Tutorial 3 – MATH 4043

1. Would you accept a gamble that offers a 10% chance to win \$95 and a 90% chance of losing \$5?

2. Would you pay \$5 to participate in a lottery that offers a 10% chance to win \$100 but a 90% chance to win nothing?

3. (Insurance companies) Loss aversion and cost versus loss sustain the insurance industry: people pay more in premiums than they get back in claims on average (otherwise the industry wouldn't be sustainable), but they buy insurance anyway to protect themselves against substantial losses.

   (a) Suppose you pay \$1 each year to protect yourself against a 1 in 1000 chance of losing \$100 that year. Calculate the expected value of this scheme and interpret your answer?

4. We toss a fair coin 3 times. Let X be a random variable that counts the number of heads in 3 coin tosses.

   (a) Write down the sample space S for this experiment.

   (b) Show the mapping from the sample space S to the various values of X and tabulate the probability mass function of X.

   (c) Show the mapping from the sample space S to the various values of X and tabulate the probability mass function of X.

5. The random variable X has probability mass function X as given in the table below.

x	-1	1	2
$p_X(x)$	k	2k	k

   (a) Find the value of k.

   (b) Tabulate and sketch the cumulative distribution function $F_X(x)$ of X.

   (c) Compute $E[X]$ and $\text{Var}[X]$.

6. Two fair dice (with 6 faces each) are tossed together. Let Y denote the sum of the numbers on the top faces of each die.

   (a) Write down the sample space S. You may use a short cut notation for this.

   (b) Tabulate all the values in the range of Y and the corresponding probability mass function $p_Y(y)$.

(c) Compute $E[Y]$ and $\mathrm{Var}[Y]$.

7. The random variable X has probability mass function X as given in the table below. Let the random variable $Y = X^2$. This means that, e.g., $\{Y = 1\} = \{X = -1\} \cup \{X = 1\}$ and so on.

x	-2	-1	0	1	2
$p_X(x)$	0.1	0.3	0.2	0.3	0.1

(a) Tabulate the range of Y and its probability mass function $p_Y(y)$.

(b) Computer $E[Y]$ and $\mathrm{Var}[Y]$.

SOLUTIONS

1. We toss a fair coin 3 times. Let X be a random variable that counts the number of heads in 3 coin tosses.

   Solution.

   (a) Write down the sample space S for this experiment.

   $$S = \{TTT, THH, HTH, HHT, TTH, THT, HTT, HHH\}$$

   (b) Show the mapping from the sample space S to the various values of X and tabulate the probability mass function of X.

   The mappings between the subsets of the sample space to the events $\{X = x\}$ are as follows. where the subsets containing the sample points on the LHS are mapped to the corresponding points on the real line by X (on RHS).

$\{TTT\}$	$\{X = 0\}$
$\{HTT, THT, TTH\}$	$\{X = 1\}$
$\{HHT, THH, HTH\}$	$\{X = 2\}$
$\{HHH\}$	$\{X = 3\}$

   Since the coin is fair, the probability of each elementary event containing one sample point is $\frac{1}{8}$, and so from the table above, we can tabulate the probability mass function

x	0	1	2	3
$p_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

   (c) Compute $E[X]$ and $\text{Var}[X]$.

   $$E[X] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}.$$

   Now the second moment is

   $$E[X^2] = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3.$$

   Therefore

   $$\text{Var}[X] = E[X^2] - (E[X])^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}.$$

Note – after Chapter 5, we see that the distribution of X is actually binomial $\text{Bin}(3,\ \frac{1}{3})$ and using the formulas directly for the mean and variance of a binomial distributed random variable, we also get the same answers.

2. The random variable X has probability mass function X as given in the table below.

x	-1	1	2
$p_X(x)$	k	2k	k

Solution.

(a) Find the value of k.

$$1 = k + 2k + k = 4k,$$

so therefore $k = 0.25$. Substituting back into the table, we get

x	-1	1	2
$p_X(x)$	0.25	0.5	0.25

(b) Tabulate and sketch the cumulative distribution function $F_X(x)$ of X.

x	-1	1	2
$p_X(x)$	0.25	0.5	0.25
$F_X(x)$	0.25	0.75	1

(c) Compute $E[X]$ and $\mathrm{Var}[X]$.

$$E[X] = (-1) \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = \frac{3}{4}.$$

Now the second moment is

$$E[X^2] = (-1)^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} = \frac{7}{4}.$$

Thus

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{7}{4} - \left(\frac{3}{4}\right)^2 = \frac{19}{16}.$$

3. Two fair dice (with 6 faces each) are tossed together. Let Y denote the sum of the numbers on the top faces of each die.

Solution.

(a) Write down the sample space S. You may use a short cut notation for this.

$$S = \{(1,1),(1,2),(1,3),\cdots,(6,6)\}$$

If you list out every possible combination, you will have 36 sample points in total. Alternatively

$$S = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}.$$

(b) Tabulate all the values in the range of Y and the corresponding probability mass function $p_Y(y)$.

The mappings between the subsets of the sample space to the events $\{X = x\}$ are as follows. where the subsets containing the sample points on the LHS are mapped to the corresponding points on the real line by X (on RHS).

$\{(1.1)\}$	$\{Y = 2\}$
$\{(1.2),(2,1)\}$	$\{Y = 3\}$
$\{(1.3),(2,2),(3,1)\}$	$\{Y = 4\}$
$\{(1.4),(2,3),(3,2),(4,1)\}$	$\{Y = 5\}$
$\{(1.5),(2,4),(3,3),(4,2),(5,1)\}$	$\{Y = 6\}$
$\{(1.6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$	$\{Y = 7\}$
$\{(2.6),(3,5),(4,4),(5,3),(6,2)\}$	$\{Y = 8\}$
$\{(3.6),(4,5),(5,4),(6,3)\}$	$\{Y = 9\}$
$\{(4.6),(5,5),(6,4)\}$	$\{Y = 10\}$
$\{(5.6),(6,5)\}$	$\{Y = 11\}$
$\{(6,6)\}$	$\{Y = 12\}$

Now note that the probability of any elementary event is $P\{(i,j)\} = \frac{1}{36}$ since any elementary event containing any single sample point are equally likely. So based on the table above, the probability mass function of Y is

y	2	3	4	5	6	7	8	9	10	11	12
$p_Y(y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$p_Y(y)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

The 3rd row is just the probability mass function of Y expressed in fractions reduced to the lowest terms.

(c) Compute $E[Y]$ and $\text{Var}[Y]$.

$$E[Y] = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36}$$
$$+ 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} = 7.$$

Now the second moment is

$$E[Y^2] = 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + 6^2 \times \frac{5}{36} + 7^2 \times \frac{6}{36} + 8^2 \times \frac{5}{36}$$
$$+ 9^2 \times \frac{4}{36} + 10^2 \times \frac{3}{36} + 11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{329}{6}.$$

Thus

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{329}{6} - 7^2 = \frac{35}{6}.$$

4. The random variable X has probability mass function X as given in the table below. Let the random variable $Y = X^2$. This means that, e.g., $\{Y = 1\} = \{X = -1\} \cup \{X = 1\}$ and so on.

x	-2	-1	0	1	2
$p_X(x)$	0.1	0.3	0.2	0.3	0.1

(a) Tabulate the range of Y and its probability mass function $p_Y(y)$.
First note the equivalence of the following events involving Y and X in the table below.

$\{Y = 0\}$	$\{X = 0\}$
$\{Y = 1\}$	$\{X = -1\} \cup \{X = 1\}$
$\{Y = 4\}$	$\{X = -2\} \cup \{X = 2\}$

Hence we calculate the probability mass function of Y as in the table below.

y		$p_Y(y)$
0	$P\{Y = 0\} = P\{X = 0\}$	0.2
1	$P\{Y = 1\} = P\{X = -1\} + P\{X = 1\}$	0.6
4	$P\{Y = 4\} = P\{X = -2\} + P\{X = 2\}$	0.2

(b) Computer $E[Y]$ and $\text{Var}[Y]$.

$$E[Y] = 0 \times 0.2 + 1 \times 0.6 + 4 \times 0.2 = 1.4.$$

$$E[Y^2] = 0^2 \times 0.2 + 1^2 \times 0.6 + 4^2 \times 0.2 = 3.8.$$

Thus

$$\text{Var}[Y] = 3.8 - (1.4)^2 = 1.84.$$