Mathematics for Data Science Tutorial 4 (week 8)

Semester 2, 2019

1. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

   (a) 2 is an eigenvalue of A. Use the trace and determinant to determine the other eigenvalue(s).

   (b) Find the eigenspace corresponding to 2.

Solution:

(a) We have $trace(A) = 3 + 1 + 1 = 5$ and $\det A = -1 \times \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + 1 \times \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 4.$

Since 2 is an eigenvalue, we have $2+\lambda_1+\lambda_2 = 5$, i.e. $\lambda_1+\lambda_2 = 3$, and $2\lambda_1\lambda_2 = 4$, i.e. $\lambda_1\lambda_2 = 2$. Therefore, $\lambda_1 = 1$ and $\lambda_2 = 2$.

(b)

$$\begin{array}{rrr|r} -1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Therefore the eigenspace corresponding to $\lambda = 2$ is $E_2 = \{(-1, 1, 1)t | t \in \mathbb{R}\}$.

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2. Consider the matrices

$$A_1 = \begin{bmatrix} -12 & 7 \\ -7 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 3 & 9 \\ 0 & -7 & -18 \\ 0 & 2 & 5 \end{bmatrix}.$$

For each matrix A_i above

(a) Determine the eigenvalues and eigenvectors.

(b) For each eigenvalue, state its multiplicity and give the dimension of its associated eigenspace.

(c) Hence determine whether the matrix A_i is diagonalisable, stating the reason for your answer.

(d) For each diagonalisable matrix A_i, determine a matrix P such that $P^{-1}A_iP = D$, where D is a diagonal matrix. What is D? (For the purposes of this exercise, order your eigenvalues from smallest to largest i.e. $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$.)

(e) Determine if each matrix A_i is invertible and, where possible, use your results from (d) to write the inverse matrix A_i^{-1} in the form $P\Delta P^{-1}$, where Δ is a diagonal matrix. (You do not need to find A^{-1} if not possible by this method.)

Solution:

A_1: (a) A_1 has eigenvalue -5 with multiplicity 2. Eigenvectors are $\mathbf{x} = \mathbf{t(1, 1), t \neq 0}$.

(b) $E_1 = \{(1, 1)t | t \in \mathbb{R}\}$, $\dim(E_1) = 1$.

(c) A_1 is not diagonalisable as it does not have two linearly independent eigenvectors.

(d) No answer required as A_1 is not diagonalisable.

(e) $\det(A) = 25 \neq 0$ so A is invertible though not diagonalisable.

A_2: (a) A_2 has eigenvalues 2 and 8 with multiplicities 2 and 1 respectively. For $\lambda = 2$ the eigenvectors are $\mathbf{x} = \mathbf{s(-1, 1, 0)} + \mathbf{t(-1, 0, 1)}$, s and t not both zero. For $\lambda = 8$ the eigenvectors are $\mathbf{x} = \mathbf{t(1, 1, 1)}$, $t \neq 0$.

(b) $E_2 = \{(-1, 1, 0)t + (-1, 0, 1)s | s, t \in \mathbb{R}\}$, $\dim(E_2) = 2$. $E_8 = \{(1, 1, 1)t | t \in \mathbb{R}\}$, $\dim(E_8) = 1$.

(c) A_2 is diagonalisable as it has 3 linearly independent eigenvectors.

(d) $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

(e) $\det(A_2) = 32 \neq 0$ so A_2 is invertible. $\Delta = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$ with P as above.

A_3: (a) A_3 has eigenvalue -1 with multiplicity 3. The eigenvectors are $\mathbf{x} = t(1, 0, 0) + s(0, -3, 1)$, s and t not both zero.

(b) $E_3 = \{(1, 0, 0)t + (0, -3, 1)s | s, t \in \mathbb{R}\}$, $\dim(E_3) = 2$.

(c) A_3 is not diagonalisable as it does not have three linearly independent eigenvectors.

(d) No answer required as A_3 is not diagonalisable.

(e) $\det(A_3) = -1 \neq 0$ so A_3 is invertible though not diagonalisable.
