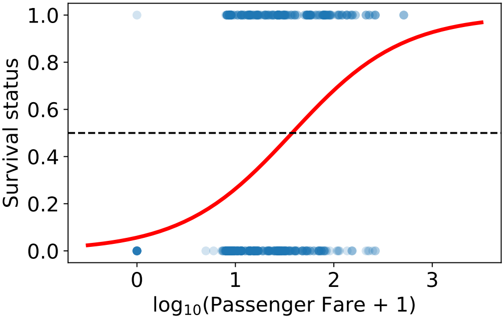
Course outline

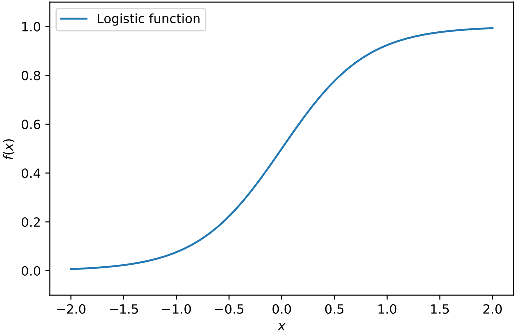
1. Fundamentals
   - Notation
   - Functions
   - Approximation
2. Series
   - Summation
   - Taylor series
3. Linear algebra
   - Representing big, complex, data
   - Systems of equations
   - Dimension reduction
4. Probability
   - Discrete random variables
   - Continuous random variables & integration
5. Optimisation



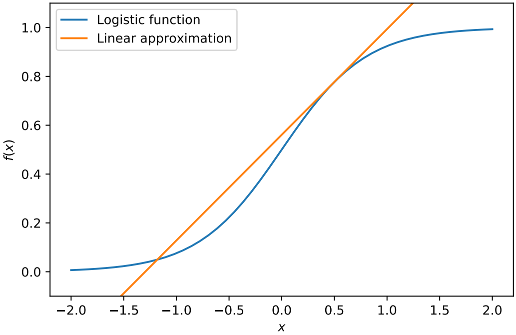
How does a computer represent a logistic (or any non-trivial) function?

$$y = f(x) = \frac{1}{1 + e^{-(ax+b)}} = \frac{e^{ax+b}}{1 + e^{ax+b}}.$$

Linear approximation



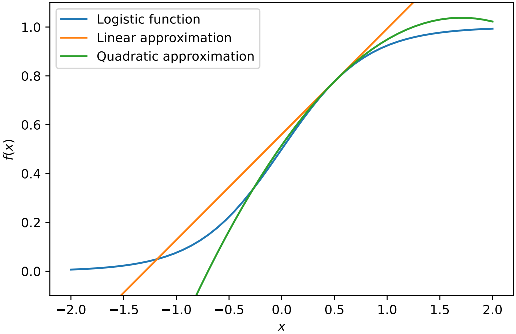
Linear approximation



Orange line is the tangent line at $x_0 = 0.5$:

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

Quadratic approximation



Green line is a quadratic function:

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Taylor polynomials

It turns out we can keep going!

$$
\begin{aligned}
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\
+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
\end{aligned}
$$

And these sums can be made exact if we add an *infinite* number of terms.

Summation notation

If $a_1, a_2, \ldots, a_n$ are real numbers, the sum of $a_1, \ldots, a_n$ is written

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \sum_{i=1}^{n} a_i \\ &= \sum_{k=1}^{n} a_k \\ &= \sum_{q=1}^{n} a_q \\ &= \sum_{x=8}^{n+7} a_{x-7} \\ &= \sum_{1 \leq k \leq n} a_k. \end{aligned}$$

Summation notation

Example

1. $\sum_{i=1}^{4} i^2$
2. $\sum_{i=3}^{6} i$
3. $\sum_{j=0}^{3} 2^j$
4. $\sum_{i=1}^{4} 2$

Summation notation

Example

What is $2^3 + 3^3 + \cdots + n^3$ in sigma notation?

Properties of $\sum$

1. $$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i \quad \text{for } c \text{ a constant}$$

2. $$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

3. $$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

4. But!

$$\sum_{i=m}^{n}(a_i \times b_i) \quad \neq \quad \left(\sum_{i=m}^{n} a_i\right) \times \left(\sum_{i=m}^{n} b_i\right)$$

5. $$\sum_{i=m}^{n}\frac{a_i}{b_i} \quad \neq \quad \frac{\sum_{i=m}^{n} a_i}{\sum_{i=m}^{n} b_i}$$

An important concept!

1. $$\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i \quad \text{for } c \text{ a constant}$$

2. $$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

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Definition (Linear operator)

An operator L is *linear* if for all functions f and g, and every scalar $c \in \mathbb{R}$,

$$L(cf) = cL(f)$$
$$L(f+g) = L(f) + L(g)$$

We will encounter many linear operators in this course!

Some important sums

1. $$\sum_{i=1}^{n} 1 = \underbrace{1 + \cdots + 1}_{n} = n$$

2. Geometric sum:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = a\frac{1 - r^{n+1}}{1 - r}$$

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3. $$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

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4. $$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n}{6}(2n^2 + 3n + 1) = \frac{n}{6}(2n+1)(n+1)$$

Example

$$\sum_{i=3}^{10}(i+2)^2$$

Proof by Induction

How else to prove statements about sums?

One (non-constructive) way is the

Principle of Mathematical Induction

Consider a statement $P(n)$ to be proved.

1. Basis step: show that $P(a)$ is true.
2. Inductive step: assume $P(k)$ is true, use this to prove $P(k+1)$.

Then $P(n)$ is true for all integers $n \geq a$.

Proof by Induction

Example

Prove that:

1. $$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

2. $$\sum_{j=1}^{n} 2^{j-1} = 2^n - 1$$

Induction can be used to prove more results than just about sums, but it's particularly applicable here.

Multiple sums

We might see something like

$$\sum_{1 \le (j,k) \le 3} a_j b_k = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} a_j b_k \right)$$

Multiple sums

Definition (Generalised associativity & distributivity)

1. $$\sum_{j\in J}\sum_{k\in K} a_{j,k} = \sum_{k\in K}\sum_{j\in J} a_{j,k}$$

2. $$\sum_{j\in J,k\in K} a_j b_k = \left(\sum_{j\in J} a_k\right)\left(\sum_{k\in K} b_k\right)$$

Note: these are specific to *finite* sums!

Example

Compute

$$\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j)$$

Example

On which line(s) does the following derivation go wrong?

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} a_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_j}{a_k} \tag{1}$$

$$= \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{a_k}{a_k} \tag{2}$$

$$= \sum_{k=1}^{n} n \tag{3}$$

$$= n^2 \tag{4}$$

Infinite series

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_i + \cdots$$

where the a_i are real numbers.

Infinite series

The N*th partial sum* S_N is the sum of the first N terms

$$S_N = a_1 + a_2 + \cdots + a_N$$

We say the infinite series $\sum_{n=1}^{\infty} a_n$ is *convergent* with *sum* S provided

$$S = \lim_{N\to\infty} S_N.$$

If $\lim\limits_{N\to\infty} S_N$ does not exist we say that the series *diverges*.

Definition (Infinite series)

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

(provided the limit exists.)

Side note: some infinite limits to know

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{x\to\infty} e^{-x} = 0$$

Example

The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example

The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ diverges.

The geometric series

Definition (Geometric series)

The series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

diverges if $|x| > 1$ and converges to $1/(1-x)$ if $|x| < 1$.

How to tell if an infinite series converges or not?

The ratio test

Consider a series $\sum_{n=0}^{\infty} a_n$, with each $a_n \neq 0$, such that

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

either exists or is infinite.
Then

$$\begin{array}{lll} \text{if} & r < 1 & \text{the series converges,} \\ \text{if} & r > 1 & \text{the series diverges,} \\ \text{if} & r = 1 & \text{the ratio test is inconclusive.} \end{array}$$

The ratio test

Example

Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}$ converges.

The ratio test

Example

Find if $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ converges or diverges.