THE UNIVERSITY of ADELAIDE

Student ID: __________

Family name: __________

Other names: __________

Desk number: __________ Date: __________

Signature: __________

Examination in the School of Mathematical Sciences

Practice exam

109685	**MATHS 1004**	**Mathematics for Data Science I**
109685	**MATHS 1004UAC**	**Mathematics for Data Science – University of Adelaide College**

Time for completing booklet: 120 mins (plus 10 mins reading time).

Question	Marks	
1	/12	
2	/14	
3	/9	
4	/11	
5	/10	
6	/14	
Total	/70	

Instructions to candidates

- Attempt all questions and write your answers in the space provided below that question.
- If there is insufficient space below a question, then use the space to the *right* of that question, indicating clearly which question you are answering.
- Only work written in this question and answer booklet will be marked.
- Examination materials must not be removed from the examination room.

Materials

- Calculators without remote communications or CAS capability are allowed.
- You may bring in one double-sided A4 formula sheet into the examination.

Do not commence writing until instructed to do so.

12 Total

Question 1.

Answer true or false to each of the following assertions. You must also provide a *very brief* (one to two lines) justification for each of your answers.

/2 marks

1(a) The matrix

$$\begin{bmatrix} 2 & 4 & -1 & 8 \\ -4 & 3 & 5 & 6 \\ 3 & 2 & 0 & 4 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

is invertible.

. . . .

. . . .

. . . .

/2 marks

1(b) A set of vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent if there is no non-trivial solution to $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = \boldsymbol{0}$.

. . . .

. . . .

. . . .

/2 marks

1(c) An even function has an inverse.

. . . .

. . . .

. . . .

/2 marks

1(d) Two die are rolled. The events described by ‘both numbers are odd’ and ‘the sum greater than 10’ are independent.

. . . .

. . . .

. . . .

Please turn over for page 3.

/2 marks

1(e) A biased coin is tossed. The probability of flipping heads is 0.7 and the probability of tails is 0.4.

. . . .

. . . .

. . . .

/2 marks

1(f) E, F are two events in a sample space S with probabilities $\Pr(E \cap F) = 0.8$, $\Pr(E) = 0.6$.

. . . .

. . . .

. . . .

Please turn over for page 4.

14 Total

Question 2.

2(a) Consider the following two matrices X and Y:

$$X = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}; \quad Y = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 3 \end{bmatrix}$$

If possible, calculate the following. If not possible, give a reason for your answer.

/1 mark

(i) $X + Y$

....

....

....

....

....

/1 mark

(ii) XY

....

....

....

....

....

(iii) YX

/1 mark

....

....

....

....

....

/1 mark

(iv) $Y^{\intercal}$

/1 mark

(v) the determinant $|X|$

/1 mark

(vi) the determinant $\det(Y)$

/1 mark

(vii) The eigenvalues of Y

Please turn over for page 6.

2(b) Consider the system of equations

$$\begin{aligned} 2x - z - 4 &= 0 \\ y - 2z &= 0 \\ -x + 3y + z &= 1 \end{aligned}$$

/1 mark

(i) Write down the matrix A and vector $\boldsymbol{b}$ such that the above system of equations can be written in matrix form $A\boldsymbol{x} = \boldsymbol{b}$.

. . . .

. . . .

. . . .

. . . .

. . . .

/5 marks

(ii) Determine the inverse of

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}$$

by wrtiting in the augmented form $[A|I]$ and performing the required row operations.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Please turn over for page 7.

/1 mark

(iii) Hence, or otherwise, find the solution $\boldsymbol{x}$.

. . . .

. . . .

. . . .

. . . .

. . . .

Please turn over for page 8.

9 Total

Question 3.

3(a) Consider the matrix

$$A = \begin{bmatrix} -7 & 5 \\ -3 & 9 \end{bmatrix}.$$

/2 marks

(i) Solve the characteristic equation to show that the eigenvalues for this matrix are $\lambda = 8$ and $\lambda = -6$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/4 marks

(ii) Hence, find the eigenvectors of A.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/3 marks

3(b) Consider a principal component analysis of some dataset X. This dataset has the covariance matrix

$$C = \begin{bmatrix} 1 & 16 & -32 \\ -4 & -41 & 88 \\ -2 & -20 & 43 \end{bmatrix}$$

The largest eigenvalue of C is $\lambda = 3$. Use this to determine a principal component of X.

....

....

....

....

....

....

....

....

11 Total

Question 4.

4(a) A company employs 7 engineers, 4 data scientists and 2 mathematicians. The director randomly chooses 3 of these employees to work on a new research project.

/1 mark

(i) How many different types of teams could be chosen (thinking only of the expertise of each member)?

. . . .

. . . .

. . . .

/1 mark

(ii) Now taking account of individuals in the company, how many different possible teams are there?

. . . .

. . . .

. . . .

/1 mark

(iii) To give some structure to the team, the director decides to appoint one member as research leader. How many compositions of teams are there now (thinking only of the expertise of each member)?

. . . .

. . . .

. . . .

/1 mark

(iv) Repeat the previous question but accounting for the individuals in the team.

. . . .

. . . .

. . . .

/1 mark

(v) The research project becomes much bigger than anticipated and so a fourth team member is added to the team. How many different team compositions are there now (thinking only of the expertise of each member)?

. . . .

. . . .

. . . .

/2 marks

4(b) (i) Suppose the probability of passing the final exam having completed the practice exam is 0.9 and the probability of passing the final exam without completing the practice exam is 0.45. If 80 students in the class complete the practice exam and 40 do not, what is the probability that a student chosen at random will pass the final exam.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/1 mark

(ii) Suppose that 80% of students who completed all assignments also completed the practice exam. Is this enough information to determine the probability of a student passing the final exam given they have completed all of the assignments.

. . . .

. . . .

. . . .

. . . .

/3 marks

4(c) A researcher studying the genes of a particular plant species. They have found that 40% of their plant samples are significantly taller than the other. In 80% of the tall plants they have identified a particular gene they believe is a marker for the plant being tall. However, 30% of the short plants are also found to have this gene marker. Suppose a new seedling has been analysed and is found to have the gene marker, what is the probability it will grow tall?

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

10 Total

Question 5.

5(a) Consider the probability density function defined by

$$f(x) = \begin{cases} a - cx & \text{if } 0 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$$

/2 marks

(i) What restrictions must a and c satisfy? Give a reason for your answer.

. . . .

. . . .

. . . .

. . . .

/1 mark

(ii) Find the expected value of x (for any a, c).

. . . .

. . . .

. . . .

. . . .

/2 marks

(iii) Is there a valid a, b for which $\mathbb{E}[X] = 3$?

. . . .

. . . .

. . . .

. . . .

5(b) Consider a function $f(x)$ for which

- $f(x) = x^{-4}$ for $x \geq 1$
- $f(x) \geq 0$ for $x \in [-2, 1]$
- $f(x) < 0$ for $x < -2$

/1 mark

(i) Write down, using correct set notation, the largest domain over which this function could be a valid probability density function.

For the remainder, suppose that the valid domain $\mathcal{D}$ of f which makes X a probability density function contains $[1, \infty)$.

/2 marks

(ii) What is the probability that X is between 2 and 4?

/2 marks

(iii) What is $\Pr(X \in \mathcal{D} \cap [-2, 1))$?

14 Total

Question 6.

/3 marks

6(a) Use the priniciple of mathematical induction to prove that

$$\frac{\partial^n}{\partial x^n} x^2 e^x = (n(n-1) + 2nx + x^2)e^x$$

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/3 marks

6(b) Using the result above and mathematical induction, show that

$$\sum_{k=0}^{n} \frac{\partial^k}{\partial x^k} x^2 e^x = \frac{1}{3}(n+1)(n(n-1) + 3nx + 3x^2)e^x \, .$$

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Please turn over for page 16.

/3 marks

6(c) Using the result above, or otherwise, calculate the limit

$$\lim_{n\to\infty} \frac{e^{-x}}{n^3} \sum_{k=0}^{n} \frac{\partial^k}{\partial x^k} x^2 e^x = \frac{1}{3}\,.$$

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/2 marks

6(d) Recall the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using this series, write down the Maclaurin series for $\frac{\partial^3}{\partial x^3} x^2 e^x$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

/3 marks

6(e) Using Taylor's theorem, which says that the remainder in an nth-order Maclaurin series polynomial is

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$$

for some z between 0 and x, along with the fact that the nth derivative e^x is bounded between $e^0 = 1$ and $e^1 = e$ for $x \in [0, 1]$, determine the number of terms n such that the error in a Maclaurin polynomial approximation of $\frac{\partial^3}{\partial x^3}x^2e^x$ at $x = 1$ is no greater than 0.005.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

End of examination questions.

Please turn over for page 18.