Solutions. Question 1

1. The prior distribution is just as in the question here:

$$P(\mu = 55) = 0.8 \quad \text{and} \quad P(\mu = 45) = 0.2.$$

2. Now we have a sample of size $n = 25$ and $\bar{x} = 50$ and $\sigma = 4/\sqrt{25}$. We can compute the likelihoods by

$$P(H = \bar{x} = 50 \mid \mu = 45, \sigma = 4/\sqrt{25}) = P(Z = (50-45)/(4/5)) = P(Z = 6.25) = 0.0003$$

$$P(H = \bar{x} = 50 \mid \mu = 55, \sigma = 4/\sqrt{25}) = P(Z = (50-55)/(4/5)) = P(Z = -6.25) = 0.0003.$$

3. If we calculate the posterior distribution then by using Bayes Rule,

$$\begin{aligned} P(\mu = 45 \mid \bar{x} = 50, \sigma = 4/\sqrt{25}) &= \frac{P(H = \bar{x} = 50 \mid \mu = 45, \sigma = 4/\sqrt{25}) \times P(\mu = 45)}{P(\bar{x} = 50)} \\ &= \frac{P(H = \bar{x} = 50 \mid \mu = 45, \sigma = 4/5)P(\mu = 45)}{P(\bar{x} = 50 \mid \mu = 45, \sigma = 4/5)P(\mu = 45) + P(\bar{x} = 50 \mid \mu = 55, \sigma = 4/5)P(\mu = 55)} \\ &= \frac{0.2}{0.2 + 0.8} \end{aligned}$$

Thus, the probability does not change from the prior. This is the same as the other calculation.

4. This means the expected values are the same and thus $E(B) = 44 + 9 = 53$.

Question 2

1. We know that $\bar{\mu}$ is Normally Distributed and the prior distribution can be written as $N(1000, \bar{\sigma})$. The credible interval allows us to compute the $\bar{\sigma}$ with $n = 1$, 95% confidence:

$$\bar{\sigma} z = 100$$

$$\bar{\sigma} = 100/1.96 = 51.02.$$

The sample taken gives us a sampling prior distribution which we can use to update our mean $\bar{\mu}$, which is $N(900, \sqrt{50000/10})$. To compute the posterior distribution, we use the sample data with $N(900, 70.71)$. Let the sample mean and standard deviation be denoted by

$$\bar{x} = 900, \textit{and} \;\; \bar{s} = 223.607, \;\; n_1 = 10.$$

Now by using the following expressions for updating the mean and standard deviation, we have

$$\tilde{\mu} = \frac{\mu(\bar{s}/\sqrt{n_1})^2 + \bar{x}(\bar{\sigma}/\sqrt{n})^2}{(\bar{s}/\sqrt{n_1})^2 + (\bar{\sigma}/\sqrt{n})^2} = 965.76$$

and

$$\tilde{\sigma} = \sqrt{\frac{(\bar{s}/\sqrt{n_1})^2 \times (\bar{\sigma}/\sqrt{n})^2}{(\bar{s}/\sqrt{n_1})^2 + (\bar{\sigma}/\sqrt{n})^2}} = 41.37$$