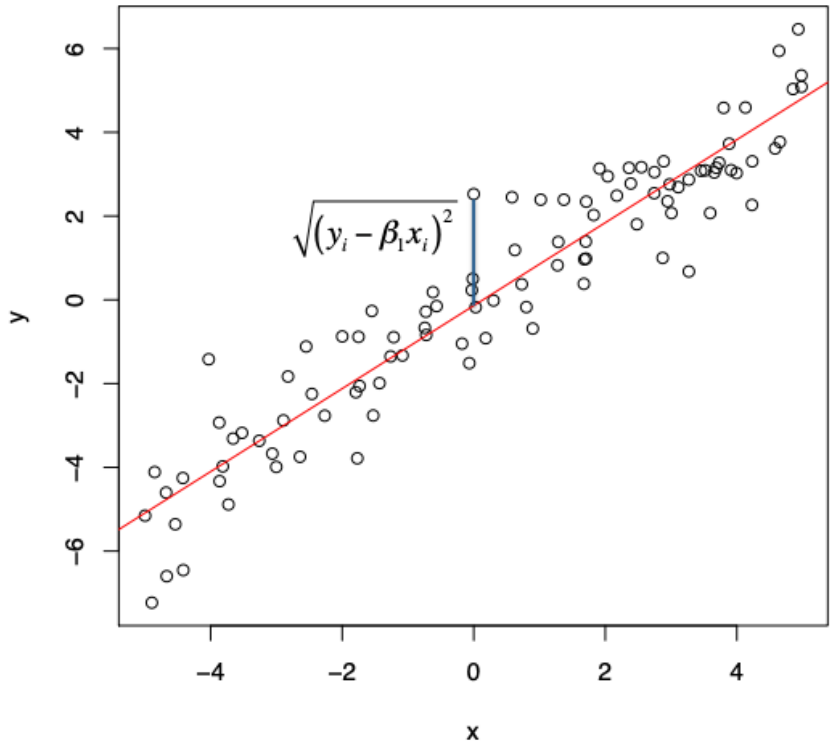
BUS 243

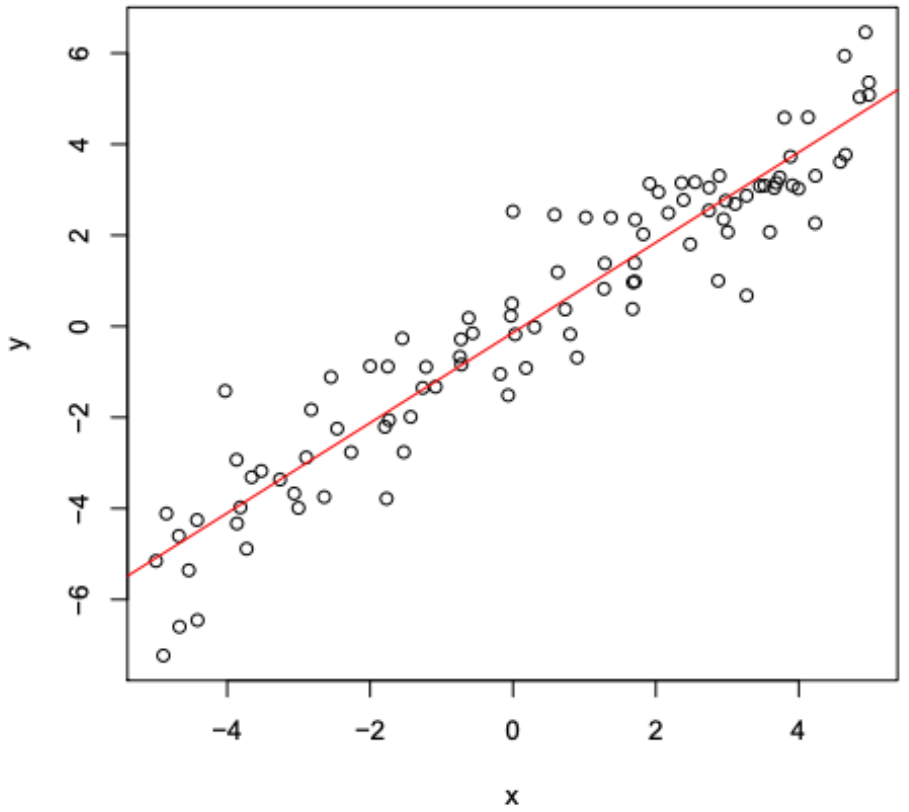
Lecture 4: Discriminative classifiers

FITTING A LINEAR REGRESSION

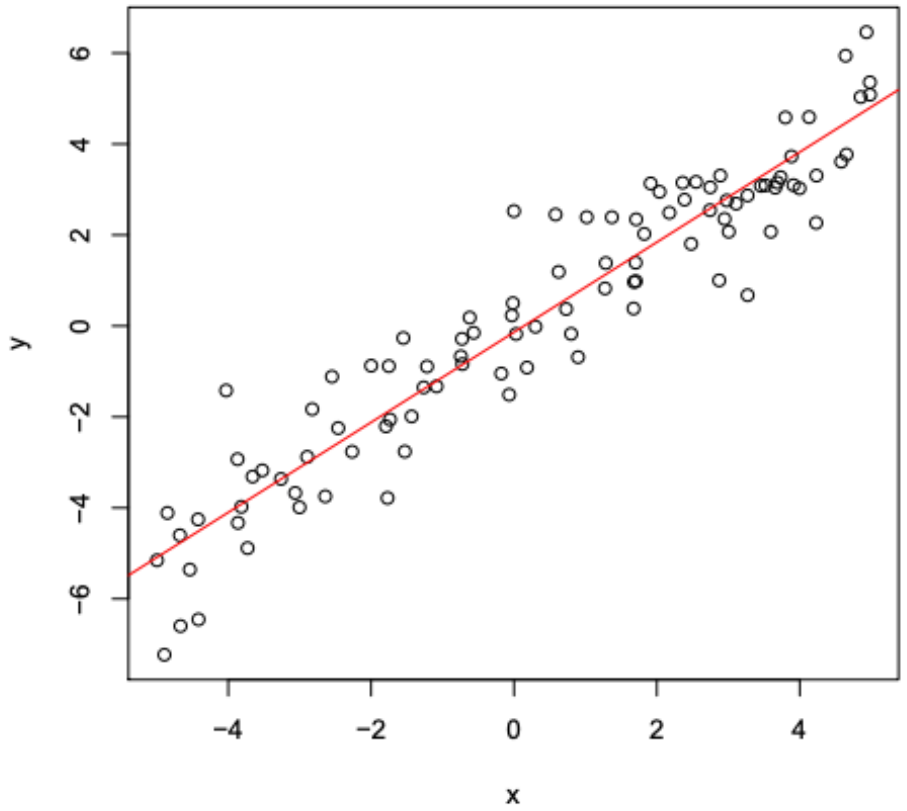
- Idea: minimize the Euclidean distance between data and fitted line
  - $RSS(\beta) = \frac{1}{n}\sum_i (y_i - \beta \boldsymbol{x}_i)^2$



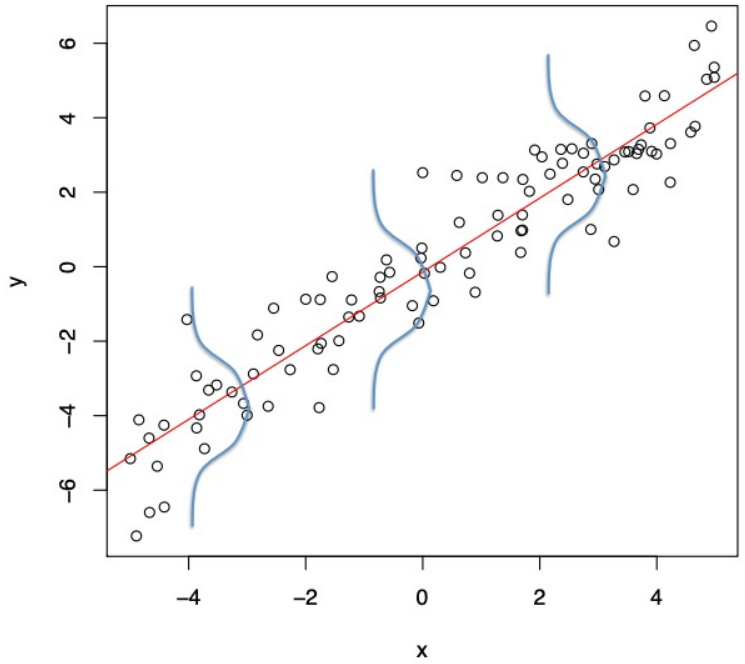
- How to find β
  - Use calculus to find the value of β that minimizes the RSS



- Probabilistic Interpretation
  - Our analysis so far has not included any probabilities
  - Linear regression does have a probabilistic (probability model-based) interpretation



- Linear regression assumes that response values have a Gaussian distribution around the linear mean function
  - $Y_i|\boldsymbol{x}_i, \beta \sim N(\boldsymbol{x}_i\beta, \sigma^2)$
- This is discriminative model where inputs x are not modeled



- Minimizing RSS is equivalent to maximizing conditional likelihood

CLASSIFICATION: LOGISTIC REGRESSION

- What are we talking about?
  - Statistical classification: $P(y|x)$
  - Examples?
    - Discriminative classifiers vs. generative classifier
  - Building block of other machine learning methods

GENERATIVE AND DISCRIMINATIVE CLASSIFIERS

- Suppose we're distinguishing cat from dog images

GENERATIVE CLASSIFIER

- Build a model of what's in a cat image
  - Knows about whiskers, ears, eyes
  - Assigns a probability to any image: how cat-y is this image?

- Also build a model for dog images

- Now given a new image
  - Run both models and see which one fits better

DISCRIMINATIVE CLASSIFIER

- Just try to distinguish dogs from cats

- Oh, look. Dogs have collars! Let's ignore everything else

- **Naïve Bayes**
  - $\hat{c} = \text{argmax}_{c \in C}\ P(d|c)P(c)$
- **Logistic Regression**
  - $\hat{c} = \text{argmax}_{c \in C}\ P(c|d)$
  - Directly compute the posterior

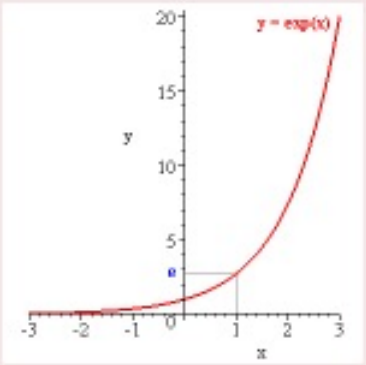
LOGISTIC REGRESSION

- Discriminative classifiers learn what features from the input are most useful to discriminate between the different possible classes!
  - Train a classifier that can make a binary decision about the class of a new input
  - To create a probability, we need to transform!

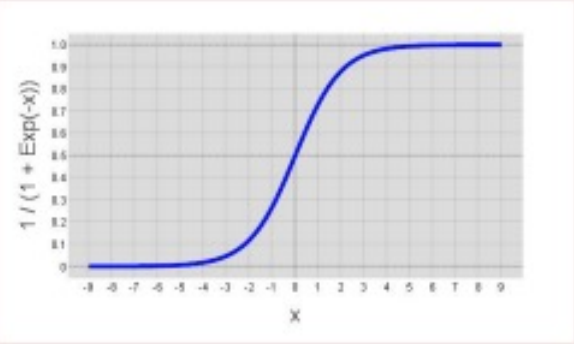
- Train a classifier that can make a binary decision about the class of a new input
  - Weight vector β_i with bias (intercept) β_0
  - Observations X_i
  - $P(Y = 0|X) = \frac{1}{1+\exp(\beta_0+\sum_i \beta_i X_i)}, P(Y = 1|X) = \frac{\exp(\beta_0+\sum_i \beta_i X_i)}{1+\exp(\beta_0+\sum_i \beta_i X_i)}$
- For shorthand, we'll say that
  - $P(Y = 0|X) = \sigma(-(\beta_0 + \sum_i \beta_i X_i))$
  - $P(Y = 1|X) = 1 - \sigma(-(\beta_0 + \sum_i \beta_i X_i))$
  - Where $\sigma(z) = \frac{1}{1+\exp(-z)}$

- What is this "exp" doing?
  - e is a special number, 2.71..
    - e^x is the limit of compound interest formula as compounds become infinitely small
    - It's the function whose derivative is itself
    - The logistic function is $\sigma(z) = \frac{1}{1+e^{-z}}$
    - S-shape and between 0 and 1
      - Model probabilities

Exponential

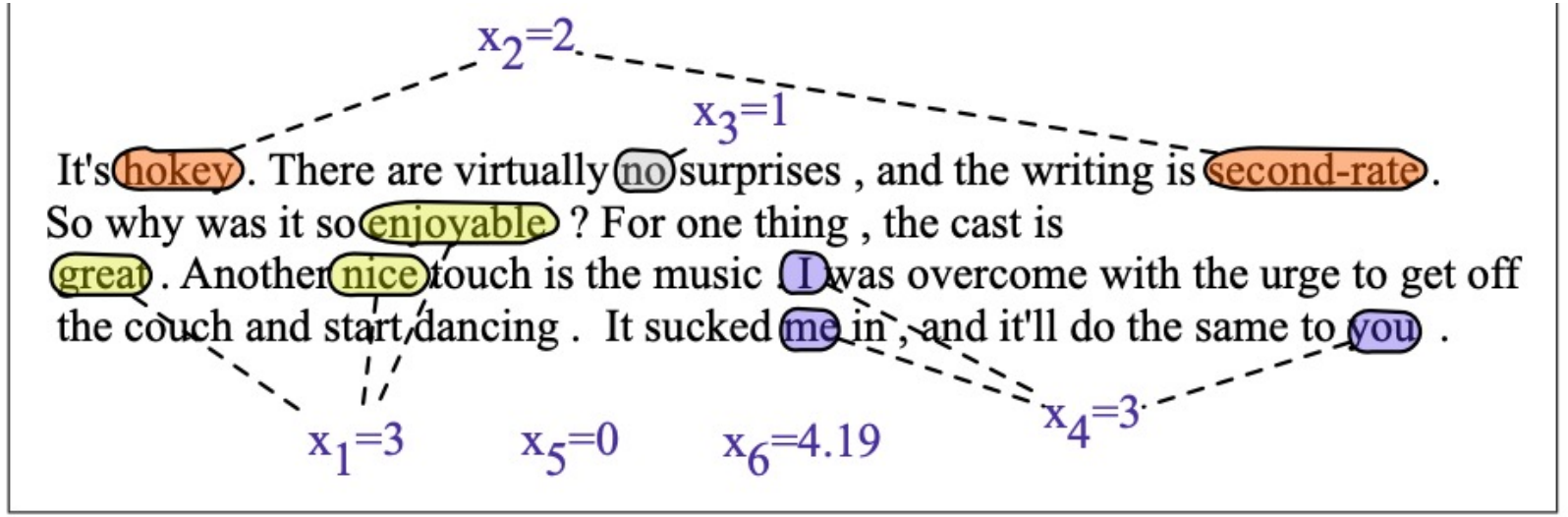


Logistic



SENTIMENT EXAMPLE: Y=1 OR Y=0?

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .



Var	Definition	Value in Fig. 5.2
x_1	count(positive lexicon) $\in$ doc)	3
x_2	count(negative lexicon) $\in$ doc)	2
x_3	$\begin{cases} 1 & \text{if "no"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	1
x_4	count(1st and 2nd pronouns $\in$ doc)	3
x_5	$\begin{cases} 1 & \text{if "!"} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	0
x_6	log(word count of doc)	$\ln(66) = 4.19$

Var	Definition	Val	5.2
x_1	count(positive lexicon) $\in$ doc)	3	
x_2	count(negative lexicon) $\in$ doc)	2	
x_3	$\begin{cases} 1 & \text{if “no”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	1	
x_4	count(1st and 2nd pronouns $\in$ doc)	3	
x_5	$\begin{cases} 1 & \text{if “!”} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$	0	
x_6	log(word count of doc)	$\ln(66) = 4.19$	

$$\text{Suppose } w = [2.5, -5.0, -1.2, 0.5, 2.0, 0.7]$$

$$b = 0.1$$

$$
\begin{aligned}
p(+|x) = P(Y=1|x) &= \sigma(w \cdot x + b) \\
&= \sigma([2.5, -5.0, -1.2, 0.5, 2.0, 0.7] \cdot [3, 2, 1, 3, 0, 4.19] + 0.1) \\
&= \sigma(.833) \\
&= 0.70
\end{aligned}
$$

$$
\begin{aligned}
p(-|x) = P(Y=0|x) &= 1 - \sigma(w \cdot x + b) \\
&= 0.30
\end{aligned}
$$

- # How is Logistic Regression Used?
  - Given a set of weights $\vec{\beta}$, we know how to compute the conditional likelihood $P(y|\beta, x)$
  - Find the set of weights $\vec{\beta}$ that maximize the conditional likelihood on training data (soon!)
  - Intuition: higher weights mean that this feature implies that this feature is a good this is the class you want for this observation
  - Naïve Bayes is a special case of logistic regression that uses Bayes rule and conditional probabilities to set these weights
    - $\text{Argmax}_{c_j}\ [\ln \hat{P}(c_j) + \sum_i \ln \hat{P}(w_i|c_j)]$

- Contrasting Naïve Bayes and Logistic Regression
  - Naïve Bayes easier
  - Naïve Bayes better on smaller datasets
  - Logistic regression better on medium-sized datasets
  - On huge datasets, it does not really matter (data always win)
    - Optional reading by Ng and Jordan has proofs and experiments
  - Logistic regression allows arbitrary features (biggest difference!)
  - Don't need to memorize (or work through) previous slide - just understand that naïve Bayes is a special case of logistic regresion

- Wait, where did the $\beta's$ come from?
  - Supervised classification: correct label y (either 0 or 1) for each x
  - But what the system produces is an estimate $\hat{y}$
  - We need
    - a loss function or a cost function
    - An optimization algorithm to update βs to minimize the loss

- A loss function
  - Cross-entropy loss
- An optimization algorithm
  - Stochastic gradient descent

LOSS FUNCTION

- We want to know how far is the classifier output from the true output
  - $L(\hat{y}, y)$: how much they are different
- Since there are only two discrete outcomes, this is a Bernoulli distribution
  - $p(y|x) = \hat{y}^y(1-\hat{y})^{1-y}$
  - Take the log of both sides, flip the sign, and plug in the definition of $\hat{y}$
  - $L_{CE} = -\left[y \log \sigma(wx+b) + (1-y) \log\left(1-\sigma(wx+b)\right)\right]$
- In information theory, when a probability dist is intended to approximate some true distribution, the cross entropy of the two distributions is a measure of how different they are

- Let's see if this works for our sentiment example
- We want loss to be
  - Smaller if the model estimate is closed to correct
  - Bigger if model is confused
- Let's first suppose the true label of this is y=1 (positive)

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

- True value is y=1. How well is our model doing?

$$
\begin{aligned}
p(+|x) = P(Y=1|x) &= \sigma(w\cdot x+b) \\
&= \sigma([2.5,-5.0,-1.2,0.5,2.0,0.7]\cdot[3,2,1,3,0,4.19]+0.1) \\
&= \sigma(.833) \\
&= 0.70
\end{aligned}
$$

$$
\begin{aligned}
p(-|x) = P(Y=0|x) &= 1-\sigma(w\cdot x+b) \\
&= 0.30
\end{aligned}
$$

What's the loss?

$$
\begin{aligned}
L_{\text{CE}}(\hat{y},y) &= -[y\log\sigma(w\cdot x+b)+(1-y)\log(1-\sigma(w\cdot x+b))] \\
&= -[\log\sigma(w\cdot x+b)] \\
&= -\log(.70) \\
&= .36
\end{aligned}
$$

- Suppose true value instead was y=0

$$p(-|x) = P(Y=0|x) = 1-\sigma(w\cdot x+b)$$
$$= 0.30$$

What's the loss?

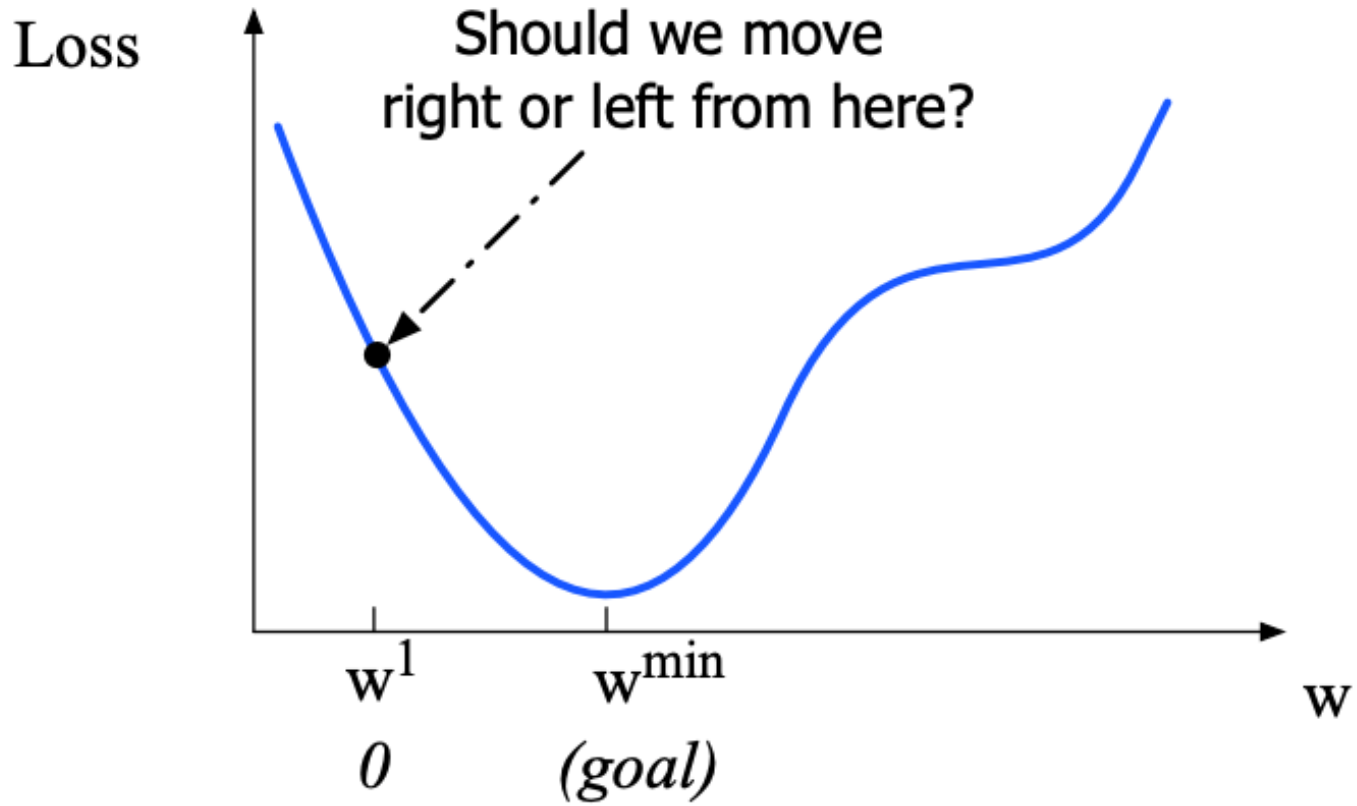
$$L_{CE}(\hat{y},y) = -[y\log\sigma(w\cdot x+b)+(1-y)\log(1-\sigma(w\cdot x+b))]$$
$$= -[\log(1-\sigma(w\cdot x+b))]$$
$$= -\log(.30)$$
$$= 1.2$$

Sure enough, loss was bigger when model was wrong!

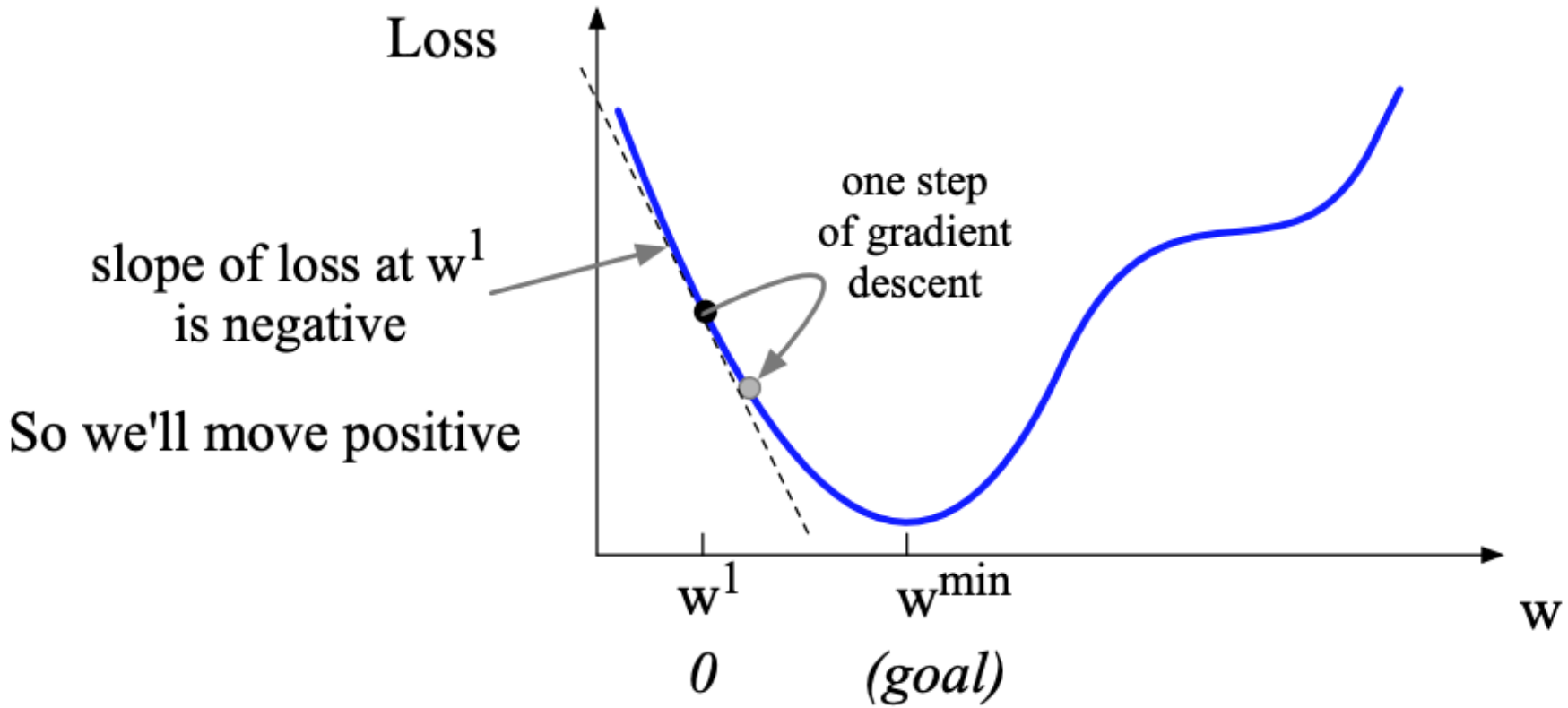
- Goal: minimize the loss
- Let's make explicit that the loss function is parameterized by weights $\theta = (w, b)$
- And, we'll represent $\hat{y}$ as $f(x; \theta)$ to make the dependence on θ more obvious
- We want the weights that minimize the loss, averaged over all examples
  - $\hat{\theta} = \text{argmin}_\theta \frac{1}{m} \sum_i L_{CE}(f(x^{(i)}; \theta), y^{(i)})$

- For logistic regression, loss function is convex
  - A convex function has just one minimum
  - Gradient descent starting from any point is guaranteed to find the minimum

Q: Given current w, should we make it bigger or smaller?
A: Move *w* in the reverse direction from the slope of the function



Q: Given current w, should we make it bigger or smaller?
A: Move *w* in the reverse direction from the slope of the function



- The **gradient** of a function of many variables is a vector pointing in the direction of the greatest increase in a function
- **Gradient Descent**
  - Find the gradient of the loss function at the current point and move in the **opposite** direction

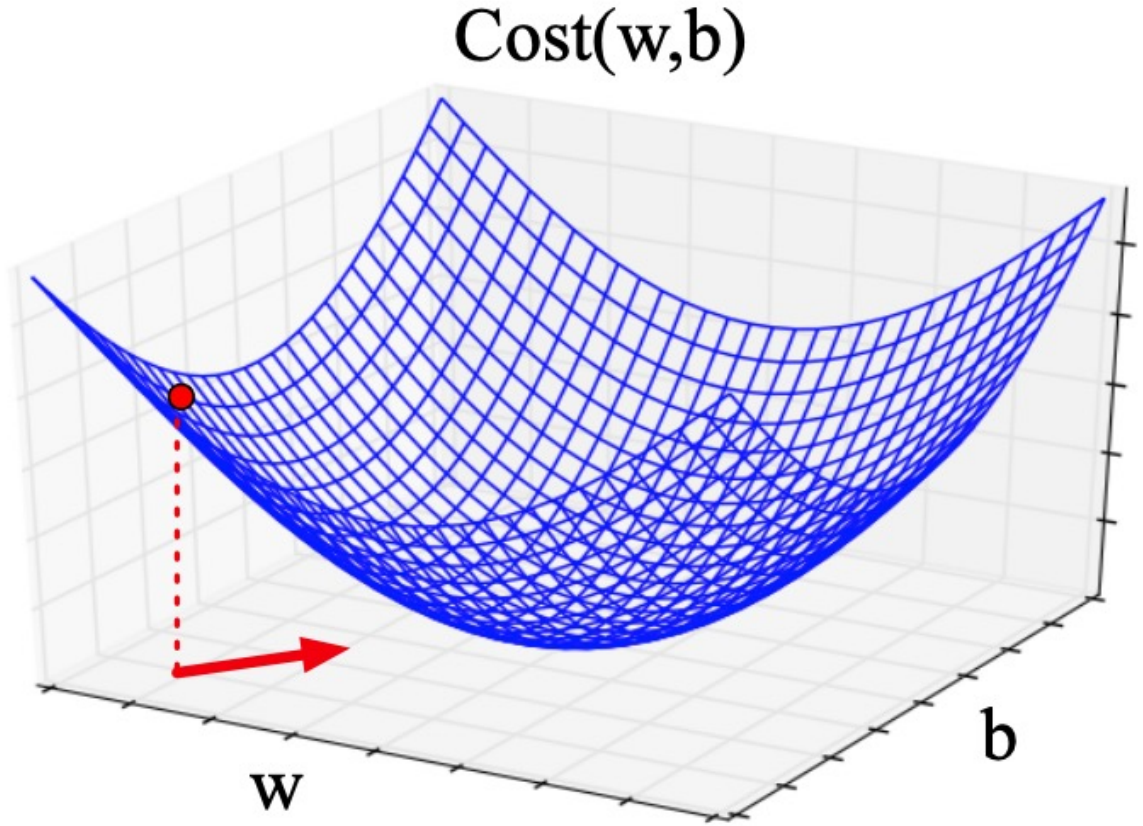
- **How much do we move in that direction?**
  - The value of the gradient (slope in our example) $\frac{d}{dw} L(f(x; w), y)$ weighted by a learning rate η
  - Higher learning rate means move w faster
    - $w^{t+1} = w^t - \eta \frac{d}{dw} L(f(x; w), y)$

- Now let's consider N dimensions
  - We want to know where in the N-dimensional space (of the N parameters that make up θ) we should move
  - The gradient is just such a vector; it expresses the directional components of the sharpest slope along each of the N dimensions

Imagine 2 dimensions, w and b

Visualizing the gradient vector at the red point

It has two dimensions shown in the x-y plane



We'll represent $\hat{y}$ as $f(x; \theta)$ to make the dependence on θ more obvious:

$$\nabla_\theta L(f(x;\theta),y)) = \begin{bmatrix} \frac{\partial}{\partial w_1} L(f(x;\theta),y) \\ \frac{\partial}{\partial w_2} L(f(x;\theta),y) \\ \vdots \\ \frac{\partial}{\partial w_n} L(f(x;\theta),y) \end{bmatrix}$$

The final equation for updating θ based on the gradient is thus

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta),y)$$

The loss function

$$L_{\text{CE}}(\hat{y}, y) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log (1 - \sigma(w \cdot x + b))]$$

The elegant derivative of this function (see textbook 5.8 for derivation)

$$\frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y] x_j$$

EXAMPLE

- One step of gradient descent
- A mini-sentiment example, where the true y=1 (positive)
- Two features
  - $x_1 = 3$ (count of positive lexicon words)
  - $x_2 = 2$ (count of negative lexicon words)
- Assume 3 parameters (2 weights and 1 bias) in Θ^0 are zero:
  - $w_1 = w_2 = b = 0$
  - $\eta = 0.1$

- Update step for update θ is:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y)$$

where $$\frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y] x_j$$

Gradient vector has 3 dimensions:

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_1} \\ \frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_2} \\ \frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial b} \end{bmatrix}$$

- Update step for update θ is:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y)$$

where $$\frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y] x_j$$

Gradient vector has 3 dimensions:

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_1} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_2} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot x + b) - y) x_1 \\ (\sigma(w \cdot x + b) - y) x_2 \\ \sigma(w \cdot x + b) - y \end{bmatrix}$$

- Update step for update θ is:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y)$$

where $$\frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y] x_j$$

Gradient vector has 3 dimensions:

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_1} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_2} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot x + b) - y) x_1 \\ (\sigma(w \cdot x + b) - y) x_2 \\ \sigma(w \cdot x + b) - y \end{bmatrix}$$

- Update step for update θ is:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y)$$

where $$\frac{\partial L_{\text{CE}}(\hat{y}, y)}{\partial w_j} = [\sigma(w \cdot x + b) - y] x_j$$

Gradient vector has 3 dimensions:

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_1} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial w_2} \\ \frac{\partial L_{\text{CE}}(\hat{y},y)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot x + b) - y) x_1 \\ (\sigma(w \cdot x + b) - y) x_2 \\ \sigma(w \cdot x + b) - y \end{bmatrix} = \begin{bmatrix} (\sigma(0) - 1) x_1 \\ (\sigma(0) - 1) x_2 \\ \sigma(0) - 1 \end{bmatrix} = \begin{bmatrix} -0.5 x_1 \\ -0.5 x_2 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix}$$

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{CE}(\hat{y},y)}{\partial w_1} \\ \frac{\partial L_{CE}(\hat{y},y)}{\partial w_2} \\ \frac{\partial L_{CE}(\hat{y},y)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot x + b) - y)x_1 \\ (\sigma(w \cdot x + b) - y)x_2 \\ \sigma(w \cdot x + b) - y \end{bmatrix} = \begin{bmatrix} (\sigma(0) - 1)x_1 \\ (\sigma(0) - 1)x_2 \\ \sigma(0) - 1 \end{bmatrix} = \begin{bmatrix} -0.5x_1 \\ -0.5x_2 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix}$$

Now that we have a gradient, we compute the new parameter vector θ^1 by moving θ^0 in the opposite direction from the gradient:

$$\theta_{t+1} = \theta_t - \eta \nabla L(f(x;\theta), y) \qquad \eta = 0.1;$$

$$\theta^1 = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} - \eta \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} .15 \\ .1 \\ .05 \end{bmatrix}$$

Note that enough negative examples would eventually make w_2 negative