $\det B = -3$. Find each of the following determinants.

    (a) $\det(AB)$

    (b) $\det(A^2)$

    (c) $\det(B^{-1}A)$

    (d) $\det(2A)$

    Solution:

    (a) $\det(A)\det(B) = -12$.

    (b) $\det(A)\det(A) = 16$.

    (c) $\frac{1}{\det(B)}\det(A) = -4/3$.

    (d) $2^n \det(A) = 4.2^n = 2^{n+2}$.

---

13. * What can you say about $\det(A)$ if the square matrix A satisfies:

    (a) $A^2 = A$ (such a matrix is called idempotent).

    (b) $A^m = O$ for some $m > 1$ (such a matrix is called nilpotent).

    Solution:

    (a) We have $\det(A)^2 = \det(A)$, so $\det(A)(\det(A) - 1) = 0$ and either $\det(A) = 0$ or $\det(A) = 1$.

    (b) $\det(A)^m = 0$ so we must have $\det(A) = 0$.

---

14. (a) Let A be a 2×3 matrix and B be a 3×2 matrix. Then BA is a 3×3 matrix. Show that $\det(BA) = 0$.

(b) Is it necessarily true that $\det(AB) = 0$?

Solution:

(a) Since A is a 2×3 matrix, there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$ (fewer equations than variables). Thus there exists $\mathbf{x} \neq \mathbf{0}$ with $B(A\mathbf{x}) = \mathbf{0}$ or $(BA)\mathbf{x} = \mathbf{0}$. This means that the square matrix BA is not invertible, that is, $\det(BA) = 0$.

(b) No. Take for example $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $AB = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ has $\det(AB) = 2 \neq 0$; but $BA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ has $\det(BA) = 0$.
