Mathematics for Data Science I
Practice Questions (week 2)

Semester 2, 2018

1. Explain the difference between

   (a) $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$

   (b) $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} a_j$

   (c) $\sum_{n=1}^{i} a_n$ and $\sum_{n=1}^{j} a_n$

   Solution:

   (a) The first of these is $a_1 + a_2 + a_3 + \cdots + a_n$, because the i takes on the values $1, 2, 3, \cdots, n$. The second of these sums is exactly the same thing! The i and j in these two expressions are examples of *dummy variables*; they are actually not in the expressions, but are used as placeholder to indicate that the index goes over the range 1 to n.

   (b) The first of these is $a_1 + a_2 + a_3 + \cdots + a_n$, as explained in (a). However, in the second, we have i taking on the values $1, 2, \cdots, n$, whereas a_j does not depend on this placeholder! Therefore we get for $i = 1$, the term a_j. For $i = 2$, we also get a_j. Thus, we get $a_j + a_j + \cdots + a_j = na_j$, because we get n terms. In this case, the final answer depends on j (there is a genuine j in the expression), but not on i (which is a placeholder).

   (c) The first of these is $a_1 + a_2 + \cdots + a_i$, because n is a dummy variable which runs from 1 to i. In contrast, the second of these is $a_1 + a_2 + \cdots + a_j$. So these two (finite) series are in general different from each other (they are only equal if $i = j$).

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2. Do the following series converge or diverge? Provide explanations.

   (a) $\sum_{n=4}^{\infty} \frac{1}{n}$

   (b) $\sum_{j=2}^{\infty} \frac{3}{j-1}$

(c) $\sum_{n=3}^{\infty} 2^n$

(d) $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

Solution: In all these cases, we note that the beginning index is irrelevant with regards to the question of whether the series converges.

(a) This is the harmonic series (missing its first few terms), which has been shown to diverge in the course notes and in lectures.

(b) We can reindex this sum by setting $m = j - 1$ to get

$$\sum_{j=2}^{\infty} \frac{3}{j-1} = \sum_{m=1}^{\infty} \frac{3}{m} = 3 \sum_{m=1}^{\infty} \frac{1}{m}.$$

This is simply three times the harmonic series, and so it diverges.

(c) This is a geometric series with general term r^n, where $r = 2$. Because $|r| = 2 > 1$, this geometric series diverges.

(d) This is also a geometric series, but here $r = 1/3$. Because $|r| = 1/3 < 1$, this series converges. Indeed, this is a geometric series in the standard form, and therefore it converges to $1/(1-1/3) = 3/2$.

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3. By utilising the fact that you know the sum of a geometric series, find the explicit values of each of the following series.

Note: that sum is:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

if $|x| < 1$.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

(b) $\sum_{n=2}^{\infty} \frac{2^{n+1}}{5^n}$

(c) $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

Solution: For all these, we will use the fact that we can sum a geometric series according to

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for} \quad |r| < 1\,.$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{1-(-1/4)} = \frac{4}{5}.$

(b) We need to be careful in applying the geometric series formula here, because the summation does not start at zero. Consequently, we can 'subtract off' the first two terms corresponding to $n = 0$ and $n = 1$ when evaluating the summation. Thus,

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{2^{n+1}}{5^n} &= \sum_{n=2}^{\infty} 2\left(\frac{2}{5}\right)^n = 2\sum_{n=2}^{\infty}\left(\frac{2}{5}\right)^n \\
&= 2\left[\frac{1}{1-2/5} - \left(\frac{2}{5}\right)^0 - \left(\frac{2}{5}\right)^1\right] \\
&= \frac{8}{15}\,.
\end{aligned}$$

(c) This is not in the form of a geometric series, but we can split the numerator to come up with two different geometric series:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty} \frac{3^{n-1}}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \\
&= \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1} \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m} - \sum_{m=0}^{\infty} \left(\frac{1}{6}\right)^{m} \\
&= \frac{1}{1-1/2} - \frac{1}{1-1/6} = \frac{4}{5}\,,
\end{aligned}$$

where in the middle step, we have 'reindexed' the summations by setting $m = n - 1$; this resulted in geometric series in the standard form.

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4. In this problem, we learn the 'telescoping sum' process to determine certain types of infinite sums.

(a) For the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, confirm that the general term $a_n = \displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

(b) Hence, observe that the partial sums s_m obey

$$
\begin{aligned}
s_m &:= \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m} \right) + \left(\frac{1}{m} - \frac{1}{m+1} \right) \\
&= 1 - \frac{1}{m+1}
\end{aligned}
$$

because the intermediate terms cancel with adjacent terms (the sum collapses like a telescope). By determining $\lim\limits_{m\to\infty} s_m$, determine $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(c) * Use this telescoping sum idea to determine the sum $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$, by considering

$$\frac{1}{n-1} - \frac{1}{n+1}.$$

(d) * Use this idea to also determine the sum $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

Solution:

(a) Making the expression on the right into one fraction we get

$$a_n = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)},$$

as required.

This can also be done using *partial fractions*: Setting $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$, we get $1 = A(n+1) + Bn$. Setting $n = 0$ gives $1 = A$, and setting $n = -1$ gives $1 = -B$. Hence $A = 1$ and $B = -1$, and the partial fraction decomposition is as stated.

(b) After the sum 'telescopes,' we get $s_m = 1 - 1/(m+1)$. Clearly, $\lim_{m\to\infty} s_m = 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n(n+1)} = \lim_{m\to\infty} s_m = 1\,.$$

(c)

$$
\begin{aligned}
s_m &:= \sum_{n=2}^{m} \frac{2}{n^2-1} = \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\
&= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots \\
&\quad + \left(\frac{1}{m-3} - \frac{1}{m-1} \right) + \left(\frac{1}{m-2} - \frac{1}{m} \right) + \left(\frac{1}{m-1} - \frac{1}{m+1} \right).
\end{aligned}
$$

The telescoping here is not quite adjacent terms, but terms which are separated by two terms. So $-1/3$ cancels with $1/3$, $-1/4$ with $1/4$, etc. This leaves the terms $1/1$ and $1/2$ with no counterparts to cancel with. Similarly, it leaves the terms $-1/m$ and $-1/(m+1)$ in the final two terms with no cancellation. Thus

$$s_m = \frac{1}{1} + \frac{1}{2} - \frac{1}{m} - \frac{1}{m+1}.$$

It is clear that $\lim_{m\to\infty} s_m = 1 + 1/2 = 3/2$. Thus

$$\sum_{n=1}^{\infty} \frac{2}{n^2-1} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{2}{n^2-1} = \lim_{m\to\infty} s_m = \frac{3}{2}.$$

(d) The relevant partial fractions decomposition would be $\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$, and hence $3 = A(n+3) + Bn$. Setting $n = 0$, we get $A = 1$. Setting $n = -3$, we get $B = -1$. Thus, we have

$$\begin{aligned} s_m &:= \sum_{n=1}^{m} \frac{3}{n(n+3)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+3}\right) \\ &= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots \\ &+ \left(\frac{1}{m-3} - \frac{1}{m}\right) + \left(\frac{1}{m-2} - \frac{1}{m+1}\right) + \left(\frac{1}{m-1} - \frac{1}{m+2}\right) + \left(\frac{1}{m} - \frac{1}{m+3}\right). \end{aligned}$$

The cancellations are a little more difficult to spot. The first negative term $(-1/4)$ will cancel with a corresponding positive term in a term which is 'three steps away' from it. Thus, except for some terms in the first three and last three of the terms in parentheses, all the terms cancel. This leaves us with

$$s_m = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{m+1} - \frac{1}{m+2} - \frac{1}{m+3}.$$

Thus, $\lim_{m\to\infty} s_m = 1 + (1/2) + (1/3) = 11/6$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n(n+3)} = \lim_{m\to\infty} s_m = \frac{11}{6}.$$

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5. Do the following series converge or diverge? Provide explanations.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(1 + \frac{2}{n^2}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{2n^2+n+7}$

Solution: In all these cases, we note that the beginning index is irrelevant with regards to the question of whether the series converges.

(a) This series has a general term $a_n = 1/n^p$, where $p = 1/2$. By the p-series test in your notes, this series diverges.

(b) This has a general term $a_n = 1+2/n^2$. As $n \to \infty$, $a_n \to 1$. Now, a necessary condition for convergence is that $a_n \to 0$. Thus, this series does not converge.

(c) Here,
$$a_n = \frac{n^2-1}{2n^2+n+7} = \frac{1-1/n^2}{2+1/n+7/n^2} \to \frac{1}{2}$$
as $n \to \infty$. By the same reason as for the previous problem, this series is divergent.

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6. Apply the ratio test to each of the following series to investigate convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{R}$.

Solution: We recall that the ratio test for convergence of a series $\sum_{n=1}^{\infty} a_n$ relates to the quantity
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$
if it exists. If $L < 1$, the series is absolutely convergent. If $L > 1$, the series is divergent, and if $L = 1$, the test is inconclusive. (This is Property 2.5.) We will apply this to each given series.

(a) In this case, we have
$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^n(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right| = \left| \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3} \right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3.$$
Because in this case $L = \lim_{n\to\infty}(1/3)(1+1/n) = 1/3 < 1$, this series is absolutely convergent.

(b) Recall that $n! = (1)(2)(3)\cdots(n-1)(n)$. So in this case, we have that $a_n = (2n)!/(n!n!)$. So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{(n+1)!(n+1)!}\frac{n!n!}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \to 4$$

as $n \to \infty$. Since $4 > 1$, this series is divergent.

(c) This is the p-series, whose behaviour we already know. However, in this problem you are asked to use the ratio test on it. So we get

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p = \left(\frac{1}{1+1/n}\right)^p.$$

In the limit $n \to \infty$, this goes to 1^p, which is equal to 1 for any $p \in \mathbb{R}$. Thus, the ratio test is *inconclusive* for the p-series.