Mathematics for Data Science I
Practice Questions (week 4)

Semester 2, 2019

These questions are all about matrices. Difficult questions are starred.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 4 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -4 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 5 & 7 \end{bmatrix}$ and $E = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.
   Evaluate each of the following matrices if they exist:
   (i) AB (ii) BAC (iii) $A + C$ (iv) ED (v) AED (vi) C^3

   Solution: (i) $AB = \begin{bmatrix} 10 & 8 & 8 \\ 16 & 8 & 4 \\ -16 & -20 & -26 \end{bmatrix}$ (ii) $BAC = \begin{bmatrix} 70 & -56 \\ 20 & -16 \end{bmatrix}$
   (iii) $A + C$ is not defined. (iv) $ED = \begin{bmatrix} -6 & -10 & -14 \\ -3 & -5 & -7 \end{bmatrix}$
   (v) $AEDB^tC^t = \begin{bmatrix} -9 & -15 & -21 \\ -12 & -20 & -28 \\ 18 & 30 & 42 \end{bmatrix}$. (vi) $C^3 = \begin{bmatrix} 125 & -100 \\ 0 & 0 \end{bmatrix}$.

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2. Let A, B, C and D be matrices which satisfy the following conditions: ABC^2D is defined and is a 3×3 matrix, A has 7 columns, and B has 5 columns. Find the size of each of the matrices A, B, C and D.

   Solution: Note that for C^2 to be defined then C must be square, let its size be $n \times n$. Then for C^2D to be defined then D must be $n \times 3$, since the overall product is 3×3. Also, since B has 5 columns, so for BC^2 to be defined, we must have $m = 5$. Now B must have 7 rows since AB is defined and A has 7 columns. Also, A must have 3 rows since the overall product is 3×3. Putting all of this together we have that A is 3×7, B is 7×5, C is 5×5 and D is 5×3.

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3. Let A be an $m \times n$ matrix and suppose there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $C = D$.

   Solution: Consider the product $CAD = C(AD) = CI_m = C$ (by question 4), but also $CAD = (CA)D = I_nD = D$. Thus $C = D$.

4. * Two square matrices A and B are said to *commute* if $AB = BA$. Find all 2×2 matrices A with commute with *every* 2×2 matrix.

   Hint: let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; choose some very simple matrices B, for instance matrices with most entries equal to 0, and investigate what conditions must be satisfied by a, b, c and d in order to have $AB = BA$.

   Solution: Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with every 2×2 matrix. In particular A commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \implies b = c = 0$$

   Similarly A commutes with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \implies \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ a & 0 \end{bmatrix} \implies a = d$$

   So $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Note that such a matrix commutes with every 2×2 matrix (because such an A is a scalar multiple of the identity matrix I_2). Therefore A commutes with every 2×2 matrix precisely when A is of the form $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for any $a \in \mathbb{R}$.

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5. Let A, B and C be invertible $n \times n$ matrices. Find expressions in terms of A, B, C and their inverses for the inverses of the following matrices. (i) ABC (ii) $AB^{-1}A$ (iii) $3ABC^2$ (iv) $-BA^{-1}CA$

   Solution: (i) $C^{-1}B^{-1}A^{-1}$. (ii) $A^{-1}BA^{-1}$ (iii) $\frac{1}{3}(C^{-1})^2B^{-1}A^{-1}$ (iv) $-A^{-1}C^{-1}AB^{-1}$.

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6. Suppose that A is an invertible $n \times n$ matrix satisfying $A^3 - 3A + 2I = 0$. Find an expression for A^{-1} in terms of A and I.

   Solution: $A(A^2 - 3I) = -2I$ so $A\left(\frac{-1}{2}(A^2 - 3I)\right) = I$ thus $A^{-1} = \frac{-1}{2}(A^2 - 3I)$.
