Tutorial 4 – MATH 4043

1. Suppose a balanced die is tossed once. Consider the bivariate random variable $Y = (Y_1, Y_2)$ where $Y_1 = 0$ if the number on the top face is even and $Y_1 = 1$ if odd, and $Y_2 = 0$ if the number on the top face is 1, 2, or 3, and $Y_2 = 1$ if 4, 5, or 6.

   (a) What is the sample space?

   (b) Show the mapping of the sample points to the various possible values of $Y = (Y_1, Y_2)$.

   (c) Tabulate the joint probability mass function of Y and the marginal probability mass function of its individual components.

   (d) Are Y_1 and Y_2 independent?

   (e) Tabulate the conditional probability mass function $p_{Y_2|Y_1}(y_2|1) = P[Y_2 = y_2|Y_1 = 1]$ over the range of Y_2.

   (f) Calculate $\mathrm{Cov}[Y_1, Y_2]$.

2. The bivariate random variable $X = (X_1, X_2)$ have joint probability mass function given in the table below

$X_2 \setminus X_1$	0	1	2
0	0.1	0.3	0.1
1	0.2	0.1	0.2

   (a) Calculate the marginal probability mass function of each of the components X_1 and X_2.

   (b) Are the components independent?

   (c) Tabulate the conditional probability mass function

$$p_{X_1|X_2}(x_1|0) = P[X_1 = x_1|X_2 = 0]$$

   over the range of X_1.

   (d) Calculate $\mathrm{Cov}[X_1, X_2]$.

3. Consider a coin tossing experiment where a biased coin is tossed 2 times. Assume that the tosses are independent of one another and that the probability of getting a head is 0.75. Let $X = (X_1, X_2)$ where X_1 counts the number of heads in the 2 tosses and X_2 counts the number of tails in the 2 tosses.

(a) Write down the sample space and show the mappings of the sample points to the various values of $X = (X_1, X_2)$.

(b) Find the marginal probability mass function of the individual components.

(c) Are they independent?

(d) Calculate $\text{Cov}[X_1, X_2]$.

4. (Employment dynamics). A researcher is interested in modelling the employment dynamics of young people using a Markov chain. She determines that at age 18 a person is either a student with probability 0.9 or an intern with probability 0.1. Each year that passes is denoted as one time period. After that she estimates the following transition probabilities:

$$\begin{array}{cc} \begin{array}{cccc} \text{Student} & \text{Intern} & \text{Employed} & \text{Unemployed} \end{array} & \\ \begin{pmatrix} 0.8 & 0.5 & 0 & 0 \\ 0.1 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0 & 0 & 0.1 & 0.6 \end{pmatrix} & \begin{array}{c} \text{Student} \\ \text{Intern} \\ \text{Employed} \\ \text{Unemployed} \end{array} \end{array}$$

(a) Write down the initial probability vector and the transition matrix for the problem.

(b) The first two states, student and intern are called transient states. Show that the probability of revisiting those states after visiting the third state, employed is zero.

(c) Calculate the probability that if someone starts off as a student (state 1) then after 4 years they will be employed.

Solutions.

1. Suppose a balanced die is tossed once. Consider the bivariate random variable $Y = (Y_1, Y_2)$ where $Y_1 = 0$ if the number on the top face is even and $Y_1 = 1$ if odd, and $Y_2 = 0$ if the number on the top face is 1, 2, or 3, and $Y_2 = 1$ if 4, 5, or 6.

   Solution.

   (a) What is the sample space?
   $S = \{1, 2, 3, 4, 5, 6\}$

   (b) Show the mapping of the sample points to the various possible values of $Y = (Y_1, Y_2)$.
   In the table below, first column gives the subset of S, 2nd column is where it maps to on $\mathbb{R}^2$ and 3rd column the corresponding joint probability mass function.

		$p_Y(y_1, y_2)$
$\{2\}$	$\{Y_1 = 0, Y_2 = 0\}$	$\frac{1}{6}$
$\{4, 6\}$	$\{Y_1 = 0, Y_2 = 1\}$	$\frac{1}{3}$
$\{1, 3\}$	$\{Y_1 = 1, Y_2 = 0\}$	$\frac{1}{3}$
$\{5\}$	$\{Y_1 = 1, Y_2 = 1\}$	$\frac{1}{6}$

   (c) Tabulate the joint probability mass function of Y and the marginal probability mass function of its individual components.

$Y_2 \backslash Y_1$	0	1	$p_{Y_2}(y_2)$
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
$p_{Y_1}(y_1)$	$\frac{1}{2}$	$\frac{1}{2}$	

   Both Y_1 and Y_2 have the same marginal probability mass function.

   (d) Are Y_1 and Y_2 independent?
   No. For example check $p_Y(0, 0) = \frac{1}{6}$, but $p_{Y_1}(y_1) \times p_{Y_2}(y_2) = \frac{1}{2} \times \frac{1}{2} \neq \frac{1}{6}$.

   (e) Tabulate the conditional probability mass function $p_{Y_2|Y_1}(y_2|1) = P[Y_2 = y_2|Y_1 = 1]$ over the range of Y_2.
   Note that $p_{Y_2}(y_2|1) = \frac{p_Y(1,y_2)}{p_{Y_1}(1)}$.
   Thus $p_{Y_2}(0|1) = \frac{p_Y(1,0)}{p_{Y_1}(1)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$ and
   $p_{Y_2}(1|1) = \frac{p_Y(1,1)}{p_{Y_1}(1)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$.

   (f) Calculate $\mathrm{Cov}[Y_1, Y_2]$.

We can calculate $E[Y_1]$ and $E[Y_2]$ first. note that since they have the same marginal probability mass functions, they must have the same mean, so

$$E[Y_i] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}, \text{ for } i = 1, 2.$$

Now to calculate $E[Y_1Y_2]$. First we need to tabulate the values of $y_1 \times y_2$ in the table below.

$Y_2 \backslash Y_1$	0	1
0	0	0
1	0	1

As all but 1 entry is zero, then we have $E[Y_1Y_2] = 0 + 0 + 0 + 1 \times \frac{1}{6} = \frac{1}{6}$.
Hence $\text{Cov}[Y_1, Y_2] = E[Y_1Y_2] - E[Y_1]E[Y_2] = \frac{1}{6} - \frac{1}{2} \times \frac{1}{2} = -\frac{1}{12}$.

2. The bivariate random variable $X = (X_1, X_2)$ have joint probability mass function given in the table below

$X_2 \setminus X_1$	0	1	2
0	0.1	0.3	0.1
1	0.2	0.1	0.2

Solution.

(a) Calculate the marginal probability mass function of each of the components X_1 and X_2.
Add across rows and down the columns to get the marginal pmf.

$X_2 \setminus X_1$	0	1	2	$p_{X_2}(x_2)$
0	0.1	0.3	0.1	0.5
1	0.2	0.1	0.2	0.5
$p_{X_1}(x_1)$	0.3	0.4	0.3	

(b) Are the components independent?
No. For example, $p_X(1, 1,) = 0.1$ but $p_{X_1}(1) \times p_{X_2}(1) = 0.4 \times 0.5 = 0.2$.

(c) Tabulate the conditional probability mass function
$p_{X_1|X_2}(x_1|0) = P[X_1 = x_1|X_2 = 0]$ over the range of X_1.
Note that $p_{X_1|X_2}(x_1|0) = \frac{p_X(x_1,0)}{p_{X_2}(0)} = \frac{p_X(x_1,0)}{0.5}$. Thus making use of the table in the solution to part (a), we get

x_1	0	1	2
$p_{X_1\|X_2}(x_1\|0)$	0.2	0.6	0.2

(d) Calculate $\mathrm{Cov}[X_1, X_2]$.

We calculate $E[X_1]$ and $E[X_2]$ first.

$$E[X_1] = 0 \times 0.3 + 1 \times 0.4 + 2 \times 0.3 = 1 \text{ and } E[X_2] = 0 \times 0.5 + 1 \times 0.5 = 0.5.$$

To calculate $E[X_1X_2]$, we need to tabulate the values of X_1X_2 as in the table below first.

$X_2 \setminus X_1$	0	1	2
0	0	0	0
1	0	1	2

So $E[X_1X_2] = 1 \times 0.1 + 2 \times 0.2 = 0.5$. Thus

$$\mathrm{Cov}[X_1, X_2] = 0.5 - 1 \times 0.5 = 0.$$

3. Consider a coin tossing experiment where a biased coin is tossed 2 times. Assume that the tosses are independent of one another and that the probability of getting a head is 0.75. Let $X = (X_1, X_2)$ where X_1 counts the number of heads in the 2 tosses and X_2 counts the number of tails in the 2 tosses.

Solution

(a) Write down the sample space and show the mappings of the sample points to the various values of $X = (X_1, X_2)$.

The sample space is $S = \{TT, TH, HT, HH\}$

The mappings can be summarised in the table below

Event in sample space	Point in $\mathbb{R}^2$	$p_X(x_1, x_2)$
$\{TT\}$	$\{X_1 = 0, X_2 = 2\}$	$\frac{1}{16}$
$\{TH, HT\}$	$\{X_1 = 1, X_2 = 1\}$	$\frac{3}{8}$
$\{HH\}$	$\{X_1 = 2, X_2 = 0\}$	$\frac{9}{16}$

(b) Find the marginal probability mass function of the individual components.

Using the joint probabilities from part (a) and tabulating in another form so that we can sum out rows and columns to get the marginal probability mass functions, we get:

$X_2 \setminus X_1$	0	1	2	$p_{X_2}(x_2)$
0	0	0	$\frac{9}{16}$	$\frac{9}{16}$
1	0	$\frac{3}{8}$	0	$\frac{3}{8}$
2	$\frac{1}{16}$	0	0	$\frac{1}{16}$
$p_{X_1}(x_1)$	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{9}{16}$	

(c) Are they independent?

No. For example $p_X(1,1) = \frac{3}{8}$, but $p_{X_1}(x_1) \times p_{X_2}(x_2) = \left(\frac{3}{8}\right)^2$, which are clearly not equal.

(d) Calculate $\mathrm{Cov}[X_1, X_2]$.

Standard calculations for mean will give $E[X_1] = 1.5$ and $E[X_2] = 0.5$

Now for $E[X_1 X_2]$, it turns out that the only non-zero term in the double sum is $1 \times 1 \times p_X(1,1)$, as all others either have zero probability or $x_1 x_2 = 0$. Thus

$$E[X_1 X_2] = 1 \times 1 \times \frac{3}{8} = \frac{3}{8}.$$

Therefore

$$\mathrm{Cov}[X_1, X_2] = E[X_1 X_2] - E[X_1]E[X_2] = \frac{3}{8} - \frac{3}{2} \times \frac{1}{2} = -\frac{3}{8}.$$

4. (Employment dynamics). A researcher is interested in modelling the employment dynamics of young people using a Markov chain. She determines that at age 18 a person is either a student with probability 0.9 or an intern with probability 0.1. Each year that passes is denoted as one time period. After that she estimates the following transition probabilities:

$$\begin{array}{c c}
\begin{array}{cccc} \text{Student} & \text{Intern} & \text{Employed} & \text{Unemployed} \end{array} & \\
\begin{pmatrix} 0.8 & 0.5 & 0 & 0 \\ 0.1 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0 & 0 & 0.1 & 0.6 \end{pmatrix} & \begin{array}{c} \text{Student} \\ \text{Intern} \\ \text{Employed} \\ \text{Unemployed} \end{array}
\end{array}$$

(a)

$$p_0 = \begin{pmatrix} 0.9 \\ 0.1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 0.8 & 0.5 & 0 & 0 \\ 0.1 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0 & 0 & 0.1 & 0.6 \end{pmatrix}$$

(b) The first two states, student and intern are called transient states. Show that the probability of revisiting those states after visiting the third state, employed is zero.

(c) Calculate the probability that if someone starts off as a student (state 1) then after 4 years they will be employed.