Proofs

Direct Proofs

Proving $p \rightarrow q$(implications)

Step 1: Assume p (is true)

Step 2: Show that q logically follows

Direct Proofs: Example 1

If n is odd then $3n + 2$ is odd

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If n is odd then $3n + 2$ is odd

p: n is odd
q: $3n + 2$ is odd

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p: n is odd
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Step1: Assume, p is true.

Hence, n is odd

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If n is odd then $3n + 2$ is odd

p: n is odd
q: $3n + 2$ is odd

Step1: Assume, p is true.

Hence, n is odd

$\therefore n = 2k + 1$

Direct Proofs: Example 1

Step2:

$3n + 2$

$= 3 * (2k + 1) + 2$

$= 6k + 5$

$= 6k + 4 + 1$

$= 2 (3k + 2) + 1$

$= 2 m+ 1$

Direct Proofs: Example 1

Step2:

$3n + 2$

$= 3 * (2k + 1) + 2$

$= 6k + 5$

$= 6k + 4 + 1$

$= 2 (3k + 2) + 1$

$= 2m + 1$

$\therefore$ $3n + 2$ is odd

Direct Proofs: Example 2

If $0 \leq x \leq 2$, then prove that $-x^3 + 4x + 1 > 0$

Direct Proofs: Example 2

If $0 \leq x \leq 2$, then prove that $-x^3 + 4x + 1 > 0$

p: $0 \leq x \leq 2$
q: $-x^3 + 4x + 1 > 0$

Direct Proofs: Example 2

If $0 \leq x \leq 2$, then prove that $-x^3 + 4x + 1 > 0$

p: $0 \leq x \leq 2$
q: $-x^3 + 4x + 1 > 0$

Step1: Assume, p is true.

Hence, $0 \leq x \leq 2$ is true

Direct Proofs: Example 2

Step2:

$$\begin{aligned}
&-x^3 + 4x + 1 \\
&= x(-x^2 + 4) + 1 \\
&= x(2 + x)(2 - x) + 1
\end{aligned}$$

Direct Proofs: Example 2

Step2:

$$
\begin{aligned}
&-x^3 + 4x + 1 \\
&= x(-x^2 + 4) + 1 \\
&= x(2 + x)(2 - x) + 1
\end{aligned}
$$

As $0 \leq x \leq 2$

$x = 0: x(2 + x)(2 - x) = 0$
$x = 1: x(2 + x)(2 - x) = 3$
$x = 2: x(2 + x)(2 - x) = 0$

Direct Proofs: Example 2

Step2:

$$-x^3 + 4x + 1$$
$$= x(-x^2 + 4) + 1$$
$$= x(2 + x)(2 - x) + 1$$

As $0 \leq x \leq 2$

$x = 0: x(2 + x)(2 - x) = 0$

$x = 1: x(2 + x)(2 - x) = 3$

$x = 2: x(2 + x)(2 - x) = 0$

$\therefore x(2 + x)(2 - x) + 1 > 0$

$\therefore -x^3 + 4x + 1 > 0$

Proof by Contraposition

$p \rightarrow q$(implications)

Contrapositive: $\neg q \rightarrow \neg p$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Proving one is as good as proving the other
proving the contrapositive is some-times easier than proving the implication

Proof by Contraposition

Contrapositive: $\neg q \rightarrow \neg p$

Step 1: Assume $\neg q$ (is true)

Step 2: Show that $\neg p$ logically follows

Proof by Contraposition: Example 1

If $3n + 2$ is odd then n is odd

Proof by Contraposition: Example 1

If $3n + 2$ is odd then n is odd

p: $3n + 2$ is odd
q: n is odd

Proof by Contraposition: Example 1

If $3n + 2$ is odd then n is odd

p: $3n + 2$ is odd
q: n is odd

$\neg$ p: $3n + 2$ is even
$\neg$ q: n is even

Proof by Contraposition: Example 1

If $3n + 2$ is odd then n is odd

p: $3n + 2$ is odd
q: n is odd

$\neg$ p: $3n + 2$ is even
$\neg$ q: n is even

Step1: Assume, $\neg$ q is true.

Hence, n is even

Proof by Contraposition: Example 1

If $3n + 2$ is odd then n is odd

p: $3n + 2$ is odd
q: n is odd

$\neg p$: $3n + 2$ is even
$\neg q$: n is even

Step1: Assume, $\neg q$ is true.
Hence, n is even

$\therefore n = 2k$

Proof by Contraposition: Example 1

Step2:

$3n + 2$

$= 3 * (2k) + 2$

$= 6k + 2$

$= 2\ (3k + 1)$

$= 2\ m$

Proof by Contraposition: Example 1

Step2:

$3n + 2$

$= 3 * (2k) + 2$

$= 6k + 2$

$= 2 (3k + 1)$

$= 2 m$

$\therefore 3n + 2$ is even

Proof by Contraposition: Example 1

Step2:

$3n + 2$

$= 3 * (2k) + 2$

$= 6k + 2$

$= 2 (3k + 1)$

$= 2 m$

$\therefore$ 3n + 2 is even

$\therefore \neg q \rightarrow \neg p$

$\therefore p \rightarrow q$

Proof by Contraposition: Example 2

If $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

p: $n = ab$
q: $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Proof by Contraposition: Example 2

If n = ab then a $\leq \sqrt{n}$ or b $\leq \sqrt{n}$

p: n = ab

q: a $\leq \sqrt{n}$ or b $\leq \sqrt{n}$

$\neg$ p: n $\neq$ ab

$\neg$ q: $\neg$(a $\leq \sqrt{n}$ or b $\leq \sqrt{n}$)

$\equiv \neg$(a $\leq \sqrt{n}$) and $\neg$(b $\leq \sqrt{n}$)

$\equiv$ (a $> \sqrt{n}$) and (b $> \sqrt{n}$)

Proof by Contraposition: Example 2

Step1: Assume, $\neg$ q is true.

Hence, (a $> \sqrt{n}$) and (b $> \sqrt{n}$) is true

Proof by Contraposition: Example 2

Step1: Assume, $\neg\ q$ is true.

Hence, (a $> \sqrt{n}$) and (b $> \sqrt{n}$) is true

Step2:

(a $> \sqrt{n}$) and
(b $> \sqrt{n}$)

$\therefore$ ab $> \sqrt{n}\sqrt{n}$
$\therefore$ ab > n

Proof by Contraposition: Example 2

Step1: Assume, $\neg$ q is true.

Hence, (a $> \sqrt{n}$) and (b $> \sqrt{n}$) is true

Step2:

(a $> \sqrt{n}$) and
(b $> \sqrt{n}$)

$\therefore$ ab $> \sqrt{n}\sqrt{n}$

$\therefore$ ab > n

n $\neq$ ab

$\therefore \neg q \rightarrow \neg p$

$\therefore p \rightarrow q$

Proof by Contradiction

$p \rightarrow q$(implications)

Step1: Assume, p is true.

Step2: Assume, $\neg$ q is true.

Step3: Prove that $\neg$p is true from $\neg$ q

Proof by Contradiction: Example 1

If $3n + 2$ is odd then n is odd

p: $3n + 2$ is odd
q: n is odd

$\neg$ p: $3n + 2$ is even
$\neg$ q: n is even

Proof by Contradiction: Example 1

If 3n + 2 is odd then n is odd

p: 3n + 2 is odd
q: n is odd

$\neg$ p: 3n + 2 is even
$\neg$ q: n is even

Step1: Assume, p is true.

Hence, 3n + 2 is odd

Proof by Contradiction : Example 1

Step2: Assume, $\neg$ q is true.

Hence, n is even

$\therefore n = 2k$

Proof by Contradiction : Example 1

Step2: Assume, $\neg q$ is true.
Hence, n is even

$\therefore n = 2k$

Step3: Prove that $\neg p$ is true

$3n + 2$
$= 3 * (2k) + 2$
$= 6k + 2$
$= 2(3k + 1)$
$= 2m$

Proof by Contradiction : Example 1

Step2: Assume, $\neg q$ is true.

Hence, n is even

$\therefore n = 2k$

Step3: Prove that $\neg p$ is true

$3n + 2$

$= 3 * (2k) + 2$

$= 6k + 2$

$= 2 (3k + 1)$

$= 2 m$

$\therefore 3n + 2$ is even

$\therefore \neg p$ is true

Proof by Contradiction: Example 1

From Step1: p is true.

Hence, $3n + 2$ is odd

Proof by Contradiction: Example 1

From Step1: p is true.

Hence, $3n + 2$ is odd

From Step3: $\neg p$ is true

$3n + 2$ is even

Proof by Contradiction: Example 1

From Step1: p is true.
Hence, $3n + 2$ is odd

From Step3: $\neg p$ is true
$3n + 2$ is even

This is a contradiction

$\therefore$ If $3n + 2$ is odd then n is odd

Proof by Contradiction

p (propositional statement)

Step1: Assume, $\neg$ p is true.

Proof by Contradiction: Example 2

Prove that $\sqrt{2}$ is irrational

p: $\sqrt{2}$ is irrational

$\neg$ p: $\sqrt{2}$ is rational

Step1: Assume, $\neg$ p is true.

Hence, $\sqrt{2}$ is rational

Proof by Contradiction: Example 2

$$\sqrt{2} = \frac{a}{b}$$
$$\text{or}, 2 = \frac{a^2}{b^2}$$
$$\text{or}, 2b^2 = a^2$$
$$\text{or}, a^2 = 2b^2$$

∴ a^2 is even

∴ a is even

∴ a = 2c

Proof by Contradiction: Example 2

$$(2c)^2 = 2b^2$$
$$\text{or, } 4c^2 = 2b^2$$
$$\text{or, } 2c^2 = b^2$$
$$\text{or, } b^2 = 2c^2$$

$\therefore$ b^2 is even

$\therefore$ b is even

$\therefore$ b = 2d

Proof by Contradiction: Example 2

$$\sqrt{2} = \frac{a}{b}$$

$$\text{or,} \sqrt{2} = \frac{2c}{2d}$$

$\therefore \sqrt{2}$ can not be a rational number

$\therefore \sqrt{2}$ irrational number

Proof by Induction

$\forall n p(n)$, n is a positive integer

Step 1(Basis step): verify that $p(1)$ is true.

Step 2(Inductive step): Show that

$\forall k\, p(k) \rightarrow p(k+1)$ is true, k is a positive integer

Proof by Induction: Example 1

$p(n) ::= [1 + 2 + \ldots + n = n(n+1) / 2]$

Proof by Induction: Example 1

$p(n) ::= [1 + 2 + \ldots + n = n(n+1) / 2]$

Step 1(Basis step):

$p(1) ::= [1 = 1(1+1) / 2] = \text{True}$

Proof by Induction: Example 1

Step 2(Inductive step):
Show that
$\forall k\ p(k) \rightarrow p(k+1)$ is true, k is a positive integer

$p(k)$: $1 + 2 + \ldots + k = k(k + 1) / 2$
$p(k+1)$: $1 + 2 + \ldots + (k+1) = (k+1)(k + 2) / 2$

Proof by Induction: Example 1

Assume, p(k) is true

$\therefore 1 + 2 + \ldots + k = k(k + 1) / 2$

Proof by Induction: Example 1

Assume, $p(k)$ is true

$\therefore 1 + 2 + \ldots + k = k(k + 1) / 2$

or, $1 + 2 + \ldots + k + (k+1) = k(k + 1) / 2 + (k+1)$

Why add $(k+1)$?
→ Check $p(k+1)$

Proof by Induction: Example 1

Show that, p(k+1) is true

$\therefore$ p(k + 1) is true

$$1 + 2 + \ldots + k = k(k + 1) / 2$$

or, $1 + 2 + \ldots + k + (k+1) = k(k + 1) / 2 + (k+1)$

or, $1 + 2 + \ldots \ldots \quad + (k+1) = k(k + 1) / 2 + (k+1)$

or, $1 + 2 + \ldots \ldots \quad + (k+1) = (k + 1)\ (k+2)/ 2$

Proof by Induction: Example 1

Show that, $p(k+1)$ is true

$$1 + 2 + \dots + k = k(k+1)/2$$

or, $1 + 2 + \dots + k + (k+1) = k(k+1)/2 + (k+1)$

or, $1 + 2 + \dots \dots + (k+1) = k(k+1)/2 + (k+1)$

or, $1 + 2 + \dots \dots + (k+1) = (k+1)(k+2)/2$

$\therefore p(k+1)$ is true

$\forall k\, p(k) \rightarrow p(k+1)$

Proof by Induction: Example 1

$$p(1) \text{ and } \forall k\ p(k) \rightarrow p(k+1)$$
$$\therefore \forall n\ p(n)$$

Proof by Induction: Example 2

$p(n) ::= [2^n < n!]$ for $n >= 4$

Step 1(Basis step):

$p(4) ::= [16 < 24] = \text{True}$

Proof by Induction: Example 2

Step 2(Inductive step):
Show that
$p(k) \rightarrow p(k+1)$ is true, $k >= 4$

$p(k): 2^k < k!$
$p(k+1): 2^{k+1} < (k+1)!$

Proof by Induction: Example 2

Assume, p(k) is true

$\therefore 2^k < k!$

or, $2^k \, 2 < k! \, 2$

or, $2^{k+1} < 2 \, k!$

or, $2^{k+1} < (k+1) \, k!$ [k > = 4]

or, $2^{k+1} < (k+1)!$

Proof by Induction: Example 2

Assume, p(k) is true

$\therefore 2^k < k!$

or, $2^k\, 2 < k!\, 2$

or, $2^{k+1} < 2\, k!$

or, $2^{k+1} < (k+1)\, k!$ [k > = 4]

or, $2^{k+1} < (k+1)!$

$\therefore$ p(k + 1) is true