Probability & Data (MATH 4043)
Week 3 Exercises

1. Toss a coin 4 times. Let A = "at least three heads" and B = "first toss is tails".

    a. What is P(A | B)?

    b. What is P(B | A)?

2. (Independence & Bayes) There are one thousand cats on the UniSA campus which are calm and friendly. But ten of them are pure evil. An engineering student and data scientist student develops an app to identify the evil cats. As you are the data scientist, you want to test the reliability of the alarm app by conducting trails.
    1. When encountering an evil cat, the alarm goes off 95% of the time.
    2. When encountering a friendly cat, the alarm goes off 1% of the time.

    a. If a cat sets off the alarm, what is the probability that it is evil?

    b. Should you as a data scientist support and believe the system is reliable?

3. (Conditional Probability) Your cousin Marvin from Adelaide always complains about taxis in New York. From his many visits to JFK, he has calculated that

$$P(taxi\ |rain) = 0.1, P(taxi \mid no\ rain) = 0.6,$$

where *taxi* denotes the event of finding a free taxi after picking up your luggage.

From lecture notes, we know that

$$P(rain) = 0.2, \qquad P(no\ rain) = 0.8,$$
$$P(late \mid rain) = 0.75, \quad P(late|no\ rain) = 0.125.$$

Given the events rain and no rain, it is reasonable to model the events plane arrived late and taxi as conditionally independent,

$$P(taxi \cap\ late|rain) = \ P(taxi\ |rain)\ P(late\ |rain)$$

$$P(taxi \cap late|no\ rain) = P(taxi\ |no\ rain)\ P(late\ |no\ rain)$$

The logic behind this is that the availability of taxis after picking up your luggage depends on whether it's raining or not, but not on whether the plane is late or not (we assume that availability is constant throughout the day).

a) Calculate the $P(taxi)$ and $P(taxi\ |late)$. Does this assumption imply that the events are independent?

4. $P(A \cap B \cap C) = P(A \mid B \cap C)\ P(B \mid C)\ P(C)$

5. Show that $P(A^C\ |B) = 1 - P(A\ |B)$ assuming that $P(B) > 0$.

6. The table below gives the probabilities of events A and B. k is a positive constant which you need to determine.

	B	$\bar{B}$	
A	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$
$\bar{A}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
	$\frac{3}{8}$	$\frac{5}{2}$	

Determine whether these events A and B independent?

7. (Bayes) For a test for a certain type of cancer, there is a probability of 0.02 that the test result is positive if it is randomly applied to a randomly chosen person on the street. The probability that a person actually has that type of cancer if tested positive is 0.98 and the probability that the person actually does not have that type of cancer if tested negative is 0.95. If I randomly pick a person and apply the test, what is the probability that the person actually has that particular cancer?

8. There are 3 fuses A, B and C and a light bulb D arranged in a circuit as shown in the diagram below. The light bulb and battery work all the time. The probability of any of the fuses failing is 0.05. Assume all components work independently of each other.

(a) What is the probability that the bulb is lighted up?

(b) What is the probability that the bulb is lighted up given that fuse A has failed?

(c) What is the probability that the bulb is lighted up given that fuse C has failed?

(d) What is the probability that fuse C has failed given that the bulb is lighted up?

Solutions

Question 1.

Define the sample (outcomes) first:

$$S = \{ HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, HTTT, TTTH, TTTT, THHT, TTHT, THTT, THTH, HTHT\}$$

The outcome space of A, B have the following outcomes:

$$A \coloneqq \{HHHH, HHHT, HHTH, HTHH, THHH\},$$
$$B \coloneqq \{THHH, TTHH, TTTH, TTTT, THHT, TTHT, THTT, THTH\}$$

Therefore, the conditional probabilities are

$$P(A \mid B) = \frac{\frac{1}{16}}{\frac{1}{2}} = \frac{1}{8} \qquad and \qquad P(B \mid A) = \frac{\left(\frac{1}{16}\right)}{\frac{5}{16}} = \frac{1}{5}$$

Question 2.

a) We are interested in $P(Evil \mid Alarm)$. We know the following information from the question,

$$P(Evil) = \frac{1}{100} = 0.01, \qquad P(Nice) = \frac{99}{100} = 0.99, \qquad P(Alarm \mid Evil) = 0.95,$$
$$P(Alarm \mid Nice) = 0.01$$

By Bayes Theorem, we have

$$P(Evil \mid Alarm) = \frac{(P(Alarm \mid Evil) \times P(Evil))}{P(Alarm \mid Evil) \times P(Evil) + P(Alarm \mid Nice)P(Nice)}$$
$$= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.01 \times 0.99} = 0.489$$

b) The probability given the alarm goes off when it is evil is too low, you would be expecting a higher certainty in such a situation.

Question 3.

P (taxi) = P (taxi,rain) + P (taxi, no rain) (by the law of total probability)
= P (taxi|rain) P (rain) + P (taxi|no rain) P (no rain)

$= 0.1 \cdot 0.2 + 0.6 \cdot 0.8 = 0.5,$

$$\mathrm{P}\ (\mathrm{taxi}|\mathrm{late}) = \frac{\mathrm{P}(\mathrm{taxi} \cap \mathrm{late})}{P(late)} = \frac{P\ (late \mid taxi)\ P(taxi)}{P(late)}$$

But

$$P(late) = P(late|rain)P(rain) + P(late\ |no\ rain)\ P(no\ rain)$$
$$= 0.75 * 0.2 + 0.125 * 0.8 = 0.25.$$

And

$$P(late \cap taxi) = P(late \cap taxi \mid rain)\ P(rain) + P(late \cap taxi\ |no\ rain)\ P(no\ rain)$$
$$= (\ 0.1\ \cdot\ 0.75\ ) \cdot\ 0.2\ + (0.6\ \cdot\ 0.125\ ) \cdot\ 0.8\ =\ 0.075$$
$$P(taxi\ |late\ ) = \frac{0.075}{0.25} = 0.3$$

$\mathrm{P}\ (\mathrm{taxi}) \neq \mathrm{P}\ (\mathrm{taxi}|\mathrm{late})$, so the events are not independent.

Question 4.

By the definition of conditional probability, the RHS

$$P(A \mid B \cap C)\ P(B \mid C)P(C) = P(A\ |B \cap C\ )\ \ P(B \cap C)$$
$$= [P(A \cap B \cap C)/\ P(B \cap C)\ ]\ P(B \cap C)$$
$$= P(A \cap B \cap C)$$

Question 5.

By using the definition of conditional probability or Venn diagrams then

$$P(A^c \mid B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - P(A\ |B)$$

Question 6.

We are required to prove $P(A \mid B) = P(A)$ if A is independent of B.

$$\frac{1}{2} = P(A) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

This is impossible and thus A and B are dependent.

Question 7.

Continuous Assessment (solution online)

Question 8.

Fuse solution online under tutorial questions ☺