ARCH-GARCH Models

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- First find the trend equation for the level L_t.
- $T_t = 11.2260.0897t + 0.0007t^2$
- Form $R_t = L_t - T_t$.

Autoregressive Process

- General $AR(2): R_t = \phi_1 R_{t-1} + \phi_2 R_{t-2} + Z_t$
- $R_t = 0.9642R_{t-1} - 0.3139R_{t-2} + Z_t$
- where $Z \sim N(0, 0.654^2)$.

One Step Ahead Forecast of $AR(2)$ Process

- $\hat{R}_t = E(R_t)$ where $\hat{R}_t$ is the one step ahead forecast at time $t-1$.
- $E(R_t) = E(0.9642R_{t-1}) - E(0.3139R_{t-2}) + E(Z_t)$.
- $\hat{R}_t = E(R_t) = 0.9642R_{t-1} - 0.3139R_{t-2}$.

Stationarity

- As stated previously, weak stationarity implied that the mean and variance remain constant over time.
- One could say that the fact that a series follows an *ARMA* process means that the mean varies locally, reverting back to the global mean.
- What if the variance does a similar thing? And how do we check to see if it does?
- This is a problem since we only have a single observation at each time t. So how are we to know anything about the variance at time t?

The ARCH Effect

- If the variance does not change over time, the series is termed Homoscedastic.
- If it does change, the series is Heteroscedastic. We say the series exhibits **A**uto**R**egressive **C**onditional **H**eteroscedasticity.
- Even though we cannot test the variance directly for this effect, we can examine its proxy, the error squared, Z_t^2.
- If the SACF for Z_t^2 has significant spikes then this is evidence for the so-called *ARCH* effect.

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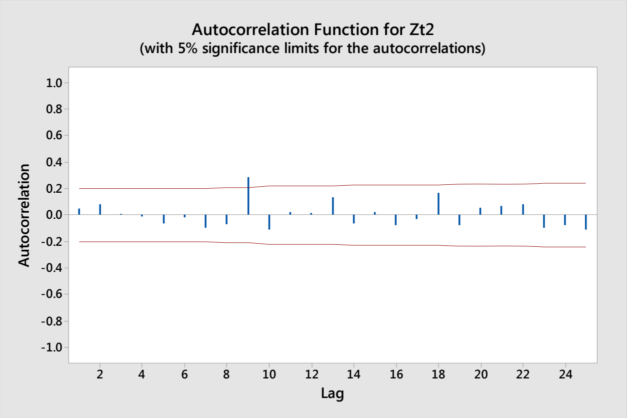


Autocorrelation Function for Zt2
(with 5% significance limits for the autocorrelations)
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
Autocorrelation
2
4
6
8
10
12
14
16
18
20
22
24
Lag

Daily Solar Radiation

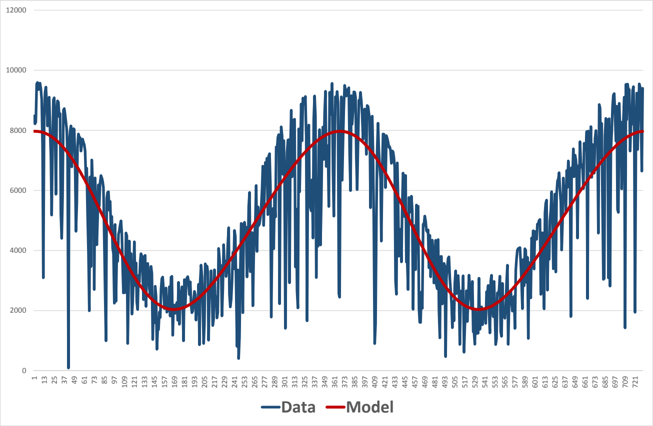


Figure: Fourier Series Model

Daily Solar Radiation

The one step ahead forecast for the residual series is $\hat{R}_t = 0.257 R_{t-1}$. This is added to the Fourier series model.

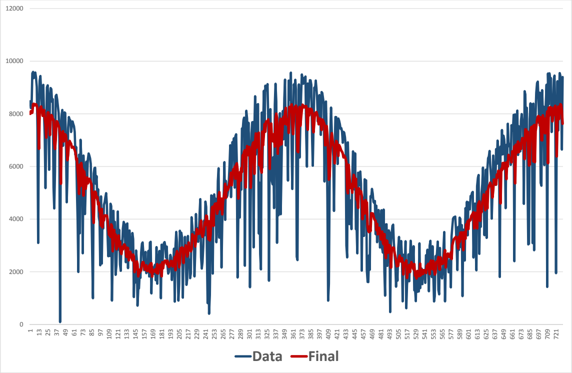
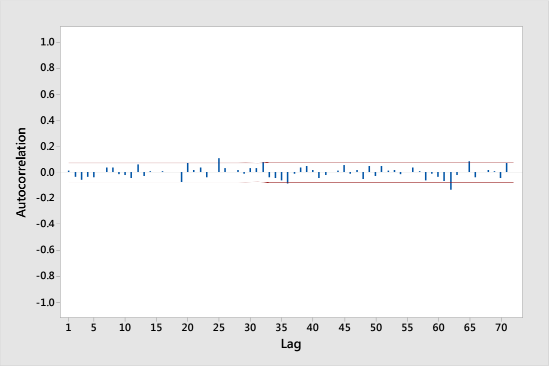


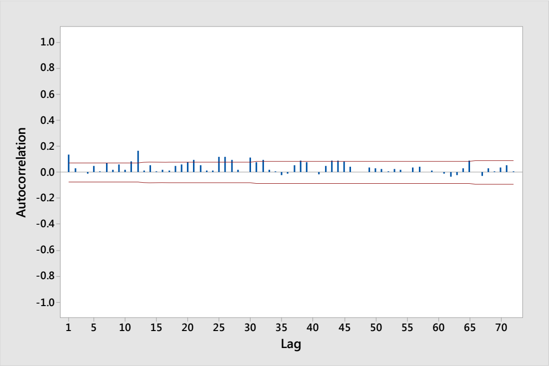
Figure: Final Model

Z_t SACF



Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1 5 10 15 20 25 30 35 40 45 50 55 60 65 70
Lag

Z_t^2 SACF

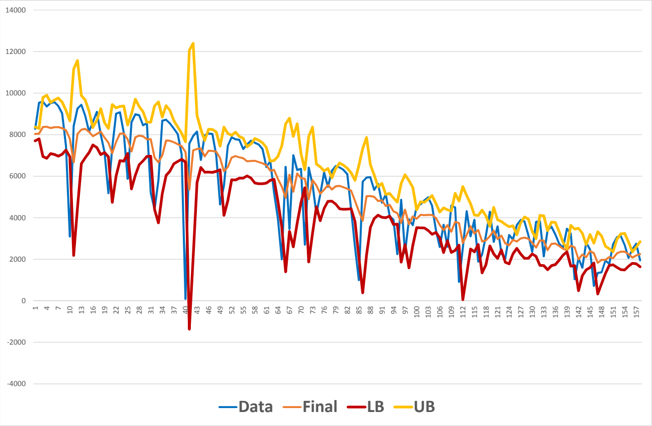


Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
5
10
15
20
25
30
35
40
45
50
55
60
65
70
Lag

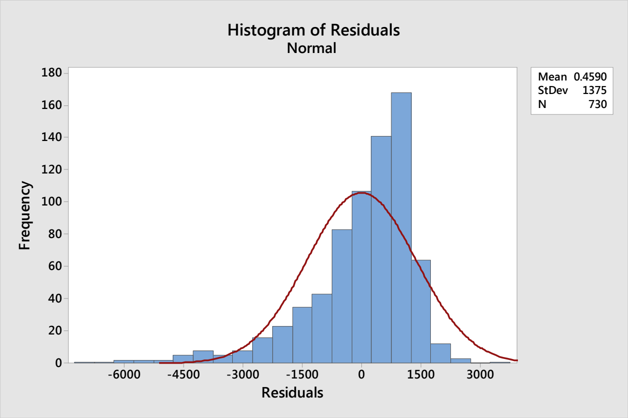
ARCH Model and Confidence Intervals

- The basic concept is that as well as forecasting the mean through the *ARMA* model, we have to forecast the variance through an *ARCH* model for the solar radiation in order to construct confidence intervals around the forecast.
- $Z_t = \sigma_t \epsilon_t, \ldots \sigma_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2 + \ldots + \alpha_m Z_{t-m}^2$
- $\{\epsilon_t\}$ is a sequence of independent and identically distributed random variables with mean 0 and variance 1.
- In this case, $\sigma_t^2 = 0.234 Z_{t-1}^2 + 0.115 Z_{t-2}^2$.

Confidence Intervals - it does not seem to work!



And Here is Why

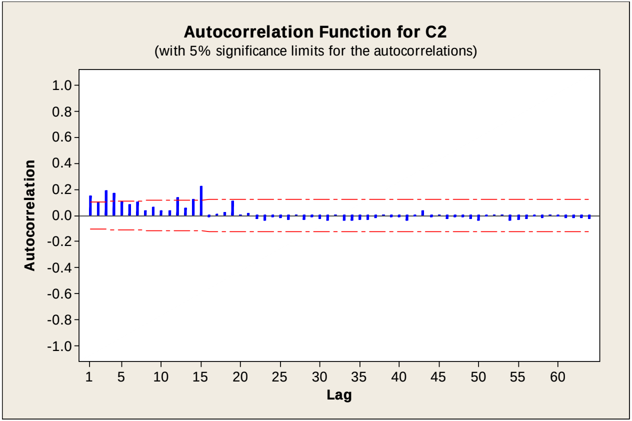


There is a method to deal with this situation of skewed residuals and that would be a good Project topic.

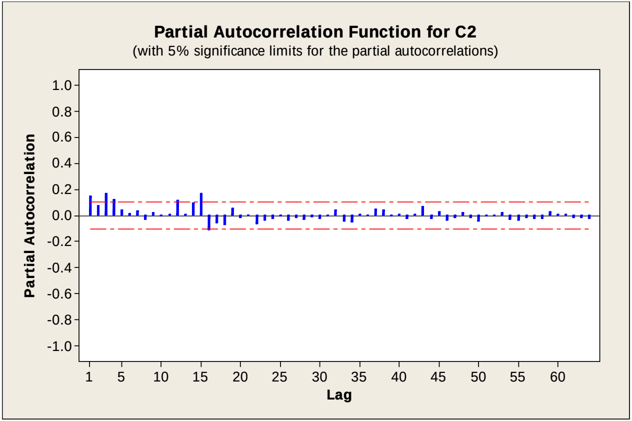
Well Behaved Example

- We will examine the log stock returns of Intel Corporation from January 1973 to December 2003.
- The series itself does not have any autocorrelation. This is in part since it is of log returns, $r_t = ln\frac{P_t}{P_{t-1}}$, where P_t is the stock price at time t.
- We will examine the SACF and SPACF of r_t^2.

Intel SACF for Squared Residuals



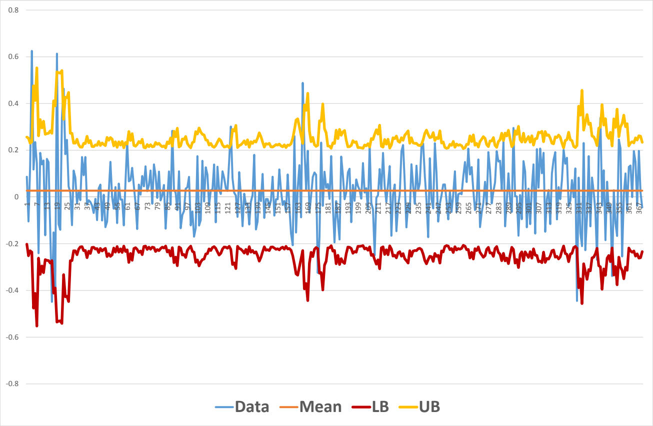
Intel SPACF for Squared Residuals



The Probabilistic Forecast Model for the Intel log returns

- $r_t = \mu_t + Z_t$
- $\sigma_t^2 = 0.0106 + 0.1054 Z_{t-1}^2 + 0.0493 Z_{t-2}^2 + 0.153 Z_{t-3}^2 + 0.1251 Z_{t-4}^2$

Performance



Quantifying Performance

- One measure of how well a prediction interval model works is Coverage.
- If one constructs a 95% prediction interval for example, then approximately 95% of the actual values of the series should lie within in.
- In this example, for a 95% PI, 95.7% of the series values fall in the PI.

Problems with ARCH

- The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks. This is not reasonable.
- For an ARCH(1) model, α_1 must be in the interval $[1, 1/3]$. This will restrict the ability to deal with leptokurtic series.
- It often requires many parameters to describe the volatility process of a series.

GARCH

- In 1982, Engel developed the ARCH model for volatility, and when the limitations stated above were noted, there was a further development.
- In 1986, Bollerslev developed the Generalised ARCH model to rectify the situation.
- It relies on constructing an ARMA model for the squared errors and then performing a slight alteration to get a variance forecast model.

The GARCH Model

- $Z_t = \sigma_t \epsilon_t$
- $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$
- ϵ_t is i.i.d. with mean zero and variance unity.
- $\alpha_i \geq 0, \forall i, \beta_j \geq 0, \forall j$ and $\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) < 1$.

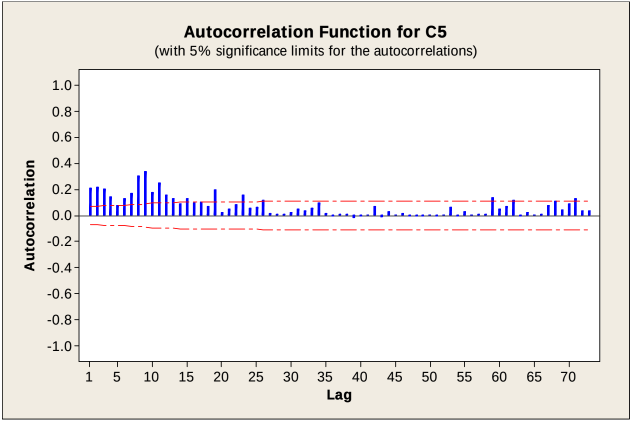
Estimating a GARCH Model

- First estimate an *ARMA* model for the level of the series.
- Then find an *ARMA* model for the squared residuals, $Z_t^2 = \phi_0 + \sum_{i=1}^{p} \phi_i Z_{t-i}^2 + \sum_{j=1}^{q} \theta_j \eta_{t-j}^2$.
- Then, the GARCH estimates are given by $\beta_i = \theta_i$ and $\alpha_i = \phi_i - \theta_i$.

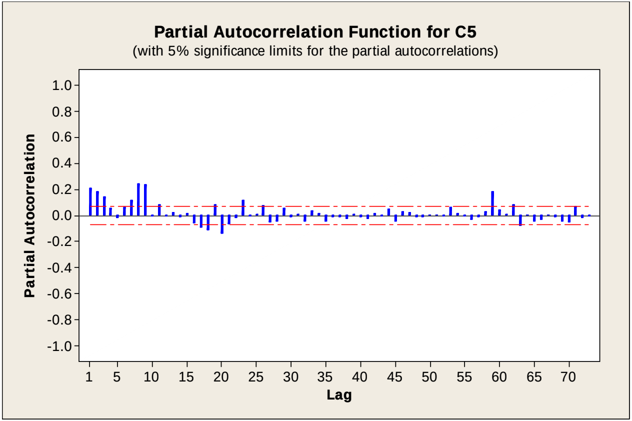
Example

- Monthly excess returns of the $S\&P$ 500 Index.
- This series has 792 observations from 1926.
- Series follows
  $r_t = 0.0065 + 0.0892 r_{t-1} - 0.0239 r_{t-2} - 0.1233 r_{t-3} + z_t$.

SACF for Squared residuals



SPACF for Squared residuals



Fitting the GARCH model

- $z_t^2 = 0.001 + 0.9676 z_{t-1}^2 - 0.8691 \eta_{t-1}^2 + \eta_t$
- Therefore, $\hat{\beta}_1 = 0.8691$ and $\hat{\alpha}_1 = 0.9676 - 0.8691 = 0.0985$.
- $\sigma_t^2 = 0.0985 z_{t-1}^2 + 0.8691 \sigma_{t-1}^2$.

GARCH Error Bounds

Coverage for 95% **PI is** 95.2%**.**