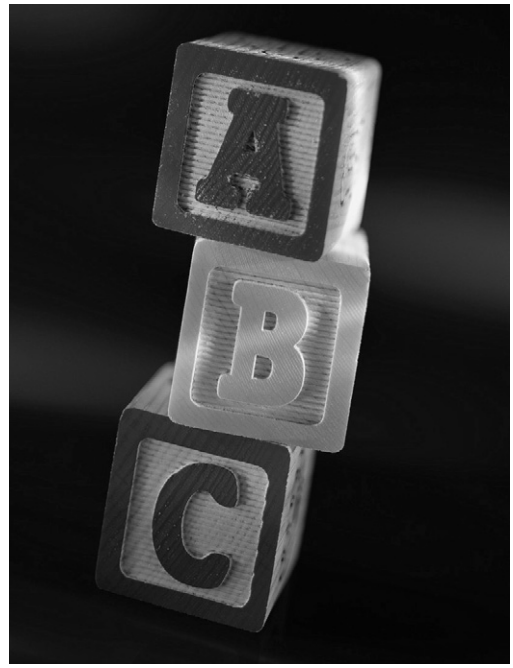
Chapter 7

Learning Bayesian Network Parameters



Until the early 1990s the DAG in a Bayesian network was ordinarily hand-constructed by a domain expert. Then the conditional probabilities were assessed by the expert, learned from data, or obtained using a combination of both techniques. Eliciting Bayesian networks from experts can be a laborious and difficult process in the case of large networks. As a result, researchers developed methods that could learn the DAG from data. Furthermore, they formalized methods for learning the conditional probabilities from data. We discuss these latter methods in this chapter; the next chapter concerns learn-

ing the DAG. In a Bayesian network the conditional probability distributions are called the **parameters**. In Section 7.1 we address the problem of learning a single parameter; in Section 7.2 we discuss learning all the parameters in a Bayesian network. We only discuss learning discrete parameters. Neapolitan [2004] shows a method for learning the parameters in a Gaussian Bayesian network.

7.1 Learning a Single Parameter

We can only learn parameters from data when the probabilities are relative frequencies, which were discussed in Section 2.3.1. So, this discussion pertains only to such probabilities. Although the method is based on rigorous mathematical results obtained by modeling an individual's subjective belief concerning a relative frequency, the method itself is quite simple. Here, we merely present the method. See [Neapolitan, 2004] for the mathematical development. After presenting a method for learning the probability of a binomial random variable, we extend the method to multinomial random variables. Finally, we provide guidelines for articulating our prior beliefs concerning probabilities.

7.1.1 Binomial Random Variables

We illustrate learning with a sequence of examples.

Example 7.1 *Recall the discussion concerning a thumbtack at the beginning of Section 2.3.1. We noted that a thumbtack could land on its flat end, which we call "heads," or it could land with the edge of the flat end and the point touching the ground, which we call "tails." Because the thumbtack is not symmetrical, we have no reason to apply the Principle of Indifference and assign probabilities of .5 to both outcomes. So, we need data to estimate the probability of heads. Suppose we toss the thumbtack 100 times, and it lands heads 65 of those times. Then the maximum likelihood estimate (MLE) is*

$$P(heads) \approx \frac{65}{100} = .65.$$

In general, if there are s heads in n trials, the MLE of the probability is

$$P(heads) \approx \frac{s}{n}.$$

Using the MLE seems reasonable when we have no prior belief concerning the probability. However, it is not so reasonable when we do have prior belief. Consider the next example.

Example 7.2 *Suppose you take a coin from your pocket, toss it 10 times, and it lands heads all those times. Then using the MLE we estimate*

$$P(heads) \approx \frac{10}{10} = 1.$$

After the coin landed heads 10 times, we would not bet as though we were certain that the outcome of the 11th toss will be heads. So, our belief concerning the $P(heads)$ is not the MLE value of 1. Assuming we believe the coins in our pockets are fair, should we instead maintain $P(heads) = .5$ after all 10 tosses landed heads? This might seem reasonable for 10 tosses, but it does not seem so reasonable if 1000 straight tosses landed heads. At some point we would start suspecting the coin was weighted to land heads. We need a method that incorporates one's prior belief with the data. The standard way to do this is for the probability assessor to ascertain integers a and b such that the assessor's experience is equivalent to having seen the first outcome (heads, in this case) occur a times and the second outcome occur b times in $m = a + b$ trials. Then the assessor's prior probabilities are

$$P(heads) = \frac{a}{m} \qquad P(tails) = \frac{b}{m}. \tag{7.1}$$

After observing s heads and t tails in $n = s + t$ trials, the assessor's posterior probabilities are

$$P(heads|s,t) = \frac{a+s}{m+n} \qquad P(tails|s,t) = \frac{a+t}{m+n}. \tag{7.2}$$

This posterior probability is called the **maximum a posterior probability (MAP)**. Note that we have used the symbol $=$ rather than $\approx$, and we have written the probability as a conditional probability rather than as an estimate. The reason is that this is a Bayesian technique, and Bayesians say that the value is their probability (belief) based on the data rather than saying it is an estimate of a probability (relative frequency).

We developed Equalities 7.1 and 7.2 based on intuitive grounds. The following theorem is a rigorous derivation of them.

Theorem 7.1 *Suppose we are about to repeatedly toss a thumbtack (or perform any repeatable experiment with two outcomes). Suppose further we assume exchangeability, and we represent our prior belief concerning the probability of heads using a* ***Dirichlet distribution*** *with parameters a and b. Then our prior probabilities are given by Equality 7.1, and after observing s heads and t tails in $n = s + t$ trials, our posterior probabilities are given by Equality 7.2.*

Proof. *The proof can be found in [Neapolitan, 2004] .* ■

See [Neapolitan, 2004] for a discussion of the Dirichlet distribution and a derivation of Equality 7.2. Neapolitan [2004] also discusses how, when we use the Dirichlet distribution with parameters a and b to represent our prior belief concerning the probability of heads, then intuitively our experience is equivalent to having seen the heads occur a times and tails occur b times in $m = a + b$ trials. Briefly, the assumption of **exchangeability**, which was first developed by de Finetti in 1937, is that an individual assigns the same probability to all sequences of the same length containing the same number of each outcome. For

example, the individual assigns the same probability to these two sequences of heads (H) and tails (T):

$$H, T, H, T, H, T, H, T, T, T \qquad \text{and} \qquad H, T, T, T, T, H, H, T, H, T.$$

Furthermore, the individual assigns the same probability to any other sequence of ten tosses that has four heads and six tails.

Next, we show more examples. In these examples we only compute the probability of the first outcome because the probability of the second outcome is uniquely determined by it.

Example 7.3 *Suppose you are going to repeatedly toss a coin from your pocket. Since you would feel it highly probable that the relative frequency is around .5, you might feel your prior experience is equivalent to having seen 50 heads in 100 tosses. Therefore, you could represent your belief with $a = 50$ and $b = 50$. Then $m = 50 + 50 = 100$, and your prior probability of heads is*

$$P(heads) = \frac{a}{m} = \frac{50}{100} = .5.$$

After seeing 48 heads in 100 tosses, your posterior probability is

$$P(heads|48, 52) = \frac{a + s}{m + n} = \frac{50 + 48}{100 + 100} = .49.$$

The notation $48, 52$ on the right of the conditioning bar in $P(heads|48, 52)$ represents the event that 48 heads and 52 tails have occurred.

Example 7.4 *Suppose you are going to repeatedly toss a thumbtack. Based on its structure, you might feel it should land heads about half the time, but you are not nearly so confident as you were with the coin from your pocket. So, you might feel your prior experience is equivalent to having seen 3 heads in 6 tosses. Then your prior probability of heads is*

$$P(heads) = \frac{a}{m} = \frac{3}{6} = .5.$$

After seeing 65 heads in 100 tosses, your posterior probability is

$$P(heads|65{,}35) = \frac{a + s}{m + n} = \frac{3 + 65}{6 + 100} = .64.$$

Example 7.5 *Suppose you are going to sample individuals in the United States and determine whether they brush their teeth. In this case, you might feel your prior experience is equivalent to having seen 18 individuals brush their teeth out of 20 sampled. Then your prior probability of brushing is*

$$P(brushes) = \frac{a}{m} = \frac{18}{20} = .9.$$

After sampling 100 individuals and learning that 80 brush their teeth, your posterior probability is

$$P(brushes|80, 20) = \frac{a + s}{m + n} = \frac{18 + 80}{20 + 100} = .82.$$

You could feel that if we have complete prior ignorance as to the probability, we should take $a = b = 0$. However, consider the next example.

Example 7.6 *Suppose we are going to sample dogs and determine whether or not they eat the potato chips we offer them. Since we have no idea whether a particular dog would eat potato chips, we assign $a = b = 0$, which means $m = 0 + 0 = 0$. Since we cannot divide a by m, we have no prior probability. Suppose next that we sample one dog, and that dog eats the potato chips. Our probability of the next dog eating potato chips is now*

$$P(eats|1, 0) = \frac{a+s}{m+n} = \frac{0+1}{0+1} = 1.$$

This belief is not very reasonable, since it means that we are know certain that all dogs eat potato chips. Owing to difficulties such as this and more rigorous mathematical results, prior ignorance to a probability is usually modeled by taking $a = b = 1$, which means $m = 1 + 1 = 2$. If we use these values instead, our posterior probability when the first sampled dog was found to eat potato chips is given by

$$P(eats|1, 0) = \frac{a+s}{m+n} = \frac{1+1}{2+1} = \frac{2}{3}.$$

Sometimes we want fractional values for a and b. Consider this example.

Example 7.7 *This example is taken from [Berry, 1996]. Glass panels in high-rise buildings sometimes break and fall to the ground. A particular case involved 39 broken panels. In their quest for determining why the panels broke, the owners wanted to analyze the broken panels, but they could only recover three of them. These three were found to contain nickel sulfide (NiS), a manufacturing flaw that can cause panels to break. To determine whether they should hold the manufacturer responsible, the owners then wanted to determine how probable it was that all 39 panels contained NiS. So, they contacted a glass expert.*

The glass expert testified that among glass panels that break, only 5% contain NiS. However, NiS tends to be pervasive in production lots. So, given that the first panel sampled, from a particular production lot of broken panels, contains NiS, the expert felt the probability was .95 that the second panel sampled also contains NiS. It was known that all 39 panels came from the same production lot. So, if we model the expert's prior belief using values of a, b, and $m = a + b$, as discussed previously, we must have that the prior probability is given by

$$P(NiS) = \frac{a}{m} = .05.$$

Furthermore, the expert's posterior probability after finding that the first panel contains NiS must be given by

$$P\left(NiS|1, 0\right) = \frac{a+1}{m+1} = .95.$$

Solving these last two equations for a and m yields

$$a = \frac{1}{360} \qquad m = \frac{20}{360}.$$

This is an alternative technique for assessing a and b. Namely, we assess the probability for the first trial. Then we assess the conditional probability for the second trial, given the first one is a "success." Once we have values of a and b, we can determine how likely it is that any one of the other 36 panels (the next one sampled) contains NiS after the first three sampled were found to contain it. We have that this probability is given by

$$P(NiS|3,0) = \frac{a+s}{m+n} = \frac{1/360+3}{20/360+3} = .983.$$

Notice how the expert's probability of NiS quickly changed from being very small to being very large. This is because the values of a and m are so small. We are really most interested in whether all 36 remaining panels contain NiS. It is left as an exercise to show that this probability is given by

$$\prod_{i=0}^{35} \frac{1/360+3+i}{20/360+3+i} = .866.$$

7.1.2 Multinomial Random Variables

The method just discussed readily extends to multinomial random variables. We have the following theorem.

Theorem 7.2 *Suppose we are about to repeatedly perform an experiment with k outcomes $x_1, x_2, \ldots, x_k$. Suppose further we assume exchangeability, and we represent our prior belief concerning the probability of heads using a* ***Dirichlet distribution*** *with parameters $a_1, a_2, \ldots, a_k$. Then our prior probabilities are*

$$P(x_1) = \frac{a_1}{m} \qquad P(x_2) = \frac{a_2}{m} \qquad \cdots \qquad P(x_k) = \frac{a_k}{m},$$

where $m = a_1 + a_2 + \cdots + a_k$. After seeing x_1 occur s_1 times, x_2 occur s_2 times, ..., and x_n occur s_n times in $n = s_1 + s_2 + \cdots + s_k$ trials, our posterior probabilities are as follows:

$$P(x_1|s_1, s_2, \ldots, s_k) = \frac{a_1+s_1}{m+n}$$

$$P(x_2|s_1, s_2, \ldots, s_k) = \frac{a_2+s_2}{m+n}$$

$$\vdots$$

$$P(x_k|s_1, s_2, \ldots, s_k) = \frac{a_k+s_k}{m+n}.$$

Proof. *The proof can be found in [Neapolitan, 2004] .* ■

We ascertain the numbers $a_1, a_2, \ldots, a_k$ by equating our experience to having seen the first outcome occur a_1 times, the second outcome occur a_2 times, ..., and the last outcome occur a_k times.

Example 7.8 *Suppose we have an asymmetrical-, six-sided die, and we have little idea of the probability of each side coming up. However, it seems that all sides are equally likely. So, we assign*

$$a_1 = a_2 = \cdots = a_6 = 3.$$

Then our prior probabilities are as follows:

$$P(1) = P(2) = \cdots = P(6) = \frac{a_i}{n} = \frac{3}{18} = .16667.$$

Suppose next we throw the die 100 *times, with the following results:*

Outcome	Number of Occurrences
1	10
2	15
3	5
4	30
5	13
6	27

We then have

$$\begin{aligned}
P(1|10,15,5,30,13,27) &= \frac{a_1 + s_1}{m+n} = \frac{3+10}{18+100} = .110 \\
P(2|10,15,5,30,13,27) &= \frac{a_2 + s_2}{m+n} = \frac{3+15}{18+100} = .153 \\
P(3|10,15,5,30,13,27) &= \frac{a_3 + s_3}{m+n} = \frac{3+5}{18+100} = .067 \\
P(4|10,15,5,30,13,27) &= \frac{a_4 + s_4}{m+n} = \frac{3+30}{18+100} = .280 \\
P(5|10,15,5,30,13,27) &= \frac{a_5 + s_5}{m+n} = \frac{3+13}{18+100} = .136 \\
P(6|10,15,5,30,13,27) &= \frac{a_6 + s_6}{m+n} = \frac{3+27}{18+100} = .254.
\end{aligned}$$

7.1.3 Guidelines for Articulating Prior Belief

Next we give some guidelines for choosing the values that represent our prior beliefs.

Binary Random Variables

The guidelines for binary random variables are as follows:

1. $a = b = 1$: We use these values when we feel we have no knowledge at all concerning the value of the probability. We might also use these values to try to achieve objectivity in the sense that we impose none of our beliefs concerning the probability on the learning algorithm. We only impose the fact that we know, at most, two things can happen. An example might be when we are learning the probability of lung cancer given smoking from data, and we want to communicate our result to the scientific community. The scientific community would not be interested in our prior belief; rather, it would be interested only in what the data had to say. Essentially, when we use these values, the posterior probability represents the belief of an individual who has no prior belief concerning the probability.

2. $a, b > 1$: These values mean that we feel it is likely that the probability of the first outcome is a/m. The larger the values of a and b, the more we believe this. We would use such values when we want to impose our beliefs concerning the relative frequency on the learning algorithm. For example, if we were going to toss a coin taken from a pocket, we might take $a = b = 50$.

3. $a, b < 1$: These values mean that we feel it is likely that the probability of one of the outcomes is high, although we are not committed to which one it is. If we take $a = b \approx 0$ (almost 0), then we are almost certain that the probability of one of the outcomes is very close to 1. We would also use values like these when we want to impose our beliefs concerning the probability on the learning algorithm. Example 7.7 shows a case in which we would choose values less than 1. Notice that such prior beliefs are quickly overwhelmed by data. For example, if $a = b = .1$, and we saw the first outcome x_1 occur in a single trial, we have

$$P(x_1|1, 0) = \frac{.1 + 1}{.2 + 1} = .917. \tag{7.3}$$

   Intuitively, we thought *a priori* that the probability of one of the outcomes was high. The fact that it took the value x_1 once makes us believe that it is probably that outcome.

Multinomial Random Variables

The guidelines are essentially the same as those for binomial random variables, but we restate them for the sake of clarity:

1. $a_1 = a_2 = \cdots = a_k = 1$: We use these values when we feel we have no knowledge at all concerning the probabilities. We might also use these values to try to achieve objectivity in the sense that we impose none of our beliefs concerning the probability on the learning algorithm. We only impose the fact that we know, at most, k things can happen. An example might be learning the probability of low, medium, and high blood pressure from data, which we want to communicate to the scientific community.

2. $a_1 = a_2 = \cdots = a_k > 1$: These values mean that we feel it is likely that the probability of the kth value is around a_k/m. The larger the values of a_k are, the more we believe this. We would use such values when we want to impose our beliefs concerning the probability on the learning algorithm. For example, if we were going to toss an ordinary die, we might take $a_1 = a_2 = \cdots = a_6 = 50$.

3. $a_1 = a_2 = \cdots = a_k < 1$: These values mean that we feel it is likely that only a few outcomes are probable. We would use such values when we want to impose our beliefs concerning the probabilities on the learning algorithm. For example, suppose we know there are 1,000,000 different species in the world, and we are about to land on an uncharted island. We might feel it probable that not very many of the species are present. So, if we considered the probabilities with which we encountered different species, we would not consider likely probabilities that resulted in a lot of different species. Therefore, we might take $a_i = 1/1{,}000{,}000$ for all i.

7.2 Learning Parameters in a Bayesian Network

The method for learning parameters in a Bayesian network follows readily from the method for learning a single parameter. We illustrate the method with binomial variables. It extends readily to the case of multinomial variables (see [Neapolitan, 2004]). After showing the method, we discuss equivalent sample sizes.

7.2.1 Procedure for Learning Parameters

Consider the two-node network in Figure 7.1 (a). We call such a network a **Bayesian network for parameter learning**. For each probability in the network there is a pair (a_{ij}, b_{ij}). The i indexes the variable; the j indexes the value of the parent(s) of the variable. For example, the pair (a_{11}, b_{11}) is for the first variable (X) and the first value of its parent (in this case there is a default of one parent value since X has no parent). The pair (a_{21}, b_{21}) is for the second variable (Y) and the first value of its parent, namely x_1. The pair (a_{22}, b_{22}) is for the second variable (Y) and the second value of its parent, namely x_2. We have attempted to represent prior ignorance as to the value of all probabilities by taking $a_{ij} = b_{ij} = 1$. We compute the prior probabilities using these pairs, just as we did when we were considering a single parameter. We have the following:

$$\begin{aligned} P(x_1) &= \frac{a_{11}}{a_{11} + b_{11}} = \frac{1}{1+1} = \frac{1}{2} \\ P(y_1|x_1) &= \frac{a_{21}}{a_{21} + b_{21}} = \frac{1}{1+1} = \frac{1}{2} \\ P(y_1|x_2) &= \frac{a_{22}}{a_{22} + b_{22}} = \frac{1}{1+1} = \frac{1}{2}. \end{aligned}$$

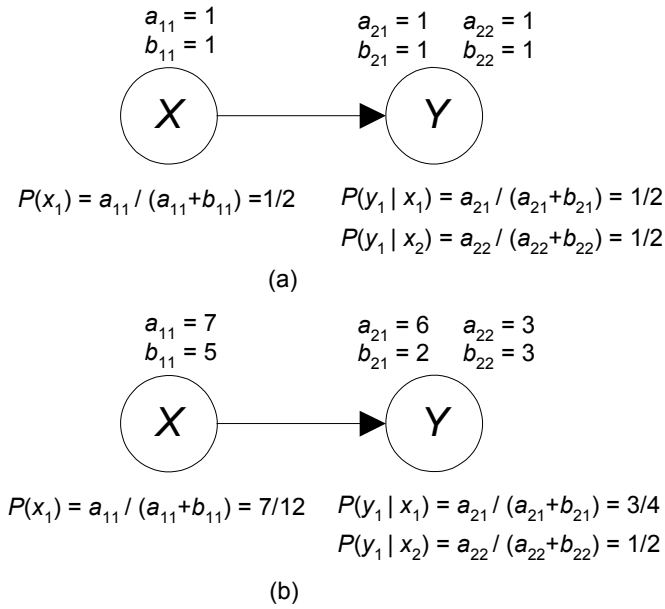


Figure 7.1: A Bayesian network for parameter learning appears in (a); the updated network based on the data in Figure 7.2 appears in (b).

When we obtain data, we use an (s_{ij}, t_{ij}) pair to represent the counts for the ith variable when the variable's parents have their jth value. Suppose we obtain the data in Figure 7.2. The values of the (s_{ij}, t_{ij}) pairs are shown in that figure. We have that $s_{11} = 6$ because x_1 occurs six times, and $t_{11} = 4$ because x_2 occurs four times. Of the six times that x_1 occurs, y_1 occurs five times and y_2 occurs one time. So, $s_{21} = 5$ and $t_{21} = 1$. Of the four times that x_2 occurs, y_1 occurs two times and y_2 occurs two times. So, $s_{22} = 2$ and $t_{22} = 2$. To determine the posterior probability distribution based on the data, we update each conditional probability with the counts relative to that conditional probability. Since we want an updated Bayesian network, we recompute the values of the (a_{ij}, b_{ij}) pairs. We therefore have the following:

$$\begin{aligned} a_{11} &= a_{11} + s_{11} = 1 + 6 = 7 \\ b_{11} &= b_{11} + t_{11} = 1 + 4 = 5 \end{aligned}$$

$$\begin{aligned} a_{21} &= a_{21} + s_{21} = 1 + 5 = 6 \\ b_{21} &= b_{21} + t_{21} = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} a_{22} &= a_{22} + s_{22} = 1 + 2 = 3 \\ b_{22} &= b_{22} + t_{22} = 1 + 2 = 3. \end{aligned}$$

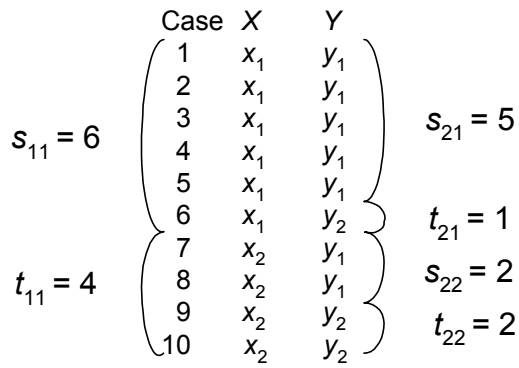


Figure 7.2: Data on 10 cases.

We then compute the new values of the parameters:

$$\begin{aligned} P(x_1) &= \frac{a_{11}}{a_{11}+b_{11}} = \frac{7}{7+5} = \frac{7}{12} \\ P(y_1|x_1) &= \frac{a_{21}}{a_{21}+b_{21}} = \frac{6}{6+2} = \frac{3}{4} \\ P(y_1|x_2) &= \frac{a_{22}}{a_{22}+b_{22}} = \frac{3}{3+3} = \frac{1}{2}. \end{aligned}$$

The updated network is shown in Figure 7.1 (b).

7.2.2 Equivalent Sample Size

There is a problem with the way we represented prior ignorance in the preceding subsection. Although it seems natural to set $a_{ij} = b_{ij} = 1$ to represent prior ignorance of all the conditional probabilities, such assignments are not consistent with the metaphor we used for articulating these values. Recall that we said the probability assessor is to choose values of a and b such that the assessor's experience is equivalent to having seen the first outcome occur a times in $a + b$ trials. Therefore, if we set $a_{11} = b_{11} = 1$, the assessor's experience is equivalent to having seen x_1 occur one time in two trials. However, if we set $a_{21} = b_{21} = 1$, the assessor's experience is equivalent to having seen y_1 occur one time out of the two times x_1 occurred. This is not consistent. First, we are saying x_1 occurred once; then we are saying it occurred twice. Aside from this inconsistency, we obtain odd results if we use these priors.

Consider the Bayesian network for parameter learning in Figure 7.3 (a). If we update that network with the data in Figure 7.2, we obtain the network in Figure 7.3 (b). The DAG in Figure 7.3 (a) is Markov equivalent to the one

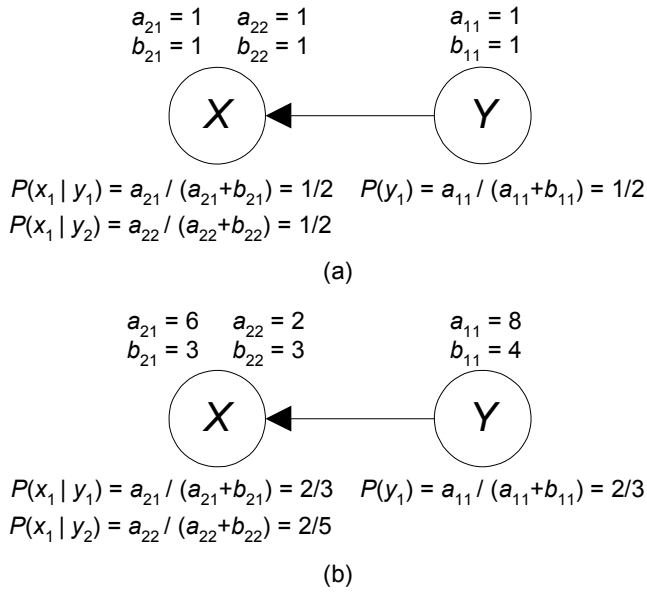


Figure 7.3: A Bayesian network initialized for parameter learning appears in (a); the updated network based on the data in Figure 7.2 appears in (b).

in Figure 7.1 (b). It seems that if we represent the same prior beliefs with equivalent DAGs, then the posterior distributions based on data should be the same. In this case we have attempted to represent prior ignorance as to all probabilities with the networks in Figure 7.1 (a) and Figure 7.3 (a). So, the posterior distributions based on the data in Figure 7.2 should be the same. However, from the Bayesian network in Figure 7.1 (b) we have

$$P(x_1) = \frac{7}{12} = .583,$$

whereas from the Bayesian network in Figure 7.3 (b) we have

$$\begin{aligned} P(x_1) &= P(x_1|y_1)P(y_1) + P(x_1|y_2)P(y_2) \\ &= \frac{2}{3} \times \frac{2}{3} + \frac{2}{5} \times \frac{1}{3} = .578. \end{aligned}$$

We see that we obtain different posterior probabilities. Such results are not only odd, but unacceptable since we have attempted to model the same prior belief with the Bayesian networks in Figures 7.1 (a) and 7.3 (a), but we end up with different posterior beliefs.

We can eliminate this difficulty by using a prior equivalent sample size. That is, we specify values of a_{ij} and b_{ij} that could actually occur in a prior sample that exhibit the conditional independencies entailed by the DAG. For example, given the network $X \rightarrow Y$, if we specify that $a_{21} = b_{21} = 1$, this

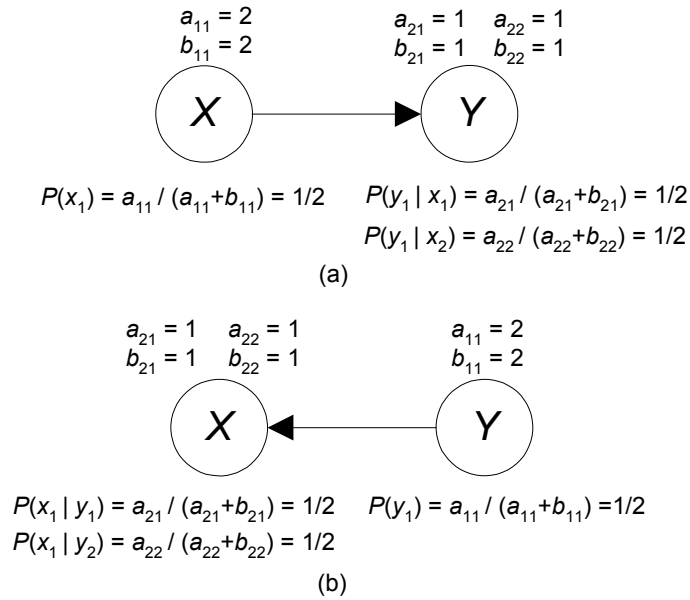


Figure 7.4: Bayesian networks for parameter learning containing prior equivalent sample sizes.

means our prior sample must have x_1 occurring two times. So, we need to specify $a_{11} = 2$. Similarly, if we specify that $a_{22} = b_{22} = 1$, this means that our prior sample must have x_2 occurring two times. So, we need to specify $b_{11} = 2$. Note that we are not saying we actually have a prior sample. We are saying that the probability assessor's beliefs are represented by a prior sample. Figure 7.4 shows prior Bayesian networks using equivalent sample sizes. Notice that the values of a_{ij} and b_{ij} in these networks represent the following prior sample:

Case	X	Y
1	x_1	y_1
2	x_1	y_2
3	x_2	y_1
4	x_2	y_2

It is left as an exercise to show that if we update both the Bayesian networks in Figure 7.4 using the data in Figure 7.2, we obtain the same posterior probability distribution. This result is true, in general. We state it as a theorem, but first we give a formal definition of a prior equivalent sample size.

Definition 7.1 *Suppose we specify a Bayesian network for parameter learning in the case of binomial variables. If there is a number N such that for all i and j*

$$a_{ij} + b_{ij} = P(\mathsf{pa}_{ij}) \times N,$$

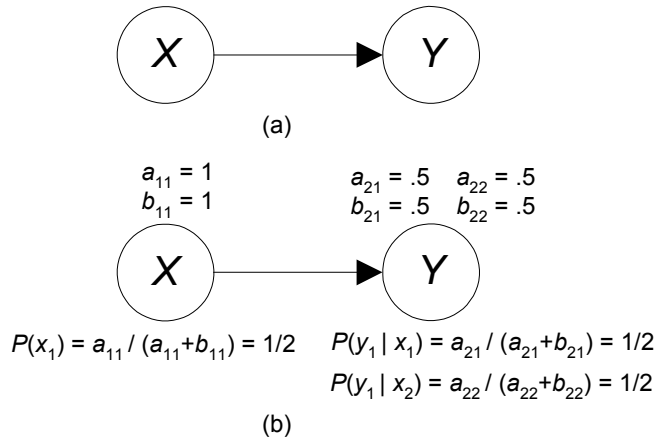


Figure 7.5: Given the DAG in (a) and that X and Y are binomial variables, the Bayesian network for parameter learning in (b) represents prior ignorance.

where pa_{ij} *denotes the jth instantiation of the parents of the ith variable, then we say the network has* ***prior equivalent sample size*** N.

This definition is a bit hard to grasp by itself. The following theorem, whose proof can be found in [Neapolitan, 2004], yields a way to represent uniform prior distributions, which is often what we want to do.

Theorem 7.3 *Suppose we specify a Bayesian network for parameter learning in the case of binomial variables and assign for all i and j*

$$a_{ij} = b_{ij} = \frac{N}{2q_i}$$

where N is a positive integer and q_i is the number of instantiations of the parents of the ith variable. Then the resultant Bayesian network has equivalent sample size N, and the joint probability distribution in the Bayesian network is uniform.

We can represent prior ignorance by applying the preceding theorem with $N = 2$. The next example illustrates the technique.

Example 7.9 *Suppose we start with the DAG in Figure 7.5 (a) and X and Y are binary variables. Set $N = 2$. Then, using the method in Theorem 7.3, we have*

$$a_{11} = b_{11} = \frac{N}{2q_1} = \frac{2}{2 \times 1} = 1$$

$$a_{21} = b_{21} = a_{22} = b_{22} = \frac{N}{2q_2} = \frac{2}{2 \times 2} = .5.$$

We obtain the Bayesian network for parameter learning in Figure 7.5 (b).

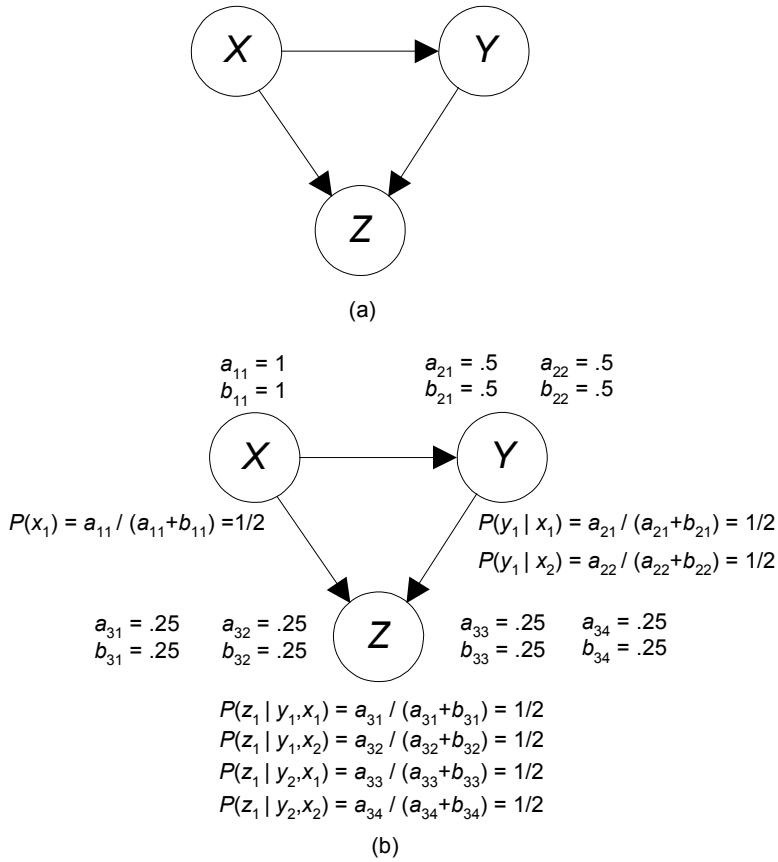


Figure 7.6: Given the DAG in (a) and that X, Y, and Z are binomial variables, the Bayesian network for parameter learning in (b) represents prior ignorance.

Note that we obtained fractional values for a_{21}, b_{21}, a_{22}, and b_{22} in the preceding example, which might seem odd. However, the sum of these values is

$$a_{21} + b_{21} + a_{22} + b_{22} = .5 + .5 + .5 + .5 = 2.$$

So, these fractional values are consistent with the metaphor that says we represent prior ignorance of the $P(Y)$ by assuming that the assessor's experience is equivalent to having seen two trials (see Section 7.1.3). The following is an intuitive justification for the reason that these values should be fractional. Recall from Section 7.1.3 that we said we use fractional values when we feel it is likely that the probability of one of the outcomes is high, although we are not committed to which one it is. The smaller the values, the more likely we feel that this is the case. Now the more parents a variable has, the smaller are the values of a_{ij} and b_{ij} when we set $N = 2$. Intuitively, this seems reasonable because when a variable has many parents and we know the values of the parents,

we know a lot about the state of the variable, and therefore, it is more likely that the probability of one of the outcomes is high.

Example 7.10 *Suppose we start with the DAG in Figure 7.6 (a), and X and Y are binary variables. If we set $N = 2$ and use the method in Theorem 7.3, then*

$$\begin{aligned} a_{11} = b_{11} &= \frac{N}{2q_1} = \frac{2}{2 \times 1} = 1 \\ a_{21} = b_{21} = a_{22} = b_{22} &= \frac{N}{2q_2} = \frac{2}{2 \times 2} = .5 \\ a_{31} = b_{31} = a_{32} = b_{32} = a_{33} = b_{33} = a_{34} = b_{34} &= \frac{N}{2q_3} = \frac{2}{2 \times 4} = .25. \end{aligned}$$

We obtain the Bayesian network for parameter learning in Figure 7.6 (b).

EXERCISES

Section 7.1

Exercise 7.1 For some two-outcome experiment that you can repeat indefinitely (such as the tossing of a thumbtack), determine the number of occurrences a and b of each outcome that you feel your prior experience is equivalent to having seen. Determine the probability of the first outcome.

Exercise 7.2 Assume that you feel your prior experience concerning the relative frequency of smokers in a particular bar is equivalent to having seen 14 smokers and 6 nonsmokers.

1. You then decide to poll individuals in the bar and ask them if they smoke. What is your probability of the first individual you poll being a smoker?

2. Suppose that after polling 10 individuals, you obtain these data (the value 1 means the individual smokes and 2 means the individual does not smoke):
$$\{1, 2, 2, 2, 2, 1, 2, 2, 2, 1\}.$$
What is your probability that the next individual you poll is a smoker?

3. Suppose that after polling 1000 individuals (it is a big bar), you learn that 312 are smokers. What is your probability that the next individual you poll is a smoker? How does this probability compare to your prior probability?

Exercise 7.3 In Example 7.7 it was left as an exercise to show that the probability that all 36 remaining window panels contain NiS is given by

$$\prod_{i=0}^{35} \frac{1/360 + 3 + i}{20/360 + 3 + i} = .866.$$

Show this.

Exercise 7.4 Suppose that you are about to watch Sam and Dave race several times, and Sam looks substantially athletically inferior to Dave. So, you give Sam a probability of .1 of winning the first race. However, you feel that if Sam wins once, he should usually win. So, given that Sam wins the first race, you give him a .8 probability of winning the next one.

1. Using the method shown in Example 7.7, compute your prior values of a and b.
2. Determine your probability that Sam will win the first five races.
3. Suppose next that Sam wins four of the first five races. Determine your probability that Sam will win the sixth race.

Exercise 7.5 Find a rectangular block (not a cube) and label the sides. Determine values of $a_1, a_2, \ldots, a_6$ that represent your prior probability concerning each side coming up when you throw the block.

1. What is your probability of each side coming up on the first throw?
2. Throw the block 20 times. Compute your probability of each side coming up on the next throw.

Exercise 7.6 Suppose that you are going to sample individuals who have smoked two packs of cigarettes or more daily for the past 10 years. You will determine whether each individual's systolic blood pressure is ≤ 100, 101-120, 121-140, 141-160, or ≥ 161. Determine values of $a_1, a_2, \ldots, a_5$ that represent your prior probability of each blood pressure range.

1. Next you sample such smokers. What is your probability of each blood pressure range for the first individual sampled?
2. Suppose that after sampling 100 individuals, you obtain the following results:

Blood Pressure Range	# of Individuals in This Range
≤ 100	2
101-120	15
121-140	23
141-160	25
≥ 161	35

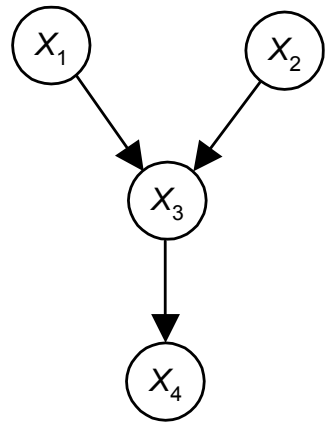


Figure 7.7: A DAG.

Compute your probability of each range for the next individual sampled.

Section 7.2

Exercise 7.7 Suppose that we have the Bayesian network for parameter learning in Figure 7.6 (b), and we have the following data:

Case	X	Y	Z
1	x_1	y_2	z_1
2	x_1	y_1	z_2
3	x_2	y_1	z_1
4	x_2	y_2	z_1
5	x_1	y_2	z_1
6	x_2	y_2	z_2
7	x_1	y_2	z_1
8	x_2	y_1	z_2
9	x_1	y_2	z_1
10	x_1	y_1	z_1
11	x_1	y_2	z_1
12	x_2	y_1	z_2
13	x_1	y_2	z_1
14	x_2	y_2	z_2
15	x_1	y_2	z_1

Determine the updated Bayesian network for parameter learning.

Exercise 7.8 Use the method in Theorem 7.3 to develop Bayesian networks for parameter learning with equivalent sample sizes 1, 2, 4, and 8 for the DAG in Figure 7.7.