Tutorial 11 – MATH 4043 SP2 2018

1. Refer to Question 3, Tutorial Week 8.
   The number of visitors clicking an online shopping website is distributed as Poisson with a mean visit rate of 7 per hour. Any individual visitor who goes onto the website has a 65% chance of making a purchase. Assume each individual visitor acts independently of each other, and assume that the visitor only makes 1 click to the website within that hour. If the ith visitor makes a purchase, the amount spent Y_i is independently distributed as exponential $\text{Exp}(\lambda_e)$ with $\lambda_e = 0.05$. Simulate the number of visitors in a two hour period, the number from these visitors who make a purchase, and if they make a purchase, their purchase amounts.

2. Refer to Q1 above.
   Suppose now that if the ith visitor makes a purchase, the log of the amount spent $\ln(Y_i)$ is normally distributed as $N(3, 2)$. Simulate the number of visitors in a two hour period, the number from these visitors who make a purchase, and if they make a purchase, their purchase amounts.

3. Consider a drunk man walking on a straight line. He starts off at point 0. For each step that he takes, the length X_i of his ith stride is distributed as continuous uniform $U[-1, 1]$. We assume that each stride is independent of each other and a negative value means a step backwards. Let $S_t = \sum_{i=1}^{t} X_i$ be his position at time t. Simulate his first 30 steps and plot his path.

4. Suppose the daily log returns of a stock is distributed as normal with annual drift 0.2 and annual volatility 75. Using the Black-Scholes-Merton Geometric Brownian motion model for stock price
   $$S_t = S_0 e^{\mu t + \sigma B_t},$$
   with $S_0 = 2$, simulate a stock price path for a 3 year period (356×3 days).

5. Suppose $B(t)$ is a standard Brownian Motion.
   (a) Find the probability that $0 < B(t) < 1$.
   (b) Find the probability that $B(6) \leq 4$ given that $B(1) = 1$.

6. Let Z be a Normally distributed random variable with mean 0 and variance 1, such that $Z \sim N(0, 1)$. Consider the stochastic process $G(t) \sim \sqrt{t}Z$.
   (a) By using **R**, generate values for $G(t)$ for $t = 1, 2, ..., 5$.

(b) Show that $G(t)$ is not a Brownian Motion by using the formal definition.

7. Suppose you are high frequency trading of stocks, if the current price of the stock is 5% higher than the purchase price then you will buy. Suppose that a stock follows a Standard Geometric Brownian Motion lifted by the current market price of \$20? What is the probability that the stock price is ready to buy after T units of time later? Evaluate for for multiple values of T and the plot the distribution with respect to T.

Tutorial 11 Solutions – MATH 4043

1. Refer to Question 3, Tutorial Week 8.
   The number of visitors clicking an online shopping website is distributed as Poisson with a mean visit rate of 7 per hour. Any individual visitor who goes onto the website has a 65% chance of making a purchase. Assume each individual visitor acts independently of each other, and assume that the visitor only makes 1 click to the website within that hour. If the ith visitor makes a purchase, the amount spent Y_i is independently distributed as exponential $\text{Exp}(\lambda_e)$ with $\lambda_e = 0.05$. Simulate the number of visitors in a two hour period, the number from these visitors who make a purchase, and if they make a purchase, their purchase amounts.
   Solution. Let N be the number of visitors and X be the number of visitors that make a purchase. Note that there are several ways in which you can set up or organise your simulation.

```
> N <- rpois(1,7)
> N
[1] 4
> X <- rbinom(N,1,0.65)
> X
[1] 0 1 0 1
> Y <- rgamma(N,1,0.05)
> Y
[1]  4.448685  8.826599  3.917239 27.486900
> X*Y
[1]  0.000000  8.826599  0.000000 27.486900
> sum(X*Y)
[1] 36.3135
```

   In my case, I have 4 visits, 2 of which end up with a purchase, namely the 2nd and 4th visitors. $X * Y$ blanks out the dollar amounts of the 4.448685 and 3.917239, since they are not needed as the 1st and 3rd customers did not make any purchases. The total amount spent by the visitors is $33.3135.

2. Refer to Q1 above.
   Suppose now that if the ith visitor makes a purchase, the log of the amount spent $\ln(Y_i)$ is normally distributed as $N(3, 2)$. Simulate the number of visitors in a two hour period, the number from these visitors who make a purchase, and if they make a purchase, their purchase amounts.
   Solution.

```
> N <- rpois(1,7)
> N
[1] 7
> X <- rbinom(N,1,0.65)
> X
[1] 1 1 1 0 0 1 1
> Y <- rnorm(N,3,2)
> Y
[1] 4.974824 5.449090 5.830086 1.957981 6.927325 3.700344 3.171369
> Y <- exp(Y)
> Y
[1]  144.72335  232.54640  340.38795    7.08501 1019.76255   40.46122   23.84010
> X*Y
[1] 144.72335 232.54640 340.38795   0.00000   0.00000  40.46122  23.84010
> sum(X*Y)
[1] 781.959
```

In my case, I have 7 visits, 5 of which end up with a purchase. $X * Y$ blanks out the dollar amounts of those visitors who did not make any purchases. The total amount spent by the visitors is $781.959.

3. Consider a drunk man walking on a straight line. He starts off at point 0. For each step that he takes, the length X_i of his ith stride is distributed as continuous uniform $U[-1, 1]$. We assume that each stride is independent of each other and a negative value means a step backwards. Let $S_t = \sum_{i=1}^{t} X_i$ be his position at time t. Simulate his first 30 steps and plot his path.

   Solution.

```
> x <- runif(30,-1,1)
> x
 [1] -0.80905080 -0.64458524  0.19193057 -0.54704420  0.20171838  0.55522137
 [7] -0.71243902  0.68010013 -0.57144244 -0.36232702  0.85162975  0.75256204
[13] -0.06818395 -0.03725815  0.85887024 -0.40325540  0.33174796  0.02106271
[19]  0.64785444 -0.30181538 -0.08235259 -0.12395346  0.14933213 -0.56677161
[25]  0.78981634  0.98547851 -0.02709574 -0.57275557  0.67655929 -0.46596014
> s <- cumsum(x)
> s <- c(0,s)
> t <- seq(0,30)
> plot(t,s,"l")
```

4. Suppose the daily log returns of a stock is distributed as normal with annual drift 0.2 and annual volatility 75. Using the Black-Scholes-Merton Geometric Brownian motion model for stock price

$$S_t = S_0 e^{\mu t + \sigma B_t},$$

with $S_0 = 2$, simulate a stock price path for a 3 year period (356×3 days). **Solution**. Note that the model is the same as

$$S_t = S_0 e^{X_t},$$

where $X_t \sim N(\mu t, \sqrt{t}\sigma)$ where μ and σ must be converted to the daily mean and daily volatility figures respectively.

```
> S0 <- 2
> N <- 365
> mu <- 0.2
> sigma <- 75
> X <- rnorm(N*3, mu/N, sigma/N)
> X <- c(0,X)
> X <- cumsum(X)
> St <- S0*exp(X)
> t <- seq(0,N*3)
> plot(t,St,"l")
```