Practice Questions (week 11)

Semester 2, 2019

These questions are about discrete random variables, Bernoulli and binomial random variables, and continuous random variables. Difficult questions are starred. Good places to go for further questions on this topic include the exercises in:

- Ross, *A first course in Probability* (6th Ed.), Chapters 4 and 5.

1. Consider a game in which a fair six-sided die is rolled. If you roll a multiple of 2 you lose \$1, but if you roll a multiple of 3 you gain \$1. Describe a random variable that denotes your winnings and determine its expectation?

2. Consider rolling two fair (six-sided) dice and let X be a random variable denoting the difference between the two numbers rolled, specifically if a and b are rolled we take $X = |a - b|$.

   (a) Describe the sample space for the experiment and determine the possible values that X can take on.

   (b) What is $\Pr(X = 2)$?

   (c) What is $\mathbb{E}[X]$?

   (d) What is $\mathrm{Var}(X)$?

3. Suppose a box contains 5 blue blocks and 5 red blocks. Consider an experiment where blocks are taken out of the box randomly, one at a time, without replacement.

   (a) Suppose the experiment stops when all of the blue blocks have been withdrawn and let X denote the total number of blocks withdrawn.

      i. What is $\Pr(X = 6)$?

      ii. What is $\Pr(X \leq 7)$?

      iii. Determine $\mathbb{E}[X]$.

      iv. What is $\mathrm{Var}(X)$?

   (b) Now, consider the case where the experiment stops as soon as at least one red and one blue block have been withdrawn and let Y denote the total number of blocks that have been withdrawn.

      i. What is $\Pr(Y \leq 8)$?

      ii. What is $\Pr(Y = 4)$?

      iii. Determine $\mathbb{E}[Y]$.

      iv. What is $\mathrm{Var}(Y)$?

4. Let X be a non-negative integer valued random variable.

(a) Show that

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr(X \geq k)\,.$$

(Hint: express $\Pr(X \geq k)$ as a sum and interchange the order of summation.)

(b) Come up with a similar expression in the case that X is an integer valued random variable (i.e. it may take on negative values).

5. Let X_1, X_2, X_3 be independent Bernoulli random variables each with chance p_1, p_2, p_3 of success. Let $Y = X_1 + X_2 + X_3$ and $Z = X_1 \times X_2 \times X_3$.

   (a) What is $\mathbb{E}[Y]$ and $\mathbb{E}[Z]$?

   (b) What is $\mathrm{Var}(Y)$ and $\mathrm{Var}(Z)$?

6. Consider an egg farm where the chickens have a 1 in 10,000 chance of laying a 'double yolker', that is an egg with two yolks. Assume the occurrence of these is independent and that eggs are randomly packaged into cartons.

   (a) What is the probability of at least one double yolker in a carton of 6 eggs?

   (b) What is the probability of at least one double yolker in a carton of 12 eggs?

   (c) How many eggs would you have to buy so that the chance of having at least one double yolker is 90%?

   (d) Given the number of eggs determined from the previous question, what is the probability of obtaining exactly 2 double yolkers, and what is the expected number of double yolkers?

7. Consider an experiment where a coin, which is possibly biased, is flipped some number of times. Let p be the probability of heads coming up on each flip of the coin.

   (a) Suppose the coin is flipped 10 times and comes up heads for 6 of those. What value of p is most likely to produce this outcome?

   (b) Generalise this for k heads out of n flips.

8. Consider a continuous random variable X with probability density function

$$f(x) = \begin{cases} ax(1-x) & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   for some $a \in \mathbb{R}$.

   (a) Determine the value of a.

   (b) Determine $\mathbb{E}[X]$.

(c) Determine $\mathrm{Var}(X)$.

9. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} c & \text{for } 0 \le x \le 1 \\ ax^{-3} & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose X is a random variable with f as its probability distribution function.

(a) What must the value of a be?

(b) What is $\Pr(X \le 2)$?

(c) Is $\frac{1}{10} < X < 10$ more likely than $X \le 2$?

(d) What is $\mathbb{E}[X]$?

(e) What is $\mathrm{Var}(X)$?

10. Consider the continuous random variable X with probability distribution function

$$f(x) = \begin{cases} 0 & x \le -\pi \\ \frac{1}{2\pi}\cos(x)^2 & \text{for } x \in (-\pi, \pi) \\ 0 & x \ge \pi \end{cases}$$

Let $F(x) = \int_{-\infty}^{x} f(x)\,dx$ (which is the cumulative distribution function of X), specifically

$$F(x) = \begin{cases} 0 & x \le -\pi \\ \frac{1}{2} + \frac{1}{2\pi}(x + \cos(x)\sin(x)) & \text{for } x \in (-\pi, \pi) \\ 1 & x \ge \pi \end{cases}$$

(a) Confirm that $\frac{d}{dx}F(x) = f(x)$.

(b) What is $\Pr(-1 \le X \le 1)$?

(c) What is $\mathbb{E}[X]$?

11. Suppose a company produces light bulbs with a lifetime L (in hours) which is exponentially distributed with parameter $\lambda = 1/5000$.

(a) What is the probability density function f of L?

(b) What is the probability a light bulb will work continuously for 365 days?

(c) What is the probability a light bulb will work continuously for at least 730 days if it has already operated for 365 days?

(d) What is the mean and variance of bulb lifetime?

12. Consider the following game. You pick a positive real number a. A random number X is then generated which happens to be exponentially distributed with parameter $\lambda = 1/10$. You are then assigned a score $S = 20 - |X - a|$.

    (a) For some fixed $a > 0$ is the probability of a score of at least 15?

    (b) Which choice of a will maximise the probability of obtaining a score of at least 15?

    (c) Do you expect this answer to change for different λ?

    (d) Suppose you pick $a = \mathbb{E}[X]$ and play this game several times with the same a. Describe the random variable which describes how many times you get a score of at least 15 after playing 8 times, then determine the expectation.

13. Suppose the total annual solar irradiation in the Adelaide region, denoted X is normally distributed with mean $1750\,\mathrm{kWh/m^2}$ and standard deviation $100\,\mathrm{kWh/m^2}$. Let Z be the standard normal distribution.

    (a) What is X in terms of Z, and further, given some $x \in \mathbb{R}$, then what is the corresponding z such that $\Pr(X \le x) = \Pr(Z \le z)$?

    (b) What is the probability that the total solar irradiation exceeds $1900\,\mathrm{kWh/m^2}$?

    (c) What is the probability the total solar irradiation is between $1700\,\mathrm{kWh/m^2}$ and $1900\,\mathrm{kWh/m^2}$?

    (d) What range of solar irradiance is in the top 42.07%?

    (e) A household that uses 4000 kWh of energy annually decides to install $12\mathrm{m}^2$ of solar panels that are rated to be 20% efficient at converting solar radiation to electricity. Determine the distribution which describes the total energy produced by the panels annually. Using this, find the probability that solar irradiation is not enough to provide all of the energy used by the household?

14. $*$ Suppose X is a random variable with $\mathbb{E}[X] = 5$ and $\mathrm{Var}(X) = 2$.

    (a) Let $Y = 3X - 8$, what is $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$?

    (b) Let $Z = 6X^2 - 4X + 7$, what is $\mathbb{E}[Z]$?

15. Let $a < b$ be real numbers and consider a random variable X with uniform distribution over the interval $[a, b]$.

    (a) What is the probability density function $f(x)$ of X?

    (b) What is $\mathbb{E}[X]$ and $\mathrm{Var}(X)$?

(c) $*$ Let $Y = e^X$. What is the probability density function $g(x)$ of Y?

(d) $*$ Still with $Y = e^X$, what is $\mathbb{E}[Y]$ and $\text{Var}(Y)$?