Mathematics for Data Science I
Practice Questions (week 3)

Semester 2, 2018

These questions are all about Taylor and Maclaurin series, and power series more generally. Difficult questions are starred. Good places to go for further questions on this topic include the exercises in:

- Stewart, *Calculus* (7th Ed.), Sections 11.10 and 11.11,
- Morris & Stark, *Fundamentals of Calculus*, Sections 9.2 and 9.3.

1. (a) Write down, *from memory*, the Taylor series for the function $\sin x$, $\cos x$ and e^x around the point $x = 0$.

   (b) By manipulating the above series (rather than by actually computing derivatives), determine the power series (around $x = 0$) for each of functions (i) $x \cos x$ and (ii) $\sin(2x)$.

Solution:

(a) You should try to remember these! They are all valid for $x \in \mathbb{R}$, and are given by

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{m=0}^{\infty} \frac{x^m}{m!}
\end{aligned}$$

(b) For (i), we simply multiply the Taylor series of $\cos x$ through by x to get

$$x \cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m)!} .$$

For (ii), we replace x by $2x$ in the Taylor series for sine to get

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m+1} x^{2m+1}}{(2m+1)!} .$$

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2. Find the Taylor series around $x = 0$ for the functions $\sinh x$ and $\cosh x$.

Solution: We note that

$$\frac{d}{dx}[\sinh x] = \cosh x \qquad \text{and} \qquad \frac{d}{dx}[\cosh x] = \sinh x\,.$$

Therefore, the derivatives of each of these functions simply go back and forth between each other. Moreover, $\sinh 0 = 0$ and $\cosh 0 = 1$. Thus, applying the Taylor series to $\sinh x$ we get

$$\begin{aligned} \sinh x &= \sum_{n=0}^{\infty} \frac{d^n}{dx^n}[\sinh x]\Big|_{x=0} \frac{x^n}{n!} \\ &= \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 \cdots \\ &= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}\,. \end{aligned}$$

Similarly, when applied to $\cosh x$ the odd-indexed terms disappear, and so

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}\,.$$

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3. Find the leading terms (up to the fourth power) of the Taylor series for each of the following functions.

   (a) $e^x \sin x$, around $c = 0$

   (b) $\left(\tan^{-1} x\right)^2$, around $c = 0$

   (c) $\dfrac{1}{x}$, around $c = 2$

   (d) $\sin x$, around $c = \pi/4$

Solution: We recall the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

for a (sufficiently differentiable) function f, around a point c.

(a) If $f(x) = e^x \sin x$, then we need to compute derivatives until the fourth (here, we will find one more term just for fun!).

$$\begin{array}{l|l}
f(x) = e^x \sin x & f^{(0)}(0) = 0 \\
f'(x) = e^x \cos x + e^x \sin x & f'(0) = 1 \\
f''(x) = 2e^x \cos x & f''(0) = 2 \\
f^{(3)}(x) = 2e^x \cos x - 2e^x \sin x & f^{(3)}(0) = 2 \\
f^{(4)}(x) = -4e^x \sin x & f^{(4)}(0) = 0 \\
f^{(5)}(x) = -4e^x \cos x - 4e^x \sin x & f^{(5)}(0) = -4
\end{array}$$

Therefore, applying the Taylor series formula, we get that

$$e^x \sin x \approx \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}.$$

(b) If $f(x) = \left(\tan^{-1} x\right)^2$, then we need to compute derivatives until the fourth one:

$$\begin{array}{l|l}
f(x) = \left(\tan^{-1} x\right)^2 & f^{(0)}(0) = 0 \\
f'(x) = \frac{2\tan^{-1} x}{1+x^2} & f'(0) = 0 \\
f''(x) = \frac{2}{(1+x^2)^2} - \frac{4x\tan^{-1} x}{(1+x^2)^2} & f''(0) = 2 \\
f^{(3)}(x) = -\frac{12x}{(1+x^2)^3} + \frac{16x^2\tan^{-1} x}{(1+x^2)^3} - \frac{4\tan^{-1} x}{(1+x^2)^2} & f^{(3)}(0) = 0 \\
f^{(4)}(x) = \frac{88x^2}{(1+x^2)^4} - \frac{16}{(1+x^2)^3} - \frac{96x^3\tan^{-1} x}{(1+x^2)^4} + \frac{48x\tan^{-1} x}{(1+x^2)^3} & f^{(4)}(0) = -16
\end{array}$$

Therefore, applying the Taylor series formula, we get that

$$\left(\tan^{-1} x\right)^2 \approx \frac{2}{2!}x^2 - \frac{16}{4!}x^4 = x^2 - \frac{2}{3}x^4.$$

(c) In this case, $f(x) = x^{-1}$, and so $f(2) = 1/2$. Moreover, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$ and $f^{(4)}(x) = 24x^{-5}$. Thus, $f'(2) = -1/4$, $f''(2) = 1/4$, $f^{(3)}(2) = -3/8$ and $f^{(4)}(2) = 3/4$. Thus, the Taylor series about $c = 2$ is

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{2} + \frac{-1/4}{1!}(x-2) + \frac{1/4}{2!}(x-2)^2 - \frac{3/8}{3!}(x-2)^3 + \frac{3/4}{4!}(x-2)^4 + \cdots \\
&\approx \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4
\end{aligned}$$

(d) Here, $f(x) = \sin x$ and so $f(\pi/4) = \sqrt{2}/2$. Moreover, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x$. Therefore, $f'(\pi/4) = \sqrt{2}/2$, $f''(\pi/4) = -\sqrt{2}/2$, $f'''(\pi/4) = -\sqrt{2}/2$ and $f^{(4)}(\pi/4) = \sqrt{2}/2$. Thus, the Taylor series about $c = \pi/4$ is

$$\begin{aligned}
\sin x &= \frac{\sqrt{2}}{2}\left[1 + \frac{1}{1!}\left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots\right] \\
&\approx \frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24}\left(x - \frac{\pi}{4}\right)^4\right]
\end{aligned}$$

4. For what values of x can we replace $\sin x$ by $x - x^3/3!$ with an error of magnitude no greater than 3×10^{-4}?

Solution: This question is related to the error of a Taylor series, i.e., Taylor's remainder once we use a Taylor polynomial to approximate the function. We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

so in this instance we have only used the first two terms above. Actually, we can say more: we have actually used terms upto *fourth-order* in x when using $x - \frac{x^3}{3!}$ to approximate $\sin x$, because the coefficient of x^4 in this Taylor series is zero. In other words, we can say that by doing this, we have used a Taylor polynomial $T_4(x)$, and the error is therefore $R_4(x)$. Now, the formula given in is

$$R_n(x) = \frac{f^{(n+1)}(z)\,(x-c)^{n+1}}{(n+1)!} \qquad \Rightarrow \qquad R_4(x) = \frac{f^{(5)}(z)x^5}{5!}$$

because here the Taylor series is about $c = 0$. The quantity z is a (potentially unknown) number between x and c. We need the fifth derivative of $f(x) = \sin x$, which is the same as the first derivative (the derivatives keep repeating with period 4), which is therefore $\cos x$. Thus,

$$|R_4(x)| = \left|\frac{\cos\xi\, x^5}{5!}\right| \leq \left|\frac{x^5}{120}\right| ,$$

because $|\cos z| \leq 1$ for any z. We want the error to be less than 3×10^{-4}, which we can ensure by setting

$$\left|\frac{x^5}{120}\right| < 3 \times 10^{-4} \qquad \Rightarrow \qquad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514 .$$

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5. Find the radius and intervals of convergence for each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2)(4)(6)\cdots(2n)} x^n$

Solution: The basic test for this is the ratio test.

(a) The nth term here is $a_n = (-1)^{n+1}\, x^n/n$, and so we require

$$1 > \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1}\,\frac{n}{x^n}\right| = |x| \lim_{n\to\infty} \frac{n+1}{n} = |x|$$

to guarantee convergence. Thus, this converges when $|x| < 1$, and so the radius of convergence is 1. We now need to test the two endpoints of this interval separately (the ratio test gives inconclusive information when $|x| = 1$). When $x = 1$, the series is the alternative harmonic series, which we know converges. On the other hand, when $x = -1$, then $(-1)^{n+1}x^n = (-1)^{2n+1} = -1$, because the power is odd. Thus, we are left with the series $-\sum 1/n$, which is negative the harmonic series, which diverges. Thus, the power series converges for $-1 < x \le 1$, and the interval of convergence is $(-1, 1]$.

(b) This is a power series not centred at 0, but the ratio test still applies. Since $a_n = n(x+2)^n/3^{n+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\,\frac{3^{n+1}}{n(x+2)^n}\right| = \left|\left(1+\frac{1}{n}\right)\frac{|x+2|}{3}\right| \to \frac{|x+2|}{3}$$

as $n \to \infty$. Thus, the series converges for $|x+2|/3 < 1$, i.e., for $|x+2| < 3$. Thus, the radius of convergence is 3. Moreover, this corresponds to $-3 < x+2 < 3$, i.e., $x > -5$ and $x < 1$. So the power series converges for $x \in (-5, 1)$. We now need to test the series using an alternative method at the endpoints. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n\, n\,,$$

which diverges because the terms a_n do not converge to 0. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n3^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n\,,$$

which also has a_n not converging to zero, and thus $\sum a_n$ cannot converge. Hence, the series diverges at both endpoints. The interval of convergence is therefore $x \in (-5, 1)$.

(c) Here, $a_{n+1} = x^{n+1}/[(2)(4)(6)\cdots(2(n+1))]$, and so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(2)(4)(6)\cdots(2n)(2n+2)}\,\frac{(2)(4)(6)\cdots(2n)}{x^n}\right| = \frac{|x|}{2n+2}\,.$$

As $n \to \infty$, this converges to 0 for all x. Thus, the radius of convergence is ∞, and the interval of convergence is $(-\infty, \infty) = \mathbb{R}$.

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6. We know (from the geometric series) that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \quad ; \quad |x| < 1.$$

By manipulating this series in various ways, determine the power series for the following functions. Make sure you mention the domain of validity for your power series.

(a) $\dfrac{1}{1+x}$

(b) $\dfrac{1}{1-2x}$

(c) $\dfrac{1}{x+2}$

(d) $\ln(1+x)$ (Hint: integrate the geometric series)

Solution:

(a) We simply replace x with $-x$, to get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^5 - \cdots \quad ; \quad |x| < 1\,.$$

(b) We can replace x with $2x$, but note that this means that the validity is for $|2x| < 1$, or $|x| < 1/2$:

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = 1+2x+4x^2+8x^3+16x^4+\cdots \quad ; \quad |x| < 1/2\,.$$

(c) The 2 in the denominator is problematic, because the original geometric series does not have it. We therefore proceed as follows:

$$\frac{1}{x+2} = \left(\frac{1}{2}\right)\frac{1}{1+x/2} = \left(\frac{1}{2}\right)\frac{1}{1-(-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \quad ; \quad |x| < 2\,,$$

in which we have used the geometric series with $r = -x/2$, and the domain of validity obeys $|-x/2| < 1$, which converts to $|x| < 2$.

(d) We know that $\int \frac{\mathrm{d}x}{1+x} = \ln|1+x| + C$, where C is a constant. We therefore integrate the given geometric series term by term to get

$$\ln(1+x) + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ,$$

and note that this is valid for $|x| < 1$, because that was the initial geometric series which we then integrated. This allows us to replace the absolute value signs within the logarithm with parentheses, because $1 + x > 0$ in this regime. However, there is a problem here: we don't know the C. Substituting $x = 0$ on both sides above, we see that $\ln 1 + C = 0 - 0 + 0 - \cdots$, and hence $C = 0$. Thus, the power series is

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad ; \quad |x| < 1 .$$

Fun fact: It turns out that term-by-term integration of power series is always a valid operation, but term-by-term differentiation is *not* always valid.

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7. The series $\displaystyle\sum_{n=0}^{\infty} (e^x - 4)^{-n}$ is not, technically speaking, a power series because it is not expressed in powers of x. On the other hand, it *is* some sort of series. Determine the values of x for which it converges.

Solution: This is an infinite series with $a_n = (e^x - 4)^{-n}$ and thus $a_{n+1} = (e^x - 4)^{-n-1}$. We can therefore apply the ratio test in the usual way.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(e^x - 4)^n}{(e^x - 4)^{n+1}} \right| = \frac{1}{|e^x - 4|} .$$

Convergence occurs when $1/|e^x - 4| < 1$, or when $|e^x - 4| > 1$. Thus, we need either $e^x - 4 < -1$ or $e^x - 4 > 1$. In the former situation, we get $e^x < 3$, and by taking the logarithm of both sides (which we are entitled to do because the logarithm is a monotone function, and thereby preserves ordering), we get $x < \ln 3$. Taking the latter situation, we get $e^x > 5$, and thus $x > \ln 5$. Therefore, we know that the series converges when $x < \ln 3$ or when $x > \ln 5$. It is interesting to note that this is not an interval, but the complement of an interval. The ratio test is inconclusive when we get 1, and thus we need to examine the endpoints separately. When $x = \ln 3$, the series is

$$\sum_{n=0}^{\infty} \left(e^{\ln 3} - 4 \right)^{-n} = \sum_{n=0}^{\infty} (3-4)^{-n} = \sum_{n=0}^{\infty} (-1)^n .$$

Since the general term a_n does not decay to zero, this series does not converge. Similarly, when $x = \ln 5$ the series is $\sum_{n=0}^{\infty} 1$, which also does not converge for the same reason. Thus, the values of x for which this series converge are given by the set $\{x \,:\, x < \ln 3 \quad \text{or} \quad x > \ln 5\}$, i.e., the set $\mathbb{R} \setminus [\ln 3, \ln 5]$.

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8. * Find the power series for the function $\tan^{-1} x$. (Hint: term-by-term integration, and the geometric series, may be useful ideas.)

Solution: We know that $\int \frac{1}{1+x^2} \mathrm{d}x = \tan^{-1} x + C$, and so

$$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} \mathrm{d}x + C = \int \frac{1}{1-(-x^2)} \mathrm{d}x + C \\ &= C + \int \sum_{n=0}^{\infty} \left(-x^2\right)^n = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, \mathrm{d}x \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots , \end{aligned}$$

where we have used the geometric series at the third equality, valid for $\left|-x^2\right| < 1$. Putting in $x = 0$, we get $C = \tan^{-1} 0 = 0$. Thus,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad ; \quad |x| < 1\,.$$

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9. Use the Taylor series expansion for e^x to determine the value of

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}\,.$$

Solution: We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

where the (unspecified) terms have at least a power of x^4. Thus

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\,.$$

Dividing by x^2, we get

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{3!} + \cdots$$

where the remaining terms now have at least a power of x^2. Now, imagine taking the limit $x \to 0$ of the above. The term $x/3!$ on the right-hand side goes to zero. The remaining terms will all go to zero because of the minimal power of x^2 in them. Thus,

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}\,.$$
