Probabilities & Data MATH 4043

SP2 2020

Tutorial Solutions Week 4/5
(Discrete Probability Distribution)

1. The number of messages sent per hour over a computer network has the following distribution:

x = number of messages	10	11	12	13	14	15
$p(x)$	0.08	0.15	0.30	0.20	0.20	0.07

The mean and standard deviation of the number of messages sent per hour are calculated as follows:

$$E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5$$

and

$$V(X) = E(X^2) - \mu^2 = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85$$

therefore

$$\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36.$$

Figure 1 (left hand side) show the probability mass function for the number of messages sent per hour over the network and the cumulative probability function is shown on the right hand side.

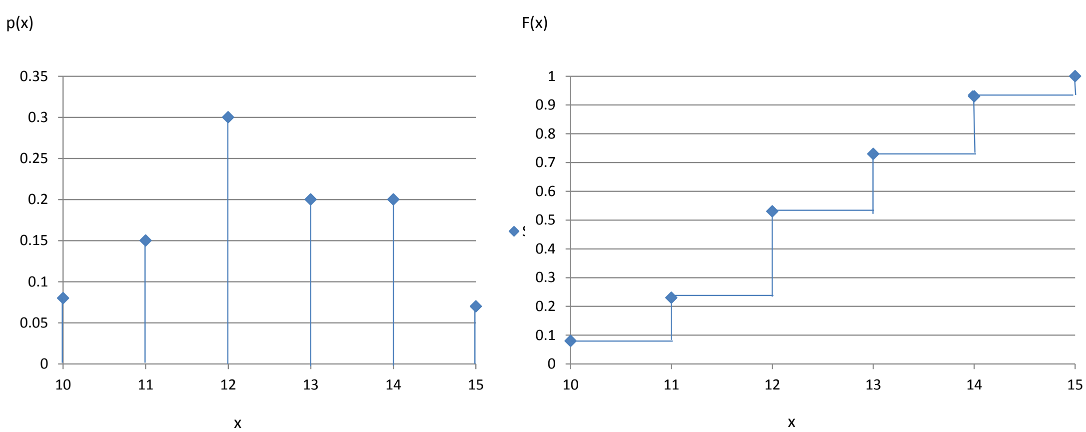

Figure 1: Probability mass function (left hand side) and Cumulative probability function (right hand side)

The probability of receiving 13 or fewer messages per hour over the computer network is given by

$$\begin{aligned} P(X \leq 13) &= 1 - [P(X = 14) + P(X = 15)] \\ &= 1 - P(X = 14) - P(X = 15) \\ &= 1 - 0.20 - 0.07 \\ &= 0.73. \end{aligned}$$

2. *Shape of the Binomial Distribution*: Figure 1 compares the shapes of the distributions for p equal to 0.05, 0.50 and 0.95, all for n equal to 5. Describe the effect of varying probability of success in a single trial when the number of trials is 5.

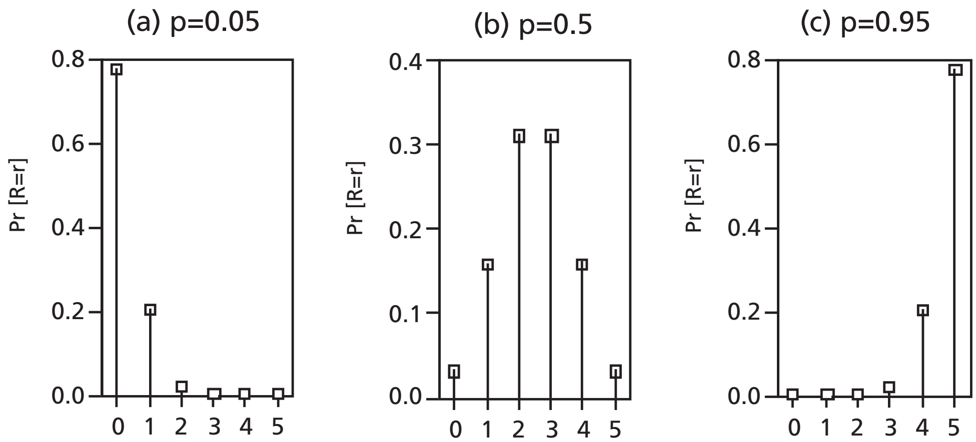

Figure 2: The effect of varying probability of success in a single trial when the number of trials is 5.

Answer: When p is close to zero (i.e. p =0.05) or one (i.e. p=0.95), the distribution is very skewed to the right and left respectively. When $p = 0.5$ the distribution is symmetical.

3. A company is considering drilling four oil wells. The probability of success for each well is 0.4, independent of the results for any other well. The cost of each well is \$200,000. Each well that is sucessful will be worth \$600,000. The binomial distribution applies.

Answer: The binomial distribution applieds here. Let us start by calculating the probability of each possible result. We use $n = 4$, $p = 0.40$ and $(1 - p) = 0.6$. That is $P(X = x) = C_x^n p^x (1 - p)^{n-x}$.

No. of Successes	**Prob**	**Answer**
0	$C_0^4(0.40)^0(0.60)^4$	$= 0.1296$
1	$C_1^4(0.40)^1(0.60)^3$	$= 0.3456$
2	$C_2^4(0.40)^2(0.60)^2$	$= 0.3456$
3	$C_3^4(0.40)^3(0.60)^1$	$= 0.1536$
4	$C_4^4(0.40)^4(0.60)^0$	$= 0.0256$
		$= 1.000$ (**check**)

Calcuate the probability of each possible result and answer the following:

(a) What is the probability that one or more wells will be successful?

$$\begin{aligned} P[\text{one ormore successful wells}] &= 1 - P[\text{no successful wells}] \\ &= 1 - 0.1296 \\ &= 0.8704 \end{aligned}$$

(b) What is the expected number of sucesses?
Expected number of successes $= (0)(0.1296) + (1)(0.3456) + (2)(0.3456) + (3)(0.1536) + (4)(0.0256)$ $= 1.6$

(c) What is the expected gain?
Expected gain = 1.6($600,000) - (4)($200,000) = $160,000.

(d) Considering all possible results, what is the probability of a loss rather than a gain?
There will be a loss if 0 or 1 well is sucessful, so the probability of a loss is

$$\begin{aligned} P[0 \text{ successful wells}] + P[1 \text{ successful well}] &= 0.1296 + 0.3456 \\ &= 0.4752. \end{aligned}$$

(e) What is the standard deviation of the number of sucesses?
Using the equation $\sigma^2 = E(X^2) - \mu^2$ we have

$$\begin{aligned} \sigma^2 &= (0)^2(0.1296) + (1)^2(0.3456) + (2)^2(0.3456) + (3)^2(0.1536) + (4)^2(0.0256) - 1.6^2 \\ &= 3.5200 - 1.6^2 \\ &= 0.9600. \end{aligned}$$

4. The number of meteors found by a radar system in any 30-second interval under specified conditions averages 1.81. Assume the meteors appear randomly and independently.

(a) What is the probability that no meteors are found in a one-minute interval?
$\lambda = (1.81)/(0.50 \text{ minute}) = 3.62/\text{minute}$. For a one-minute interval, $\mu = \lambda t = 3.62$.

$$P[\text{none in one minute}] = e^{-\lambda t} = e^{-3.62} = 0.0268.$$

(b) What is the probability of observing at least five but not more than eight meteors in two minutes of observation?
For two minutes, $\mu = \lambda t = (3.62)(2) = 7.24$.

$$P[R = r] = \frac{(\lambda t)^r e^{-\lambda t}}{r!}$$

Then

$$P[R = 5] = \frac{(7.24)^5}{e}^{-7.24} 5! = 0.1189.$$

Similarly,

$$P[R = 6] = \frac{(7.24)^6}{e}^{-7.24} 6! = 0.1435.$$

$$P[R = 7] = \frac{(7.24)^7}{e}^{-7.24} 7! = 0.1484.$$

$$P[R = 8] = \frac{(7.24)^8}{e}^{-7.24} 8! = 0.1343.$$

So the

$$\begin{aligned} &P[\text{at least five but no more than eight meteors in two minutes}] \\ &= 0.1198 + 0.1435 + 0.1484 + 0.1343 \\ &= 0.545. \end{aligned}$$

5. The average number of collisions occurring in a week during the summer months at a particular intersection is 2.00. Assume that the requirements of the Poisson distribution are satisfied. Answer: $\lambda = 2.00/$ week, $t = 1$ week, so $\lambda t = 2.00$.

   (a) What is the probability of no collisions in any particular week?

$$P[R = 0] = e^{-\lambda t} = e^{-2.00} = 0.135.$$

   (b) What is the probability that there will be exactly one collision in a week?

$$\begin{aligned} P[\text{exactly one collision in a week}] &= P[R = 1] = (\lambda t)e^{-\lambda t} \\ &= 2.00e^{-2.00} = 0.271. \end{aligned}$$

   (c) What is the probability of exactly two collisions in a week?

$$\begin{aligned} P[\text{exactly two collisions in a week}] &= P[R = 2] = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \\ &= \frac{(2.00)^2 e^{-2.00}}{2!} = 0.271. \end{aligned}$$

   (d) What is the probability of finding not more that two collisions in a week?

$$\begin{aligned} P[\text{not more than two collisions in a week}] &= P[R \le 2] \\ &= P[R = 0] + P[R = 1] + P[R = 2] \\ &= 0.135 + 0.271 + 0.271 \\ &= 0.677 \end{aligned}$$

   (e) What is the probability of finding more that two collisions in a week?

$$\begin{aligned} P[\text{more than two collisions in a week}] &= P[R > 2] \\ &= 1 - P[R \le 2] \\ &= 1 - 0.677 \\ &= 0.323 \end{aligned}$$

   (f) What is the probability of exactly two collisions in a particular two-week interval?
   Now we still have $\lambda = 2.00/\text{week}$, but $t = 2$ weeks, so $\lambda t = 4.00$. Then

$$\begin{aligned} P[\text{exactly two collisions in a two week interval}] &= \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \\ &= \frac{(4.00)^2 e^{-4.00}}{2!} = 0.147. \end{aligned}$$

6. We are told that 5% of the tools produced by a certain process are defective. Find the probability that in the sample of 40 tools chosen at random, exactly three will be defective. Calculate

   (a) using the binominal with $n = 40$, $p = 0.05$ and $(1 - p) = 0.95$, we have

$$\begin{aligned} P[R = 3] &= C_3^{40}(0.05)^3(0.95)^{40-3} \\ &= \frac{(40)(39)(38)}{(3)(2)(1)}(0.05)^3(0.95)^{37} \\ &= 0.185. \end{aligned}$$

   (b) using the Poisson distribution as an approximation we have $\mu = np = (40)(0.05) = 2.00$ therefore

$$P[R = 3] = \frac{(2.00)^3 e^{-2.00}}{(3)(2)(1)} = 0.180.$$