7. There are 3 fuses A, B and C and a light bulb D arranged in a circuit as shown in the diagram below. The light bulb and battery work all the time. The probability of any of the fuses failing is 0.05. Assume all components work independently of each other.

   Solution. Let

$$
\begin{aligned}
A &= \{\text{ Fuse A works }\}\\
B &= \{\text{ Fuse B works }\}\\
C &= \{\text{ Fuse C works }\}\\
L &= \{\text{ There is light }\} = (A \cap B) \cup C.
\end{aligned}
$$

   Note that all fuses work independently of one another.

   (a) What is the probability that the bulb is lighted up?

   From the diagram of the circuit, there will only be light if current can flow through the branch containing fuses A and B, or the branch containing fuse C, or both. Thus

$$
\begin{aligned}
P(L) &= P[(A \cap B) \cup C]\\
&= P(A \cap B) + P(C) - P[(A \cap B) \cap C]\\
&= P(A)P(B) + P(C) - P(A)P(B)P(C) \text{ (because of independence)}\\
&= 0.95 \times 0.95 + 0.95 - 0.95 \times 0.95 \times 0.95 = 0.995125.
\end{aligned}
$$

   (b) What is the probability that the bulb is lighted up given that fuse A has failed?

   We can to calculate $P(L|\bar{A})$. Note that from the circuit
   $L \cap \bar{A} = (\bar{A} \cap B \cap C) \cup (\bar{A} \cap \bar{B} \cap C) = \bar{A} \cap C$.
   It does not matter whether fuse B works or not but fuse C must work. Therefore

$$
\begin{aligned}
P(L|\bar{A}) &= \frac{P(L \cap \bar{A})}{P(\bar{A})}\\
&= \frac{P(\bar{A} \cap C)}{P(\bar{A})}\\
&= \frac{P(\bar{A})P(C)}{P(\bar{A})} = P(C) = 0.95.
\end{aligned}
$$

(c) What is the probability that the bulb is lighted up given that fuse C has failed?
We want to calculate $P(L|\bar{C})$. Now
$L \cap \bar{C} = (A \cap B) \cap \bar{C}$
since both fuses A and B must be working if C fails, in order to have light. Therefore

$$\begin{aligned} P(L|\bar{C}) &= \frac{P(L \cap \bar{C})}{P(\bar{C})} \\ &= \frac{P[(A \cap B) \cap \bar{C}]}{P(\bar{C})} \\ &= \frac{P(A \cap B)P(\bar{C})}{P(\bar{C})} \\ &= P(A \cap B) = P(A)P(B) = 0.95 \times 0.95 = 0.9025. \end{aligned}$$

(d) What is the probability that fuse C has failed given that the bulb is lighted up?
This is $P(\bar{C}|L)$. Hence

$$\begin{aligned} P(\bar{C}|L) &= \frac{P(L \cap \bar{C})}{P(L)} \\ &= \frac{P[(A \cap B) \cap \bar{C}]}{P(L)} \text{ (See the circuit diagram)} \\ &= \frac{P(A)P(B)P(\bar{C})}{P(L)} \\ &= \frac{0.95 \times 0.95 \times 0.05}{0.995125} = \frac{361}{7961}. \end{aligned}$$