Mathematics for Data Science I
Practice Questions (week 6)

Semester 2, 2019

These questions are all about linear algebra – matrix inverses and determinants.

1. Let A, B and C be invertible $n \times n$ matrices. Find expressions in terms of A, B, C and their inverses for the inverses of the following matrices. (i) ABC (ii) $AB^{-1}A$ (iii) $3ABC^2$ (iv) $-BA^{-1}CA$

2. For each of the following matrices, find the inverse using elementary row operations. (a) $\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

3. Find all values of α for which the following matrix is *not* invertible.

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & -7 & 6 \\ -1 & 3 & \alpha \end{bmatrix}$$

4. Let a, b, c be fixed non-zero numbers.

   (a) Find the inverse of $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ in terms of a, b, c.

   (b) Find the inverse of $\begin{bmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{bmatrix}$.

5. True or False? Examine each of the following statements carefully and decide whether they are true or false. Give a short reason for your decision in each case.

   (a) Let A and B be square matrices such that $AB = O$. If A is invertible then $B = O$.

   (b) Let A and B be invertible matrices of the same size. Then

$$(A + B)^{-1} = A^{-1} + B^{-1}$$

(c) The matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 7 & 0 \\ 0 & 1 & -2 & 0 & 1 & 5 \\ 2 & 0 & 4 & -3 & 1 & 8 \\ 1 & -1 & 2 & 3 & 7 & 0 \\ 4 & 8 & 11 & -21 & 0 & -7 \\ 3 & 5 & -6 & 2 & 1 & 4 \end{bmatrix}$$

is invertible.

6. Let A be an $n \times n$ matrix with two identical columns. Explain why A is not invertible.

7. Suppose that A is an invertible $n \times n$ matrix satisfying $A^3 - 3A + 2I = 0$. Find an expression for A^{-1} in terms of A and I.

8. Calculate the following determinants. Your answers should be formulas in terms of a and b.

(a) $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ a & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & a & b \\ b & a & 0 \\ -a & -b & -a \end{vmatrix}$

9. * Let J_n be the $n \times n$ matrix all of whose entries are equal to 1. For example,

$$J_1 = [1], \quad J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Prove that if $n > 1$, then the matrix $I_n - J_n$ is invertible with inverse

$$(I_n - J_n)^{-1} = I_n - \frac{1}{n-1} J_n.$$

Here I_n is the $n \times n$ identity matrix.

10. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, find:

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ (ii) $\begin{vmatrix} d & e & f \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{vmatrix}$ (iii) $\begin{vmatrix} a-4g & d & g \\ b-4h & e & h \\ c-4i & f & i \end{vmatrix}$

11. Find the values of c for which the matrix $A = \begin{bmatrix} c & c & 0 \\ c^2 & 2 & c \\ 0 & c & c \end{bmatrix}$ is invertible.

12. Suppose A and B are $n \times n$ matrices with $\det A = 4$ and $\det B = -3$. Find each of the following determinants.

(a) $\det(AB)$

(b) $\det(A^2)$

(c) $\det(B^{-1}A)$

(d) $\det(2A)$

13. * What can you say about $\det(A)$ if the square matrix A satisfies:

    (a) $A^2 = A$ (such a matrix is called idempotent).

    (b) $A^m = O$ for some $m > 1$ (such a matrix is called nilpotent).

14. (a) Let A be a 2×3 matrix and B be a 3×2 matrix. Then BA is a 3×3 matrix. Show that $\det(BA) = 0$.

    (b) Is it necessarily true that $\det(AB) = 0$?