1 Fourier Series Models

2 Introduction

Baron Jean-Baptiste Joseph Fourier introduced the idea that an arbitrary function, even one defined by different expressions in adjacent sections of its range as in a staircase waveform, could be represented a single expression. Fourier developed the idea as a direct result of his research into the flow of heat within solid bodies.
Fourier was apparently obsessed with heat, one anecdote tells of him receiving visitors in his rooms with a roaring fire and wearing a heavy overcoat despite the summer temperature. This obsession may have been due to his sojourn in Egypt as a member of Napoleon's Savants, 165 of the best and brightest of France's academics.
By 1807 he had completed his theory of heat conduction which depended on the idea of analysing temperature distribution of heat within a metal bar into spatially sinusoidal components. Publication was hindered by a lack of support from the great mathematicians of the day, LaPlace, Lagrange and Poisson among others, who expressed doubts as to its veracity. Nevertheless, Fourier's theory won a prestigious prize for mathematics in 1811 albeit with a caveat mentioning lack of generality and rigor. In fact publication was delayed further until 1815.
The problem lay in the fact that Fourier integrals rely on an interval stretching to infinity in all directions. In order to completely characterise the heat distribution in a metal bar the bar would have to be of infinite length as Fourier integrals are, by definition, over the interval $[\infty, \infty]$. Fourier circumvented this difficulty by considering a bar that had been bent into a circle. In this way the temperature is thus forced to be spatially periodic. There is no problem with generality if one supposes the circumference of the circular bar to be larger than the greatest distance that could be of interest on a straight bar conducting heat.

The formula

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots$$

was first published by Euler when Fourier was only a boy and forms the basis on which Fourier built his theory. This formula is correct however only for $-\pi < x < \pi$ and not for other ranges of x. The right hand side is the Fourier series for the saw tooth periodic function f defined by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{for} \quad -\pi < x < \pi \\ 0 & \text{for} \quad x = \pi \\ f(x+2\pi) & \text{for} \quad \text{all} \quad x \end{cases}$$

Fourier wrote "the equation is no longer true when the value of x is between π and 2π. However, the second side of the equation is still a convergent series but the sum is not equal to $\frac{x}{2}$. Euler who knew this equation gave it without comment" [1].

3 Fourier Series

Any periodic function satisfies the relation:$f(t) = f(t+T)$ where T is the period of the function.
The simplest and most common periodic functions are the trigonometric functions. The functions $\sin nt$ and $\cos nt,\ \ n \in 0, 1, 2, 3...$ being harmonic are by definition periodic with periods $\frac{2\pi}{n}$. Because trigonometric functions are relatively easy to work with and because they possess the important property of orthogonality they are more desirable than an arbitrary function. Accordingly it may be advantageous to expand the arbitrary function into series of trigonometric functions, the expansion being the Fourier Series.

Consider a set of functions which satisfy

$$\int_0^T \psi_{r(t)}\psi_{s(t)}dt = 0;\ \ r, s = 1, 2, \ldots; r \neq s \tag{1}$$

The set $\psi_r(t)$ is said to be orthogonal in any interval of length T. If in addition they satisfy

$$\int_0^T \psi_r(t)^2 dt = 1, r = 1, 2, \ldots \tag{2}$$

then they are orthonormal.
Hence for an orthonormal set we have

$$\int_0^T \psi_r(t)\psi_s(t)dt = \delta_{rs}, r, s = 1, 2, \ldots; \tag{3}$$

where δ_{rs} is equal to unity when r=s and zero otherwise. This is called the Krondecker delta. For example it is easy to verify that the set of functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 3t}{\sqrt{\pi}} \tag{4}$$

constitutes an orthonormal set. We can write

$$\begin{aligned} \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\frac{\sin rt}{\sqrt{\pi}}dt &= -\frac{1}{\sqrt{2\pi}}\frac{\cos rt}{r}\Big|_0^{2\pi} = 0 \\ \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\frac{\cos rt}{\sqrt{\pi}}dt &= \frac{1}{\sqrt{2\pi}}\frac{\sin rt}{r}\Big|_0^{2\pi} = 0 \\ r = 1, 2, \ldots \end{aligned} \tag{5}$$

For $r \neq s$ we have

$$\int_0^{2\pi} \frac{\sin rt}{\sqrt{\pi}}\frac{\cos rt}{\sqrt{\pi}}dt = \frac{1}{2\pi}\int_0^{2\pi}[\cos(r+s)t\sin(r-s)t]dt \tag{6}$$

$$\begin{aligned} &= -\frac{1}{2\pi}\left[\frac{\cos(r+s)t}{r+s} + \frac{\cos(r-s)t}{r-s}\right]_0^{2\pi} \\ &= 0 \end{aligned} \tag{7}$$

and for $r = s$

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin rt}{\sqrt{\pi}} \frac{\cos rt}{\sqrt{\pi}} dt &= \frac{1}{2\pi} \int_0^{2\pi} \sin(2rt) dt \\
&= -\frac{1}{4r\pi} \cos(2rt) \Big|_0^{2\pi} \\
&= 0
\end{aligned} \tag{8}$$

So the set is orthogonal, moreover since

$$\begin{aligned}
\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}} \right)^2 dt &= 1 \\
\int_0^{2\pi} \left(\frac{\sin(rt)}{\sqrt{\pi}} \right)^2 dt &= \frac{1}{r\pi} \left[\frac{rt}{2} - \frac{\sin(2rt)}{4} \right]_0^{2\pi} \\
&= 1, r = 1, 2, 3 \ldots \\
\int_0^{2\pi} \left(\frac{\cos(rt)}{\sqrt{\pi}} \right)^2 dt &= \frac{1}{r\pi} \left[\frac{rt}{2} + \frac{\sin(2rt)}{4} \right]_0^{2\pi} \\
&= 1, r = 1, 2, 3 \ldots
\end{aligned} \tag{9}$$

The set is not only orthogonal but orthonormal.
If for a set of constants c_r (r=1,2,3...) not all equal to zero $\exists$ an homogeneous linear relation

$$\sum_{r=1}^{n} c_r \psi_r(t) = 0 \tag{10}$$

for all t then the set of functions $\psi_r(t)$ is linearly dependent. The set (4) is linearly independent as

$$c_0 \frac{1}{\sqrt{2\pi}} + c_1 \frac{\sin t}{\sqrt{\pi}} + c_2 \frac{\cos t}{\sqrt{\pi}} + c_3 \frac{\sin 2t}{\sqrt{\pi}} + c_4 \frac{\cos 2t}{\sqrt{\pi}} + \cdots + c_{2p} \frac{\cos pt}{\sqrt{\pi}} = 0 \tag{11}$$

must mean that $c_1 = c_2 = \cdots = c_{2p} = 0$.

3.1 Trigonometric Series

Because the set of functions (4) is complete in the interval $0 \leq t \leq 2\pi$ every function $f(t)$ which is continuous in that interval can be represented by the *Fourier Series*

$$f(t) = \frac{1}{2} a_0 + \sum_{r=1}^{\infty} (a_r \cos(rt) + b_r \sin(rt)) \tag{12}$$

where the constants a_r and b_r are known as the Fourier coefficients. To represent a given function as a Fourier series we must calculate these coefficients. Referring to the results thus far we have

$$\begin{aligned}
\int_0^{2\pi} \cos(rt)\cos(st)dt &= 0 && (13)\\
\int_0^{2\pi} \sin(rt)\sin(st)dt &= 0,\ \ r \neq s && (14)\\
\int_0^{2\pi} \cos(rt)\sin(st)dt &= \int_0^{2\pi} \sin(rt)\cos(st)dt && (15)\\
&= 0,\ \ r, s = 1, 2, \ldots;
\end{aligned}$$

When r=s the integrals in equations (13, 14) are not zero but

$$\begin{aligned}
\int_0^{2\pi} \cos^2(rt)dt &= 2\pi, r = 0 && (16)\\
&= \pi, r = 1, 2, 3, \ldots
\end{aligned}$$

and

$$\int_0^{2\pi} \sin^2(rt)dt = \pi, r = 1, 2, 3, \ldots$$

and furthermore

$$\begin{aligned}
\int_0^{2\pi} \cos(rt)dt &= 2\pi, r = 0 && (17)\\
&= 0, r = 1, 2, 3, \ldots\\
\int_0^{2\pi} \sin(rt)dt &= 0, r = 1, 2, 3, \ldots
\end{aligned}$$

Now, if we multiply both sides of (12) by $\cos st$ and integrate we get

$$\int_0^{2\pi} f(t)\cos stdt = \frac{1}{2}a_0 \int_0^{2\pi} \cos stdt + \sum_{r=1}^{\infty} a_r \int_0^{2\pi} \cos(rt)\cos stdt + \sum_{r=1}^{\infty} b_r \int_0^{2\pi} \sin(rt)\cos stdt \tag{18}$$

For $s = 0$, equation (18) yields $a_o = 1/\pi \int_0^{2\pi} f(t)dt$ so a_o may be referred to as the average value of $f(t)$.
If $s \neq 0$ then (18) reduces to $a_r = 1/\pi \int_0^{2\pi} f(t)\cos rtdt$ from the orthogonality conditions..
If we multiply both sides of (12) by $\sin st$ and integrate we get

$$\int_0^{2\pi} f(t)\sin stdt = \frac{1}{2}a_0 \int_0^{2\pi} \sin stdt + \sum_{r=1}^{\infty} a_r \int_0^{2\pi} \cos(rt)\sin stdt + \sum_{r=1}^{\infty} b_r \int_0^{2\pi} \sin(rt)\sin stdt \tag{19}$$

This reduces to $b_r = 1/\pi \int_0^{2\pi} f(t)\sin rtdt$ when $s \neq 0$ because of the orthogonality conditions..
When $f(t)$ is an even series, that is,$f(t)$ is reflected at the origin, the b coefficients vanish and the series is known as a Fourier cosine series. Similarly when the series is an odd series, $f(t) = -f(-t)$ the a coefficients vanish and the series is known as a Fourier sine series. This is more easily demonstrated by considering the interval $-\pi \leq t \leq \pi$. If $f(t)$ contains discontinuities then the Fourier series approaches $f(t)$ on the intervals where it is continuous.