Mathematics for Data Science I
Practice Questions (week 2)

Semester 2, 2018

These questions are all about summation and series. Difficult questions are starred. Good places to go for further questions on this topic include the exercises in:

- Stewart, *Calculus* (7th Ed.), Appendix E (sigma notation),
- Stewart, *Calculus* (7th Ed.), Sections 11.2 and 11.6 (series and the ratio test),
- Morris & Stark, *Fundamentals of Calculus*, Section 9.4.

1. Explain the difference between

   (a) $\displaystyle\sum_{i=1}^{n} a_i$ and $\displaystyle\sum_{j=1}^{n} a_j$

   (b) $\displaystyle\sum_{i=1}^{n} a_i$ and $\displaystyle\sum_{i=1}^{n} a_j$

   (c) $\displaystyle\sum_{n=1}^{i} a_n$ and $\displaystyle\sum_{n=1}^{j} a_n$

2. Do the following series converge or diverge? Provide explanations.

   (a) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$

   (b) $\displaystyle\sum_{j=2}^{\infty} \frac{3}{j-1}$

   (c) $\displaystyle\sum_{n=3}^{\infty} 2^n$

   (d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

3. By utilising the fact that you know the sum of a geometric series, find the explicit values of each of the following series.

   Note: that sum is:
   $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
   if $|x| < 1$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{2^{n+1}}{5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

4. In this problem, we learn the 'telescoping sum' process to determine certain types of infinite sums.

   (a) For the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, confirm that the general term $a_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

   (b) Hence, observe that the partial sums s_m obey

$$\begin{aligned} s_m &:= \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) + \left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

   because the intermediate terms cancel with adjacent terms (the sum collapses like a telescope). By determining $\lim_{m\to\infty} s_m$, determine $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   (c) * Use this telescoping sum idea to determine the sum $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$, by considering

$$\frac{1}{n-1} - \frac{1}{n+1}.$$

   (d) * Use this idea to also determine the sum $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

5. Do the following series converge or diverge? Provide explanations.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(1 + \frac{2}{n^2}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{2n^2+n+7}$

6. Apply the ratio test to each of the following series to investigate convergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$.