Number Fields Associated to Hyperbolic 3-Manifolds[*]

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This semester our project focused on two primary goals. First, given a finite collection of complex numbers recover the number field they generate and express them in terms of the final description. The basic algorithm using LLL is described in the Exp. Math. paper of Goodman, Neumann, et. al., but subsequent related algorithms (e.g. PSLQ) may have better performance, and anyhow there's a lot of parameters that need to be tuned (e.g. different variants of LLL). We looked at a large number of examples and tested the various implementations of the algorithms to determine which worked the best. Second, using this algorithm, we tried to find evidence to support or refute a conjecture, which will be stated later.

Our general focus is on knot and link complements which are examples of 3-manifolds. Firstly we need to define a mathematical knot. The easiest way to visualize a knot is to tie a knot in a piece string and glue the ends together. Formally we define a knot as:

A knot $K \subseteq \mathbb{R}^3$ or S^3 is a subset of points homeomorphic to the circle S^1

Next we define a link is a finite disjoint union of knots. One link we examine is the Whitehead link.

Secondly our next major component we define is manifold. Manifolds are a topological space that is locally Euclidean around every point. To illustrate this idea of manifold, consider the ancient belief that the Earth was flat as contrasted with the modern evidence that it is round. The discrepancy arises essentially from the fact that on the small scales that we see, the Earth

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does indeed look flat. However from a far point in space we view the Earth as a sphere. In general, any object that is nearly “flat” on small scales is a manifold, and so manifolds constitute a generalization of objects we could live on in which we would encounter the round or flat Earth problem.

The basic example of a manifold is Euclidean space, and many of its properties carry over to manifolds. In addition, any smooth boundary of a subset of Euclidean space, like the circle or the sphere, is a manifold. Manifolds are therefore of interest in the study of geometry, topology, and analysis.

One of the goals of topology is to find ways of distinguishing manifolds. A circle is topologically the same as any closed loop, no matter how different these two manifolds may appear. Similarly, the surface of a coffee mug with a handle is topologically the same as the surface of the donut, and this type of surface is called a (one-handled) torus.

Manifolds arise naturally in a variety of mathematical and physical applications as ”global objects.” For example, in order to precisely describe all the configurations of a robot arm or all the possible positions and momenta of a rocket, an object is needed to store all of these parameters. The objects that crop up are manifolds. From the geometric perspective, manifolds represent the profound idea having to do with global versus local properties.

Next, our third component is the compliment of a knot or link. A hyperbolic link embedded in $\mathbb{R}^3$ or S^3 has a complement with a geometry of constant negative curvature. The compliment turns out to be a hyperbolic manifold. The hyperbolic volume of a hyperbolic link is the finite volume of the link’s complement with respect to its complete hyperbolic metric. The volume is an important topological invariant of the link. In particular, it is known that there are only finitely many hyperbolic knots with the same volume and hyperbolic volume has proven very effective in distinguishing knots.

We use the open source program SnapPy to examine these knot/link compliments. SnapPy can return a representation of manifolds as a set of complex numbers called shape parameters. These shape parameters in the form $a + bi$ are algebraic, so they are polynomial roots. We have created code to systematically go through multiple 3-manifold examples and find their shape parameters and then find the representative polynomial.

To calculate these polynomials we use the lattice basis reduction called the Lenstra Lenstra Lovasz algorithm. The code calculates linear dependence of a set of shape parameters and tires to find the ”shortest” vector in a given integer lattice which in turn represents the coefficients of a polynomial. Each polynomial has some degree $n \in \mathbb{N}$ in the end we are to find a number field associated with each manifold from the polynomials which is our first goal.

Lastly our second goal is to examine with evidence the conjecture that states, every set of manifolds with bounded volume and bounded degree is finite. To find evidence for refuting or supporting this conjecture we perform what is called a Dhen filling on our manifold compliment. More specifically we Dhen fill the Whitehead link. Consider a general view of a knot/link compliment of a 2-manifold with constant negative curvature. Each compliment is structured with cusps that extend to infinity. The Dhen filling procedure is essentially preforming a slice down the cusps and then re-gluing the cusp in a slightly different position but not as to destroy the cusp or the rest of the compliment. This in turn produces a new manifold that however is not a knot/link compliment. This new manifold will certainly look different from the original manifold but also have less volume. In other words, we know that the volumes of these new manifolds are bounded above by the volume of the original link compliment manifold. This comes into play when examining the various manifolds in a set in relation to the conjecture.