Tutorial 9 – MATH 4043

This tutorial contains **a** continuous assessment item.

1. This question deals with the discretisation of an exponentially distributed random variable. Let X be distributed as exponential with $\lambda = 5$. Define another random variable Y with probability mass function $p_Y(y) = P[Y = y]$ such that

$$P[Y = y] = P[y - 1 < X \leq y] \text{ for } y = 1, 2, \cdots$$

   (a) Obtain an expression for $p_Y(y)$ in terms of y and simplify.

   (b) Identify the distribution of Y from its probability mass function.

2. Let X be distributed as exponential $\text{Exp}(\lambda)$.

   (a) Show that $P[X > 7|X > 2] = P[X > 5]$.

   (b) Let $a > 0$ and $b > 0$. Show that

$$P[X > a + b|X > a] = P[X > b].$$

   This is known as the memoryless property of the exponential random variable.

3. Let Y be a binomially distributed random variable where $Y \sim \text{Bin}(100, 0.45)$. Let X be a normally distributed random variable such that $\mu = E[X] = E[Y]$ and $\sigma^2 = \text{Var}[X] = \text{Var}[Y]$. You may use **R** to assist you in your calculations.

   (a) Calculate $P[Y = 45]$ and $P[44.5 \leq X \leq 45.5]$. Note that because X is a continuous random variable, we cannot calculate $P[X = 45]$ since it will be zero.

   (b) Calculate $P[40 \leq Y \leq 45]$ and $P[39.5 \leq X \leq 45.5]$.

   (c) What do you observe in your calculations?

   This is known as the Normal approximation to the binomial. It works reasonably well when n is large and p is neither too small or too large.

4. There is another inequality that is useful for a non-negative valued random variable X. This is called Markov's inequality and it is

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

   Let X be exponential with $\lambda = 2$.

   (a) Calculate $P[X \geq 2]$.

   (b) Compare with the Markov inequality.

   (c) Calculate $P[X < 1]$.

   (d) Compare with the Markov inequality.

   (e) Challenge problem: Prove Markov's inequality. You can assume that X is any continuous random variable.

5. . Let X be distributed as Poisson with $\lambda = 16$.

   (a) Calculate $P[6 \leq X \leq 26]$.

   (b) Compare your result with an appropriate use of Chebyshev's Inequality.

6. **Continuous Assessment Item**. Suppose the waiting times (in hours) for a customer arriving in a queue shop (timed immediately after the previous customer or after time 0 in the case of the very 1st customer) is distributed as exponential Exp(6). Use **R** to simulate the arrival times of the customers within a 2 hour period. Suppose you ended up with k customers arriving in the 2 hour period, use **R** to calculate $P[S_k \leq 2]$, where S_k is the sum of the k waiting times.