15

Solutions of Algebraic and Transcendental Equations

15.1 INTRODUCTION

The equations of the form $f(x) = 0$ where $f(x)$ is purely a polynomial in x. e.g. $x^6 - x^4 - x^3 - 1 = 0$ is called an **algebraic equation.** But, if $f(x)$ involves trigonometrical, arithmetic or exponential terms in it, then it is called **transcendental equation.**
E.g. $xe^x - 2 = 0$ and $x \log_{10} x - 1.2 = 0$.

Basic Properties and Observations of an Algebraic Equation and its Roots:

(*i*) If $f(x)$ is exactly divisible by $(x - \alpha)$, then α is a root of $f(x)$.

(*ii*) Every algebraic equation of nth degree has n and only n real or imaginary roots. Conversely, if $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be the n roots of the nth degree equation $f(x) = 0$, then

$$f(x) = A(x - \alpha_1)(x - a_2)\ldots\ldots\ldots(x - \alpha_n).$$

(*iii*) If $f(x)$ is continuous in the interval $[a, b]$ and $f(a)$, $f(b)$ have different signs, then the equation have at least one root between $x = a$ and $x = b$ (oftenly known as *Intemediate Value Theorem.*)

(*iv*) In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e. if $\alpha + i\beta$ is root of $f(x) = 0$, then $\alpha - i\beta$ is also its root. Similarly, if $\alpha + \sqrt{\beta}$ is an irrational root of $f(x) = 0$, then $\alpha - \sqrt{\beta}$ is also its roots.

Some General Observations on the roots of algebraic equations are as follows:

(*i*) The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficients can never exceed the number of changes in sign from positive to negative and negative to positive.
E.g. Take $x^5 - 3x^2 + 4x - 6 = 0$
Here the coefficients are with sign + – + – i.e. they change from positive to negative, negative to positive, positive to negative and, hence the equation contains three positive roots.

(*ii*) The number of negative roots can not exceed the number of changes of sign in $f(-x)$.
E.g. $f(-x) = -x^5 - 3x^2 - 4x - 6 = 0$;

Here all the coefficients are negative and so there is no change in signs, meaning there by that the given equation does not contain any negative root.

(*Note: observations (i) & (ii) are more commonly known as* ***Descrate's sign rule.)***

(*iii*) Every equation of an odd degree has at least one real root, whereas every equation of an even degree with last term in it if negative, has at least two real roots, one positive and the other negative.

About Numerical Computation of Roots:

In Science and engineering often we require the root of algebraic, non-algebraic or trascendental equations and that in a situation when no information is available about the root, it is not necessarily be whole number and difficult to find by conventional Mathematics methods.

In general to compute numerical value of the root, we wish to find some approximate value of the root which satisfies our need without much change in its basic characters. For doing so, generally we need some rough estimate of the root to start with and the iterate it for better approximations. For that matter, we classify them in broadly in two types viz. **closed end and open end methods.** Under closed end method, as the name it self, two initial guesses are required to block the root. In these methods, root essentially converges as we move closer and closer to the root in each iteration. That is why some time Bi-section and Regula-Falsi Methods are closed bracketing or closed end methods of iteration.

In some of the methods though we need two initial estimates, however, $f(x)$ is not required to change sign between the estimates, that is why these are not termed as bracketing methods and call them open end, like Secant and Newtons Raphson methods.

Here, in this chapter, we discuss some of the methods namely Graphical, Iteration Method, Bisection Method, Regula-Falsi, Newton Raphson's, Horner's Method, Muller's Technique and Lin-Bairstow Method one by one.

15.2 GRAPHICAL METHOD

A simple method for obtaining the approximation value $x = \alpha$ (say) near to the root or to estimate the root of the equation $f(x) = 0$ is to make a plot of the function and observes where it crosses the x-axis. This point, which represents the value for which $f(x) = 0$, provides a rough approximation of the root.

More conveniently, we can express $f(x) = 0$ as $f_1(x) = f_2(x)$ and plot the graph of $f_1(x)$ and $f_2(x)$ with respect to the same axes and find the abscissa of the point of intersection which is the root of the equation $f(x) = 0$

E.g. 1) Solve the equation, $f(x) = x^n + ax + b = 0$

We draw the graph of $\begin{aligned} y_1 &= f(x_1) = x^n = 0, \\ y_2 &= f(x_2) = -(ax+b) \end{aligned}$ and then consider their point of intersection.

E.g. 2) In order to find the root of the equation $e^x \sin x = 1$, we rewrite it as $e^x e^{-x} = 1$ taking $\sin x = e^{-x}$ and then determining the intersection of $y = e^x$ with $y = e^{-x}$.

Note: *It is observed that graphical techniques are of limited practical value because they are not precise. However, they can be utilized to obtain rough estimates of roots.*

Significance: Apart from providing rough estimates of the roots, graphical interpretations are important tools for understanding the properties of the functions and anticipating the pitfalls of the numerical methods.

Example 1: Solve Graphically $\sin x = \dfrac{x^2}{4}$ for a non-zero root.

Solution: Let $y_1 = \sin x$; $y_2 = \dfrac{x^2}{4}$

TABLE 3.1: VALUE OF y_1 AND y_2 FOR STARTING VALUES OF X

x(*radian*)	y_1	y_2	$f(x) = y_1 - y_2$
0.0	0.000	0.000	0.000
1.0	0.8414	0.250	0.591 > 0
2.0	0.9090	1.000	–0.091 < 0

Table 3.1 indicate that the root lies between 1 and 2.

TABLE 3.2: VALUES OF y_1 AND y_2 FOR VARIOUS SMALL INTERVALS

x(radian)	$y_1 = \sin x$	$y_2 = \dfrac{x^2}{4}$	$f(x) = y_1 - y_2$
0.0	0.000	0.0	0.0
0.5	0.4793	0.0625	0.4168 > 0
1.0	0.8414	0.25	0.5914 > 0
1.1	0.8910	0.3025	0.5885 > 0
1.2	0.9391	0.36	0.5719 > 0
1.3	0.96346	0.4225	0.54096 > 0
1.4	0.9854	0.490	0.4954 > 0
1.5	0.9974	0.5625	0.4349 > 0
1.6	0.9996	0.640	0.3596 > 0
1.7	0.99169	0.7225	0.26919 > 0
1.8	0.97390	0.810	0.1639 > 0
1.9	**0.94638**	**0.9025**	**0.0440 > 0**
2.0	**0.9090**	**1.00**	**–0.0010 < 0**

Table 3.2 indicates the trend as there is a change in sign in values of y for $x = 1.90$ and 2.00.

TABLE 3.3:

x(radian)	$y_1 = \sin x$	$y_2 = \dfrac{x^2}{4}$	$f(x) = y_1 - y_2$
1.91	0.94638	0.9025	0.044 > 0
1.92	0.939734	0.9216	0.018 > 0
1.93	0.936229	0.931225	0.005044 > 0

Table 3.3 shows that $x = 1.93$ is better approximation.

$\therefore$ The real root of the equation $\sin x = \dfrac{x^2}{4}$ is 1.93.

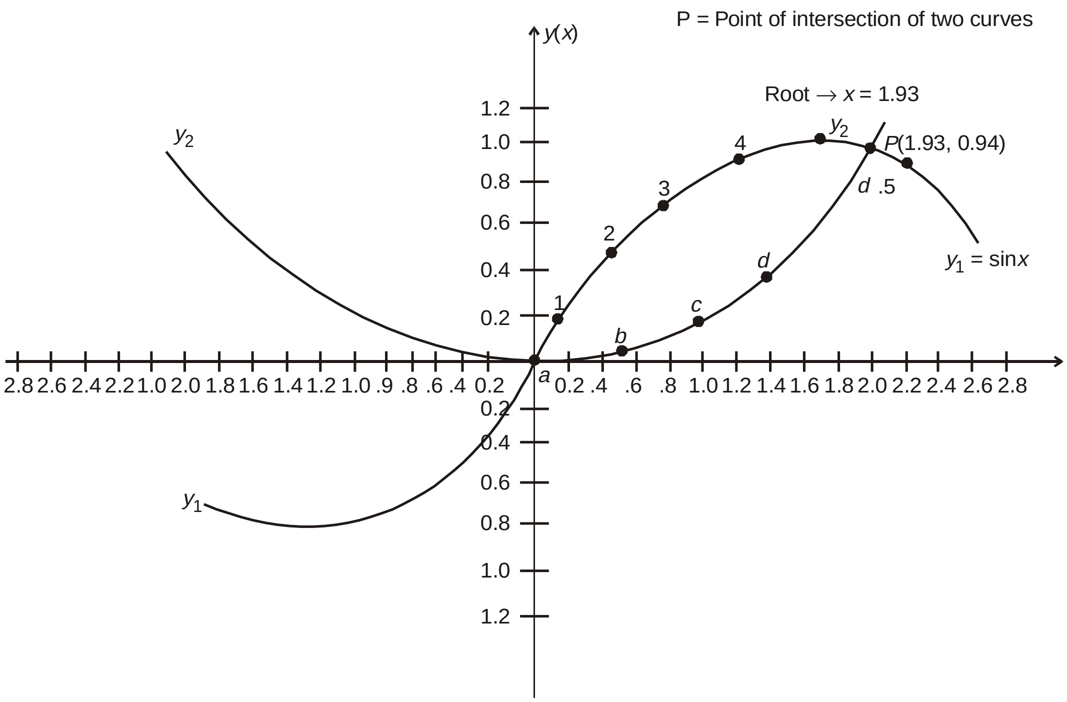

Fig. 15.1

15.3 ITERATION METHOD

This method is used finding the root of only those equations

$$f(x) = 0 \qquad \ldots(1)$$

which are expressible as $x = \phi(x)$ ...(2)

The root of the equation (1) are the same as the point of intersection of the straight line $y = x$ and $y = \phi(x)$. For that, first we find an initial approximation x_0 of the required root and then make it better value x_1 by replacing x by x_0 in the right hand side of (2) i.e.

$$x_1 = \phi(x_0) \qquad \ldots(3)$$

Likewise a still better approximation is computed by replacing $x = x_1$ in the right hand side of (3) i.e.

$$x_2 = \phi(x_1) \qquad \ldots(4)$$

This procedure is continued,

$$\left.\begin{aligned} x_3 &= \phi(x_2), \\ x_4 &= \phi(x_3), \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ x_n &= \phi(x_{n-1}) \end{aligned}\right\} \qquad \ldots(5)$$

If this sequence of approximate values $x_0, x_1, \ldots\ldots, x_{n-1}$ converges to a limit α, then α is taken as the root of the equation (1)

Geometrical Significance: Draw the graph of $y_1 = x$ and $y_2 = \phi(x)$ as shown below in the figure 3.2. Since $|\phi'(x)| < 1$ for convergence, the initial approximation of the curve $y_2 = \phi(x)$ must be less then 45° in the neighbourhood of x_0. This fact has been observed in constructing the graph.

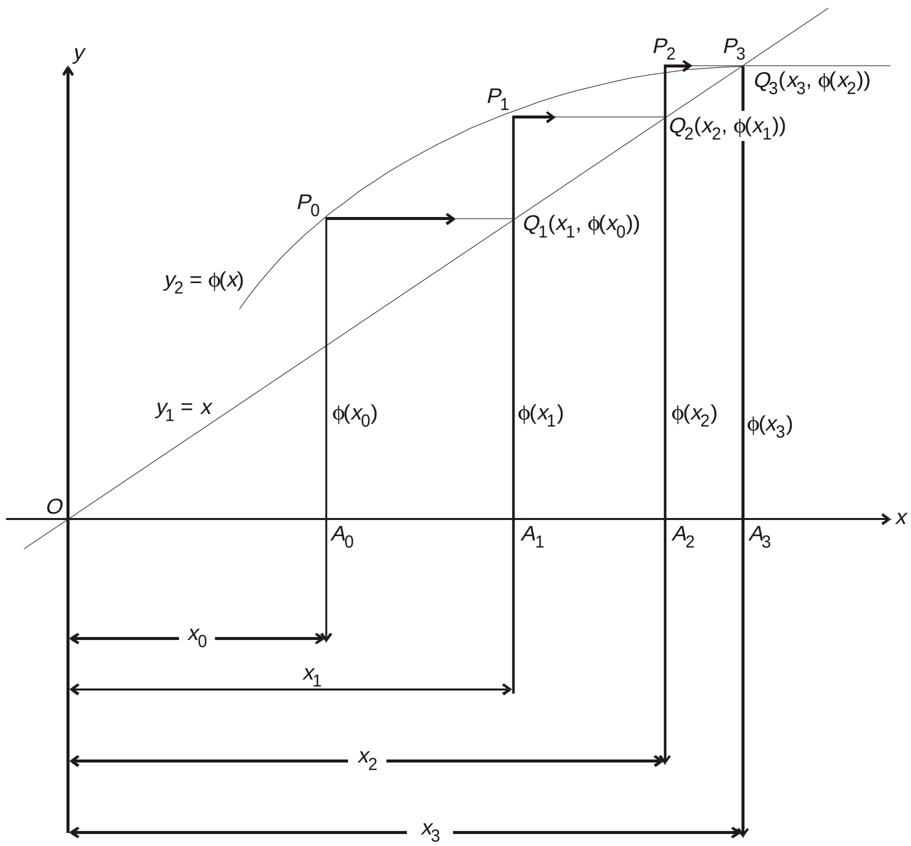

Fig. 15.2

For tracing the convergence of the iteration process, draw the ordinates $\phi(x_0)$. Then from point p_0, draw a line parallel to OX until it intersects the line $y_1 = x$ at the point $Q_1(x_1, \phi(x_0))$. This point Q_1 is the geometric representation of the first iteration $x_1 = \phi(x_0)$.

Then draw Q_1P_1, P_1Q_2, Q_2P_2, P_2Q_3 etc. as indicated by the arrows in geometry. The points Q_1, Q_2, Q_3,………, thus approaches to the point of intersection of the curves $y_1 = x$ and $y_2 = \phi(x)$ as iteration proceeds and converges to the desired root.

Conditions for Convergence:

This sequence of approximate roots does not always converge. To make its convergence sure, it have to satisfy certain conditions.

1. It be the interval containing the root $x = \alpha$, of the equation $x = \phi(x)$, then $|\phi(x)| < 1$ for all $x \in I$.
2. The initial approximation x_0 for the root lies in I.

Observations:

1. *As there could be several ways of rewriting the given equation* $f(x) = 0$ as $x = \phi(x)$, *hence care should be taken so that the condition.* $|\phi'(x)| < 1$ *for convergence is satisfied. Further, if* $|\phi'(x)| < 1$ *the root converges but oscillates about the exact value.* e.g. 1 *illustrating the above fact:*

e.g. 1. The equations $x^3 - 2 = 0$ ...(1)

can be rewritten either $x = x^3 + x - 2$ ...(2)

$$x = \frac{1}{5}(2 + 5x - x^3) \quad \text{...(3)}$$

In (1) $\phi'(x) = 3x^2 - 1$ and $\left|\phi'(x)\right|_{x=1.5} = 7.75 > 1$ ...(4)

where in (3), $\phi'(x) = \frac{1}{5}(5x - 3x^2)$ and $\left|\phi'(x)\right|_{x=1.5} = 0.35 < 1$ ...(5)

A root of the equation (1) lies in the interval (1, 2) and hence we have evaluated $\left|\phi'(x)\right|_{x=1.5} = 0.35 < 1$. Clearly equation (5) will ensure convergence, provided the initial approximation for the root is at $x = 1.5$ choosen in the interval (1, 2).

2. Further, the convergence of an algorithm can be categorized into two types viz. local convergence if the starting point x_0 is sufficiently close to the root and global convergence for any starting value. E.g.2 illustrates e.g. the above facts.

e.g. 2 Take equation $f(x) = x^2 - 2x - 3 = 0$ ...(1)

The root of this equation are $x = 3$ and $x = -1$.

Case 1: We can rewrite equation (1) as $x = \sqrt{2x + 3} = g(x)$, say ...(2)

Starting with $x_0 = 4.0$, we get

$x_1 = 3.316$ $\quad x_4 = 3.011$

$x_2 = 3.104$ $\quad x_5 = 3.004$

$x_3 = 3.034$ $\quad$ ....(3)

As $n \to \infty$, $x_\alpha = 3.000$

Case 2: $f(x) = x^2 - 2x = 3$ can be rewritten as $x = \dfrac{3}{x-2}$ ...(4)

Starting with $x_0 = 4.0$, we get

$x_1 = 1.500;$ $\quad x_5 = -0.919;$

$x_2 = -6.000$ $\quad x_6 = -1.028$

$x_3 = -0.375;$ $\quad x_7 = -0.991;$

$x_4 = -1.263$ $\quad x_8 = -1.003$ ...(5)

From above, we observe that $\{x_n\}$ sequence approximation of root converges to the root $x = \alpha$ of $f(x)$, which is -1, but iteration oscillates rather than converging monotonically.

Case 3: Further $f(x) = x^2 - 2x - 3 = 0$ can be rewritten $x = \dfrac{x^2 - 3}{2}$ ...(6)

Starting with $x_0 = 4.0$, we get $x_1 = 6.5$, $x_2 = 19.635$, $x_3 = 191.0$

Which clearly goes on diverging, resulting in astray instead of the desired result.

Hence, a care should be taken while rewriting $f(x) = 0$ as $x = g(x)$.

Order of Convergence (Rate of Convergence):

Any method is said to have convergence of order p, if p is the largest positive real number such that $|\varepsilon_{n+1}| \le k|\varepsilon_n|^p$ where k is a finite positive constant, and ε_n and ε_{n+1} are the errors in nth and $(n + 1)$ th iterated value of the root α of the equation $f(x) = 0$ respectively. When $p = 1$, the convergence is said to be linear, when $p = 2$, the convergence is said to be quadratic.

Discussion on the Order of Convergence of Iteration Method:

Let α be the root of the equation $f(x) = 0$ expressible as $x = \phi(x)$, then

$$\alpha = \phi(\alpha) \qquad \ldots(1)$$

If x_n and x_{n+1} are the nth and $(n + 1)$th approximation of the root α, then

$$x_{n+1} - \alpha = \phi(x_n) - \alpha \qquad \ldots(2)$$

From (1) and (2).

$$x_{n+1} = \phi(x_n) \qquad \ldots(3)$$

By mean value theorem of differential calculus,

$$\phi(x_n) - \alpha = (x_n - \alpha)\phi'(\theta), \text{ where } x_n < \theta < \alpha \qquad \ldots(4)$$

Using (4) in (3),

$$x_{n+1} - \alpha = (x_n - \alpha)\phi'(\theta) \qquad \ldots(5)$$

Let p be the maximum value of $|\phi'(\theta)|$ in the interval i.e. $|\phi'(\theta)| \le p$ for all x in 1.

Then from (5),

$$|x_{n+1} - \alpha| \le p|x_n - \alpha| \text{ i.e. } |\varepsilon_{n+1}| \le p|\varepsilon_n| \qquad \ldots(6)$$

where ε_n and ε_{n+1} are errors in the nth and $(n + 1)$th iterated value of the root.

Since the index of ε_n being 1, the rate of convergence of the iteration method $x_{n+1} = \phi(x_n)$ is **linear.**

Observations:

1. This method is particularly useful for finding the real root of an equation given in the form of an infinite series.
2. Smaller the value of $\phi'(x)$, the more rapidly it will converge.

Aitken's Δ^2 Method:

Let x_{n-1}, x_n, x_{n+1} be the three successive approximation to the root α of the equation $x = \phi(x)$. Then we know that

$$(\alpha - x_n) = k(\alpha - x_{n-1}) \quad \text{and} \quad (\alpha - x_{n+1}) = k(\alpha - x_n) \qquad \ldots(7)$$

On dividing the two, $\dfrac{(\alpha - x_n)}{(\alpha - x_{n-1})} = \dfrac{(\alpha - x_{n+1})}{(\alpha - x_n)}$

Implying $\quad \alpha - x_{n+1} = \dfrac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}}$

But in the sequence of approximations,

$$\Delta x_n = x_{n+1} - x_n$$

$$\Delta^2 x_n = \Delta(\Delta x_n) = \Delta(x_{n+1} - x_n) = \Delta x_{n+1} - \Delta x_n = x_{n+2} - 2x_{n+1} - x_n$$
$$\Delta^2 x_{n-1} = x_{n+1} - 2x_n - x_{n-1}$$

Hence (7) can be written as $\alpha = x_{n+1} + \dfrac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$ ...(8)

This yields successive approximations to the root α and explains the term Δ^2 process.

In numerical application of the problem, the values of the following quantities must be obtained.

$$\begin{array}{ccccc} x_{n-1} & & & & \\ & \to & \Delta x_{n-1} & & \\ x_n & & & \to & \Delta^2 x_{n-1} \\ & \to & \Delta x_n & & \\ x_{n+1} & & & & \end{array}$$

Example 2: Find the root of the equation $2x = \cos x + 3$ correct to three decimal of places. Further, test its acceleration of convergence: Aitken's Δ^2 process.

Solution: Re-write the equation in the form $x = \dfrac{1}{2}(\cos x + 3)$ ...(1)

So that $\phi(x) = \dfrac{1}{2}(\cos x + 3)$ and $\phi'(x) < \left|\dfrac{\sin x}{2}\right| < 1$ ...(2)

Hence, the iteration method can be applied to equation (1) and we start with $x_0 = \dfrac{\pi}{2}$. The successive iterations are

$x_1 = 1.500$, $x_4 = 1.526$, $x_7 = 1.523$,
$x_2 = 1.535$, $x_5 = 1.522$, $x_8 = 1.524$;
$x_3 = 1.518$; $x_6 = 1.524$;

Clearly, 1.524 can be considered as the solution correct to three decimal places.

Now, from the table of differences, using first three numbers of iterated values, fourth value is achieved.

$$\begin{array}{ccccc} x_1 = 1.500 & & & & \\ & \to & \Delta x_1 = 0.035 & & \\ x_2 = 1.535 & & & \to & \boldsymbol{\Delta^2 x_1 = -0.052} \\ & \to & \boldsymbol{\Delta x_2 = -0.017} & & \\ \boldsymbol{x_3 = 1.518} & & & & \end{array}$$

And $$x_4 = x_3 - \frac{(\Delta x_1)^2}{\Delta^2 x_2} = 1.518 - \frac{(-0.017)^2}{-0.052} = 1.524$$

Example 3: Use method of iteration to find a root of the equation $xe^x = 1$ lying between 0 and 1 correct to three decimal of places.

Solution: Rewrite the given equation as $x = e^{-x}$ ...(1)

So that $\phi(x) = e^{-x}$ and $\phi'(x) = -e^{-x}$ ...(2)

Here $|\phi'(x)|$ = for $x < 1$, hence this method ensures the condition of convergence. Define $x_{n+1} = \phi(x_n)$ and take $x_0 = 1$, we find the successive iterations are given by

$$x_1 = \frac{1}{e^{x_0}} = \frac{1}{e} = 0.3678794, \qquad x_7 = 0.5601154, \qquad x_{13} = 0.5665566$$

$$x_2 = \frac{1}{e^{x_1}} = 0.6062435, \qquad x_8 = 0.5711431, \qquad x_{14} = 0.5672762$$

$$x_3 = 0.5004735, \qquad x_9 = 0.5648793, \qquad x_{16} = 0.567157$$

$$x_4 = 0.6062435, \qquad x_{10} = 0.5684287, \qquad x_{17} = 0.567119$$

$$x_5 = 0.5453957, \qquad x_{11} = 0.5664147, \qquad x_{18} = 0.567157$$

$$x_6 = 0.5796123, \qquad x_{12} = 0.5675566,$$

So $\boldsymbol{x = 0.567}$ in the desired root correct to three decimal of places.

15.4 BOLZANO OR BISECTION METHOD

This method of solving transcendental equations, consists in locating the roots of the equation $f(x) = 0$ between two numbers say a and b such that $f(x)$ is continuous for $a \le x \le b$ and $f(a)$ and $f(b)$ are of opposite signs so that the product $f(a)f(b) < 0$ i.e. the curve cuts the x-axis between a and b. Then the desired root is approximately $x_1 = \dfrac{a+b}{2}$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise the root lies between either 'a and x_1' or 'x_1 and b' according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process to improve the result to greater accuracy.

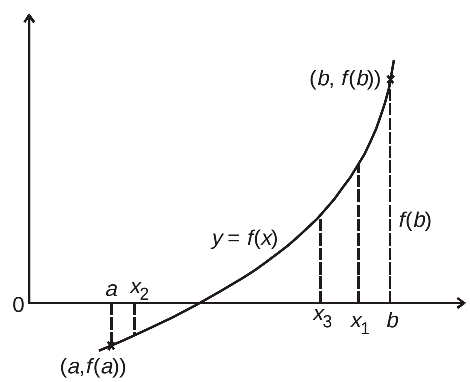

Fig. 15.3

From the figure 15.3, $f(x_1)$ is positive, so that the root lies between a and x_1 i.e. $x_2 = \dfrac{a + x_1}{2}$. If $f(x_2)$ is negative, the root lies between x_1 and x_2. Then the next approximation will be $x_3 = \dfrac{x_1 + x_2}{2}$ and so on.

The Bisection Method is also called ***Binary Chopping 'or' Incremental Search Method 'or' Interval Halving method.***

Observations:

(*i*) Though the bisection method is not rapid but it is simple and reliable.

(*ii*) Since the new interval containing the root is exactly half the length of the previous one i.e. the interval length is reduced by a factor of $\frac{1}{2}$ at each stage or operation. Therefore, after n repeated iteration, the new interval containing the root be of length $\frac{(b-a)}{2^n}$. Say after n iterations, the latest interval is as small as given $\in$, then

$$\frac{(b-a)}{2^n} \leq \in \quad \text{or} \quad \frac{(b-a)}{\in} \leq 2^n \quad \text{or} \quad \frac{\log(b-a)-\log(\in)}{\log 2} \leq n$$

which give the number of iterations required to achieve the desired accuracy, $\in$.
For example, in particular the minimum number of iterations needed for achieving a root in the interval (0, 1) for given error, $\in$ are as follows:

$\in$:	10^{-2}	10^{-3}	10^{-4}
n :	7	10	14

If $(b\text{-}a) = 1$, and $\in = 0.001$, then it can be seen that $n \geq 10$.

(*iii*) As by each iteration the interval is reduced by a factor of half meaning thereby that error is reduced by a factor of half. Or in other words, the errors in $(n + 1)^{\text{th}}$ and nth iterations are in the ratio of half i.e. $\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{2}$. Meaning there by, **the convergence of this method is linear i.e. 1.**

Working Rule:

Step 1, Practically, choose the lower end as $x_0(x = a)$ and upper end as $x_n(x = b)$ to guess for the root so that the function changes sign over the interval. This can be checked by ensuring that $f(x_0)f(x_n) < 0$

Step 2: Make the following root x_r is determined by $x_r = \frac{x_0 + x_n}{2}$

Step 3: Make the following evaluations to determine in which subinterval the root lies.

(*a*) If $f(x_0)f(x_n) < 0$ root lies in the lower subinterval and repeat.

(*b*) If $f(x_0)f(x_n) > 0$ the root lies in upper subinterval and repeat the process.

(*c*) If $f(x_0)f(x_n) = 0$, the root equals x; terminate the computation.

Example 4: Solve $x^3 - 9x + 1 = 0$ for the root between $x = 2$ and $x = 4$ by using Bisection Method.

Solution: Here $f(x) = x^3 - 9x + 1$ *so that f(x) so continuous in* $2 \leq x \leq 4$.

Further, $\left.\begin{matrix} f(2) = -9 \\ f(4) = 29 \end{matrix}\right\}$ and $f(2)\cdot f(4) < 0$ *so that a* root lies between $x = 2$ and $x = 4$.

Now $\quad x_1 = \frac{a+b}{2} = \frac{2+4}{2} = 3$ and $f(3) = 1$ so that $f(2)\cdot f(3) < 0$.

Hence, $\quad x_2 = \frac{2+3}{2} = 2.5$ and $f(2.5) = -5.87$

For next iteration, we see that $f(3)\ f(2.5) < 0$ and the root lies between 2.5 and 3.

$\therefore \quad x_3 = \frac{2.5+3}{2} = 2.75$

Similarly, $x_4 = 2.875$ and $x_5 = 2.9375$ and the process is continued for better accuracy.

Example 5: Solve $f(x) = x^3 - 2x^2 - 4$ by the method of interval halving procedure.
[KUK, MCA 2004]

Solution: For given $f(x) = x^3 - 2x^2 - 4$, we find two numbers $a = 2$ and $b = 3$ such that $f(a) = f(2) = -4$ and $f(b) = f(3) = 5$ are of opposite sign

Hence the root lies between $a = 2$ and $b = 3$. Then for iteration purpose, we follows as:
An alternate way of expressing the problem in tabular form:

Iteration No.	*a*	*b*	$x = \frac{(a+b)}{2}$	*f(a)*	*f(b)*	*f(x)*
1.	2	3	2.50000	–ve	+ve	–ve (1.8750)
2.	2.50000	3.00000	2.75000	–ve	+ve	+ve (1.6719)
3.	2.50000	2.75000	2.62500	–ve	+ve	+ve (0.3066)
4.	2.50000	2.62500	2.56250	–ve	+ve	–ve (0.3.640)
5.	2.56250	2.62500	2.59375	–ve	+ve	–ve (0.0055)
6.	2.59375	2.62500	2.60938	–ve	+ve	+ve (0.1488)
7.	2.59375	2.60938	2.60157	–ve	+ve	+ve (0.0719)
8.	2.59375	2.60157	2.59765	–ve	+ve	+ve (0.0329)
9.	2.59375	2.59760	2.59570	–ve	+ve	+ve (0.0136)
10.	2.59375	2.59570	2.59470	–ve	+ve	–ve (0.0040)

Hence, the root correct to three decimal places is $x = 2.5947$.

15.5 REGULA FALSI OR FALSE POSITION METHOD

This is the oldest method of computing the real root of a numerical equation $f(x) = 0$ and it is almost a replica of bisection method.

Draw the graph of the curve $y = f(x)$ through $x = a$ and $x = b$ such that $f(a)$ and $f(b)$ are of opposite signs. Then $x = \alpha$ is the abscissa of the point C where the graph of $y = f(x)$ meets the x-axis.

Let the chord AB meets the x-axis at a point whose abscissa is x_1 which is taken as the first approximation of the root.

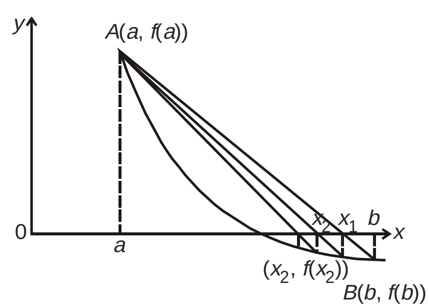

Fig. 15.4

To obtain the value of x, consider the equation of the line AB (two point form) through $(a, f(a))$ as bellow:

$$y - f(a) = m(x - a)$$

or $$y - f(a) = \frac{f(b) - f(a)}{(b-a)}(x - a) \quad \text{as} \quad m = \frac{f(b) - f(a)}{(b-a)}$$

But it meets the x-axis at $x = a$ where $y = 0$

i.e. $$0 - f(a) = \frac{f(b) - f(a)}{(b-a)}(x_1 - a) \quad \text{or} \quad x_1 = a - \frac{(b-a)}{f(b) - f(a)} f(a) \qquad \ldots(1)$$

Successive application of this process gives $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$. Such points leading to an accurate value of $x(= \alpha)$ as shown in the figure 3.4. This iteration process is known as the method of false position and its **rate if conversance equals 1.618,** which is faster than that of Bisection Method. Some times this method is also known as **Method of Variable Secant.**

General expression representing $(n + 1)^{th}$ iterated value x_{n+1} is as follows:

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \ldots(2)$$

provide that at each step, $f(x_n) - f(x_{n-1}) < 0$.

Observations:

The method of false position is based on the principle that any portion of a smooth curve is practically straight for a short distance and therefore, it is legitimate to assume that the change in $f(x)$ is proportional to the change in x over a short interval as in the case of linear interpolation from logrithmic and trigonometric tables. For it, assume that the graph of the curve $y = f(x)$ is straight line between the pts. $(a, f(a))$ and $(b, f(b))$, these pts being on the opposite sides of the x-axis.

Pit Falls of the False-Position Method: Although false position method is preferable among all the bracketing methods, but there are certain functions on which it performs poor and bisection yields superior results, e.g. Take $f(x) = x^{10} - 1$.

In this problem, after 5 iteration, true error is reduced less than 2% in bisection method whereas in case of false position after 5 iterations results has been reduced about 59% leaving very high error.

Condition for Convergence:

Let $\{x_n\}$ be a sequence of values of x obtained by the above method and let α be the exact root of the given equation. Then the method is said to be convergent if $\lim |x_n - \alpha| = 0$

Order of Convergence

Let ε_n be the error in the nth iterated value x_n, so that

$$x_n = \alpha + \varepsilon_n, \quad x_{n+1} = \alpha + \varepsilon_{n+1},$$

On putting these values in expression $x_{n+1} = x_n - \dfrac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$, we get

$$\varepsilon_{n+1} = \frac{\varepsilon_{n-1} f(\alpha + \varepsilon_n) - \varepsilon_n f(\alpha + \varepsilon_{n-1})}{f(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_{n-1})}$$

$$= \frac{\varepsilon_{n-1}\left[f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots\right] - \varepsilon_n\left[f(\alpha) + \varepsilon_{n-1} f'(\alpha) + \frac{\varepsilon_{n-1}^2}{2!} f''(\alpha) + \ldots\right]}{\left[f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots\right] - \left[f(\alpha) + \varepsilon_{n-1} f'(\alpha) + \frac{\varepsilon_{n-1}^2}{2!} f''(\alpha) + \ldots\right]}$$

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_{n-1}\varepsilon_n \cdot \frac{f''(\alpha)}{f'(\alpha)} + O\left(\varepsilon_n^2\right), \quad \text{using } f(\alpha) = 0 \qquad \ldots(3)$$

Find a number k such that $\varepsilon_{n+1} = c\varepsilon_n^k \qquad \ldots(4)$

And therefore $\varepsilon_n = c\,\varepsilon_{n-1}^{k}$ or $(\varepsilon_n)^{\frac{1}{k}} = c^{\frac{1}{k}}\varepsilon_{n-1}$ or $\varepsilon_{n-1} = c^{\frac{-1}{k}}\varepsilon_n^{\frac{1}{k}}$ ...(5)

Thus on using values of ε_n and ε_{n-1} in (3),

$$\varepsilon_{n+1} = c^{\frac{1}{k}}\varepsilon_n^{\frac{1}{k}} \cdot \varepsilon_n \cdot p, \quad \text{where} \quad p = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \qquad ...(6)$$

On equating the powers of ε on both sides in (4), we get

$$k = 1 + \frac{1}{k} \quad \text{or} \quad k^2 - k - 1 = 0 \quad \text{or} \quad k = \frac{1 \pm \sqrt{5}}{2}.$$

The positive value of k is $\frac{1+\sqrt{5}}{2} = 1.618.$

Hence, $\varepsilon_{n+1} = c\,\varepsilon_n^{1.618}$

Example 6. Use Regula-Falsi Method to find a real root of the equation $x \log_{10} x - 1.2 = 0$ correct to five places of decimal.
[Bhopal, 2002; Madras, 2003; VTU, 2004; Kottayam, 2005; NIT Kurukshetra, 2008]

Solution: Let $f(x) = x\log_{10}x - 1.2 = 0$

Now $f(2) = 2\log_{10}2 - 1.2 = 2 \times 0.30103 - 1.2 = -0.59794$ (–ve)

$f(3) = 3\log_{10}3 - 1.2 = 3 \times 0.47712 - 1.2 = 0.23136$ (+ve)

Thus, there is at least one root of the given equation which lies in the interval (2, 3).

Now $$x_1 = a - \frac{(b-a)}{f(b) - f(a)} f(a); \quad \text{for} \quad a = 2,\, b = 3$$

$$= 2 - \frac{(3-2)}{0.23136 - (-0.59794)} \times (-0.59794)$$

$$= 2 + \frac{0.59794}{0.82930} = 2 + 0.721 = 2.721$$

$$f(x_1) = f(2.721) = 2.721 \log_{10} 2.721 - 1.2 = 2.721 \times 0.43472 - 1.2 = 0.0171$$

Now, $f(x_1)$ is –ve and $f(b)$ is +ve, so the root lies in between 2.721 and 3.

Here for the next iteration, $a = 2.721$ and $b = 3$.

$\therefore$ $$x_2 = a - \frac{(b-a)}{f(b) - f(a)} f(a)$$

$$= 2.721 - \frac{(3-2.721)}{0.23136 - (-0.0172)} \times (-0.0172) = 2.721 + 0.019201 = 2.740201$$

Now $f(x_2) = f(2.740201 = -0.0004$

Clearly, the root lies between $x(= a) = 2.740201$ and $x = (b) = 3$.

$$\therefore \qquad x_3 = 2.740201 - \frac{(3 - 2.740201)}{f(3) - f(2.740201)} f(2.740201)$$

$$= 2.74021 + .00044844 = 2.74064844 \cong 2.74065 \text{ (+ve)}$$

Now, $f(2.721201)$ is –ve and $f(2.740201)$ is + ve, so we can proceed for further iterations for more accuracy. But, here we observe that the two consecutive values of x are same up to 3 decimal place and hence, 2.74065 is the correct values of x up to 5 decimal places.

Example 7: Using Regula Falsi Method, compute the real root of the equation $xe^x - 2 = 0$ correct up to three decimals places. [SVTU, 2007; NIT Kurukshetra, 2004, 2007]

Solution: In $f(x) = xe^x - 2$; for $x = 1$ $f(1) = 1 \times e^1 - 2 = 0.718$ (+ve)
for $x = 0.5$ $f(0.5) = 0.5 \times e^5 - 2 = -1.17$ (–ve)

As the value of $f(x)$ at $x = 0.5$ is –ve and at $x = 1.0$ it is +ve. Therefore, the root lies between $x = 0.5$ and $x = 1$.

I^st Step: For $x_0 = 0.5$ and $x_1 = 1$;

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - (x_0)} f(x_0)$$

$$= 0.5 - \frac{1 - 0.5}{0.718 - (-1.17)} \times (-1.17)$$

$$= 0.5 + \frac{0.5}{.718 + 1.17} 1.17 = 0.5 + 0.3098 = 0.8098$$

$$f(x_2) = f(0.8098) = 0.8098\ e^{0.8098} - 2 = -0.18 = 0.8098$$

2^nd Step: Root lies between $x_1 = 1$ and $x_2 = 0.8098$

$$\therefore \qquad x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 0.8098 - \frac{1 - 0.8098}{0.718 + 0.18}(-0.18) = 0.8098 + \frac{0.1902}{0.898} = 0.8479$$

$$f(x_2) = f(0.8479) = 0.8479 - e^{0.8479} - 2 = -0.02(-ve)$$

3^rd Step: Next root lies between $x_3 = 0.847$ and $x_1 = 1$

$$\therefore \qquad x_4 = x_3 - \frac{(x_1 - x_3)}{f(x_1) - f(x_3)} f(x_3)$$

$$= 0.8479 - \frac{1 - 0.8479}{0.718 - (-0.02)}(-0.02) = 0.8479 + \frac{0.0032}{0.738} = 0.8519$$

$$f(x_4) = 0.8519\ e^{0.8519} - 2 = -0.0031$$

4th Step: Next value of x in between $x_1 = 1$ and $x_4 = 0.8519$

$$\therefore \qquad x_5 = x_4 - \frac{x_1 - x_4}{f(x_1) - f(x_4)} f(x_4)$$

$$= 0.8519 - \frac{1 - 0.8519}{0.718 + 0.0031} \times 0.0031 = 0.8519 + \frac{0.00046}{0.7211} = 0.8525$$

Approximate root is correct to 4 decimal of place, 0.8525

Example 8: Find the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position.

Solution: Let $f(x) = x^3 - 5x - 7$ …(1)

Now by putting $x = 2$ and $x = 3$, we get

$$f(2) = (2)^3 - 5(2) - 7 = -9 \quad \text{and} \quad f(3) = (3)^3 - 5(3) - 7 = 5$$

Hence the root lies between 2 and 3

Take $x_0 = 2$ and $x_1 = 3$.

$$\therefore \qquad x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] \times f(x_0) = 2 - \left[\frac{3-2}{5-(-9)}\right](-9)$$

$$= 2 - (-0.64286) = 2.64286 \qquad \text{…(2)}$$

and $f(2.64286) = (2.64286)^3 - 5(2.64286) - 7 = 18.45960 - 13.2143 - 7 = -0.7547$ …(3)

Hence, the root of equation lies between 2.64286 and 3.

For next iteration, $x_2 = 2.64286$ and $x_1 = 3$, so that

$$x_3 = 2.46286 - \left[\frac{3 - 2.64286}{5 - (1.7547)} \times -(1.7547)\right]$$

$$= 2.64286 - \left[\frac{0.35714}{6.7547} \times (-1.7547)\right] = 2.64286 - (-0.09278) = 2.73564$$

and $f(2.73564) = (2.73564)^3 - 5(2.73564) = -0.2054(-\text{ve})$ …(4)

For next iteration, root lies between $x_3 = 2.73564$, $x_1 = 3$

$$\therefore \qquad x_4 = 2.73564 - \left[\frac{3 - 2.73564}{5 - (-0.2054)} \times -(0.2054)\right] = 2.7356 - (-0.01043). \qquad \text{…(5)}$$

And $f(2.7461) = (2.7461)^3 - 5(2.7461) - 7 = -0.0219$ …(6)

For next iteration, take $x_4 = 2.7461$, $x_1 = 3$

$$\therefore \qquad x_5 = 2.7461 - \left[\frac{3 - 2.7461}{5 - (0.0219)} \times -0.0219\right] = 2.7461 - (0.001107) = 2.7472,$$

And $f(2.7472) = (2.7472)^3 - 5(2.7472) - 7 = -0.00258$ …(7)

Hence again root lies between 2.7472 and 3

$$\therefore \quad x_6 = 2.7472 - \left[\frac{3 - 2.7472}{5 - (-0.00258)} \times (-0.00258)\right] = 2.7472 - (0.00013) = 2.7473. \quad \text{…(8)}$$

Again putting $x = 2.7473$ in equation (1), we get

$$f(2.7473) = (2.7473)^3 - 5(2.7473) - 7 = -0.00082$$

Hence, the required root up to 4 decimal places is = 2.7473.

Example 9: Obtain the root of the equation $x^x = 100$ correct to three decimal places, using iteration method. Given the root lies between 3 and 4.

Solution: Given $x^x = 10^2$ implying $x\log_{10} = 2\log_{10}10 = 2$ …(1)

Let $f(x) = x\log_{10}x - 2$

$\therefore$ $f(3) = 3\log_{10}3 - 2 = -0.5687$ and $f(4) = 3\log_{10}4 - 2 = 0.4084$ …(2)

Using iterative Regula-Falsi Formula, the 1st approximation,

$$x_1 = a - \frac{(b-a)}{f(b) - f(a)} f(a), \quad \text{where } a = 3 \text{ and } b = 4$$

$$= -3 - \frac{1}{0.4084 - (-0.5687)} \times -0.5687 = 3 + \frac{0.5687}{0.9771} = 3.50 \quad \text{…(3)}$$

Here $f(3.58) = -0.0717(-\text{ve})$ and $f(4) = 0.4084\ (+\text{ve})$ …(4)

Clearly the root lies in between 3.58 and 4.

For 2nd approximation

$$x_2 = a - \frac{(b-a)}{f(b) - f(a)} f(a), \quad \text{where } a = 3.58 \text{ and } b = 4.00$$

$$= 3.58 + \frac{0.42 \times 0.00171}{0.4255} = 3.58 + \frac{0.0072}{0.4255} = 3.5969$$

$\therefore$ Roots of $f(x) = 3.597$

Example 10: Find the root of the equation $\tan x + \tanh x = 0$ correct to three significant figures, using an iterative formula. Given that the root lies between 2 and 3.

Solution: Given $f(x) = \tan x + \tanh x$

$\therefore$ $f(a) = f(2) = -2.1850 + 0.9640 = -1.2210$ …(1)

$f(b) = f(3) = -0.1425 + 0.9950 = 0.8525$ …(2)

Using iterative Regula-Falsi Formula, the first approximation,

$$x_1 = a - \frac{(b-a)}{f(b) - f(a)} f(a), \quad \text{where } a = 3 \text{ and } b = 4$$

$$= -3 - \frac{1}{0.4084 - (-5687)} \times -0.5687 = 3 + \frac{0.5687}{0.9771} = 3.58 \qquad \ldots(3)$$

Here $f(3.58) = -0.0717(-\text{ve})$ and $f(4) = 0.4084\ (+\text{ve})$ ...(4)

Clearly the root lies in between 3.58 and 4.

For second approximation,

$$x_2 = a - \frac{(b-a)}{f(b) - f(a)} f(a), \text{ where } a = 3.58 \text{ and } b = 4.00$$

$$= 3.58 + \frac{0.42 \times 0.00171}{0.4255} = 3.58 + \frac{0.0072}{0..4255} = 3.5969$$

$\therefore$ Root of $f(x) = 3.597$

Example 10: Find the root of the equation tan*x* + tanh*x* = 0 correct to three significance figures, using an iterative formula. Given that the root lies between 2 and 3.

Solution: Given $f(x) = \tan x + \tanh x$

$\therefore$ $f(a) = f(2) = -2.1850 + 0.9640 = -1.2210$ ...(1)

$f(b) = f(3) = -0.1425 + 0.9950 = 0.8525$ ...(2)

Applying Regula-Falsi Method, **the first approximation** is.

$$x_1 = a - \frac{(b-a)}{f(b) - f(a)} f(a)$$

$$= 2 - \frac{1}{0.8525 - (-1.2210)}(-1.2210) = 2 + \frac{1.2210}{2.0735} = 2.95 \qquad \ldots(3)$$

and $f(2.59) = -0.6153 + 0.9888 = +0.3765$, $f(2) = -1.2210$ ...(4)

Therefore, the root lies in between 2 and 2.59.

For next iteration, $a = 2$, $b = 2.59$

$$\therefore \qquad x_2 = 2 + \frac{0.59}{1.5945}(1.2210) = 2.45 \qquad \ldots(5)$$

and $f(x_2) = f(2.45) = \tan 2.45 + \tanh 2.45 = -0.8280 + 0.9852 = 0.1572;$ ...(6)

For next iteration, $a = 2$ and $b = 2.45$

$$\therefore \qquad x_3 = 2.0 - \frac{0.45}{1.3782}(-1.2210) = 2.3987 \qquad \ldots(7)$$

and $f(2.3987) = -0.9184 + 0.9836 = 0.0652\ (+\text{ve})$ ...(8)

Hence, the root lies in between 2 and 2.3987.

For **4th approximation,** $a = 2$ and $b = 2.3987$

$$\therefore \qquad x_4 = a - \frac{(b-a)}{f(b) - (a)} f(a) = 2.0 - \frac{0.3987}{1.2862}(-1.2210) = 2.3785 \qquad \ldots(9)$$

Here $\qquad f(2.3785) = -0.9563 + 0.9830 = 0.0267 \approx 0$

Hence, $f(x) = 0$ *has the root*, $x = 2.3785$.

15.6 SECANT METHOD OR CHORD METHOD

This method is an improvement over the False Position Method as it does not require the condition of $f(a)\cdot f(b) < 0$. Here in this method also the graph of the curve is approximated by the secant chord (straight line) but at each iteration step, two recent most approximations to the root are used to find next approximation with the condition that the interval must contain the root.

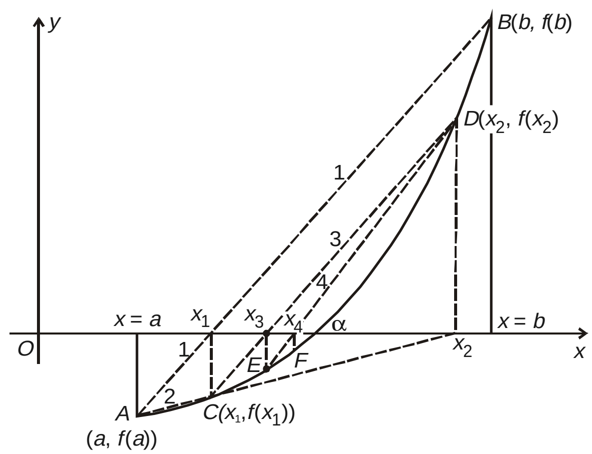

Fig. 15.5

Initially taking two points a and b, we write the equation of the chord joining these points as:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

Then the abscissa of the point where it crosses the x-axis ($y = 0$) is given by

$$x = a - \frac{(b - a)}{f(b) - f(a)} f(a)$$

which is an approximation to the root. The general formula for successive approximations is therefore given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n),\ n \ge 1$$

Observations:

This method fails if at any stage of iteration $f(x_n) = f(x_{n+1})$ which shows that it not necessarily converges. Once Secant converges, it convergences faster than Regula-Falsi with convergence rate of 1.618.

Example 11: Find the root of the equation $xe^x = \cos x$ using the Secant method correct to four decimal places.

Solution: Here $xe^x = \cos x$ for initial approximations $a = 0$, $b = 1$ gives,

$$f(0) = 1, \quad f(1) = \cos 1 - e = -2.17798$$

Then
$$x_1 = b - \frac{(b - a)}{f(b) - f(a)} f(b) = 1 - \frac{1}{-3.17798}(-2.17798) = 0.31467$$

and $\qquad f(x_1) = 0.51987$

For next iteration consider b and x_1 (being the recent most iterated values), so that

$$x_2 = x_1 - \frac{(b - x_1)}{f(b) - f(x_1)} f(x_1) = 0.31467 - \frac{(1 - 0.31467)}{-2.17798 - 0.51987}(0.51987) = 0.44673$$

and $f(x_2) = 0.20354$

For next iteration consider x_2 and x_1 (being the recent most iterated values), so that

$$x_3 = x_1 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2) = 0.44673 - \frac{(0.44673 - 0.31467)}{0.20354 - 0.51987}(0.20354) = 0.53171$$

Repeating this process, the successive iteration are $x_4 = 0.51690$, $x_5 = 0.51775$, $x_6 = 0.51776$. Hence, the root is 0.5177.

Observations: Here we see that up to 2^{nd} iterations, values are same by False-Position method as well as by Secant Method, but after 3^{rd} iteration onward values are definitely achieved faster in Secant Method than that of False Position Method as here blocking of the root is not required and we work recent most values.

15.7 NEWTON METHOD OR NEWTON-RAPHSON METHOD [KUK, 2007, 2008]

This method of successive approximation is due to English Mathematician and Physicist, Sir Issac Newton (1642-1727) is also called **fixed point method.** This technique is very useful for finding the root of the equation of the form $f(x) = 0$, where x is real and $f(x)$ is an easily differential function.

To derive the formula for computing the root by this method, let x_0 denote the approximate value of the desired root and let h denote the correction which must be applied to x_0 to give the exact value of the root, so that $x = x_0 + h$.

The equation, $f(x) = 0$ then becomes

$$f(x_0 + h) = 0 \qquad \ldots(1)$$

Expanding by Taylor's theorem, we get

$$0 = f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f'(x_0 + \theta h), \; 0 \le \theta \le 1; \qquad \ldots(2)$$

Now if h is relatively small, we may neglect the term containing h^2 and get the simple relation,

$$f(x_0) + hf'(x_0) = 0 \text{ implying } h = -\frac{f(x_0)}{f'(x_0)} \qquad \ldots(3)$$

So the 1^{st} iterated value of x i.e.

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \ldots(4)$$

The next improved value of the root is then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

And so on, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(5)$$

Observations:

(*i*) This method is generally used to improve the results obtained by other methods. It is applicable to solutions of both algebraic and transcendental having real as well for complex roots.

(*ii*) Newton formula converges provided the initial approximation x_0 is chosen of sufficiently close to the root, otherwise, it may lead to an astray. Thus, a proper choice of initial guess is very important for convergence of this method. For h to be small, $f'(x)$ has to be large. Because, when $f'(x)$ is large, then the graph of $f(x)$ while crossing the x-axis will be nearly vertical and the value is found rapidly. In other cases, when $f'(x)$ is small, the correction may take more number of iterations or even the method may fail leading to an endless cycle.

(*iii*) If the equation has a pair of double roots in the neighbourhood of x_n, the Newton Raphson method is modified as $x_{n+1} = x_n - 2\left(\dfrac{f(x_n)}{f'(x_n)}\right)$ which also shows its quadratic convergence.

(*vi*) Division by zero may occurs if $f'(x)$ is zero.

(*v*) In Newton-Raphson method, $f'(x_n)$ has to be evaluated at each iteration step, instead if we use $f'(0)$ at each iteration step, then $x_{n+1} = x_n - \left(\dfrac{f(x_n)}{f'(0)}\right)$ is called **Von Mises' Method.**

Geometrical Significance: Let x_0 be a point near the exact α of the equation, $f(x) = 0$. See the figure 15.6. then tangent at $A_0(x_0, f(x_0)$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, (since at $x = x_1, y_1 = 0$) which is the first approximation of the root α. If A_1 is the point corresponding to x_1 on the curve, on continuing, the tangent at A_1 will cut the x-axis at x_2 which is nearer to the root α, a second approximation to the root. This way, by repeating, we approaches the root quite rapidly. Clearly, here we replace the part of the curve between the point A_0 and the x-axis by means of the tangent at A_0.

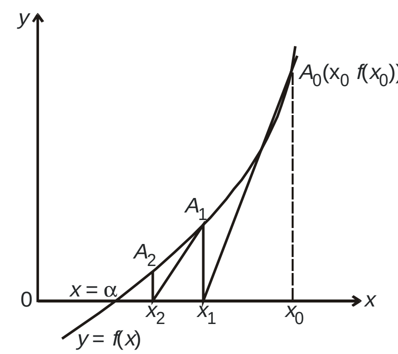

Fig. 15.6

Condition for Convergence:

From the formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ it is evident that x_{n+1} is a function of x_n **[KUK, 2007]**

i.e. $$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, $$\phi(x) = x - \frac{f(x)}{f'(x)} \quad \text{implying} \quad \phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$$

Since the iteration method (§ 15.3) converges if $|\phi'(x)| < 1$.

$\therefore$ Newton formula converges if $|f(x) \cdot f''(x)| < |f'(x)|^2$ is close interval.

Hence assuming $f(x)$, $f'(x)$, $f''(x)$ to be continuous, we can select a small interval in the vicinity of α in which the above condition is satisfied.

Order of Convergence: Let α be the exact root of the equation $f(x) = 0$ and let x_n and x_{n+1} be the n^{th} and $(n + 1)^{th}$ approximations of the root with ε_n and ε_{n+1} as the corresponding errors in them, so that **[KUK, 2007, 08, 09]**

$$x_n = \alpha + \varepsilon_n \quad \text{and} \quad x_{n+1} = \alpha + \varepsilon_{n+1}$$

Now by formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

i.e. $$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \dfrac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots\ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \dfrac{\varepsilon_n^2}{2!} f'''(\alpha) + \ldots\ldots}, \text{ Taylor's expansion}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \dfrac{\varepsilon_n^2}{2!} f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \dfrac{\varepsilon_n^2}{2!} f'''(\alpha) + \ldots\ldots}, \left(\because f(\alpha) = 0\right)$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2\left[f'(\alpha) + \varepsilon_n f''(\alpha)\right]}, \textit{ on neglecting 3rd and higher power of } \varepsilon_n$$

$$= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)\left[1 + \varepsilon_n \dfrac{f''(\alpha)}{f'(\alpha)}\right]},$$

$$= \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[1 + \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1} = \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$\cong \frac{\varepsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)}, \quad \text{where } \left[1 - \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right] \cong 1 \text{ for very small } \varepsilon$$

This is a quadratic convergence, if $\dfrac{1}{2} \dfrac{f''(\alpha)}{f'(\alpha)} < 1.$

Here it is cleat that error at each subsequent step is proportional to the square of the error in the previous step. Or in other words, error is corrected to double decimal places at every iteration at least if the factor $\dfrac{1}{2} \dfrac{f''(\alpha)}{f'(\alpha)}$ is not too large. e.g. if error is of order 0.01 after 1st iteration, means it will reduce to 0.0001 after 2nd iteration and hence as such the convergence is quadratic.

Special Forms in Particular Cases:

(*i*) $x_{n+1} = \dfrac{1}{2}\left[x_n + \dfrac{N}{x_n}\right]$ to compute the square root of N i.e. $\sqrt[2]{N}$

(*ii*) $x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$ to compute the cube root of N i.e. $\sqrt[3]{N}$

(*iii*) $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$ to compute the k^{th} root of N i.e. $\sqrt[k]{N}$

(*iv*) $x_{n+1} = x_n\left[2 - Nx_n\right]$ to compute the root of reciprocal of N i.e. $\frac{1}{N}$

(*v*) $x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right]$ to compute reciprocal of square root of N i.e. $\frac{1}{\sqrt{N}}$

Proofs:

(*i*) Let $x\sqrt{N}$ or $x^2 = N$ so that

$f(x) = x^2 - N$ and $f'(x) = 2x$

Then, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right], \quad N = 0, 1, 3, \ldots$$

(*ii*) Let $x = (N)^{\frac{1}{3}}$ or $x^3 = N \Rightarrow x^3 - N = 0$ so that

$f(x) = x^3 - N$ and $f'(x) = 3x^2$

Then, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right), \quad N = 0, 1, 2, \ldots$$

(*iii*) $x = \sqrt[k]{N}$ or $x^k = N \Rightarrow x^k - N = 0$ so that

$f(x) = x^k - N$ and $f'(x) = k\,x^{k-1}$

Then, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{k\,x_n^{k-1}} = \frac{k\,x_n^k - x_n^k + N}{k\,x_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n^k + \frac{N}{x_n^{k-1}}\right]$$

(*iv*) Let $x = \frac{1}{N}$ or $\frac{1}{x} - N = 0$ so that

$f(x) = \frac{1}{x} - N$ and $f(x) = -x^{-2}$, then by Newton's Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2$$

$$= x_n + [1 - N\,x_n]x_n = [2 - Nx_n]$$

(*v*) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$ so that

$f(x) = x^2 - \frac{1}{N}$ and then by Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \frac{1}{N}}{2x_n} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right]$$

Generalized Newton Raphson Method For Multiple Roots:

If a root α of the equation $f(x) = 0$ is repeated m times, then $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$ which is called a general Newton's formula and is reduces to Newton-Raphson formula, when $m = 1$.

Observations:

1. If α is one of the root of the equation $f(x) = 0$ with m times repetition, then α will be $(m-1)$ times repeated root of the equation $f'(x) = 0$, $(m-2)$ times repeated root of the equation $f''(x) = 0$ and so on. Further, if the initial approximation x_0 is sufficiently close to the root, then the expressions,

   $x_0 - m\frac{f(x_0)}{f'(x_0)},\ x_0 - (m-1)\frac{f(x_0)}{f'(x_0)},\ x_0 - (m-2)\frac{f(x_0)}{f'(x_0)}, \ldots\ldots$ will have the same value.
2. Generalized Newton formula has a second order convergence for multiple roots also.

Example 12: Show that the generalized Newton formula $x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$, gives a quadratic convergence when $f(x) = 0$ has a pair of double roots in the neighborhood of $x = x_n$.

Solution: Let $x = \alpha$ is a double near $x = x_n$, then $f(x) = 0$, $f'(x) = 0$

Now, $$\varepsilon_{n+1} = \varepsilon_n - \frac{2f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)},$$

where ε_n and ε_{n+1} are the errors in the x_n and x_{n+1} respectively.

$$= \varepsilon_n - \frac{2\left[\frac{\varepsilon_n^2}{2!}f''(\alpha) + \frac{\varepsilon_n^3}{3!}f'''(\alpha) + \ldots\right]}{\left[\varepsilon_n f''(\alpha) + \frac{\varepsilon_n^2}{2!}f'''(\alpha) + \ldots\right]}$$

$$= \varepsilon_n - \frac{\varepsilon_n\left[f(\alpha) + \frac{1}{3!}\varepsilon_n f'''(\alpha)\right]}{\left[f''(\alpha) + \frac{\varepsilon_n}{2}f'''(\alpha)\right]},$$ Neglecting terms with high powers of ε_n

$$= \frac{\varepsilon_n^2}{6}\frac{f'''(\alpha)}{\left[f''(\alpha) + \frac{\varepsilon_n}{2}f'''(\alpha)\right]} \cong \frac{1}{6}\left(\frac{f'''(\alpha)}{f''(\alpha)}\right)\varepsilon_n^2$$

Which clearly shows that $\varepsilon_{n+1} \propto \varepsilon_n^2$ and hence a second order convergence.

15.8 NEWTON RAPHSON EXTENDED FORMULA: CHEBYSHEV METHOD

We know that on expanding $f(x)$ in the neighborhood of x_0 by Taylor's series,

$$0 = f(x) = f(x + \overline{x - x_0}) = f(x_0) + (x - x_0)f'(x_0) \text{ to the first approximation}$$

Hence the Ist approximation to the root at $x = x_1$ is given by

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \qquad \ldots(1)$$

Again expanding $f(x_1)$ by Taylor's series to the second approximation, we get

$$f(x_1) = f(x_0 + \overline{x_1 - x_0}) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{1}{2}(x_1 - x_0)^2 f''(x_0)$$

Since x_1 is an approximation to the root, therefore, $f(x_1) = 0$

means $$f(x_0) + (x_1 - x_0)f'(x_0) + \frac{1}{2}(x_1 - x_0)^2 f''(x_0) = 0 \qquad \ldots(2)$$

Implying $$(x_1 - x_0)f'(x_0) = -f(x_0) - \frac{1}{2}(x_1 - x_0)^2 f''(x_0)$$

On dividing throughout by $f'(x_0)$ and then using (1)

$$(x_1 - x_0) = -\frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \frac{f''(x_0)}{f'(x_0)}, \qquad \ldots(3)$$

or $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \frac{f''(x_0)}{f'(x_0)} \qquad \ldots(4)$$

This is known as **Chebyshev's Formula of 3rd Order.**

Example 12: Compute the real root of $x\log_{10}x - 1.2 = 0$ upto four decimal places.
[Burdwan, 2003]

Solution: For given $f(x) = x\log_{10} x - 1.2$, find

$f(1) = 1 \log_{10} 1 - 1.2 = -1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.2313$

So the root lies between 2 and 3 since for $x = 2$, $f(x)$ is negative where as for $x = 3$, $f(x)$ is positive.

Take $x_0 = 3$, $f(x_0) = 0.2313$,

And $$f'(x_0) = 1 \cdot \log_{10} x + x\frac{1}{x}\log_{10} e = \log_{10} 3 + \log_{10} e = 0.4771 + 0.4342 = 0.9113$$

(Using, $\log_a x = \log_a e \cdot \log_e x$)

Now by Newton Raphson Method for $x_0 = 3$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{0.2313}{0.9113} = 2.7462$$

For next iteration, find $f(x_1)$ at $x_1 = 2.7462$ i.e.

$$f(x_1) = 2.7462 \times \log_{10}(2.7462) - 1.20 = 1.2048 - 1.20 = 0.0048.$$

And $$f'(x_1) = \log_{10}2.7462 + \log_{10}e = 0.4387 + 0.4342 = 0.8729$$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7462 - \frac{0.0048}{0.8729} = 2.7407$$

So 2.7407 is the real root of the given equation correct upto four decimal places.

Example 14: Using Newton-Raphson Method, compute the real root of the following equation correct to four decimal places, $x = \sqrt{28}$.

Solution: Given $x = \sqrt{28} \Rightarrow (x^2 - 28) = 0 = f(x)$ so that $f'(x) = 2x$

To start with, take $x = 5, 6$ and find

$$f(5) = 25 - 28 = -3 \quad \text{and} \quad f(6) = 36 - 28 = 8$$

As value of $f(x)$ at $x = 5$ is –ve, where as at $x = 6$ it is +ve, therefore, the root lies between 5 and 6, and near to 5.

1st Approximation:

For $$x_0 = 5, \quad f(x_0) = 3, \quad f'(x_0) = 10$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{-3}{10} = 5 + 3.0 = 5.300$$

2nd Approximation:

For $$x_1 = 5.30, \; f(x_1) = x^2 - 28 = (5.30)^2 - 28 = 0.09, \; f'(x_1) = 2 \times 5.30 = 10.60$$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5.30 - \frac{0.09}{10.60} 0.085 = 5.2915$$

3rd Approximation:

For $$x_2 = 5.0915, \; f(x_2) = (5.2915)^2 - 28 = -0.000027, \; f'(x_2) = 2 \times 5.2915 = 10.583$$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 5.2915 - \frac{0.000027}{10.583} = 5.291502$$

Which is correct upto 4 decimals.

Alternatively using the special form for finding the value of the square root of number n,

$$x_{i+1} = x_i - \frac{x_i^2 - n}{2x_i} = \frac{1}{2}\left(x_i + \frac{n}{x_i}\right), \quad i = 0, 1, 2.$$

Now to start with $x_0 = \dfrac{5+6}{2} = \dfrac{11}{2} = 5.5$

Using the above formula for $n = 28$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0}\right) = \frac{1}{2}\left(5.5 + \frac{28}{5.5}\right) = \frac{1}{2}(5.5 + 5.091) = 5.2955$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = \frac{1}{2}\left(5.2955 + \frac{28}{5.2955}\right) = \frac{1}{2}(5.2955 + 5.2875) = 5.2915$$

and $f(x) = f(5.2915) = x^2 - 28 = (5.2915)^2 - 28 = 27.9999 - 28 \approx 0$ at $x = x_2$

Hence, 5.2915 is the correct value upto 4 decimal of places.

ASSIGNMENT 1

1. Using the iteration method, find the root of the equation $x = \frac{1}{2} + \sin x$ correct to four decimal places. [MDU, 2002]

2. Find the real root of $2x - \log_{10}x = 7$ correct to four decimal places using iteration method. [MDU, 2002]

3. Using the method of iteration, find the real root of the equation $\cos x = 3x - 1$ correct to three decimal of places. Verify the result so achieved by Aitkin's Δ^2 Method.

4. Find the real root of the equation $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots = 0.463$ [MDU, 2007]

5. Using Bisection method, find the approximate root of the equation $\sin x = \frac{1}{x}$, that lies between $x = 1$ and $x = 1.5$, correct to the third decimal place. [VTU, 2003]

6. Using Bisection method find the root of the equation,

(*i*) $x - \cos x = 0$ [Madras, 2003] (*ii*) $x + \log_{10}x = 3.375$

7. Using the Regula-Falsi method, solve correct to four decimal places

(*i*) $x - \log_{10}x = 3.375$ [MDU,2003] (*ii*) $10^x + x - 4 = 0$

8. Find the root of the equation $xe^x = \cos x$, using the Regula-Falsi method correct to four decimal places. [Hint: $f(0) = -2.17798$] [KUK, 2008; NIT Kurukshetra, 2008]

9. Use the method of False Position, to find the fourth root of 32 correct to three decimal places. [Hint: Take $x^4 - 32 = 0$]

10. Use Regula Falsi method to compute real root of the equation $x^3 - 9x + 1 = 0$ if the root lies between 2 and 3. [NIT Jalandhar, 2006; 2007]

11. Find the cube of 41, using Newton-Raphson method, $\left[\text{Hint}: x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)\right]$

[Madras, 2006; NIT Kurukshetra, 2003]

12. Use Newton's method to find the smallest root of the equation $e^x \sin x = 1$ to four places of decimal. [MDU Rohtak, 2007]

13. The bacteria concentration in reservoirs as $C = 4e^{-2t} + e^{-0.1t}$, using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5.

[MDU, 2007]

[Hint: As here f(6) = 0.048836212, $f(7) = -0.0034117$, $C = 0.5$, take $x_0 = 6.80$]

14. Find by Newton's method, the root of the equation $\cos x = xe^x$. [Madras 2003; VTU, 2003]

15. Find the square root of 20 correct to three decimal of places by using recurrence formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{20}{x_n}\right).$$

[NIT Allahabad, 2006]

15.9 HORNER'S METHOD

This method consists in finding of both rational and irrational roots of the polynomial equation figure by figure first finding the integer part and then decimal part to any desired decimal place of accuracy. The step by step procedure for diminution is given below:

1. Here by trial, we find integers 'a and b' such that $f(a)$ and $f(b)$ are of opposite sign for continuous $f(x)$, thus bracketing the root. Let the bracketed root be $a.d_1d_2d_3d_4 \ldots$ where d_1, d_2, d_3, d_4 etc. are the digits in the decimal part.
2. Diminish the root of the equation $f(x) = 0$ by 'a', to get new equation $f_1(x) = 0$ with root $0.\mathrm{d}_1d_2d_3d_4\ldots$
3. Multiply the root of the equation $f_1(x) = 0$ by 10, so that new equation $g(x) = 0$ is obtained with one of the root as $d_1.\ d_2d_3d_4 \ldots$ which lies between 0 and 10.
4. Now d_1 is found such that $g(d_1)$ and $g(d_1 + 1)$ are of opposite sign. Then the root of the equation $g(x) = 0$ is dimished by d_1 and the resulting equation $g_1(x) = 0$ with root $0.d_2d_3d_4\ldots$
5. Multiply the root of the equation $g_1(x) = 0$ by 10 so that the new equation $h(x) = 0$ is obtained with one of the root as $d_2\ .\ d_3d_4\ldots$ which lies between 0 and 10. d_2 is obtained by trial as before. By continuing this process, the digits d_3 , $d_4\ldots$ are successively obtained.

Observations:

This method is best for finding the real roots of a polynomial equation but can be applied for a transcendental equation.

Example 15: Find by Horner's Method the root of the equation $x^3 + 24x - 50 = 0$ that lies between 1 and 2 to three decimal places.

Solution: Let the root of equation $f(x) = x^3 + 24x - 50 = 0$ be $1.\ d_1\ d_2\ d_3\ d_4\ldots$

Diminish the root of the above equation by 1, so that the root of the transformed equation is $0.d_1\ d_2\ d_3\ d_4\ldots$

1	1	0	24	–50
	–	1	1	25
1	1	1	25	–25
	–	1	2	
1	1	2	27	
	–	1		
1	1	3		

Thus, the transformed equation $f_1(x) = x^3 + 3x^2 + 27x - 25$ has a root $0.\ d_1\ d_2\ d_3\ d_4\ldots$ lying between 0 and 1. Now, multiply the root of $f_1(x) = 0$ by 10 so that the changed equation $g(x) = x^3 + 30x^2 + 2700x - 25000 = 0$ has one of its roots as $d_1\ .\ d_2\ d_3\ d_4\ldots$ Find $g(8) < 0$ and $g(9) > 0$ so that $d_1 = 8$.

Now diminish the root of the above equation by 8.

8	1	30	2700	–25000
	–	8	304	24032
8	1	38	3004	–968
	–	8	368	
8	1	46	3372	
	–	8		
	1	54		

Thus the transformed equation $g_1(x) = x^3 + 30x^2 + 2700x - 25000 = 0$ has a root $0.d_2\ d_3\ d_4\ldots$ lying between 0 and 1. Multiply the root of $g_1(x) = 0$ by 10.

The new equation $h(x) = x^3 + 540x^2 + 337200x - 968000 = 0$ has its root as $d_2\ .\ d_3\ d_4\ldots$ lying between 1 and 10. Find $p(8) < 0$ and $p(9) > 0$ so that this root is $8.\ d_4\ \ldots$

Now diminish the root of the equation $h(x) = 0$ by 2.

2	1	540	337200	–968000
	–	2	1084	675668
2	1	542	338284	–291432
	–	2	1088	
2	1	544	339372	
	–	2		
	1	546		

The transformed equation $h_1(x) = x^3 + 546x^2 + 33972x - 291432 = 0$ has its root $0.d_3d_4\ldots$between 0 and 1. Multiply the root of this equation by 10.

The new equation, $p(x) = x^3 + 5460x^2 + 33937200x - 291432000 = 0$ has one of the root as $d_3\ .\ d_4\ldots$ lying between 0 and 10. Find $p(9) < 0$ and $p(9) > 0$ so that this root is $8.d_4\ldots$

Now diminish the root of the above equation by 8.

8	1	5460	33937200	–291432000
	–	8	43744	27184552
8	1	5468	3380944	–19584448
	–	8	43808	
8	1	5476	34024752	
	–	8		
	1	5484		

Hence, the root of the equation $f(x) = x^3 + 24x - 50 = 0$ correct to 4 decimal of place is 1.8285.

15.10 MULLER'S METHOD

In this method first we assume three approximations x_{n-2}, x_{n-1}, x_n as the root of the equation $f(x) = 0$. The next better approximatio x_{n+1} is obtained as the root of this equation $f(x) = 0$ by approximating the polynomial $f(x)$ as a second degree parabola passing through these three points $(x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), (x_n, y_n)$.

Let $p(x) = A(x - x_n)^2 + B(x - x_n) + y_n$ ...(1)

be the desired parabola passing through these points, so that

$$y_{n-1} = A(x_{n-1} - x_n)^2 + B(x_{n-1} - x_n) + y_n \quad ...(2)$$

$$y_{n-2} = A(x_{n-2} - x_n)^2 + B(x_{n-2} - x_n) + y_n \quad ...(3)$$

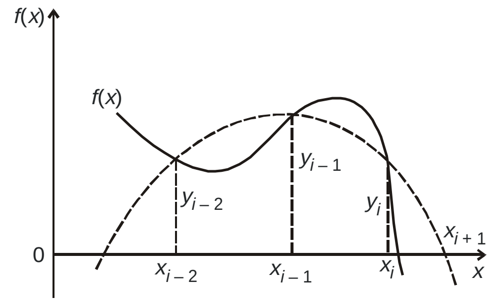

Fig. 15.7

On simplification, we get two simultaneous equations in unknowns A and B,

$$y_{n-1} - y_n = A(x_{n-1} - x_n)^2 + B(x_{n-1} - x_n)$$

$$y_{n-2} - y_n = A(x_{n-2} - x_n)^2 + B(x_{n-2} - x_n)$$

More precisely, write the above equations as:

$$d_1 = Ah_1^2 + Bh_1, \text{ where } h_1 = x_{n-1} - x_n;\ d_1 = y_{n-1} - y_n \quad ...(4)$$

$$d_2 = Ah_2^2 + Bh_2, \text{ where } h_2 = x_{n-2} - x_n;\ d_2 = y_{n-2} - y_n \quad ...(5)$$

From (4) and (5), we get

$$A = \frac{h_2 d_1 - h_1 d_2}{h_1 h_2 (h_1 - h_2)} \text{ and } B = \frac{h_1^2 d_2 - h_2^2 d_1}{h_1 h_2 (h_1 - h_2)} \quad ...(6)$$

With these values of A and B, the quadratic equation (1),

$$p(x) = A(x - x_n)^2 + B(x - x_n) + y_n = 0$$

gives the next approximate root at $x = x_{n+2}$ as

$$(x_{n+1} - x_n) = \frac{-B \pm \sqrt{B^2 - 4Ay_n}}{2A}$$

$$= \frac{-B \pm \sqrt{B^2 - 4Ay_n}}{2A} \times \frac{B \pm \sqrt{B^2 - 4Ay_n}}{B \pm \sqrt{B^2 - 4Ay_n}} = \frac{-2y_n}{B \pm \sqrt{B^2 - 4Ay_n}} \quad ...(8)$$

In (8), the sign in the denominator is so chosen that it is largest in magnitude meaning there by that the difference between x_{n+1} and x_n is very small. See the subsequent example as an illustration of the method.

Observations: This method of iteration converges quadratically almost for all initial approximations. In case, if no better approximations are known, we can take $x_{n-2} = -1$, $x_{n-1} = 0$ and $x_n = 1$. It can also be used for finding complex roots.

Example 16: Find the root of the equation $x^3 - 2x - 5 = 0$ which lies between 2 and 3 by Muller's method.

Solution: Le $x_{n-2} = 1$, $x_{n-1} = 2$ and $x_n = 3$ so that

$$y_{n-2} = -6,\ y_{n-1} = 1 \text{ and } y_n = 16$$

Then

$$h_1 = x_{n-1} - x_n = -1,\ d_1 = y_{n-1} - y_n = -17;$$

$$h_2 = x_{n-2} - x_n = -2,\ d_2 = y_{n-2} - y_n = -22;$$

So that

$$A = \frac{h_2 d_1 - h_1 d_2}{h_1 h_2 (h_1 - h_2)} = 6,\quad B = \frac{h_1^2 d_2 - h_2^2 d_1}{h_1 h_2 (h_1 - h_2)} = 23$$

And the quadratic becomes $6(x_{n+1} - 3)^2 + 23(x_{n+1} - 3) + 16 = 0$ (From (1), Multer's Method)

$\therefore$ The next approximation of the root x_{n+1} to the desired root is

$$(x_{n+1} - 3) = \frac{32}{23 + \sqrt{45}}, \text{ Since } B \text{ is positive or } x_{n+1} = 2.0868$$

This process can now be repeated with three approximations 1, 2, 2.0868 respectively for still better value.

15.11 LIN-BAIRSTOW METHOD

Here in this method, by an iteration process we resolve an nth degree polynomial $f(x)$ into quadratic factors corresponding to a pair of real or complex roots. For purpose of illustration of the method, here we take a fourth degree polynomial, however, the contention can be extended to a polynomial of nth degree.

Let the polynomial be $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$. When this polynomial will be divided by $(x^2 + px + q)$, the quotient will be a second degree polynomial of the form $(b_0x^2 + b_1x + b_2)$ and the remainder will be a first degree expression of the form $(Rx + S)$

Thus $$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = (x^2 + px + q)(b_0x^2 + b_1x + b_2) + (Rx + S) \quad \ldots(1)$$

Here values of R and S will depend on p and q and hence they are taken functions of p and q. If p and q are so chosen that R and S vanish (i.e. no remainder), then $(x^2 + px + q)$ will be factor of the given polynomial. Meaning thereby, the problem reduces to finding $p$and q such that

$$R(p, q) = 0 \quad \text{and} \quad S(p, q) = 0 \quad \ldots(2)$$

Let $(p_0\ q_0)$ be an initial approximation and $(p_0 + \Delta p_0, q_0 + \Delta q_0)$ be the actual solution of the equation (2).

Then $$R(p_0 + \Delta p_0, q_0 + \Delta q_0) = 0 \text{ and } S(p_0 + \Delta p_0, q_0 + \Delta q_0) = 0 \quad \ldots(3)$$

By Taylor's series, from (3)

$$R(p_0, q_0) + \frac{1}{1!}(R_p\,\Delta p_0 + R_q\,\Delta q_0) = 0 \text{ (app.), where } R_p = \frac{\partial R}{\partial p} \text{ and } R_q = \frac{\partial R}{\partial q}$$

i.e. $$(R_p\Delta p_0 + R_q\Delta q_0) = -R(p_0, q_0) \quad \ldots(4)$$

Similarly $$(S_p\Delta p_0 + S_q\Delta q_0) = -S(p_0, q_0) \quad \ldots(5)$$

Solving equations (4) and (5)

$$\Delta p_0 = \frac{SR_q - RS_q}{R_pS_q - R_qS_p} \quad \text{and} \quad \Delta q_0 = \frac{RS_p - SR_p}{R_pS_q - R_qS_p} \quad \ldots(6)$$

On repeating like terms in (1),

$$\begin{array}{lcl} a_0 = b_0, & & b_0 = a_0, \\ a_1 = b_0p + b_1, & & b_1 = a_1 - pb_0 \\ a_2 = b_0q + b_1p + b_2, & & b_2 = a_2 - pb_1 - qb_0, \\ a_3 = b_1q + b_2p + R, & \text{implying} & b_3 = a_3 - pb_2 - qb_1, \\ a_4 = b_2q + S, & & b_4 = a_4 - pb_3 - qb_2, \end{array} \quad \ldots(7)$$

where we have assumed that $R = b_3$, and $S = b_4 + pb_3$ $\quad \ldots(8)$

The above equations can be represented by recurrence formula

$$b_\alpha = a_\alpha - pb_{\alpha-1} - qb_{\alpha-2}, \text{ where } b_{-1} = 0 \text{ and } b_{-2} = 0;\ \alpha = 0, 1, 2, 3, 4 \quad \ldots(9)$$

Differentiating (9) partially w.r.t. p and denoting $\dfrac{\partial b_\alpha}{\partial p}$ by $-c_{\alpha-1}$, we get

$$-c_{\alpha-1} = -(-pc_{\alpha-2} + b_{\alpha-1}) - q(-c_{\alpha-3}) \text{ i.e. } c_{\alpha-1} = b_{\alpha-1} - pc_{\alpha-2} - qc_{\alpha-3} \quad \ldots(10)$$

Differentiating (9) partially w.r to q and denoting $\dfrac{\partial b_\alpha}{\partial q}$ by $-c_{\alpha-2}$, we get

$$-c_{\alpha-2} = -p(-c_{\alpha-3}) - (-qc_{\alpha-4} + b_{\alpha-2}) \text{ i.e. } c_{\alpha-2} = b_{\alpha-2} - pc_{\alpha-3} - qc_{\alpha-4} \quad \ldots(11)$$

The equations (10) and (11) can be represented by a single recurrence relation

$$c_\alpha = b_\alpha - pc_{\alpha-1} - qc_{\alpha-2},\ k = 0, 1\ 2, 3; \text{ where } c_0 = b_0 \text{ and } c_{-1} = 0 = c_{-2} \quad \ldots(12)$$

Now
$$R_p = \frac{\partial b_3}{\partial p} = -c_2 \left(\because \frac{\partial b_\alpha}{\partial p} = -c_{\alpha-1}\right),\quad S_p = \frac{\partial b_4}{\partial p} + p\frac{\partial b_3}{\partial p} + b_3 = -c_3 - pc_2 + b_3;$$

$$R_q = \frac{\partial b_3}{\partial q} = -c_1 \left(\because \frac{\partial b_k}{\partial q} = -c_{\alpha-2}\right),\quad S_q = \frac{\partial b_4}{\partial q} + p\frac{\partial b_3}{\partial q} = -c_2 - pc_1$$

On using the values of R, S, R_p, R_q, S_p, S_q in (6), we get

$$\Delta p_0 = \frac{-c_1(b_4 + pb_3) + b_3(c_2 + pc_1)}{c_2(c_2 + pc_1) - c_1(c_3 + pc_2 - b_3)} = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)}$$

$$\Delta q_0 = \frac{b_3(-c_3 - pc_2 + b_3) + c_2(b_4 + pb_3)}{c_2^2 - c_1(c_3 - b_3)} = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)}$$

The better approximations for p and q are then obtained as

$$p_1 = p_0 + \Delta p_0 \text{ and } q_1 = q_0 + \Delta q_0 \quad \ldots(13)$$

Still better approximations for p and q are obtained by iterative procedure similar to synthetic division method for computing b_α's and c_α's as below.

	a_0	a_1	a_2	a_3	a_4
$-p$		$-pb_0$	$-pb_1$	$-pb_2$	$-pb_3$
$-q$			$-qb_0$	$-qb_1$	$-qb_2$
	b_0	b_1	b_2	b_3	b_4
$-p$		$-pc_0$	$-pc_1$	$-pc_2$	
$-q$			$-qc_0$	$-qc_1$	
	c_0	c_1	c_2	c_3	

Example 17: Extract the quadratic factor of the form $x^2 + px + q$ from the polynomial $x^4 - 3x^3 - 4x^2 - 2x + 8$ using Bairstow method and assuming that $p_0 = 1.5$ and $q_0 = 1.5$.

	1	–3	–4	–2	8
$-p_0 = -1.5$		–1.5	6.75	–1.875	–4.3125
$-q_0 = -1.5$			–1.5	6.75	–1.875

	$1(b_0)$	$-4.5(b_1)$	$1.25(b_2)$	$2.875(b_3)$	$1.8125(b_4)$
$-p_0 = -1.5$		-1.5	9.0	-13.125	
$-q_0 = -1.5$			-1.5	9	
	1	$-6.0(c_1)$	$8.75(c_2)$	$-1.25(c_3)$	

$$\Delta p_0 = \frac{b_3c_2 - b_4c_1}{c_2^2 - c_1(c_3 - b_3)} = \frac{2.875 \times 8.75 + 1.8125 \times 6}{(8.75)^2 - 6(1.25 + 2.875)} = \frac{36.03125}{51.8125} = 0.6954$$

$$\Delta q_0 = \frac{b_4c_2 - b_3(c_3 - b_3)}{c_2^2 - c_1(c_3 - b_3)} = \frac{1.8125 \times 8.75 + 2.875 \times 4.125}{51.8125} = 0.5350$$

$$p_1 = p_0 + \Delta p_0 = 1.5 + 0.6954 = 2.1954$$

$$q_1 = q_0 + \Delta q_0 = 1.5 + 0.5350 = 2.0350$$

	1	-3	-4	-2	8
$-p_1 = 2.1954$		-2.1954	11.4060	-11.7915	7.0668
$-q_1 = -2.0350$			-2.0350	10.5726	-10.9300
	1	-5.1954	5.3710	$-3.2189(b_3)$	$4.1368(b_4)$
$-p_1 = -2.1954$		-2.1954	16.2258	-42.9460	
$-q_1 = -2.0350$			-2.0350	15.0402	
	1	$-7.3908\ (c_1)$	$19.5618(c_2)$	$-31.1247(c_3)$	

$$\Delta p_1 = \frac{-3.2189 \times 19.5618 + 4.1368 \times 7.3908}{(19.5618)^2 - 7.3908 \times 27.9058} = -\frac{32.3932}{176.4178} = -0.1836$$

$$\Delta q_1 = \frac{4.1368 \times 19.5618 - 3.2189 \times 27.9058}{176.4178} = -\frac{8.9027}{176.4178} = -0.0505$$

$\therefore$ $$p_2 = p_1 + \Delta p_1 = 2.1954 - 0.1836 = 2.0118$$

$$q_2 = q_1 + \Delta q_1 = 2.0350 - 0.0505 = 1.9845$$

	1	-3	-4	-2	8
$-p_2 = -2.118$		-2.0118	10.0827	-8.2448	0.6013
$-q_2 = -1.9845$			-1.9845	9.9459	-8.1329
		-5.0118	4.0982	$-0.2989(b_3)$	$0.4684(b_4)$
$-p_2 = -20118$	1	-2.0118	14.1301	-32.6793	
$-q_2 = -1.9845$			-1.9845	13.9383	
	1	$-7.0236(c_1)$	$16.2438(c_2)$	$-19.03999(c_3)$	

$$\Delta p_2 = \frac{-0.2989 \times 16.2438 + 0.4684 \times 7.0236}{(16.2438)^2 - 7.0236 \times 18.7410} = -\frac{1.5654}{132.318} = -0.0118$$

$$\Delta q_2 = \frac{0.4684 \times 16.2438 - 0.2989 \times 18.7410}{132.2318} = \frac{2.0069}{132.2318} = 0.0152$$

$$p_3 = p_2 + \Delta p_2 = 2.0118 - 0.0118 = 2$$

$$q_3 = q_2 + \Delta q_2 = 1.9845 - 0.0152 = 1.9997$$

The required quadratic factor is $x^2 + 2x + 1.9997$, while the actual factor is $x^2 + 2x + 2$.

Assignment 2

1. Apply Muller's method to obtain the root of the equation $x - xe^x = 0$, which lies between 0 and 1. [Hint: $x_0 = -1$, $x_1 = 0$, $x_2 = 1$]
2. Solve the equation $x^3 + 2x^2 + 10x - 20 = 0$ by the Muller's method.
   [Hint: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$; $h = -0.645934$]
3. Using Muller's method, find the root of the equation $x^3 - 2x - 5 = 0$, which lies between 1 and 2. [NIT Allahabad 2007]
4. Apply Muller's method to obtain the real root of the equation $x^3 - x^2 - x - 1 = 0$, which lies between 1 and 2. [MDU, 2006]
5. Find by Horner's method, the positive root of $x^3 + x^2 + x - 100 = 0$ correct to three decimal place.
6. Using Horner's method, find the root of the equation $x^3 - 3x^2 + 2.5 = 0$, which lies between 1 and 2 correct to three decimal place.
7. Find the cube root of 30 correct to 3 decimal place, using Horner's method.
8. Using Barstow's method, obtain the quadratic factors of the equation $x^4 - x^3 + 6x^2 + 5x + 10 = 0$ with $p, q = 1.14, 1.42$ (perform two iterations). [NIT Jalandhar, 2007]

15.12 COMPARATIVE STUDY

The bisection method is the most reliable one as here we shall certainly reach the root. But it is very slow since only one binary digit precision is achieved in each iteration.

If evaluation of $f'(x)$ is not difficult, then Newton-Raphson Method is recommended. The Newton-Raphson Method is considered to be the fastest method. But this method approaches to the root only when the initial guess is chosen sufficiently near to the root. Hence, its convergence must be verified using bisection method once in every 10 iteration.

If evaluation of $f'(x)$ is difficult, Secant Method (Chords Method) is recommended. It gives better results than Regula-Falsi Method, but this may not converge to the root some time. Regula-Falsi Method always converges.

For locating complex roots, Newton's Method, Muller's Method and Lin-Bairstow Methods are recommended. However, Bairstow is best among all. If all the roots of the given equation are required, then Lin-Bairstow Method is quite useful.

For finding the real roots of a polynomial, Horner's and Graeffe's Root Squaring Method are composed. But, if the roots are real and **distinct,** then Graeffe's Root Squaring Method is quite useful.

ANSWERS

Assignment 1

1. $x_1 = 1.5$, $x_2 = 1.49749$, $x_3 = 1.49731$, $x_4 = 1.49730$
2. 3.78927
3. $x_4 = 0.6067$, $x_5 = 0.6072$, $x_6 = 0.6071$
4. $x_1 = 0.4703$, $x_2 = 0.4754$, $x_3 = 0.4765$, $x_4 = 0.4774$, $x_5 = 0.4778$
5. 1.11328
6. **(*i*)** 0.937 **(*ii*)** 2.9
7. (*i*) 2.91 (*ii*) 0.695
8. 0.5177
9. 2.378
10. 2.9428
11. 3.44855
12. 0.58853
13. 6.888
14. 0.5177

Assignment 2

1. $x_3 = 0.5124$; $x_4 = 0.5177$; $x_5 = 0.5177$
2. $x_3 = 1.3540659$; $x_4 = 1.3686472$; $x_5 = 1.366808107$
3. 2.0945
4. 1.839
5. 4.264
6. 1.168
7. 3.107
8. $x^2 + 1.1446x + 1.4219$