

# Proofs

# Direct Proofs

## Proving $p \rightarrow q$ (implications)

Step 1: Assume  $p$  (is true)

Step 2: Show that  $q$  logically follows

# Direct Proofs: Example 1

If  $n$  is odd then  $3n + 2$  is odd

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$p$ :  $n$  is odd

$q$ :  $3n + 2$  is odd

Step1: Assume,  $p$  is true.

Hence,  $n$  is odd

$$\therefore n = 2k + 1$$

# Direct Proofs: Example 1

Step2:

$$3n + 2$$

$$= 3 * (2k + 1) + 2$$

$$= 6k + 5$$

$$= 6k + 4 + 1$$

$$= 2 (3k + 2) + 1$$

$$= 2 m + 1$$

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$\therefore 3n + 2$  is odd



# Direct Proofs: Example 2

If  $0 \leq x \leq 2$ , then prove that  $-x^3 + 4x + 1 > 0$

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$p$ :  $0 \leq x \leq 2$

$q$ :  $-x^3 + 4x + 1 > 0$

# Direct Proofs: Example 2

If  $0 \leq x \leq 2$ , then prove that  $-x^3 + 4x + 1 > 0$

$p: 0 \leq x \leq 2$

$q: -x^3 + 4x + 1 > 0$

Step1: Assume,  $p$  is true.

Hence,  $0 \leq x \leq 2$  is true

# Direct Proofs: Example 2

Step2:

$$\begin{aligned} & -x^3 + 4x + 1 \\ &= x(-x^2 + 4) + 1 \\ &= x(2 + x)(2 - x) + 1 \end{aligned}$$

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As  $0 \leq x \leq 2$

$$x = 0: x(2 + x)(2 - x) = 0$$

$$x = 1: x(2 + x)(2 - x) = 3$$

$$x = 2: x(2 + x)(2 - x) = 0$$

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**As  $0 \leq x \leq 2$**

$$x = 0: x(2 + x)(2 - x) = 0$$

$$x = 1: x(2 + x)(2 - x) = 3$$

$$x = 2: x(2 + x)(2 - x) = 0$$

$$\therefore x(2 + x)(2 - x) + 1 > 0$$

$$\therefore -x^3 + 4x + 1 > 0$$

# Proof by Contraposition

$p \rightarrow q$  (implications)

Contrapositive:  $\neg q \rightarrow \neg p$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Proving one is as good as proving the other  
proving the contrapositive is some-times easier than proving  
the implication

# Proof by Contraposition

Contrapositive:  $\neg q \rightarrow \neg p$

Step 1: Assume  $\neg q$  (is true)

Step 2: Show that  $\neg p$  logically follows



# Proof by Contraposition: Example 1

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If  $3n + 2$  is odd then  $n$  is odd

$p$ :  $3n + 2$  is odd

$q$ :  $n$  is odd

$\neg p$ :  $3n + 2$  is even

$\neg q$ :  $n$  is even

# Proof by Contraposition: Example 1

If  $3n + 2$  is odd then  $n$  is odd

$p$ :  $3n + 2$  is odd

$q$ :  $n$  is odd

$\neg p$ :  $3n + 2$  is even

$\neg q$ :  $n$  is even

Step1: Assume,  $\neg q$  is true.

Hence,  $n$  is even

# Proof by Contraposition: Example 1

If  $3n + 2$  is odd then  $n$  is odd

$p$ :  $3n + 2$  is odd

$q$ :  $n$  is odd

$\neg p$ :  $3n + 2$  is even

$\neg q$ :  $n$  is even

Step1: Assume,  $\neg q$  is true.

Hence,  $n$  is even

$$\therefore n = 2k$$

# Proof by Contraposition: Example 1

Step2:

$$3n + 2$$

$$= 3 * (2k) + 2$$

$$= 6k + 2$$

$$= 2 (3k + 1)$$

$$= 2 m$$

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$$= 3 * (2k) + 2$$

$$= 6k + 2$$

$$= 2 (3k + 1)$$

$$= 2 m$$

$\therefore 3n + 2$  is even

# Proof by Contraposition: Example 1

Step2:

$$3n + 2$$

$$= 3 * (2k) + 2$$

$\therefore 3n + 2$  is even

$$= 6k + 2$$

$$= 2 (3k + 1)$$

$$\therefore \neg q \rightarrow \neg p$$

$$= 2 m$$

$$\therefore p \rightarrow q$$



# Proof by Contraposition: Example 2

If  $n = ab$  then  $a \leq \sqrt{n}$  or  $b \leq \sqrt{n}$

$p$ :  $n = ab$

$q$ :  $a \leq \sqrt{n}$  or  $b \leq \sqrt{n}$

# Proof by Contraposition: Example 2

If  $n = ab$  then  $a \leq \sqrt{n}$  or  $b \leq \sqrt{n}$

$p: n = ab$

$q: a \leq \sqrt{n} \text{ or } b \leq \sqrt{n}$

$\neg q: \neg(a \leq \sqrt{n} \text{ or } b \leq \sqrt{n})$   
 $\equiv \neg(a \leq \sqrt{n}) \text{ and } \neg(b \leq \sqrt{n})$

$\neg p: n \neq ab$

$\equiv (a > \sqrt{n}) \text{ and } (b > \sqrt{n})$

# Proof by Contraposition: Example 2

Step1: Assume,  $\neg q$  is true.

Hence,  $(a > \sqrt{n})$  and  $(b > \sqrt{n})$  is true

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Step1: Assume,  $\neg q$  is true.

Hence,  $(a > \sqrt{n})$  and  $(b > \sqrt{n})$  is true

Step2:

$(a > \sqrt{n})$  and  
 $(b > \sqrt{n})$

$\therefore ab > \sqrt{n}\sqrt{n}$

$\therefore ab > n$

# Proof by Contraposition: Example 2

Step1: Assume,  $\neg q$  is true.

Hence,  $(a > \sqrt{n})$  and  $(b > \sqrt{n})$  is true

Step2:

$(a > \sqrt{n})$  and  
 $(b > \sqrt{n})$

$\therefore ab > \sqrt{n}\sqrt{n}$

$\therefore ab > n$

$n \neq ab$

$\therefore \neg q \rightarrow \neg p$

$\therefore p \rightarrow q$

# Proof by Contradiction

$p \rightarrow q$ (implications)

Step1: Assume,  $p$  is true.

Step2: Assume,  $\neg q$  is true.

Step3: Prove that  $\neg p$  is true from  $\neg q$

# Proof by Contradiction: Example 1

If  $3n + 2$  is odd then  $n$  is odd

$p$ :  $3n + 2$  is odd

$q$ :  $n$  is odd

$\neg p$ :  $3n + 2$  is even

$\neg q$ :  $n$  is even

# Proof by Contradiction: Example 1

If  $3n + 2$  is odd then  $n$  is odd

$p$ :  $3n + 2$  is odd

$q$ :  $n$  is odd

$\neg p$ :  $3n + 2$  is even

$\neg q$ :  $n$  is even

Step1: Assume,  $p$  is true.

Hence,  $3n + 2$  is odd



# Proof by Contradiction : Example 1

Step2: Assume,  $\neg q$  is true.

Hence,  $n$  is even

$$\therefore n = 2k$$

# Proof by Contradiction : Example 1

Step2: Assume,  $\neg q$  is true.

Hence,  $n$  is even

$$\therefore n = 2k$$

Step3: Prove that  $\neg p$  is true

$$\begin{aligned} & 3n + 2 \\ &= 3 * (2k) + 2 \\ &= 6k + 2 \\ &= 2 (3k + 1) \\ &= 2 m \end{aligned}$$

# Proof by Contradiction : Example 1

Step2: Assume,  $\neg q$  is true.

Hence,  $n$  is even

$$\therefore n = 2k$$

Step3: Prove that  $\neg p$  is true

$$3n + 2$$

$$= 3 * (2k) + 2$$

$$= 6k + 2$$

$$= 2 (3k + 1)$$

$$= 2 m$$

$$\therefore 3n + 2 \text{ is even}$$

$$\therefore \neg p \text{ is true}$$

# Proof by Contradiction: Example 1

From Step1:  $p$  is true.

Hence,  $3n + 2$  is odd

# Proof by Contradiction: Example 1

From Step1:  $p$  is true.

Hence,  $3n + 2$  is odd

From Step3:  $\neg p$  is true

$3n + 2$  is even

# Proof by Contradiction: Example 1

From Step1:  $p$  is true.

Hence,  $3n + 2$  is odd

From Step3:  $\neg p$  is true

$3n + 2$  is even

This is a contradiction

$\therefore$  If  $3n + 2$  is odd then  $n$  is odd

# Proof by Contradiction

$p$  (propositional statement)

Step1: Assume,  $\neg p$  is true.

# Proof by Contradiction: Example 2

Prove that  $\sqrt{2}$  is irrational

$p$ :  $\sqrt{2}$  is irrational

$\neg p$ :  $\sqrt{2}$  is rational

Step1: Assume,  $\neg p$  is true.

Hence,  $\sqrt{2}$  is rational



# Proof by Contradiction: Example 2

$$\sqrt{2} = \frac{a}{b}$$

$$\text{or, } 2 = \frac{a^2}{b^2}$$

$$\text{or, } 2b^2 = a^2$$

$$\text{or, } a^2 = 2b^2$$

$\therefore a^2$  is even

$\therefore a$  is even

$$\therefore a = 2c$$

# Proof by Contradiction: Example 2

$$\begin{aligned}(2c)^2 &= 2b^2 \\ \text{or, } 4c^2 &= 2b^2 \\ \text{or, } 2c^2 &= b^2 \\ \text{or, } b^2 &= 2c^2\end{aligned}$$

$\therefore b^2$  is even  
 $\therefore b$  is even

$$\therefore b = 2d$$

# Proof by Contradiction: Example 2

$$\sqrt{2} = \frac{a}{b}$$
$$\text{or, } \sqrt{2} = \frac{2c}{2d}$$

$\therefore \sqrt{2}$  can not be a rational number

$\therefore \sqrt{2}$  irrational number

# Proof by Induction

$\forall n p(n)$ ,  $n$  is a positive integer

Step 1(Basis step): verify that  $p(1)$  is true.

Step 2(Inductive step): Show that

$\forall k p(k) \rightarrow p(k+1)$  is true,  $k$  is a positive integer

# Proof by Induction: Example 1

$$p(n) ::= [1 + 2 + \dots + n = n(n+1) / 2]$$

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$$p(n) ::= [1 + 2 + \dots + n = n(n+1) / 2]$$

Step 1(Basis step):

$$p(1) ::= [1 = 1(1+1) / 2] = \text{True}$$

# Proof by Induction: Example 1

Step 2(Inductive step):

Show that

$\forall k \ p(k) \rightarrow p(k+1)$  is true,  $k$  is a positive integer

$$p(k): 1 + 2 + \dots + k = k(k + 1) / 2$$

$$p(k+1): 1 + 2 + \dots + (k+1) = (k+1)(k + 2) / 2$$

# Proof by Induction: Example 1

Assume,  $p(k)$  is true

$$\therefore 1 + 2 + \dots + k = k(k + 1) / 2$$



# Proof by Induction: Example 1

Assume,  $p(k)$  is true

$$\therefore 1 + 2 + \dots + k = k(k + 1) / 2$$

$$\text{or, } 1 + 2 + \dots + k + (k+1) = k(k + 1) / 2 + (k+1)$$

Why add  $(k+1)$ ?

→ Check  $p(k+1)$

# Proof by Induction: Example 1

Show that,  $p(k+1)$  is true  $\therefore p(k + 1)$  is true

$$1 + 2 + \dots + k = k(k + 1) / 2$$

$$\text{or, } 1 + 2 + \dots + k + (k+1) = k(k + 1) / 2 + (k+1)$$

$$\text{or, } 1 + 2 + \dots \dots + (k+1) = k(k + 1) / 2 + (k+1)$$

$$\text{or, } 1 + 2 + \dots \dots + (k+1) = (k + 1) (k+2) / 2$$

# Proof by Induction: Example 1

Show that,  $p(k+1)$  is true

$$1 + 2 + \dots + k = k(k + 1) / 2$$

$$\text{or, } 1 + 2 + \dots + k + (k+1) = k(k + 1) / 2 + (k+1)$$

$$\text{or, } 1 + 2 + \dots + (k+1) = k(k + 1) / 2 + (k+1)$$

$$\text{or, } 1 + 2 + \dots + (k+1) = (k + 1)(k+2) / 2$$

$\therefore p(k + 1)$  is true  
 $\forall k \ p(k) \rightarrow p(k+1)$

# Proof by Induction: Example 1

$$\begin{aligned} &p(1) \text{ and } \forall k \, p(k) \rightarrow p(k+1) \\ &\therefore \forall n \, p(n) \end{aligned}$$

# Proof by Induction: Example 2

$p(n) ::= [2^n < n!] \text{ for } n \geq 4$

Step 1(Basis step):

$p(4) ::= [16 < 24] = \text{True}$

# Proof by Induction: Example 2

Step 2(Inductive step):

Show that

$p(k) \rightarrow p(k+1)$  is true,  $k \geq 4$

$$p(k): 2^k < k!$$

$$p(k+1): 2^{k+1} < (k+1)!$$

# Proof by Induction: Example 2

Assume,  $p(k)$  is true

$$\therefore 2^k < k!$$

$$\text{or, } 2^k < k! \cdot 2$$

$$\text{or, } 2^{k+1} < 2 \cdot k!$$

$$\text{or, } 2^{k+1} < (k+1) \cdot k! \quad [k \geq 4]$$

$$\text{or, } 2^{k+1} < (k+1)!$$

# Proof by Induction: Example 2

Assume,  $p(k)$  is true

$$\therefore 2^k < k!$$

$$\text{or, } 2^k < k! \cdot 2$$

$$\text{or, } 2^{k+1} < 2 \cdot k!$$

$$\text{or, } 2^{k+1} < (k+1) \cdot k! \quad [k \geq 4]$$

$$\text{or, } 2^{k+1} < (k+1)!$$

$\therefore p(k+1)$  is true