Practice Questions (week 7)

Semester 2, 2019

These questions are all about linear algebra – linear (in)dependence, eigenvalues, and eigenvectors.

1. Determine, with reasons, whether the following statements are (A) always true, (B) always false, or (C) might be true or false.

   (a) If $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^5$ and $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is linearly dependent.

   (b) If $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathbb{R}^2$ lie on the same straight line through the origin, then the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

   (c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

   (d) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then so is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (e) If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$.

   (f) There is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ of linearly independent vectors in $\mathbb{R}^4$.

2. Consider the vectors $\mathbf{u} = (2, 1, 4, 3)$, $\mathbf{v} = (3, 4, 1, 2)$, $\mathbf{w} = (1, 2, -1, 0)$ in $\mathbb{R}^4$.

   (a) Are the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ linearly independent?

   (b) Are the vectors $\mathbf{v}$, $\mathbf{w}$ linearly independent?

   (c) Are the vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{0}$ linearly independent?

   (d) Are the vectors $\mathbf{u}$, $\mathbf{w}$, $5\mathbf{u} - 3\mathbf{w}$ linearly independent?

3. Which of the following sets of vectors are linearly independent?

   (a) $\{(3, 4),\ (4, 3)\}$

   (b) $\{(2, 1, -3, 6),\ (5, 3, 7, 8),\ (1, 1, 13, -4)\}$

   (c) $\{(2, 1, 3),\ (2, -2, -5),\ (7, 3, 9)\}$

   (d) $\{\mathbf{e}_1,\ \mathbf{e}_1 + 2\mathbf{e}_2,\ \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3,\ \ldots,\ \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + \cdots + n\mathbf{e}_n\}$ where $\mathbf{e}_i \in \mathbb{R}^n$ is the vector which has 1 in the i-th place and 0 everywhere else.

4. For which value(s) of d are the following sets of vectors linearly dependent? Justify your answers.

   (a) $\{(1, -1, 4)), (3, -5, 5), (-1, 5, d)\}$

   (b) $\{(3, 7, -2), (-6, d, 4), (9, 1, -4)\}$

5. * Let $A = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 9 & 5 \\ 1 & 3 & 1 \end{bmatrix}$ and let W be the set of all linear combinations of the columns of A.

(a) Show that $(4, 10, 2)$ is in W (you should be able to do this by inspection, that is, without any row operations).

(b) Solve the homogeneous system with augmented matrix $[A|\mathbf{0}]$.

(c) Given that $(1, 2, -1)$ is a solution of the linear system $[A|\mathbf{b}]$, what is the vector $\mathbf{b}$?

(d) Write down the general solution of $[A|\mathbf{b}]$ (with the vector $\mathbf{b}$ from part (c)), without directly solving this system.

6. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Prove that $\mathbf{w}$ is a linear combination of $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$ and $\mathbf{u}$. (Another way of saying this is that $\mathbf{w}$ is in the *span* of $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$ and $\mathbf{u}$, $\mathbf{w} \in \operatorname{span}\{\mathbf{u}, \mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}\}$.)

7. Let A be an $n \times n$ matrix.

(a) Define what it means for $\mathbf{x} \in \mathbb{R}^n$ to be an eigenvector of A with eigenvalue λ.

(b) Define the characteristic polynomial of A.

8. Find all eigenvalues and eigenvectors for the following matrices.

(a) $A_1 = \begin{bmatrix} 0 & 3 \\ 6 & -3 \end{bmatrix}$

(b) $A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(c) $A_3 = \begin{bmatrix} 1 & 5 & 5 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$

9. Show that if A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then

(a) cA has eigenvalues $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$ for any constant $c \in \mathbb{R}$.

(b) If A^{-1} exists, then A^{-1} has eigenvalues $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$

(c) A^m has eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ for $m = 1, 2, 3, \ldots$.

10. (a) Show that $\lambda = 0$ is an eigenvalue of A if and only if A is not invertible.

(b) Without calculation find one eigenvalue and two linearly independent eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

11. Suppose that for some 3×3 matrix A, we have

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) Give one eigenvalue of A.

(b) Explain why $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for this eigenvalue and hence find $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(c) If det(A)=12, then what is the multiplicity of the eigenvalue from (a)?

12. Consider the matrices

$$A_1 = \begin{bmatrix} -12 & 7 \\ -7 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 3 & 9 \\ 0 & -7 & -18 \\ 0 & 2 & 5 \end{bmatrix}.$$

For each matrix A_i above

(a) Determine the eigenvalues and eigenvectors.

(b) For each eigenvalue, state its multiplicity and give the dimension of its associated eigenspace.

(c) Hence determine whether the matrix A_i is diagonalisable, stating the reason for your answer.

(d) For each diagonalisable matrix A_i, determine a matrix P such that $P^{-1}A_iP = D$, where D is a diagonal matrix. What is D? (For the purposes of this exercise, order your eigenvalues from smallest to largest i.e. $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$.)

(e) Determine if each matrix A_i is invertible and, where possible, use your results from (d) to write the inverse matrix A_i^{-1} in the form $P\Delta P^{-1}$, where Δ is a diagonal matrix. (You do not need to find A^{-1} if not possible by this method.)

13. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

(a) Verify that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue λ_1?

(b) Use the trace of A to find a second eigenvalue λ_2.

(c) Find the eigenspace for λ_2.

(d) Write down an invertible matrix P such that $P^{-1}AP = D$, where $D = \text{diag}(\lambda_1, \lambda_2)$.

14. Show that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is not diagonalizable.