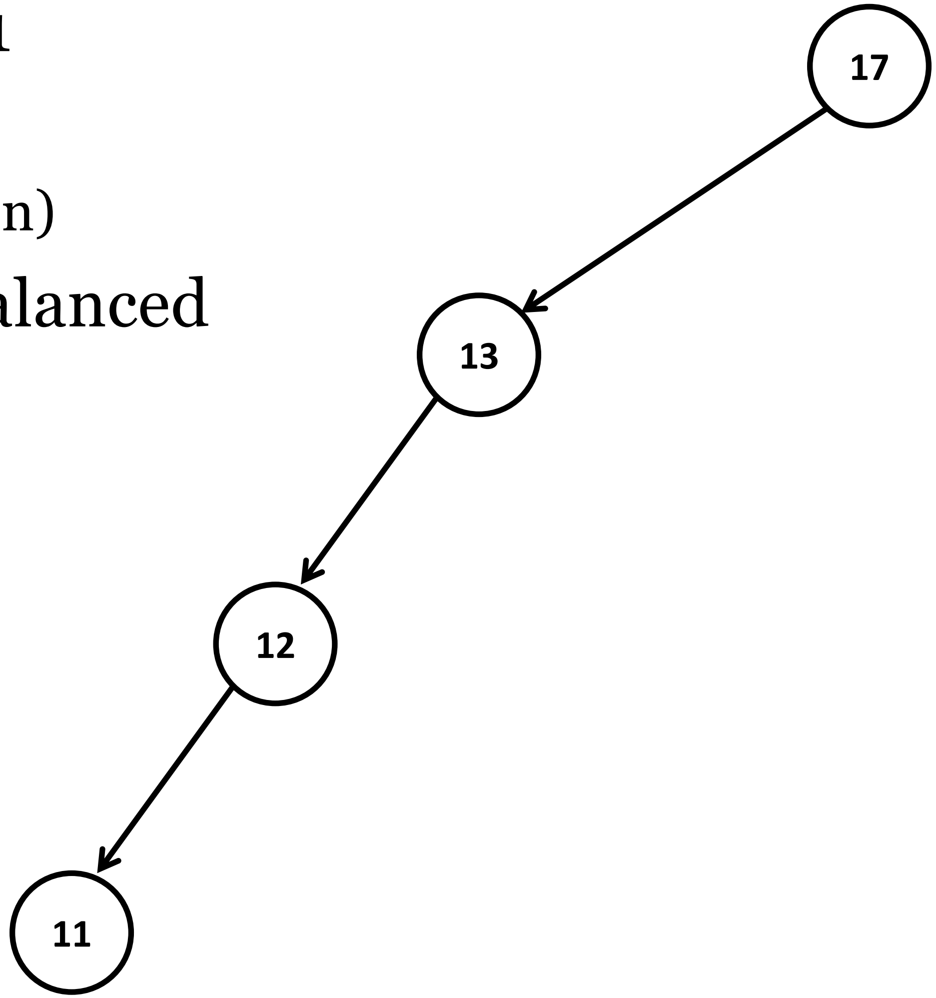
School of Computer Science

COMP SCI 1103/2103 Algorithm Design & Data Structure

BST and AVL Trees

Trees can become unbalanced

- Insert: 17, 13, 12, 11
- Increased depth
  - Search becomes O(n)
- Want trees to be balanced



Self-balancing BSTs

- A self-balancing BST automatically keeps its structure balanced.
- Example: **AVL tree**
- An AVL tree is a BST with a balance condition
  - For every node, the heights of two child subtrees can only differ by at most 1. See examples.
  - After insertion / deletion, if the above property is violated, then some housekeeping is needed to restore the property, which takes O(log n) extra time.
  - Since the tree is always fairly balanced, searching, insertion, and deletion all take logarithmic time in the worst case.
    - The height is not minimized, but still in O(log n)
- Examples of balanced trees with this definition

AVL-Tree

Observation:

- Binary search trees can get imbalanced when applying insert and/or remove operations.

Idea:

- Whenever a subtree rooted at a node v gets imbalanced, apply operations that balance it out in time O(log n).

AVL Tree

Let h(T) be the height of a tree T (maximum number of edges from root to any leaf).

Let v be a node in T and T_l and T_r be the left and right subtree of v.

We denote by $b(v) = h(T_l) - h(T_r)$ the balance degree of v.

Definition: A binary search tree T is called an AVL-tree if for each $v \in T, b(v) \in \{-1, 0, 1\}$ holds.

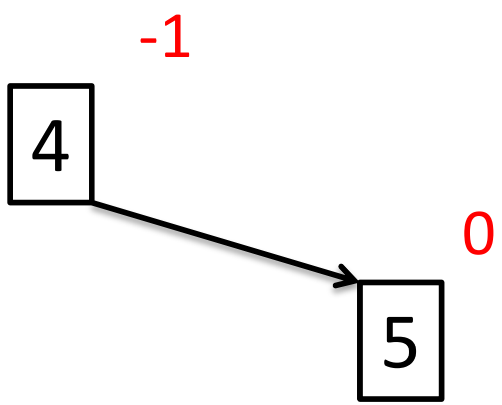
Height of an AVL-tree

Theorem(without proof) Let T be an AVL-tree consisting of n nodes. Then $h(T) \leq 1.44 \log n$

We have to consider the operations find, insert, and delete for AVL-trees.

- Find is as for Binary Search Trees.
- For insert and remove we might have to rebalance the tree.

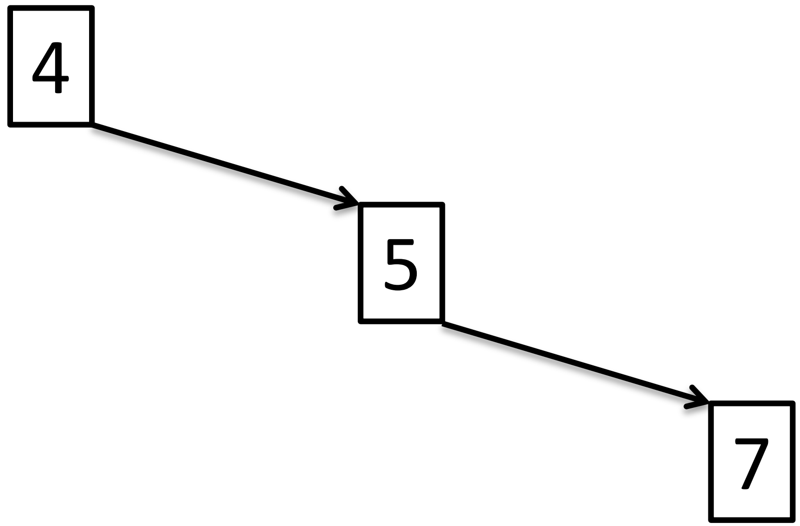
Example Insert



Balance values

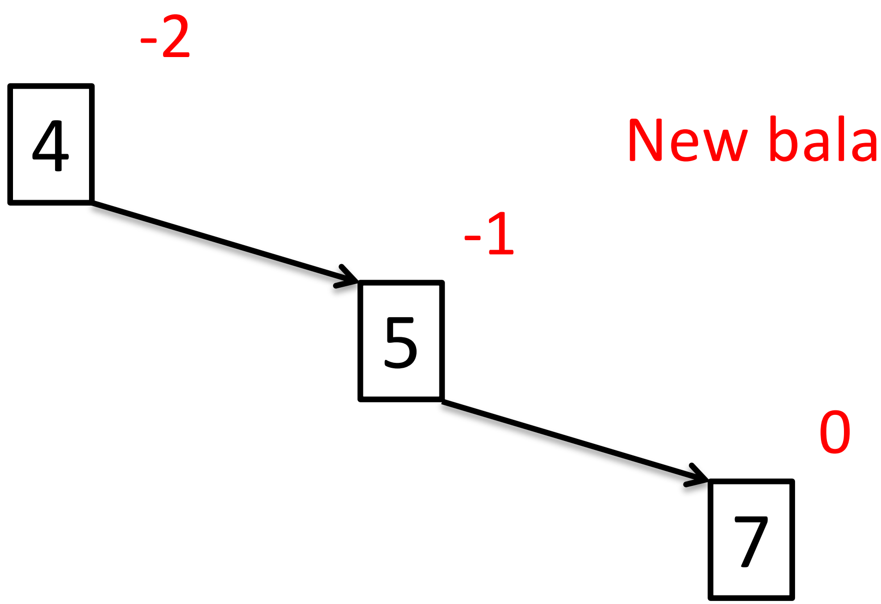
Insert 7

Example Insert



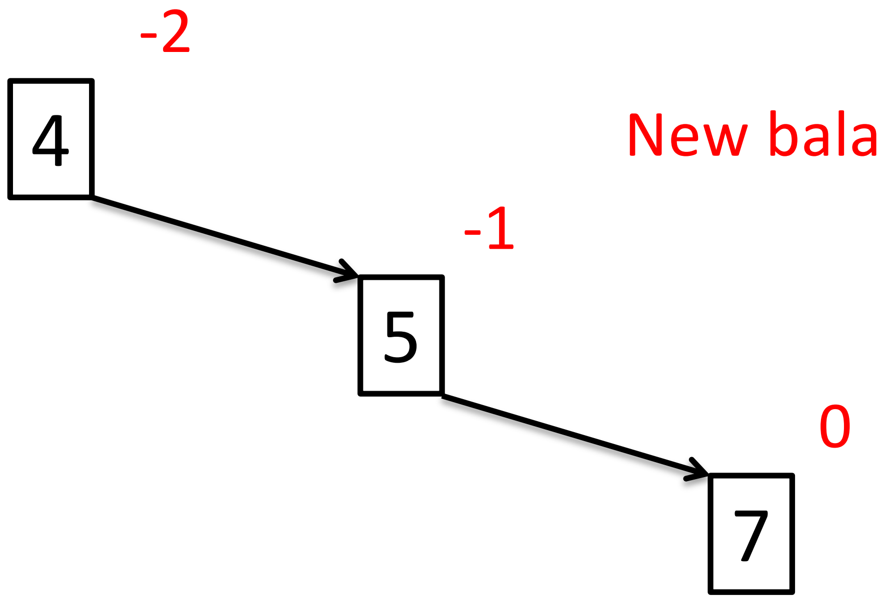
Consider path from new leaf
to the root and check balance values

Example Insert



New balance values

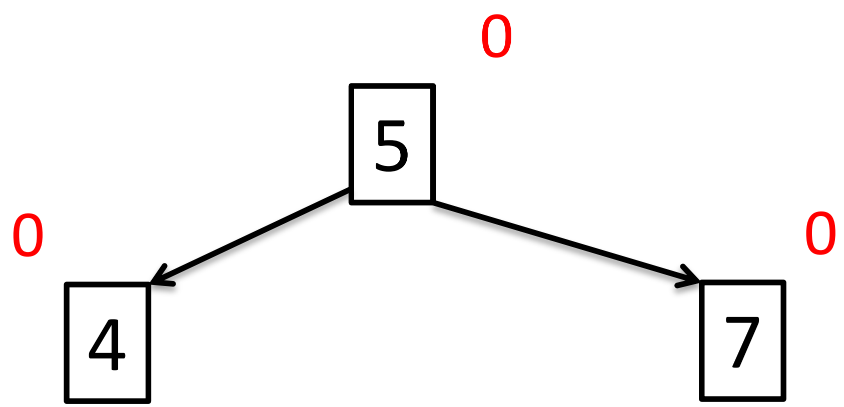
Example Insert



New balance values

AVL-property at node 4 violated

Example Insert



Rotation establishes AVL-property again

Insertion

Inserting a new element z can violate the AVL-property.

Consider path from the newly inserted leaf z to the root and repair AVL-property.

Rebalancing

Let z be the newly inserted leaf.

Consider the path from z to the root (reverse the insertion path).

Update the balance values.

Repair AVL-property (if necessary).

Insert

- we insert new node z as for Binary Search Trees.
- bal(z)=0 holds after insertion.
- bal(v) might change by 1 for a node v on the path from z to the root.
- If $b(v) \notin \{-1, -0, 1\}$ rebalance

Rebalancing

Start examining for v, where v is the parent of z, and continue with the parent of v (if necessary).

Assume that the right child x of node v is on the path from z to the root.

Before insertion -> After Insertion:

- bal(v) = 1 -> bal(v)=0 (height of tree rooted at v has not changed, stop rebalancing)
- bal(v)=0 -> bal(v) = -1 (height of tree rooted at v has increased by 1, stop rebalancing only if v is root, otherwise examine parent of v)
- bal(v)=-1 -> bal(v) = -2 (AVL-property violated, carry out rotation)

Left Rotation

Assume node v and right child x on the path.

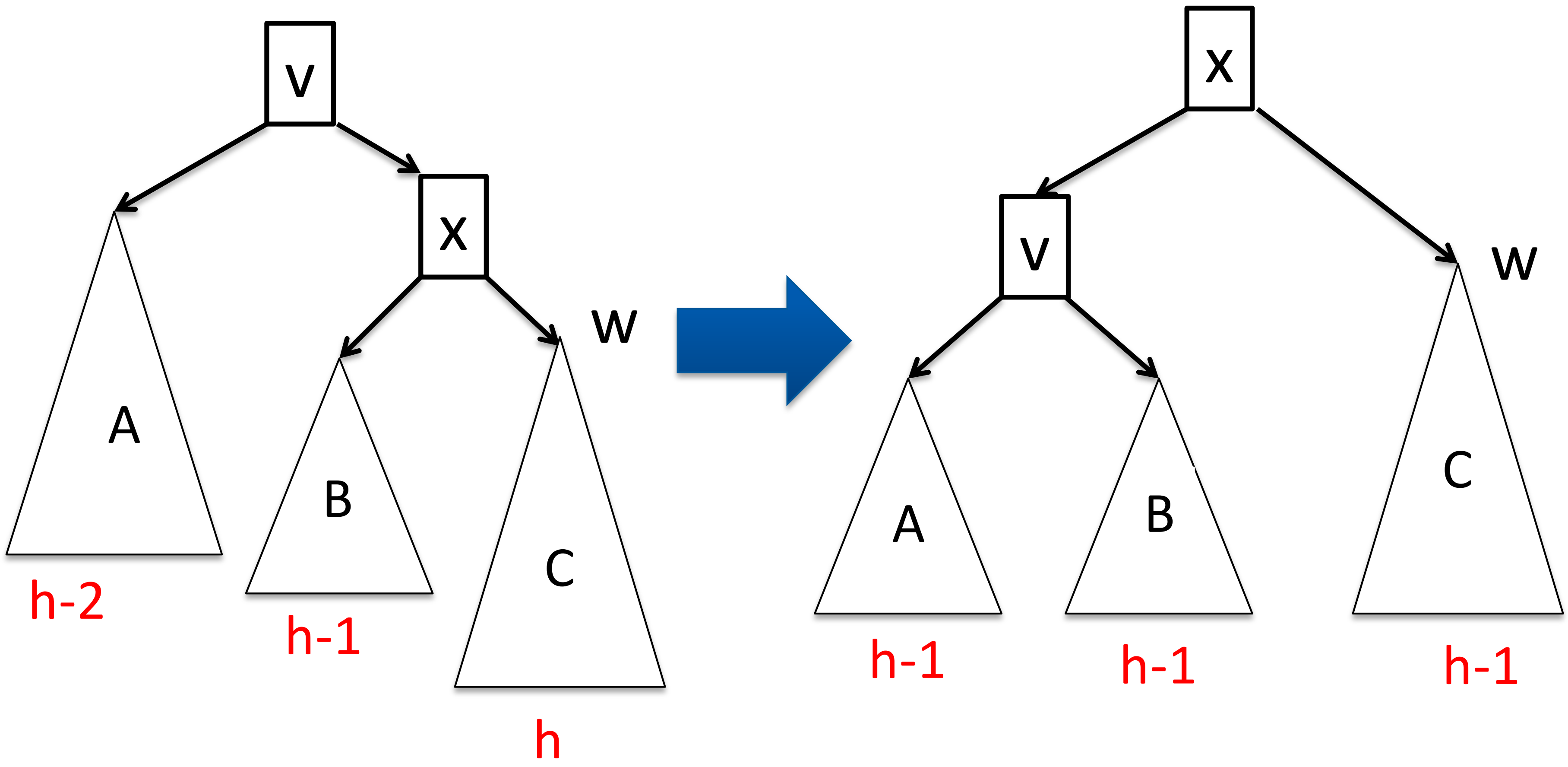
- w is right child of x on the path

  -> Left rotation

  New balance values: bal(x)=0 and bal(v)=0

Analogous: Right Rotation

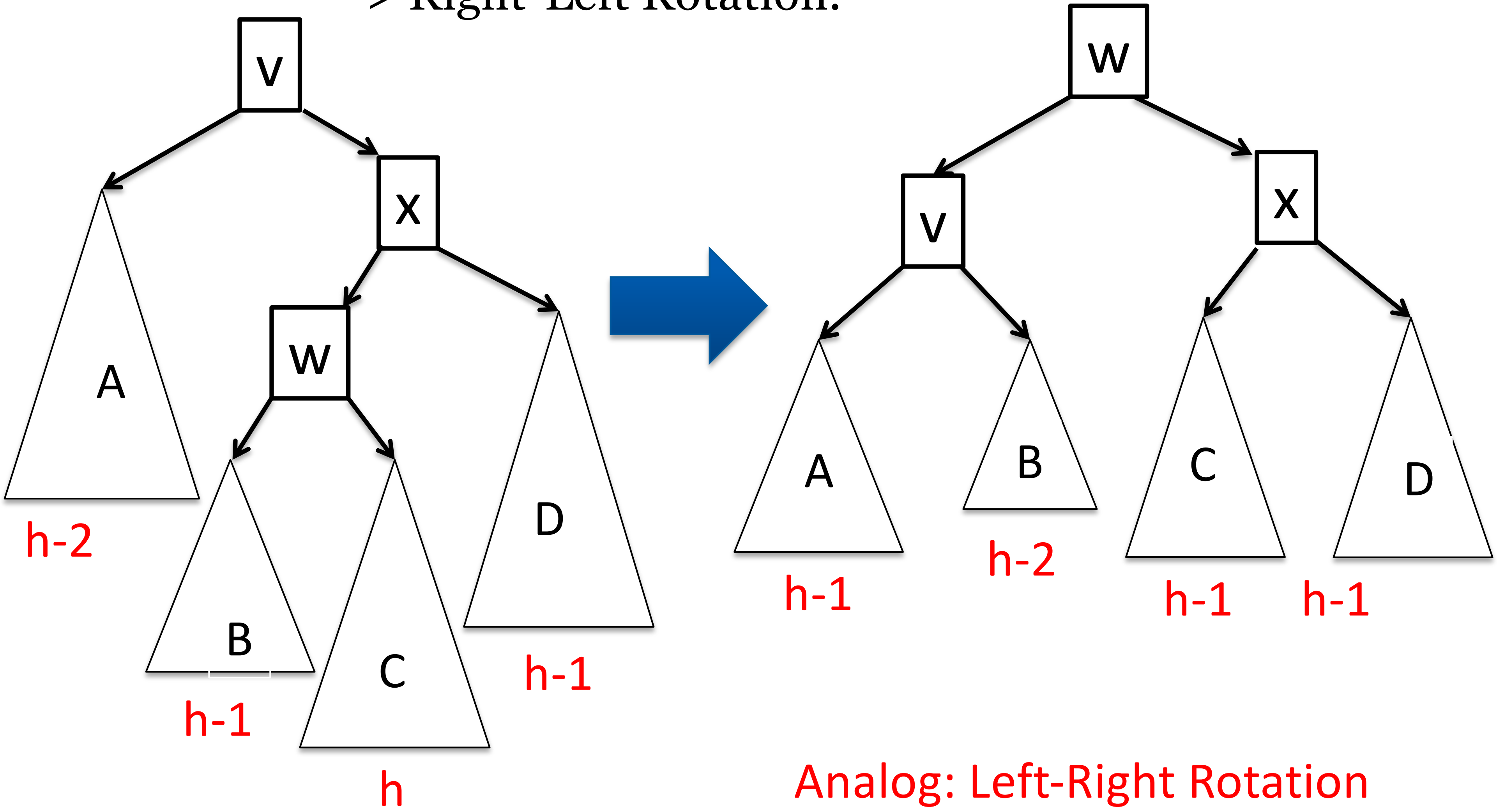
Left Rotation



Analog: Right Rotation

Right-Left Rotation

w is left child of x on the path
-> Right-Left Rotation.



Analog: Left-Right Rotation

Example Insert

Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6

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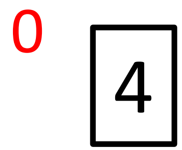
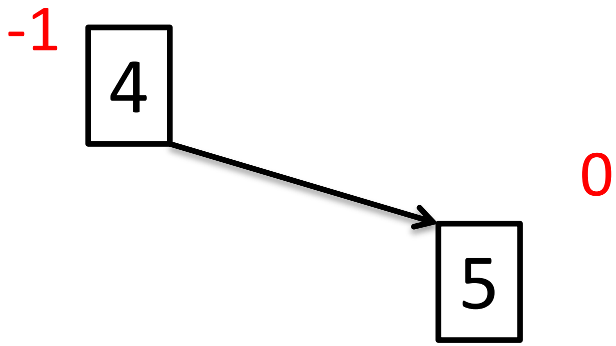


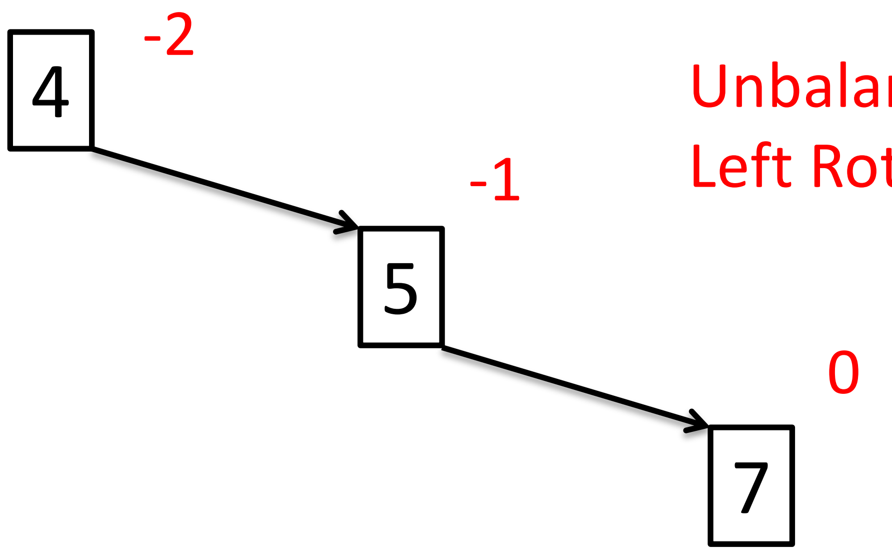
Example Insert

Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Example Insert

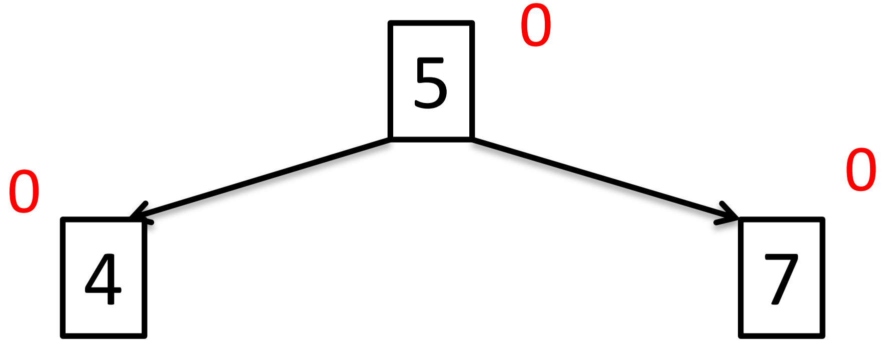
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Unbalanced
Left Rotation

Example Insert

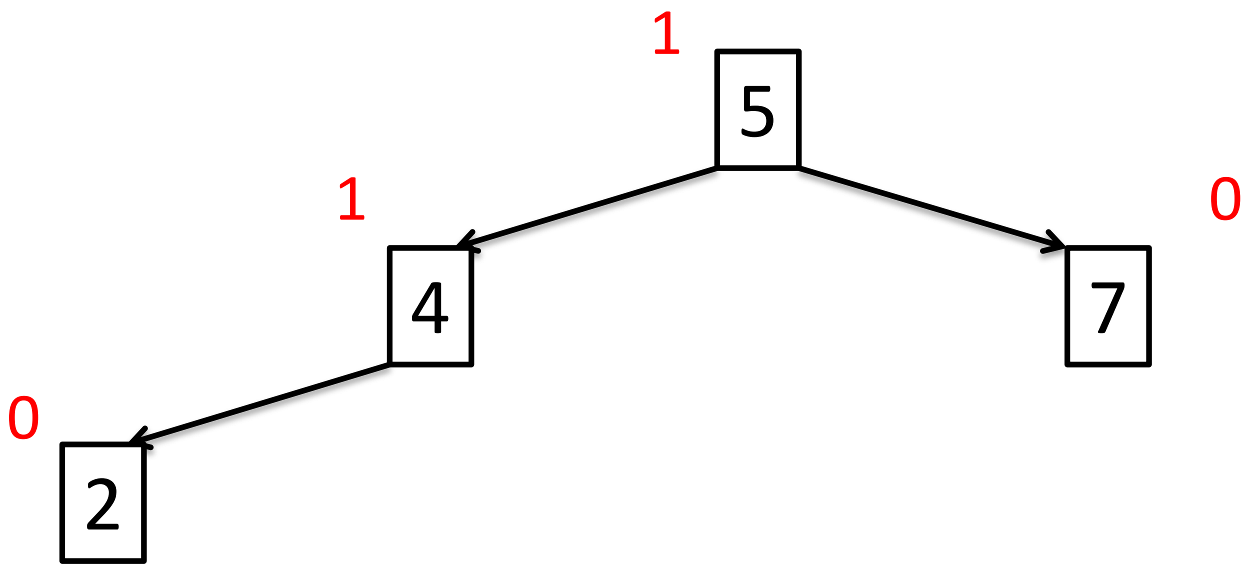
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Balance OK

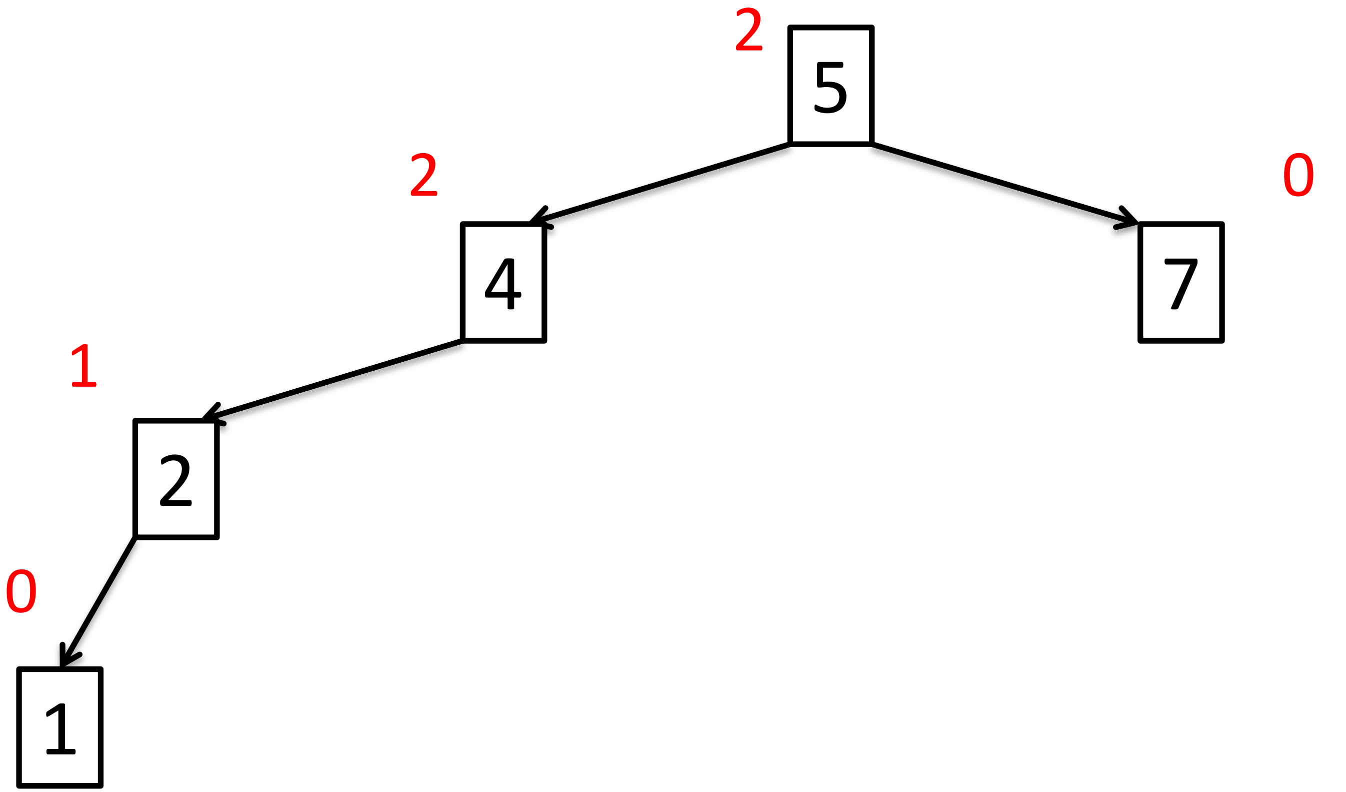
Example Insert

Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Example Insert

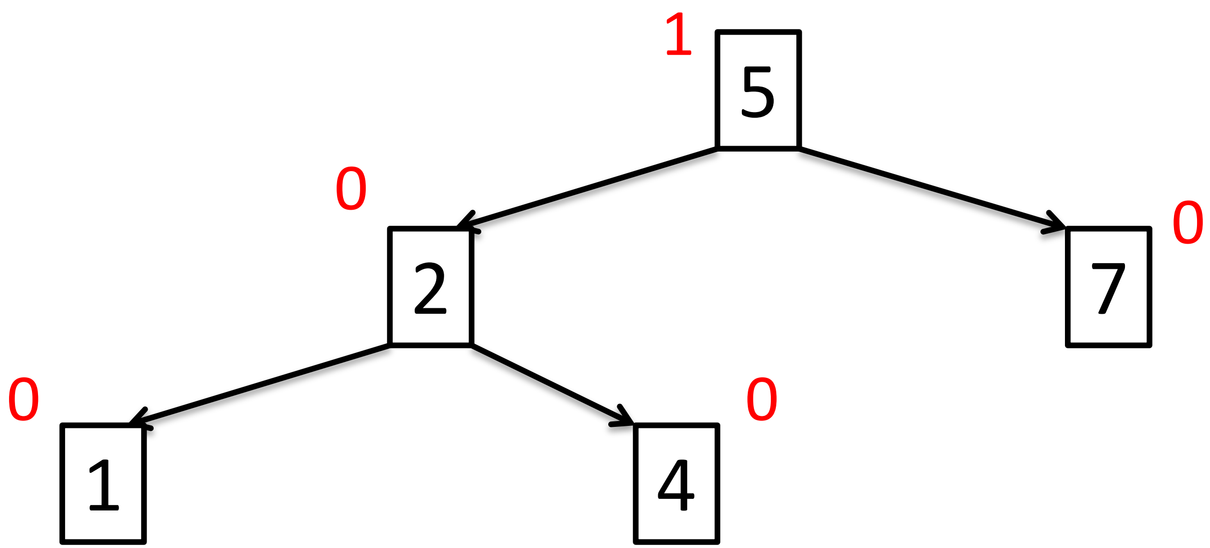
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Unbalanced
Right Rotation

Example Insert

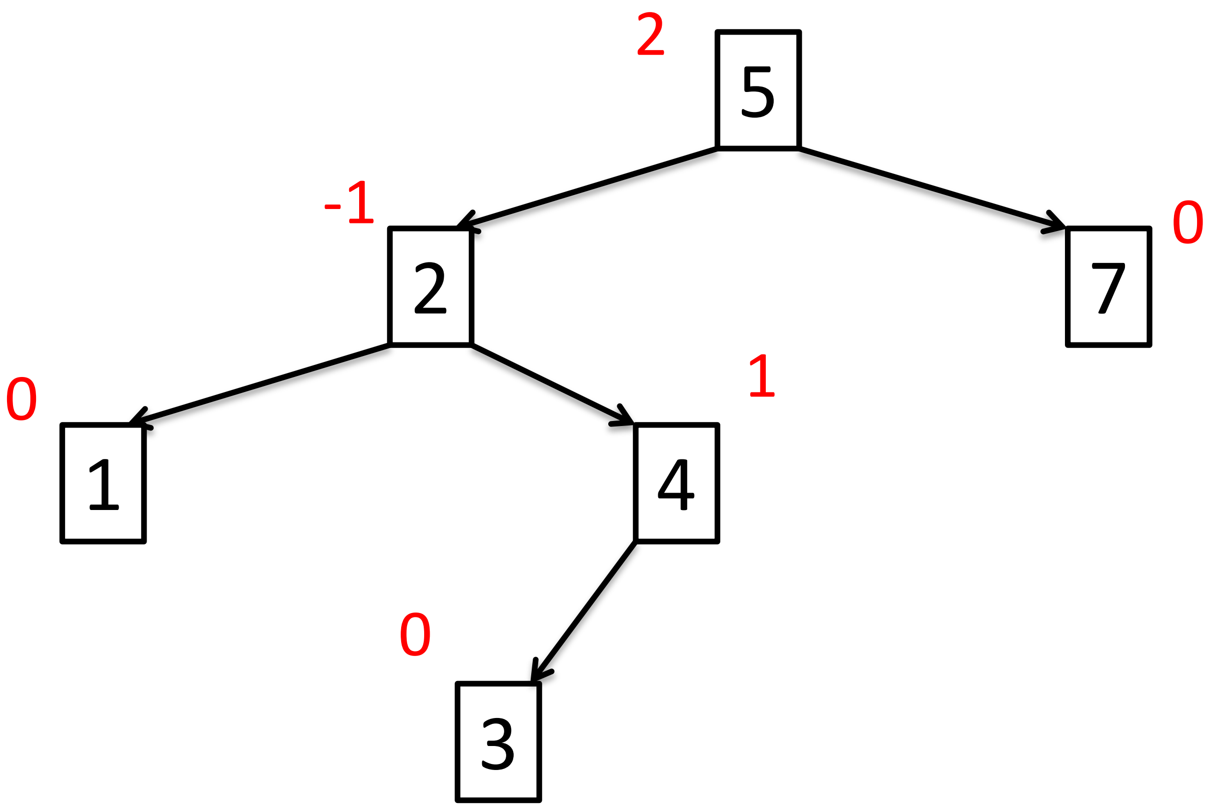
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Balance OK

Example Insert

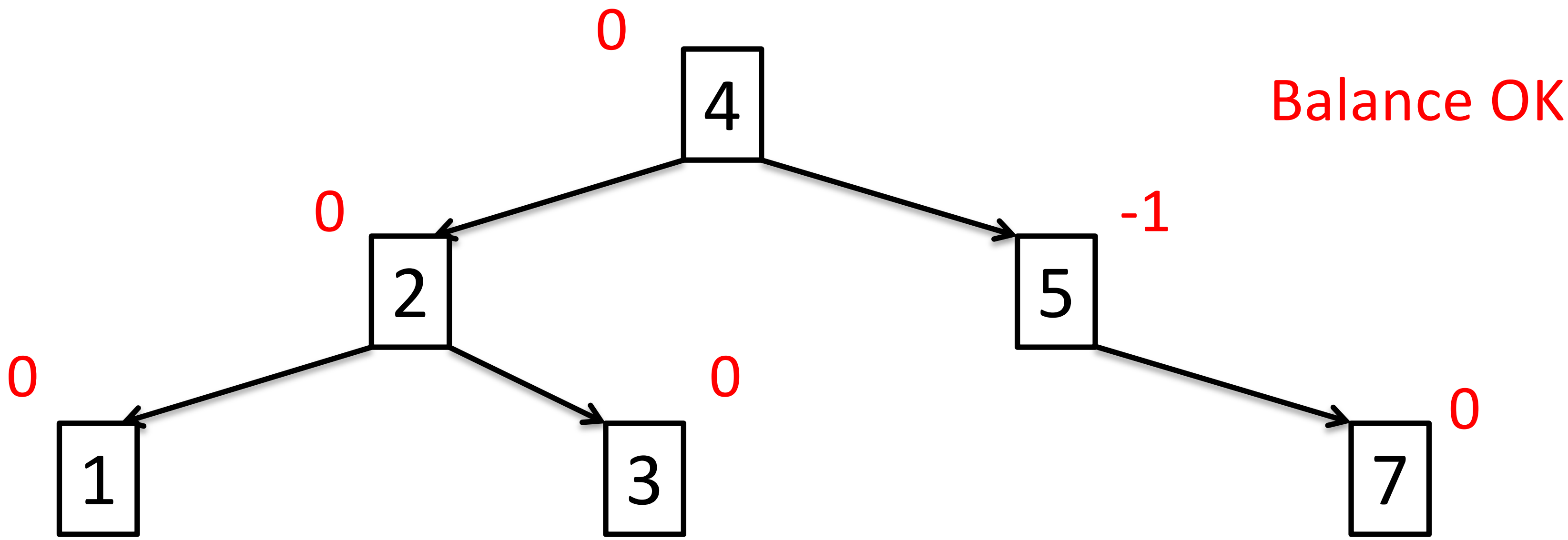
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Unbalanced
Left-right Rotation

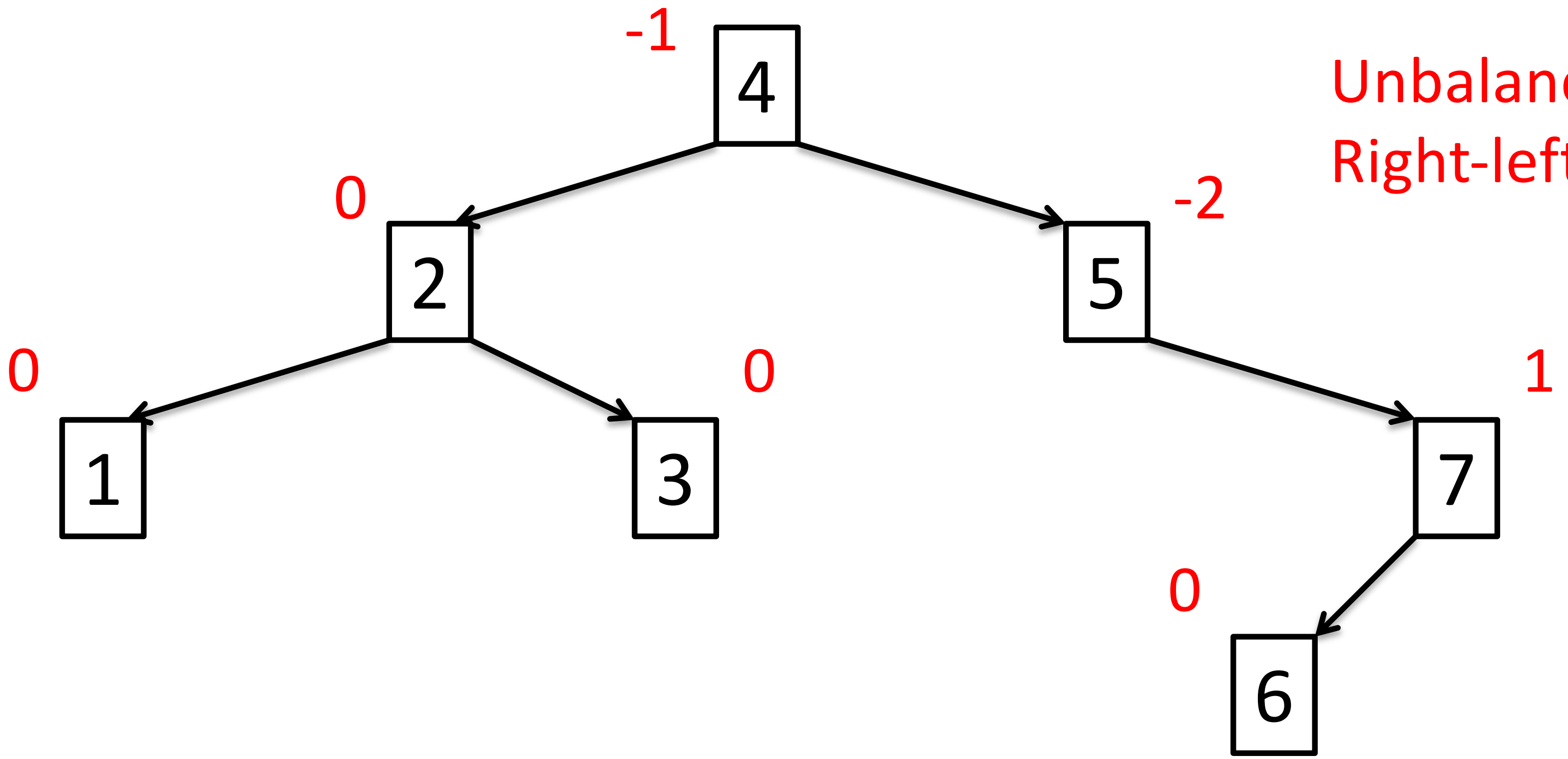
Example Insert

Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Example Insert

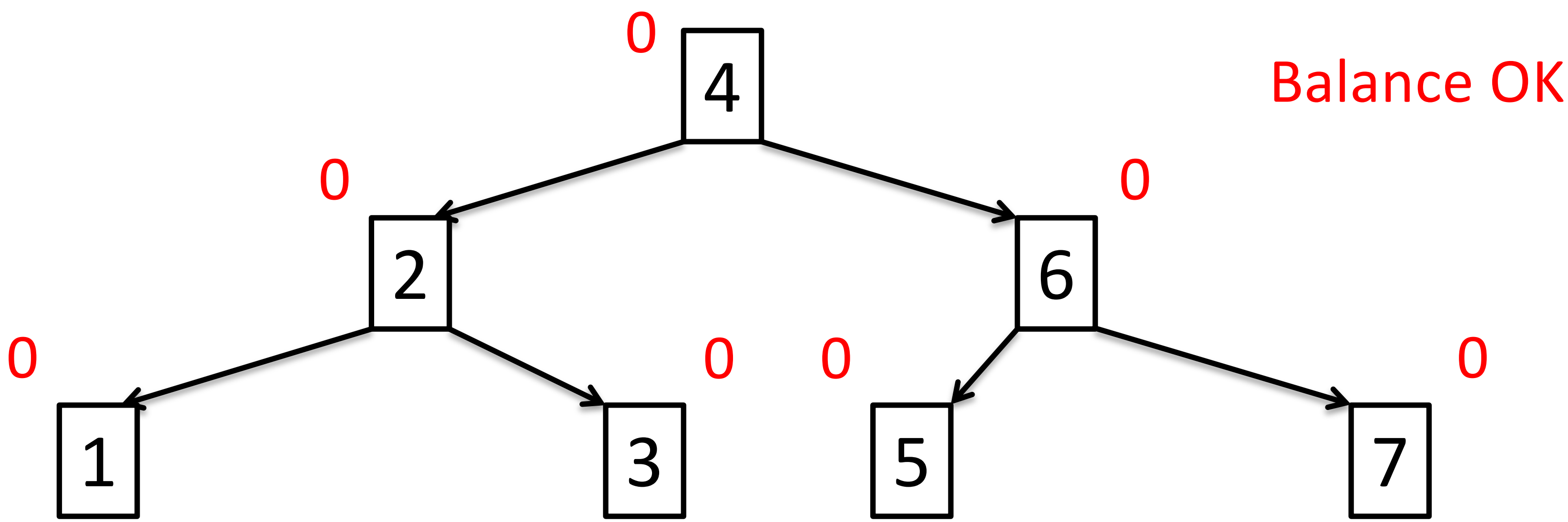
Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Unbalanced
Right-left Rotation

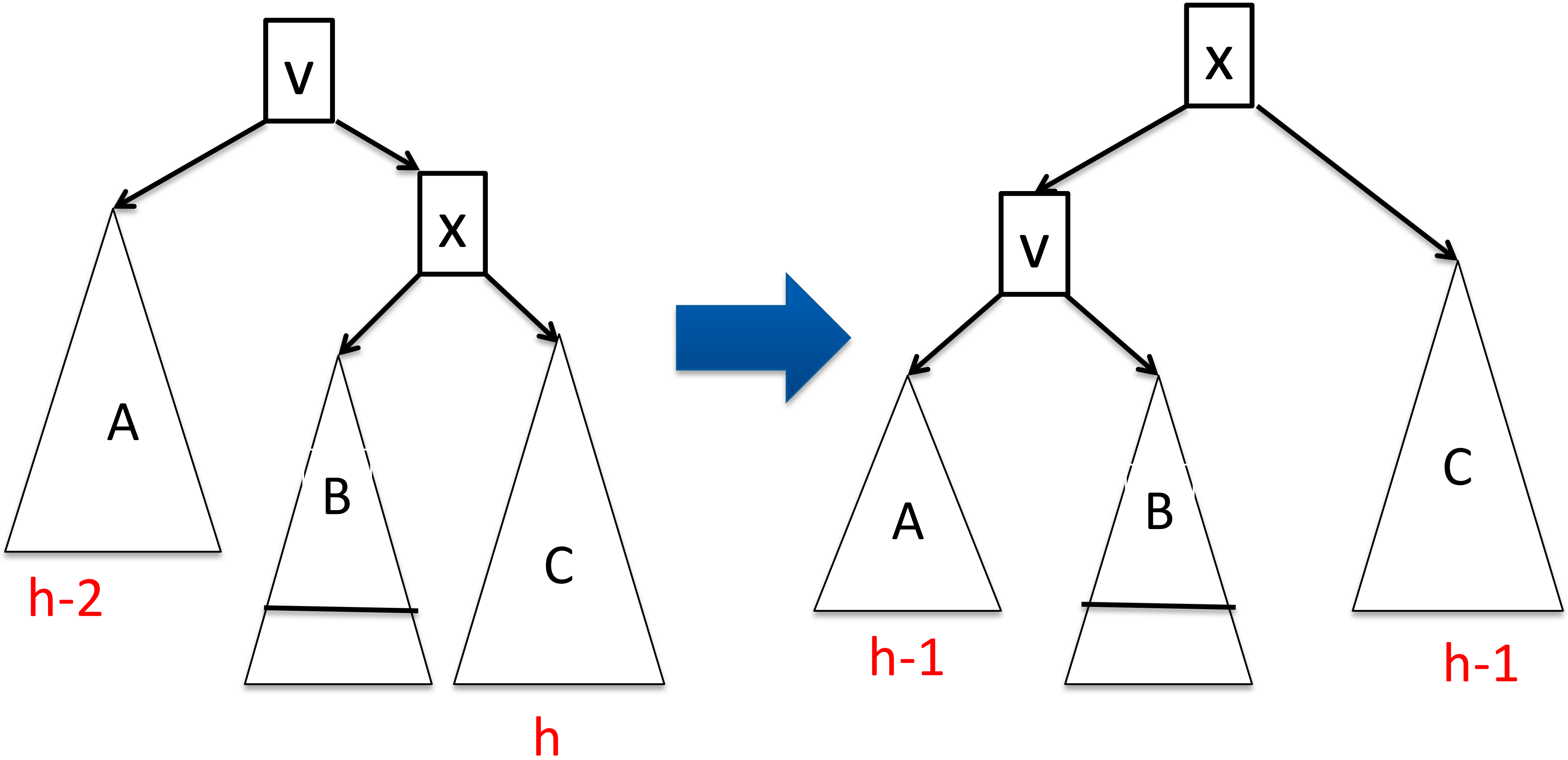
Example Insert

Create AVL-Tree for sequence 4, 5, 7, 2, 1, 3, 6



Remove – Left Rotation

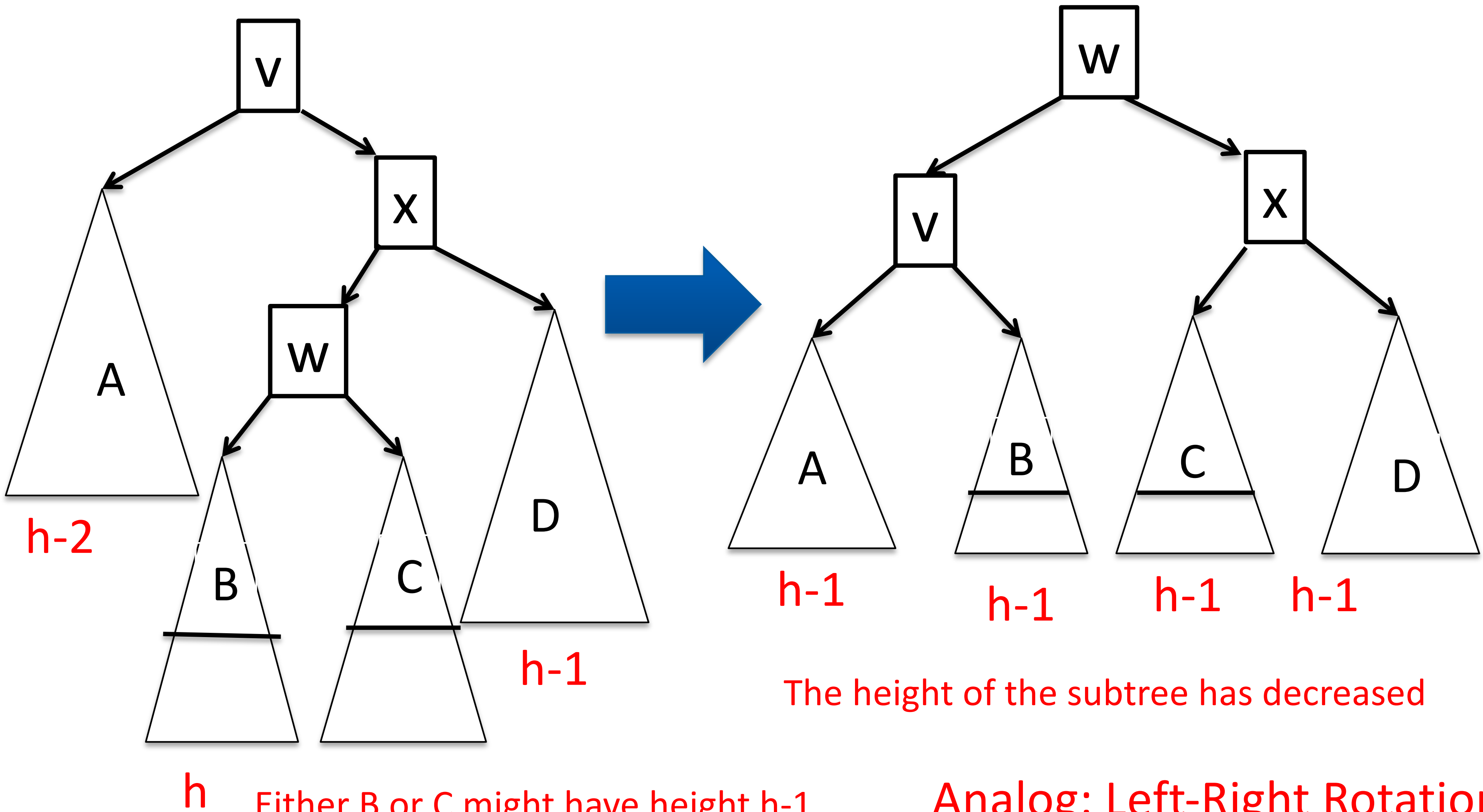
W.l.o.g. assume that the deleted node was in the left subtree of v and height of this tree has decrease by 1.



If B had height h-1 before deletion,
the height of the subtree has decreased

Analog: Right Rotation

Right-Left Rotation



The height of the subtree has decreased

Either B or C might have height h-1

Analog: Left-Right Rotation

Rebalancing after Deletion

- After having rebalanced for node v the height of the tree previously rooted at v might have decreased after deleting and rebalancing.
- If this is the case old parent of v might be imbalanced.
- We might have to continue rebalancing until the root has been reached.

Runtime AVL-trees

Theorem: The operations find, insert, and delete can be implemented for AVL-trees in worst-case time O(log n).