1.

(a) X is binomial distributed, $X \sim B(100{,}0.15)$ and

$$\mu_X = np = 100 \cdot 0.15 = 15.$$

$$\sigma_X = \sqrt{npq} = \sqrt{100 \cdot 0.15 \cdot (1-0.15)} = 3.5707$$

(b)
$$P(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X)$$
$$= P(15 - 3.5707 \leq X \leq 15 - 3.5707)$$
$$= P(11.4293 \leq X \leq 18.5707)$$
$$= \sum_{k=12}^{18} P(X = k)$$
$$= \sum_{k=12}^{18} \binom{100}{k} 0.15^k (1-0.15)^{100-k}$$
$$= 0.6737$$

(c) We approximate $B(100{,}0.15)$ with $X \sim Poisson(\lambda)$, where $\lambda = 100 \cdot 0.15 = 15.$

$$P(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X)$$
$$= \sum_{k=12}^{18} P(X = k)$$
$$= \sum_{k=12}^{18} \frac{15^k}{k!} e^{-15}$$
$$= 0.6347$$

(d) We approximate $B(100{,}0.15)$ with $X \sim N(\mu_X, \sigma_X)$,where $\mu_X = 15, \sigma_X = 3.5707.$

Note that the boundaries are not integers, continuity correction is not necessary.

$$P(\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X)$$
$$= P(11.4293 \leq X \leq 18.5707)$$
$$= P\left(\frac{11.4293-15}{3.5707} \leq \frac{X-15}{3.5707} \leq \frac{18.5707-15}{3.5707}\right)$$
$$= P\left(-1 \leq \frac{X-15}{3.5707} \leq 1\right)$$
$$= P(-1 \leq Z \leq 1)$$
$$= 1 - 2 \cdot P(Z \leq -1)$$
$$= 0.6827$$

(e) Note that the probability for values of X which lie out side of $(\mu_X - \sqrt{2}\sigma_X, \mu_X + \sqrt{2}\sigma_X)$ is

$$P(X \le \mu_X - \sqrt{2}\sigma_X \; or \; X \ge \mu_X + \sqrt{2}\sigma_X)$$

$$= 1 - P(\mu_X - \sqrt{2}\sigma_X \le X \le \mu_X + \sqrt{2}\sigma_X)$$

$$\le 1 - P(\mu_X - \sigma_X \le X \le \mu_X + \sigma_X)$$

$$= 1 - 0.6737$$

$$= 0.3263 < 0.5$$

(f) If $X \sim B(100{,}0.1)$, then

$$np = 100 \cdot 0.1 = 10 > 5$$

$$nq = 100 \cdot 0.9 = 90 > 5$$

Thus the the Normal approximation is also appropriate.

2.

(a) The prior distribution is $N(\mu_0, \sigma_0^2)$, where $\mu_0 = 10, \sigma_0^2 = 4$, thus

$$P(\mu) = \frac{1}{\sqrt{2\pi\sigma_0{}^2}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}} \propto e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

The sample likelihood

$$P(X|\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_x{}^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma_x{}^2}} \propto e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_x{}^2}}$$

Then the posterior distribution

$$P(\mu|X) \propto e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_x{}^2}} \cdot e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

$$= e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_x{}^2} - \frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

$$= e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_x{}^2} - \frac{(\mu-\mu_0)^2}{2\sigma_0{}^2}}$$

$$= e^{\frac{-\sum_{i=1}^{n} x_i{}^2 + 2n\mu\bar{x} + n\mu^2}{2\sigma_x{}^2} - \frac{\mu^2 - 2\mu_0\mu + \mu_0{}^2}{2\sigma_0{}^2}}$$

$$= e^{-\frac{\mu^2}{2}(\frac{1}{\sigma_0{}^2} + \frac{n}{\sigma_x{}^2}) + \mu(\frac{\mu_0}{\sigma_0{}^2} + \frac{n\bar{x}}{\sigma_x{}^2})}$$

$$\propto e^{-\frac{\mu^2}{2\tilde{\sigma}^2} + \frac{\mu\tilde{\mu}}{\tilde{\sigma}^2} - \frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}}$$

$$\propto e^{-\frac{(\mu-\tilde{\mu})^2}{2\tilde{\sigma}^2}} \sim N(\tilde{\mu}, \tilde{\sigma}^2)$$

Where $\tilde{\mu} = \frac{\frac{\mu_0}{\sigma_0{}^2} + \frac{n\bar{x}}{\sigma_x{}^2}}{\frac{1}{\sigma_0{}^2} + \frac{n}{\sigma_x{}^2}}$, $\tilde{\sigma}^2 = \frac{1}{\frac{1}{\sigma_0{}^2} + \frac{n}{\sigma_x{}^2}}$, thus

$$\widetilde{\sigma} = \sqrt{\frac{1}{\frac{1}{\sigma_0{}^2}+\frac{5}{\sigma_x{}^2}}} = \sqrt{\frac{1}{\frac{1}{4}+\frac{5}{0.2^2}}} = 0.0894$$

$$\tilde{\mu} = \widetilde{\sigma}^2 \left(\frac{\mu_0}{\sigma_0{}^2} + \frac{n\bar{x}}{\sigma_x{}^2}\right) = 0.007984 \cdot \left(\frac{10}{4} + \frac{5\cdot 10.5}{0.2^2}\right) = 10.499$$

(b) The new balance devices are not as good as before. However we should noticed that the size of a five observations is too small, the results is influenced by individuals.

(c) Use R to simulate the posterior distribution

```
> s <rnorm(4,10,2)
> s
[1] 11.178067  7.081635 10.495804  8.927309
[2] > mean(s)
[1] 9.420704
> sd(s)
[1] 1.821981
```

The mean is $\bar{\bar{x}}$ =9.4207, the sample standard deviation is $s = 1.8220$, the sample size is 4.

(d) The new posterior distribution is $N(\bar{\bar{\mu}}, \bar{\bar{\sigma}}^2)$,where

$$\bar{\bar{\sigma}} = \sqrt{\frac{1}{\frac{1}{\widetilde{\sigma}^2}+\frac{4}{s^2}}} = 0.0890$$

$$\tilde{\mu} = \bar{\bar{\sigma}}^2 \left(\frac{\tilde{\mu}}{\widetilde{\sigma}^2} + \frac{4\bar{\bar{x}}}{s^2}\right) = 10.4887$$

(e) The mean in (d) is smaller than the mean in (a), to obtain a more accuracy results, the sample size should be larger.

3.

(a) Let S_t be the stock price at time t, note that each

$$\ln(S_i) - \ln(S_{i-1}) \text{ for i =1,2,...,t}$$

is $\mathrm{N}(\mu, \sigma)$. $\ln(S_i) - \ln(S_{i-1}) = X_i \sim \mathrm{N}(\mu, \sigma)$, then

$$\ln(S_t) - \ln(S_0) = \sum_{i=1}^{t} X_i = \mu t + \sigma \sum_{i=1}^{t} Z_i = \mu t + \sigma B_t$$

Then

$$S_t = S_0 e^{\mu t + \sigma B_t} \qquad (1)$$

We obtain an expression for the Stock Price in terms of Brownian Motion.

(b) R code:

```
> data <-read.csv("Stock.csv")
> Price = data$Price
> summary(Price)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
88.13	113.86	126.08	122.87	132.38	147.33

Thus choose s=88.13

(c) First we need to estimate the μ and σ in (1).

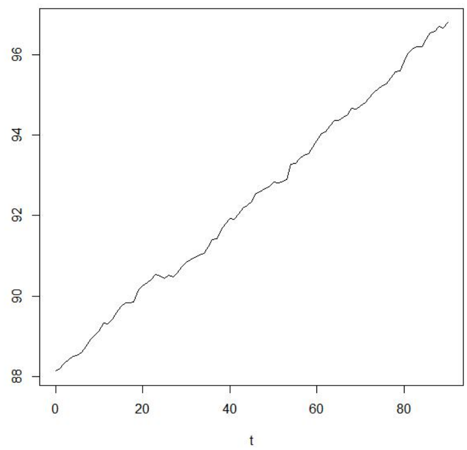
$$\mu \approx 0.0009421188\,,\ \sigma = 0.0008088511$$

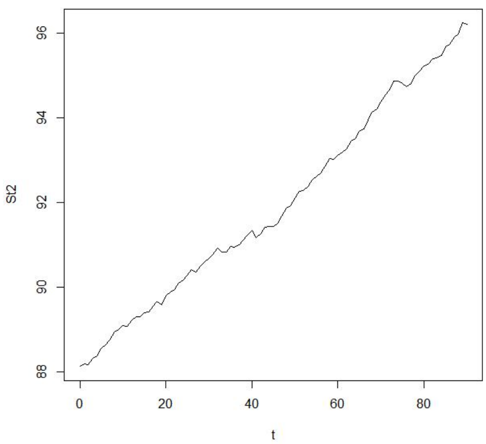
Codes:

```
> log_diff = log(Price[2:366])-log(Price[1:365])
> mu = mean(log_diff)
> sigma = sd(log_diff)/sqrt(365)
> mu
[1] 0.0009421188
> sigma
[1] 0.0008088511
> N=90
> t = seq(0,N)
> logreturns = cumsum(rnorm(N,mu,sigma))
> S0 = 88.13
> logreturns <- c(0,logreturns)
> St1 <- S0*exp(logreturns)
> plot(t,St,"l")
> logreturns = cumsum(rnorm(N,mu,sigma))
> S0 = 88.13
> logreturns <- c(0,logreturns)
> St2 <- S0*exp(logreturns)
> plot(t,St,"l")
> logreturns = cumsum(rnorm(N,mu,sigma))
```

```
> S0 = 88.13

> logreturns <- c(0,logreturns)

> St3 <- S0*exp(logreturns)

> plot(t,St,"l")

> logreturns = cumsum(rnorm(N,mu,sigma))

> S0 = 88.13

> logreturns <- c(0,logreturns)

> St4 <- S0*exp(logreturns)

> plot(t,St,"l")

> logreturns = cumsum(rnorm(N,mu,sigma))

> S0 = 88.13

> logreturns <- c(0,logreturns)

> St5 <- S0*exp(logreturns)

> plot(t,St,"l")
```

S+b =98.13



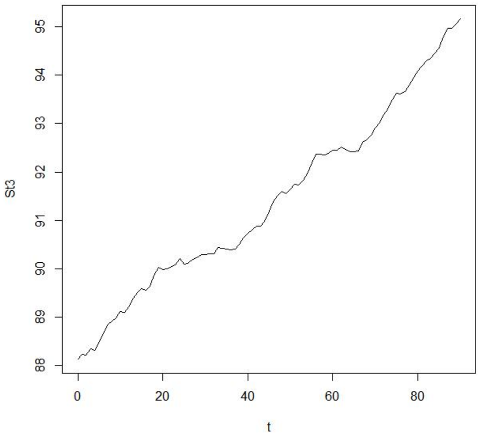


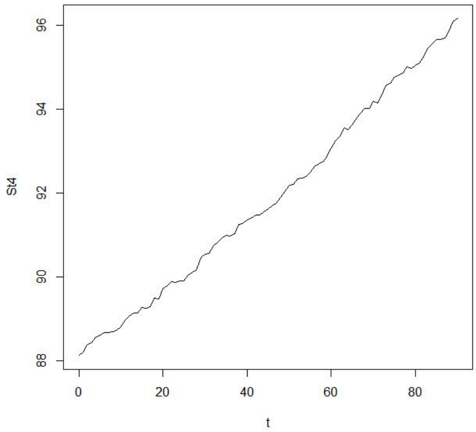
Stock price at the final day:

96.79830 loss

Stock price at the final day:

96.19409 loss



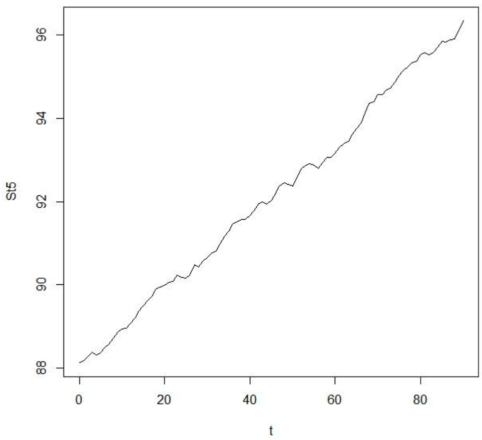


Stock price at the final day:

95.15669 loss

Stock price at the final day:

96.16582 loss

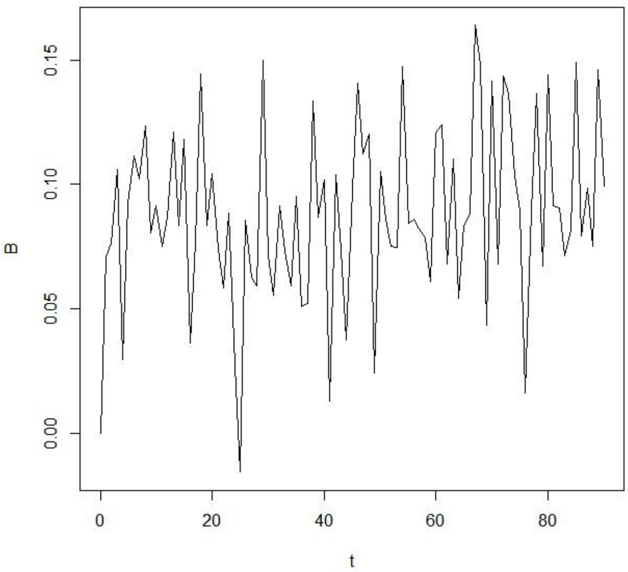


Stock price at the final day:

96.33422 loss

(d) The mean is 0.08791013, and the sd is 0.003778889.

```
> average_st = (St1+St2+St3+St4+St5)/5
> B =c(0, average_st[2:91]-average_st[1:90])
> plot(t,average_st,"l")
> mean(B)
[1] 0.08791013
>sd(B)/sqrt(91)
[1] 0.003778889
```



(e) Let

$$88.13e^{\mu 90+\sigma B_{90}} < 98.13$$

Where $\mu \approx 0.0009421188$, $\sigma = 0.0008088511$, then we have

$$P(88.13e^{\mu 90+\sigma B_{90}} < 98.13)$$

$$= P(B_{90} < \frac{log(\frac{98.13}{88.13})-90\mu}{\sigma})$$

$$= P(B_{90} < 28.0514)$$

$$= P(Z < \frac{28.0514-0}{\sqrt{90}})$$

=0.9984

(f)

$$P(88.13e^{\mu n+\sigma B_{n}} < 98.13)$$

$$= P(B_{n} < \frac{0.1074-n\mu}{\sigma})$$

$$= P(Z < \frac{0.1074-n\mu}{\sqrt{n}})$$

$= P(Z < -\infty)$ as $n \to \infty$

=0