Mathematics for Data Science I
Practice Questions (week 4-5)

Semester 2, 2019

These questions are all about linear algebra – systems of equations and matrices.

1. Which of the following equations are linear?
   (a) $x_1 - 5x_2 + x_3^2 = 0$
   (b) $x_1 - 2x_3 = 7 - x_2$
   (c) $x_3 = 0$
   (d) $x_1 - x_2x_3 = 1$
   (e) $0 = 8$

2. (a) Solve the following two linear systems.

   (i) $\begin{aligned} 2x + y &= 4 \\ -x + 3y &= 5 \end{aligned}$ (ii) $\begin{aligned} x + y &= -1 \\ 4x + y &= 2 \end{aligned}$

   (b) What does it mean to say that two linear systems (of the same size) are equivalent?

   (c) Are the two systems in (a) equivalent?

   (d) Can we transform one of the two systems into the other by elementary operations?

3. Solve the following systems of linear equations.

   (a) $\begin{aligned} 2x_1 + 6x_2 &= 12 \\ 4x_1 - 3x_2 &= 19 \end{aligned}$ (b) $\begin{aligned} x_1 - 3x_2 + 2x_3 &= -10 \\ 2x_2 - x_3 &= 6 \\ x_1 + x_3 &= 0 \end{aligned}$

4. Solve the following systems of linear equations.

   (a) $\begin{aligned} 2x_1 + 6x_2 &= 3 \\ x_1 + 3x_2 &= 1 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 - x_3 &= 2 \end{aligned}$

5. Consider the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + ax_2 &= 4 \end{aligned}$$

   (a) If the system has the unique solution $(x_1, x_2) = (5, 2)$, what is a?
   (b) If the system has no solutions, what is a?

6. How many 5×1 matrices are there in reduced row echelon form?

7. For each of the following matrices in reduced row echelon form, find the set of solutions to the corresponding system of equations (you can call the variables $x_1, x_2, \ldots$).

$$\text{(a)} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \text{(b)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{(c)} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \qquad \text{(d)} \left[\begin{array}{cccc|c} 1 & 4 & 8 & 3 & -1 \end{array}\right]$$

8. Suppose that you bring the matrix

$$\begin{bmatrix} 5 & 0 & -4 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

into reduced row echelon form. Which columns will contain a pivot? Do not calculate the reduced row echelon form. Just look at the matrix!

9. (a) Are the following two matrices in reduced row echelon form?

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Are they row equivalent?

10. Use row operations to put the following matrices into reduced row echelon form.

$$\text{(a)} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \qquad \text{(b)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\text{(c)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ \alpha & 0 & 1 & 2 \end{array}\right] \quad \text{for some } \alpha \neq 0$$

11. Find the solution sets of the following systems of linear equations.

$$\begin{aligned} \text{(a)}\ 4x + 2y + 5z &= 8 \\ 2x + 4y + 4z &= 1 \\ -2y - z &= 2 \end{aligned} \qquad \begin{aligned} \text{(b)}\ 4x + 2y + 4z &= 8 \\ x + 4y + 4z &= 1 \\ 2x - 6y - 3z &= 6 \end{aligned}$$

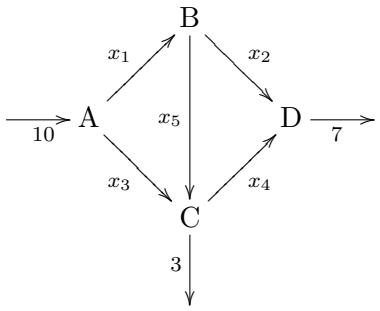
12. Suppose three foods are used to make a meal. The foods have the following units per gram of vitamin B and vitamin C respectively.

	Vitamin B	Vitamin C
Food 1	12	18
Food 2	8	10
Food 3	4	2

By setting up a system of linear equations, determine if it is possible to combine these foods into a meal of 500 grams with precisely 2000 units of Vitamin B and 1000 units of Vitamin C. If it is possible, in how many ways can it be done?

13. * Use linear equations to find a polynomial $p(x)$ with $p(0) = 1$, $p(1) = 0$, $p(2) = 5$, $p(3) = 22$.

14. * Consider the following network of irrigation channels with flows measured in, say, thousands of litres per day.

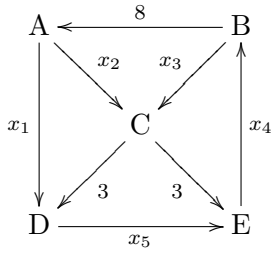


(a) Set up and solve a linear system for the possible flows x_1, x_2, x_3, x_4, x_5, using x_4 and x_5 as free variables.

(b) Can we close both channels BC and CD?

(c) Find the maximum and minimum possible flow x_4 through channel CD.

15. * Consider the following road network with traffic flows measured in, say, hundreds of cars per hour.



(a) Set up and solve a linear system for the possible flows x_1, x_2, x_3, x_4, x_5.

(b) Find the maximum and minimum possible traffic flow for each road AD, AC, BC, EB, and DE.

(c) Is it possible to close two of these five roads at once?