Math 714 – Fall 2020

**Homework 3 – Tony Yuan**

Github link for Matlab codes: https://github.com/YeTonyYuan/Math-714

A(a i) Taking total derivative w.r.t. to t on $u(x(t), t) = c$, we get $\partial_t u(x(t), t) + \partial_x u(x(t), t)x'(t) = 0$.

(a ii) $x'(t) = -\frac{\partial_t u(x(t),t)}{\partial_x u(x(t),t)} = a$.

(a iii) $x(t) = at + x_0$. Since $u(x(t), t)$ is a constant along $x(t)$, $u(x(t), t) = u(x(0), 0) = u_0(x_0)$.

(a iv) $u(at + x_0, t) = u_0(x_0)$. $u(x, t) = u(at + (x - at), t) = u_0(x - at)$.

(b i) We need $u \in L_1$. $\hat{u}_0(\xi)$ is infinitely differentiable. $u_o(\xi) \to 0$ as $|\xi| \to \infty$.

(b ii) $v(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} v(\xi)d\xi$. Taking derivative on both sides and move differentiation inside the integral, we have $v'(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} i\xi\hat{v}(\xi)d\xi$.

(b iii)

$$\begin{aligned}\mathcal{F}[v(\cdot - h)](\xi) &= \int_{\mathbb{R}} e^{-i\xi x} v(x - h)dx \\ &= \int_{\mathbb{R}} e^{-i\xi(x+h)} v(x)dx \\ &= e^{-i\xi h}\hat{v}(\xi).\end{aligned}$$

(b iv) $\partial_t \hat{u}(\xi, t) = \int_{\mathbb{R}} e^{-i\xi x} u_t(x, t)dx = \int_{\mathbb{R}} e^{-i\xi x}(-a)u_x(x, t)dx = -ai\xi\hat{u}(\xi, t)$ by part (ii).

(b v) By solving the ODE from part (iv), we have $\hat{u}(\xi, t) = e^{-ai\xi t}\hat{u}_0(\xi)$.

(b vi) By Plancherel's Theorem, $\hat{u}_0 \in L_2$. Then $\hat{u}(\cdot, t) \in L_2$ for any fixed $t$, $u(\cdot, t) \in L_2$ for any fixed $t$ by Plancherel again.

(b vii) This follows directly from taking the inverse FT.

(b viii) $\hat{u}(\xi, t) = e^{-ai\xi t}\hat{u}_0(\xi) = \mathcal{F}[u_0(\cdot - at)](\xi)$. By the inversion formula, $u(x, t) = u_0(x - at)$.

B(a) Let $u = e^{i(kx-wt)}$. Then $-iw = \pm k^4$. $u = e^{ikx}e^{\pm k^4 t}$. It is stable only when we have the minus sign.

B(b) $\frac{\rho-1}{\Delta t} = -\frac{1}{\Delta x^4}$, $|\rho| = |1 - \frac{\Delta t}{\Delta x^4}| < 1$. $\Delta t < 2\Delta x^4$.

B(c) We can use backward Euler method.

C(a)

$$\begin{aligned}&\frac{u(x,y,t+\Delta t)-2u(x,y,t)+u(x,y,t-\Delta t)}{\Delta t^2}-\Delta u\\&=(u_{tt}+\frac{\Delta t^2}{12}u_{tttt}+\cdots)-\Delta u\\&=(u_{tt}-\Delta u)+\frac{\Delta t^2}{12}\Delta^2 u+\mathcal{O}(\Delta t^4)\end{aligned}$$

since $u_{tttt}=(\Delta u)_{tt}=\Delta(u_{tt})=\Delta^2 u$.

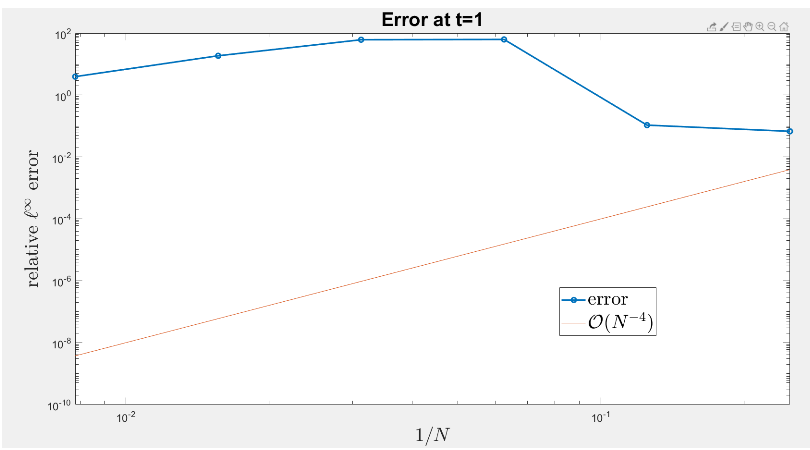
Thus, $\frac{u(x,y,t+\Delta t)-2u(x,y,t)+u(x,y,t-\Delta t)}{\Delta t^2}-\Delta u-\frac{\Delta t^2}{12}\Delta^2 u=0$ yields fourth order accuracy in time in the solution.

C(b)

$$\begin{cases}\frac{u^1-u^{-1}}{2\Delta t}=u_t^0+\frac{\Delta t^2}{6}u_{ttt}^0=u_t^0+\frac{\Delta t^2}{6}\Delta u_t^0\\ \frac{u^1-2u^0+u^{-1}}{\Delta t^2}=u_{tt}^0+\frac{\Delta t^2}{12}u_{tttt}^0=\Delta u^0+\frac{\Delta t^2}{12}\Delta^2 u^0\end{cases}$$

$$u^1=u^0+\Delta t u_t^0+\frac{\Delta t^3}{6}\Delta u_t^0+\frac{\Delta t^2}{2}\Delta u^0+\frac{\Delta t^4}{24}\Delta^2 u_t^0.$$

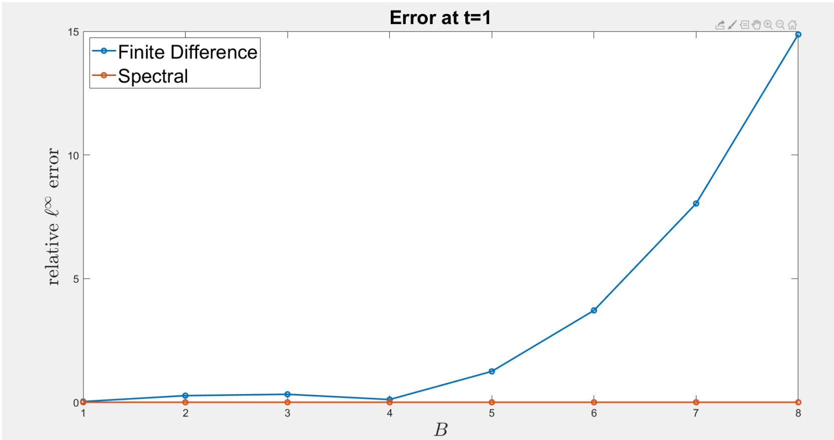
C(c)




C(d) The largest eigenvalue of $\Delta u$ in magnitude is $-0.096N^4$ and then the largest eigenvalue of $\Delta^2 u$ is $0.096^2N^8$.

The largest eigenvalue in magnitude of $\Delta u+\frac{\Delta t^2}{12}\Delta^2 u$, $\lambda=-0.096N^4+\frac{\Delta t^2}{12}\cdot 0.096^2N^8$.

Since $-4\le\lambda\Delta t^2\le 0$, then $-4\le -0.096\Delta t^2N^4+\frac{\Delta t^4}{12}\cdot 0.096^2N^8\le 0$. $0.096\Delta t^2N^4\le 12$, $\Delta t\le 11.18N^{-2}$.

C(e) 


I choose $N = 32$ due to the heavy computations. The number of points per wavelength from my finite difference method is 16 and I am unable to find it for the spectral method due to limitation of computations.

D

$$
\begin{aligned}
|\hat{f}(\xi)| &= |\int_{\mathbb{R}} e^{-i\xi x} f(x) dx| \\
&= \frac{1}{|\xi|} |\int_{\mathbb{R}} (-i\xi) e^{-i\xi x} f(x) dx| \\
&= \frac{1}{|\xi|} |\int_{\mathbb{R}} e^{-i\xi x} f'(x) dx| \text{ by integration by parts} \\
&\leq \frac{1}{|\xi|} \int_{\mathbb{R}} |f'(x)| dx \\
&= \|f\|_{TV} |\xi|^{-1}.
\end{aligned}
$$

If $f$ is not differentiable almost everywhere, we simply approximate $f$ by $C^1$ functions that converge to $f$, which is feasible since $f \in L_1$.