This course was developed as a part of the follwoing VLIR-UOS Cross-Cutting projects:

- Statistics: 2011-2016, 2017.
- Statistics: 2017.
- Statistics for development : 2018-2022.

The >eR-Biostat initative

Making R based education materials in statistics accessible for all

Basic concepts in statistical modeling using R: simple linear regression

Developed by

Legesse Kassa Debusho (UNISA, South Africa) and Ziv Shkedy (Hasselt University)

https://erbiostat.wixsite.com/erbiostat

LAST UPDATED: 2022

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Contents

- Simple linear regression:
  - Introduction and model formulation.
  - Fitting a simple linear regression model using the lm() function in R.
  - Model diagnostic.
  - Model diagnostic in R.

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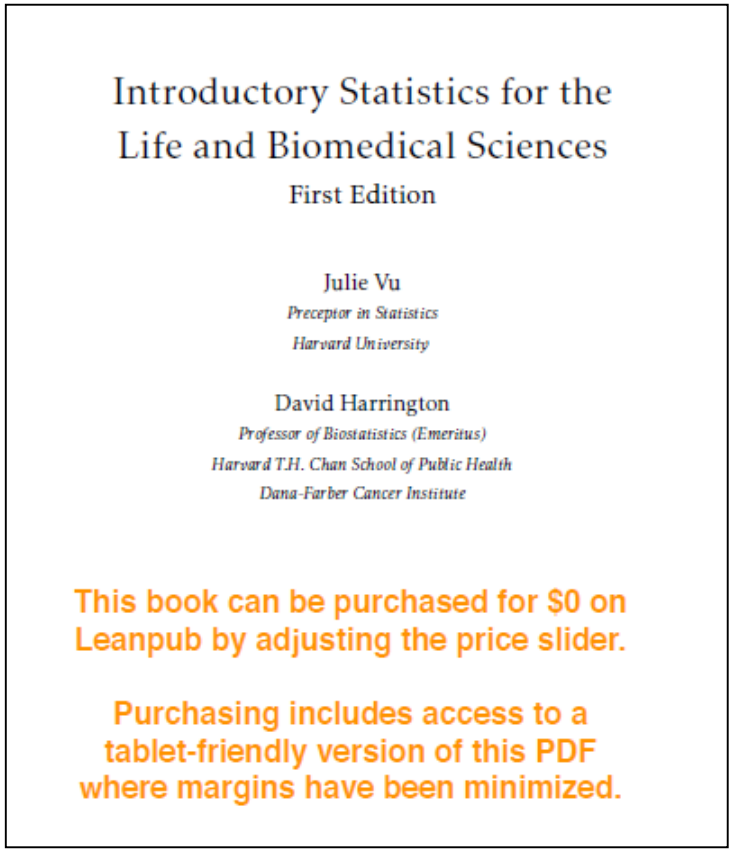
Recommended reading



Introductory Statistics for the Life and Biomedical Sciences

First Edition

Julie Vu
Preceptor in Statistics
Harvard University

David Harrington
Professor of Biostatistics (Emeritus)
Harvard T.H. Chan School of Public Health
Dana-Farber Cancer Institute

This book can be purchased for $0 on Leanpub by adjusting the price slider.

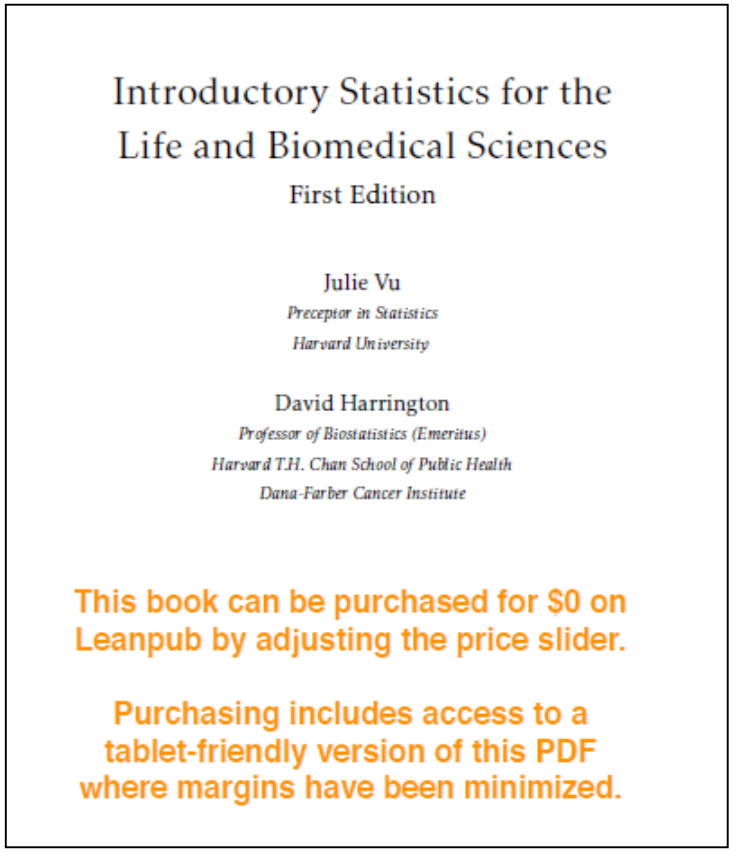
Purchasing includes access to a tablet-friendly version of this PDF where margins have been minimized.

Chapter 6: Simple linear regression

The book is available for free online:

https://www.openintro.org/book/biostat/

Recommended reading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- In this part of the course, we cover mainly Chapter 6.
- The examples that are used in the slides for illustration are not the same as the examples in the book.
- Use the R program to reproduce the examples.

Chapter 6: Simple linear regression

Part 1
Simple linear regression

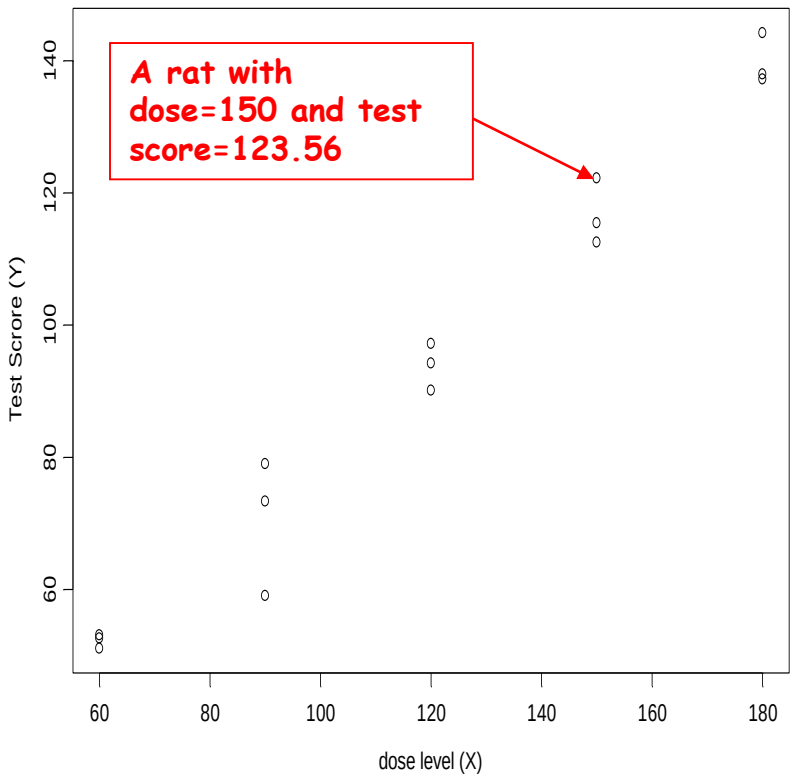
Introduction

A Biopharmaceutical problem

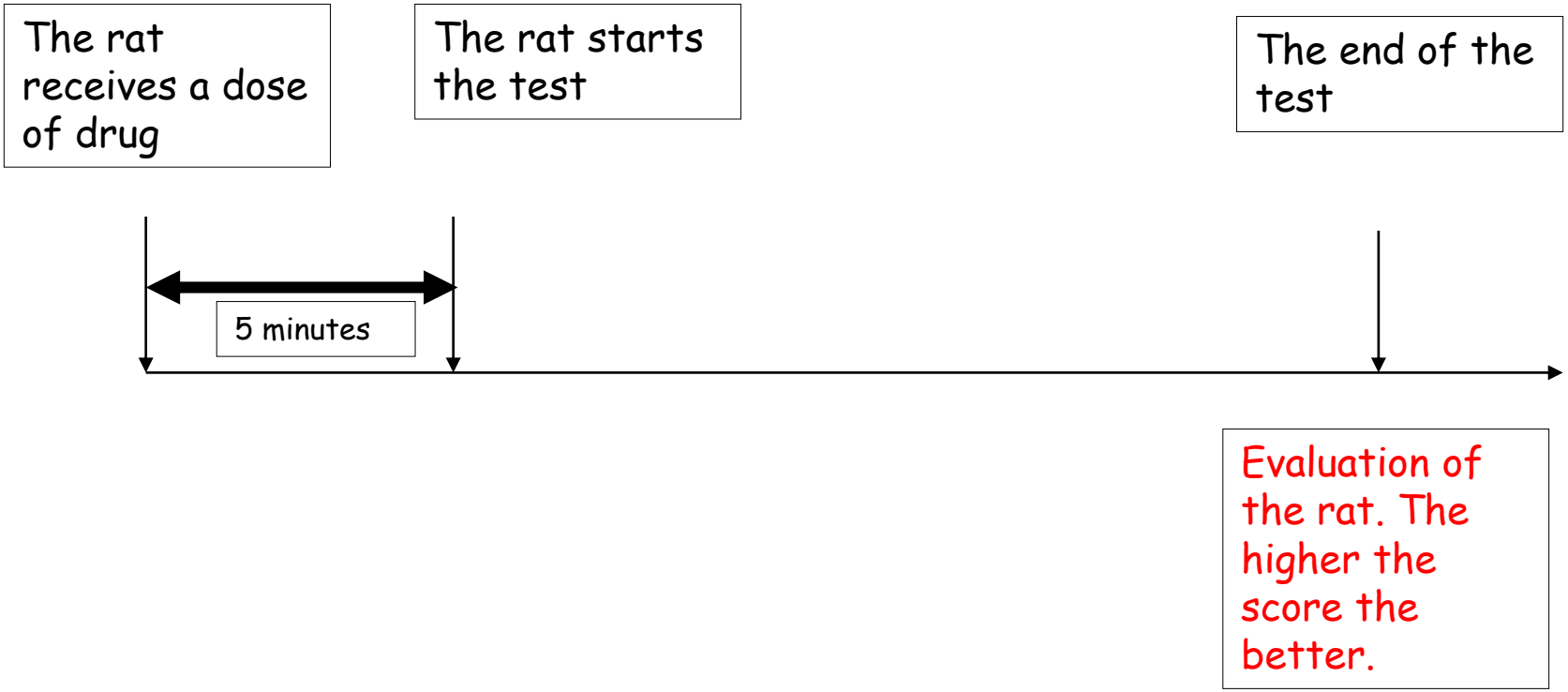
- A group of 15 rats received a dose of a drug and then had to complete a test.

- It is assumed that the performance of the rat (=test score) depends on the dose level.

The data



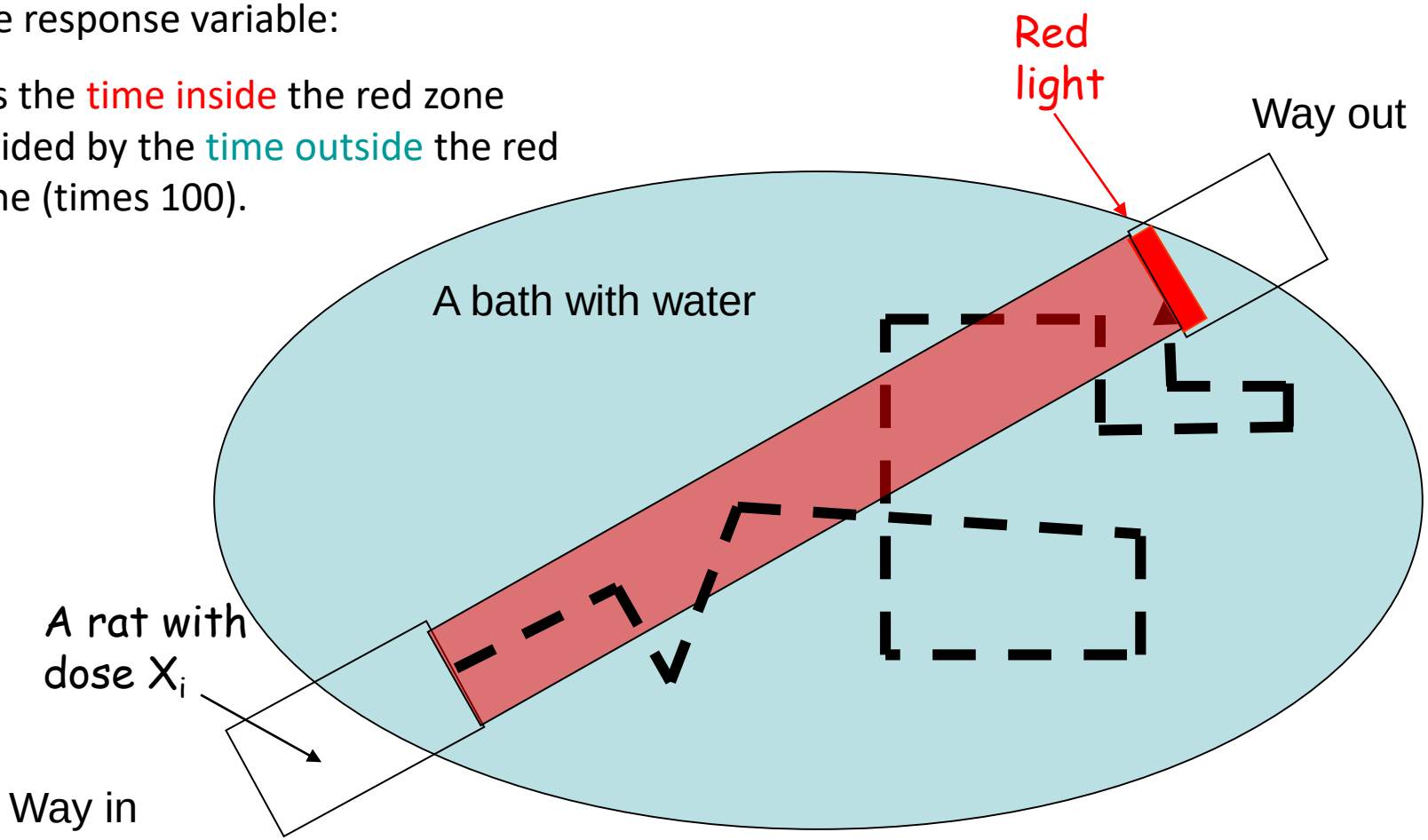
Description of the experiment



The evaluation of the rat

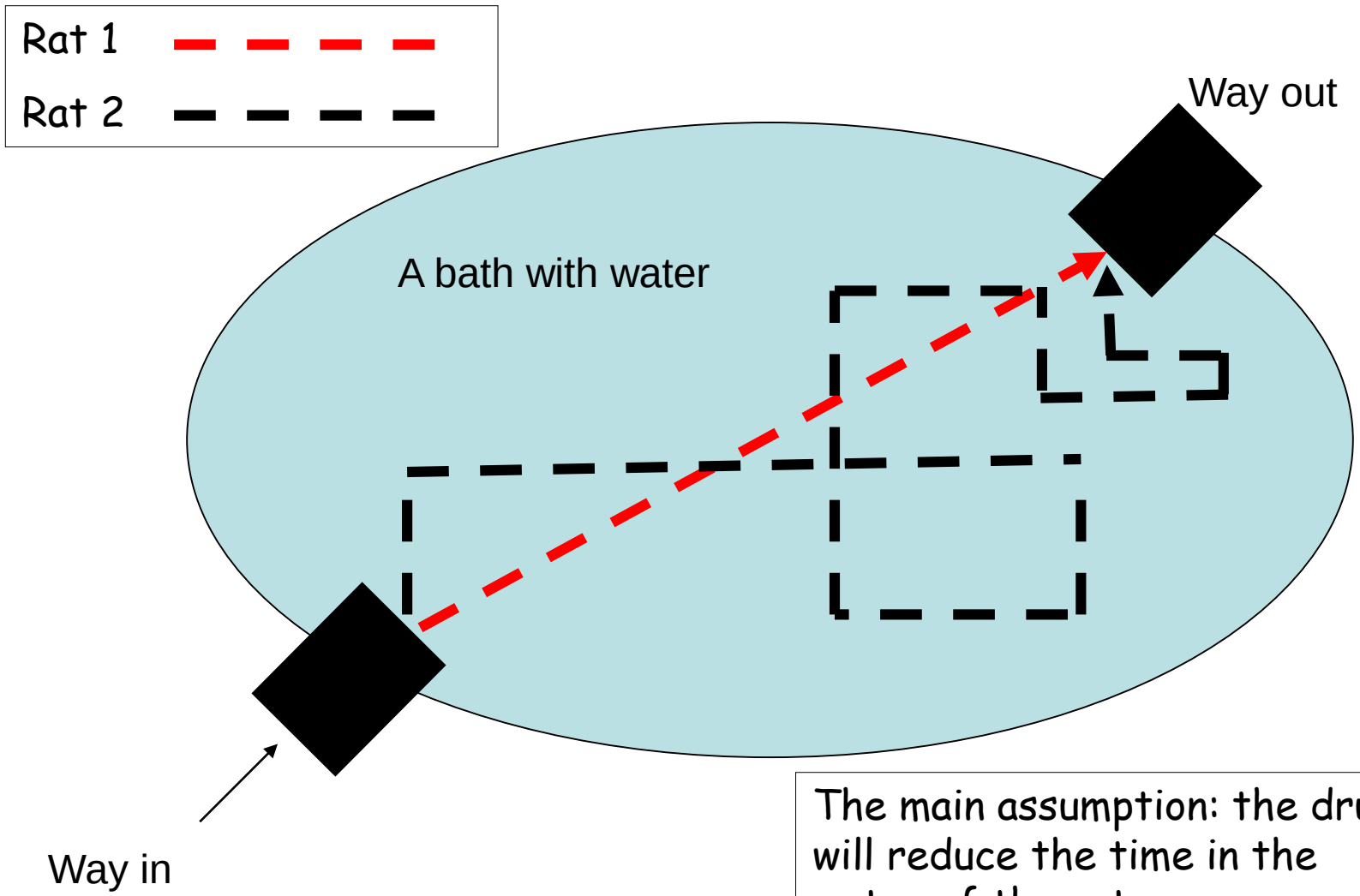
The response variable:

Y_i is the time inside the red zone divided by the time outside the red zone (times 100).



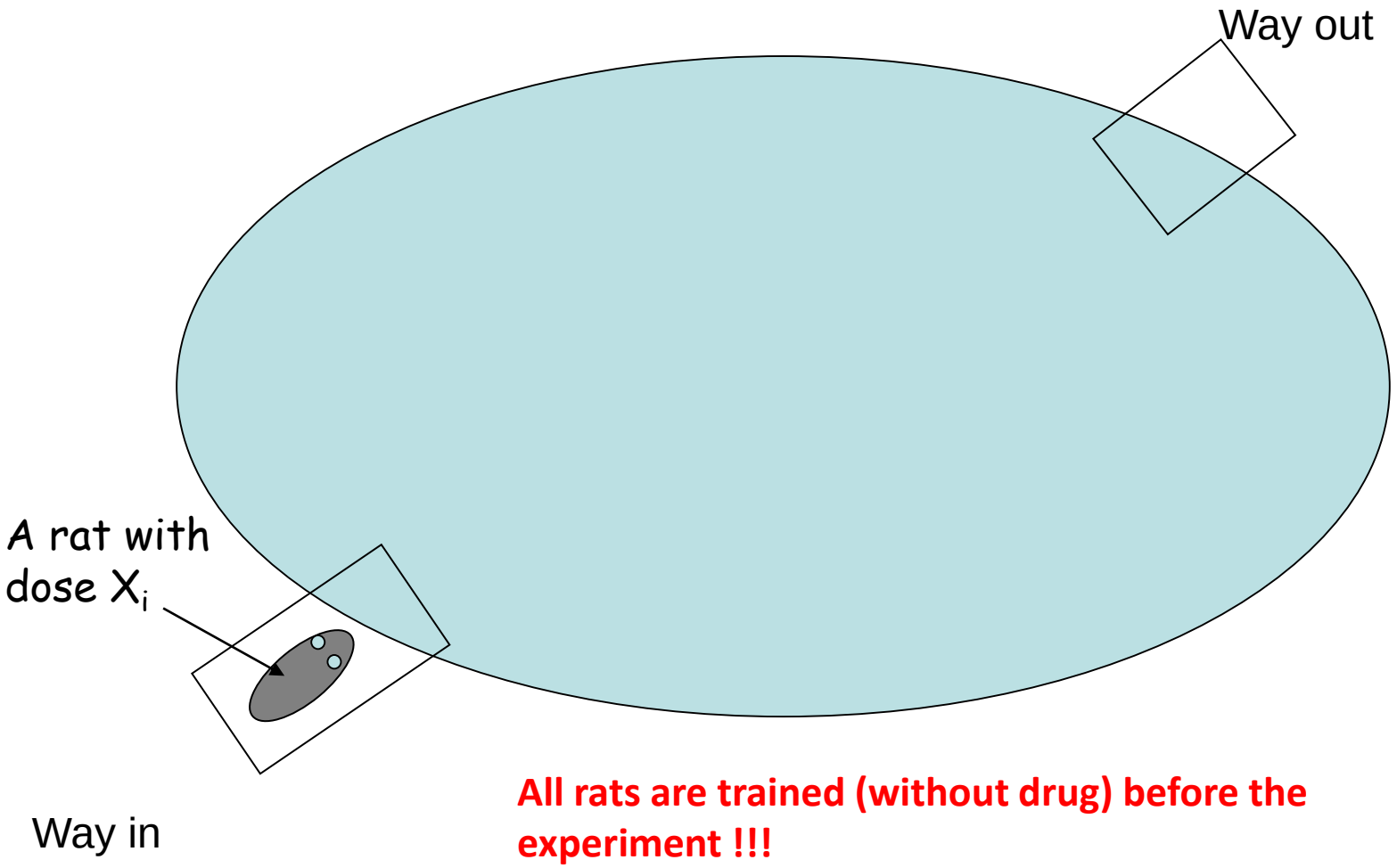
The mission of the rat: swim <u>directly</u> to the other side (the red light)

Description of the experiment



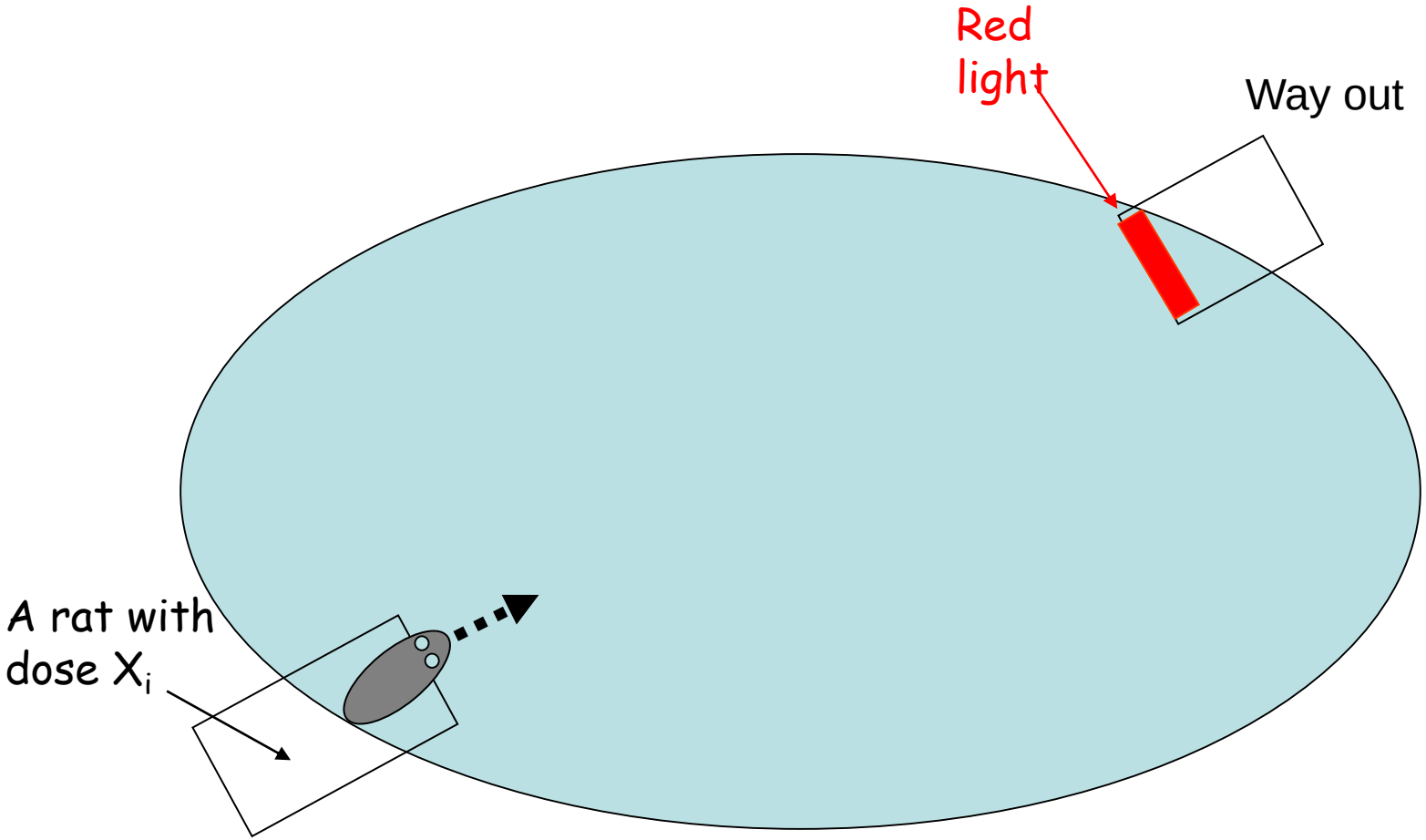
The main assumption: the drug will reduce the time in the water of the rat.

The evaluation of the eat



All rats are trained (without drug) before the experiment !!!

The evaluation of the rat



All rats are trained (without drug) before the experiment to swim to the red light!!!

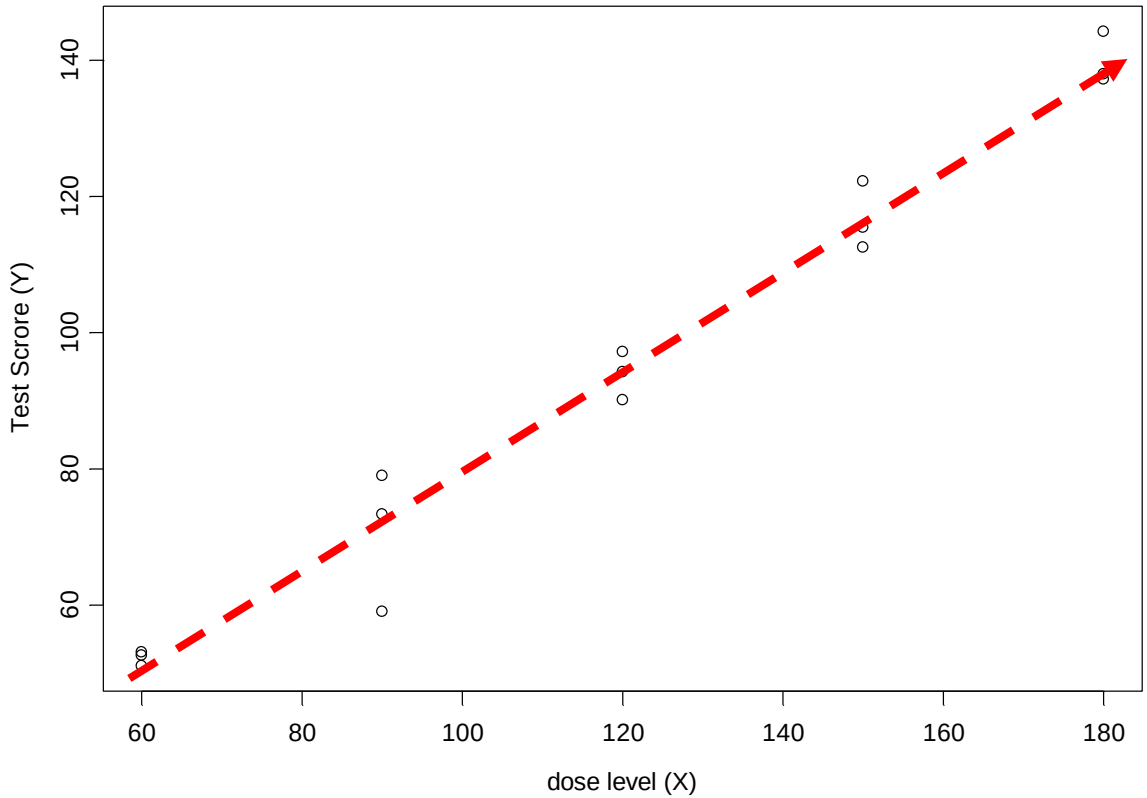
The scientific question

- Does the performance of the rat depend on the dose level ?

A good drug is expected to improve the rats' performance (swim directly to the red light)

The scientists expect that: the higher the dose the better the performance

The data

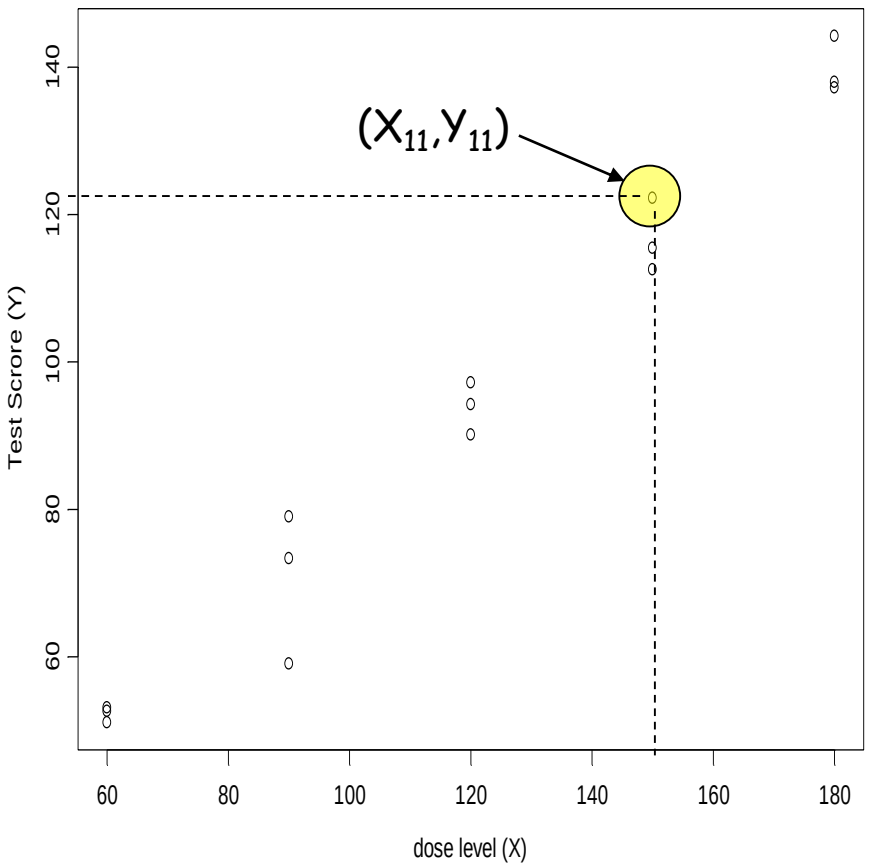


Upward trend: in general, test score increases with dose level

Regression terminology

- The test score (Y_i) is the dependent variable. It depends on the dose level (X_i).
- The dose level is called the independent variable or the predictor.
- The observation unit:

$$(X_i, Y_i),\ i=1,2,\ldots,n.$$



Data structure

(X_1,Y_1)
(X_2,Y_2)
. .
. .
(X_i,Y_i)
. .
. .
(X_n,Y_n)

The dose response data:

- 2 variables.
- 15 subjects.

Dependent variable y_i

Dose level x_i

60	56.07362
60	49.45516
60	56.07840
90	74.18539
90	73.13873
90	77.35170
120	95.37789
120	93.03198
120	92.46663
150	117.61100
150	123.56117
150	119.12260
180	130.81847
180	137.31600
180	139.09742

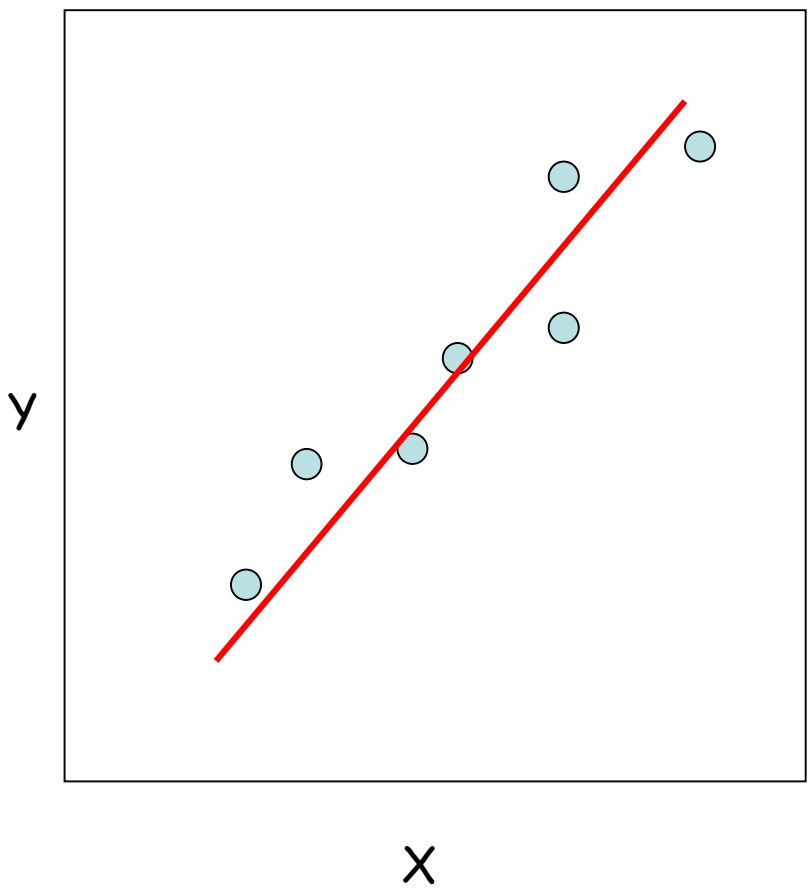
(X_{11},Y_{11}) Observation 11

Dose level x_i

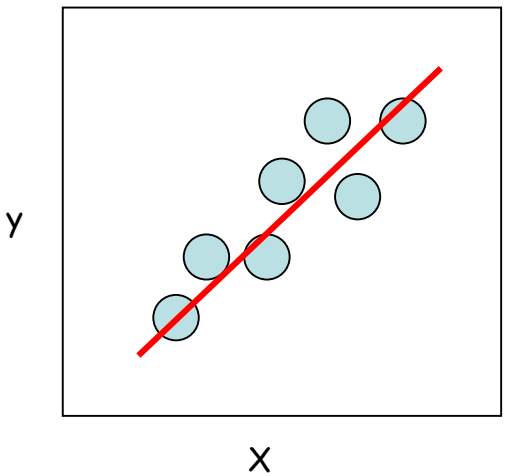
Test score y_i

What is a **Simple** Linear Regression model ?

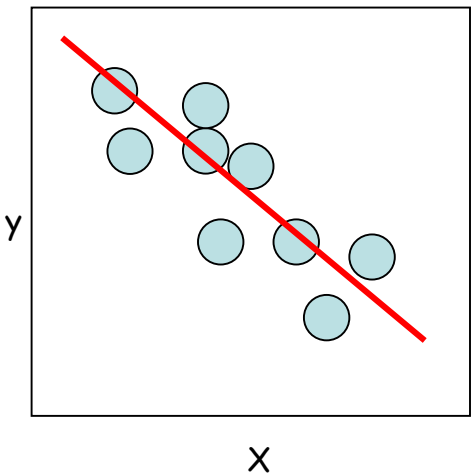
- A regression model is a statistical model which aims to describe the relationship between a predictor (the dose level) and a dependent variable (test score) with a **straight line**.



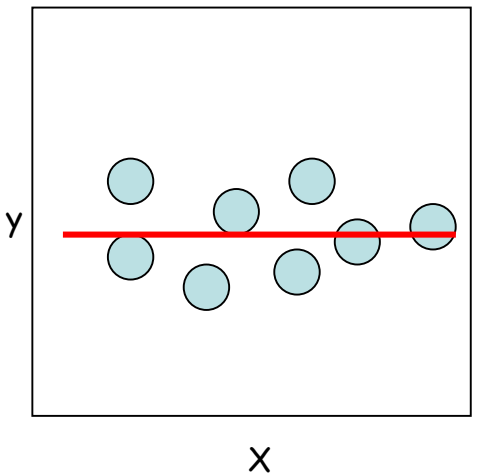
Properties of the **simple** linear regression model : trends



Upward trend



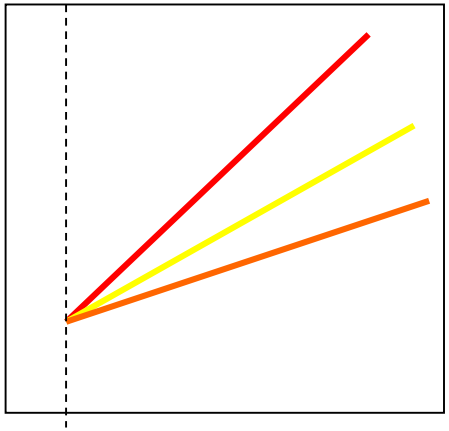
Downward trend

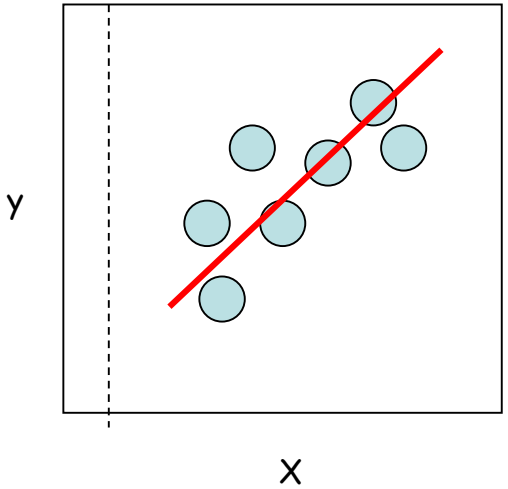


Y does not depend on X

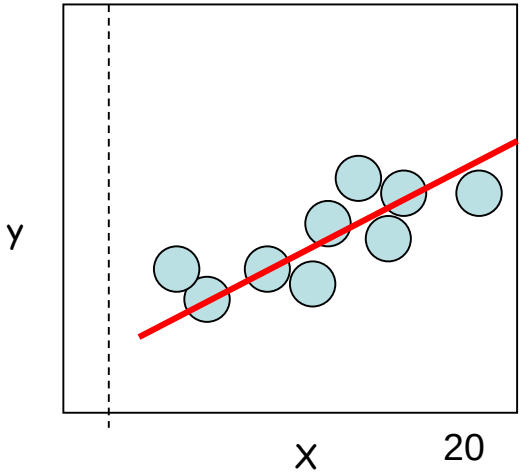
=no trend

Parameters of the **simple** linear regression model : slope

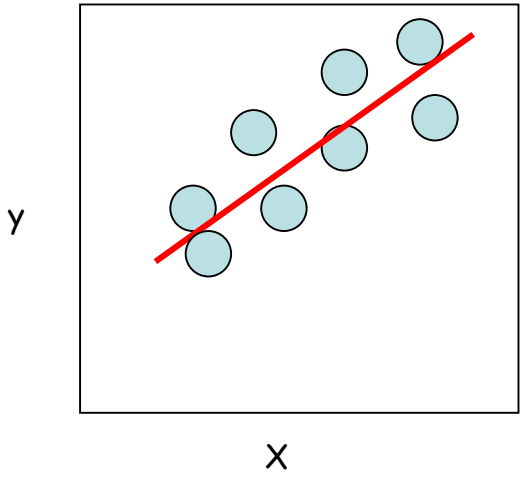


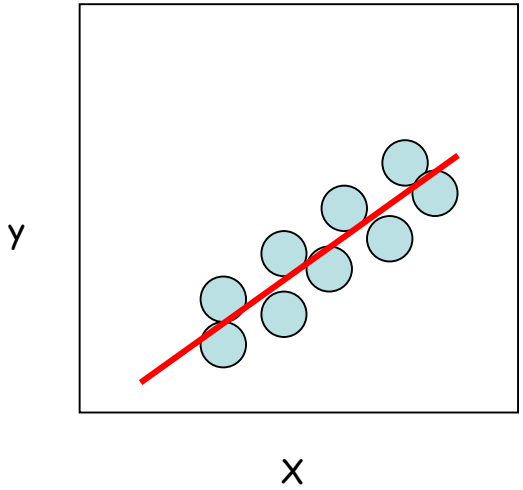


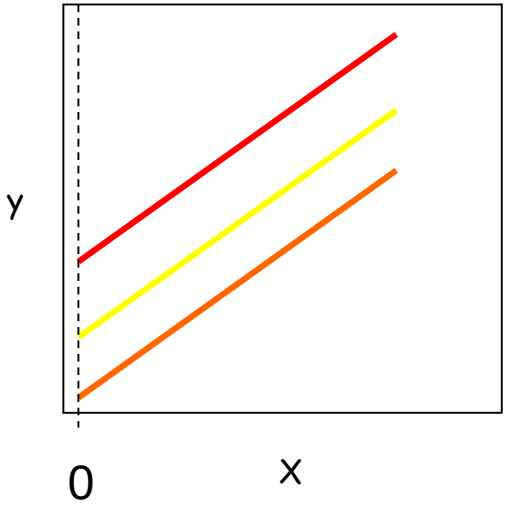
The slope is the change in the mean of Y for a unit change in X.



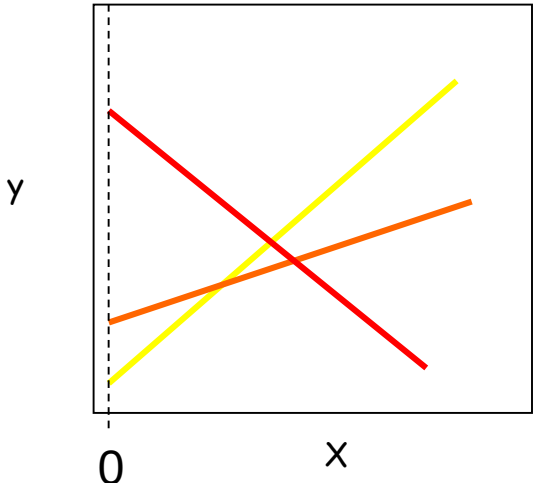
Parameters of the **simple** linear regression model : intercept







The intercept is the height of the regression line when X=0.



A **Simple** Linear Regression model: model formulation

- We assume that the relationship between the predictor and the response can be described with the model:

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

Dependent variable Y_i

Predictor X_i

observed

Two parameters: intercept and slope

Random error

unobserved

Estimation (I)

- We need to estimate the unobserved parameters of the model:

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

$$\hat{Y}_i = a + bX_i$$

- The estimator for the random error:

$$e_i = \hat{Y}_i - Y_i$$

predicted value for the test score

(the estimator for the test score)

- **a and b are the estimators for alpha and beta**
- **e_i (the residual) is the estimator for the random error**

Regression model & the data

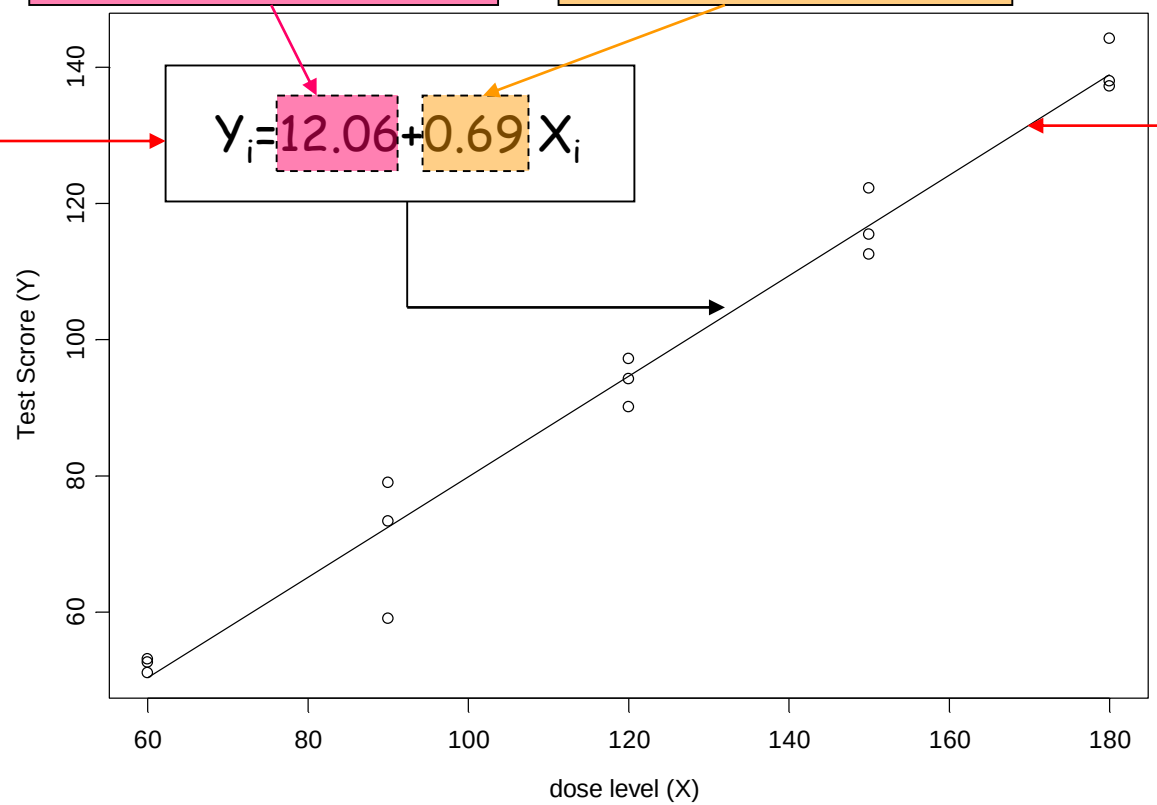
a: the estimate for alpha

b: the estimate for beta

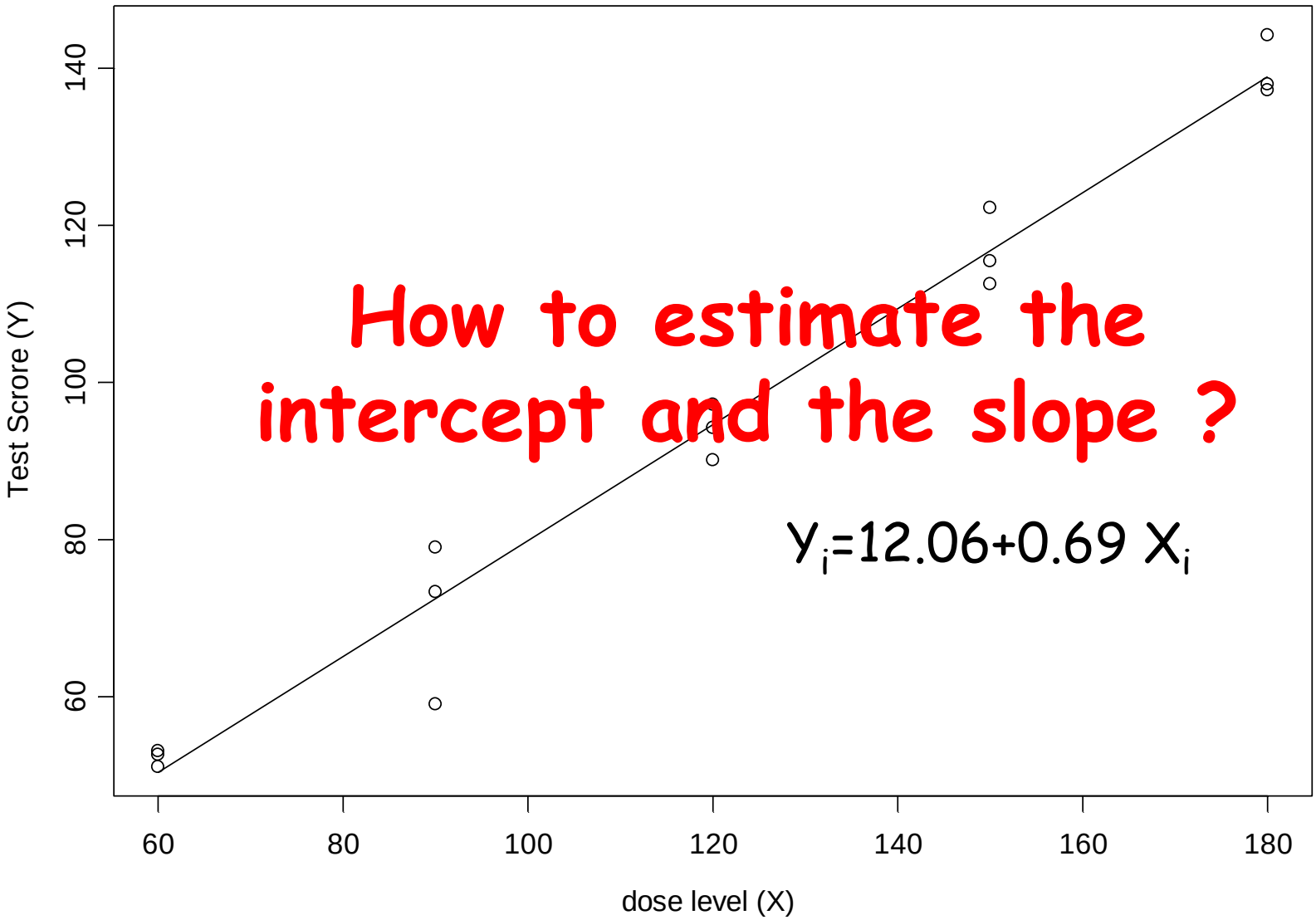
The estimated (fitted) model

$Y_i = 12.06 + 0.69\, X_i$

The regression line



$$Y_i = \alpha + \beta X_i + \varepsilon_i$$ The model

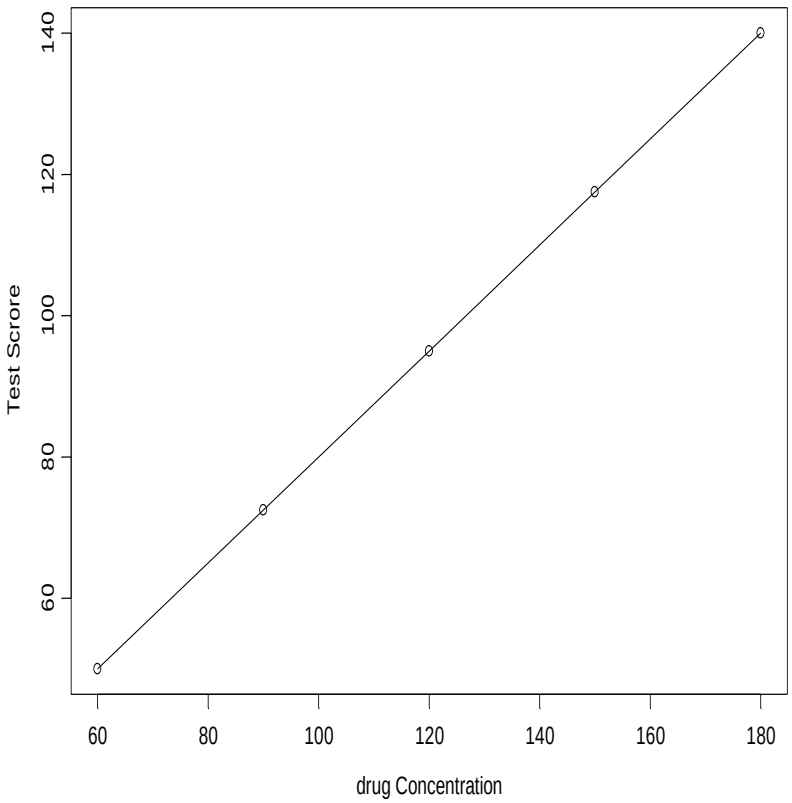


How to estimate the intercept and the slope ?

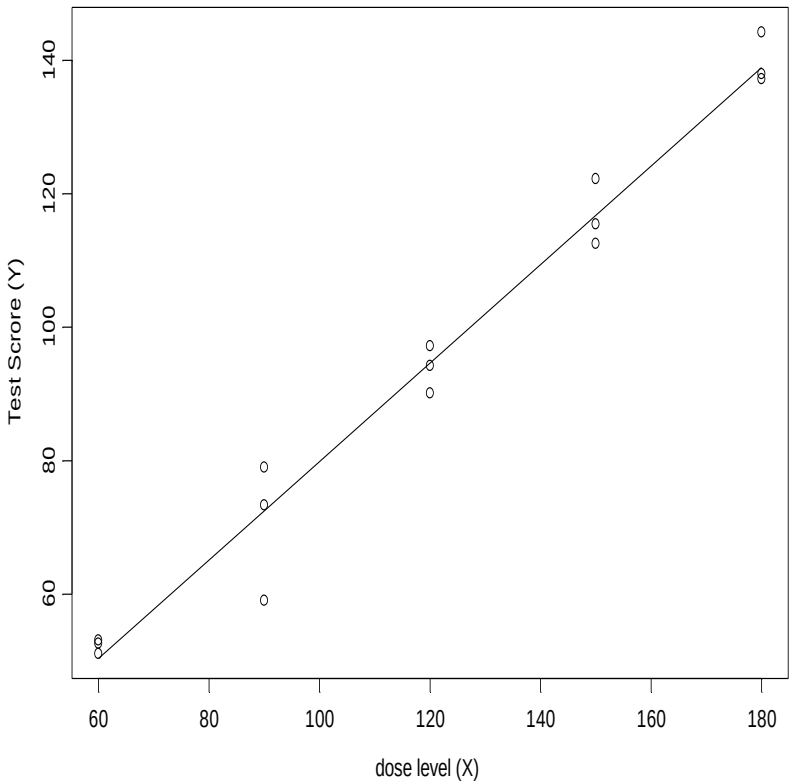
$Y_i = 12.06 + 0.69\ X_i$

Regression model & the data

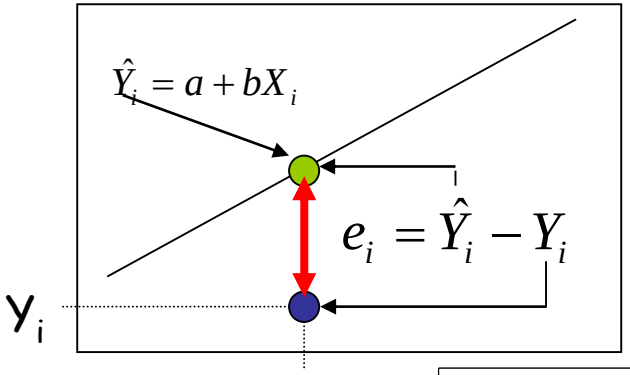
A EXAMPLE OF PERFECT FIT…



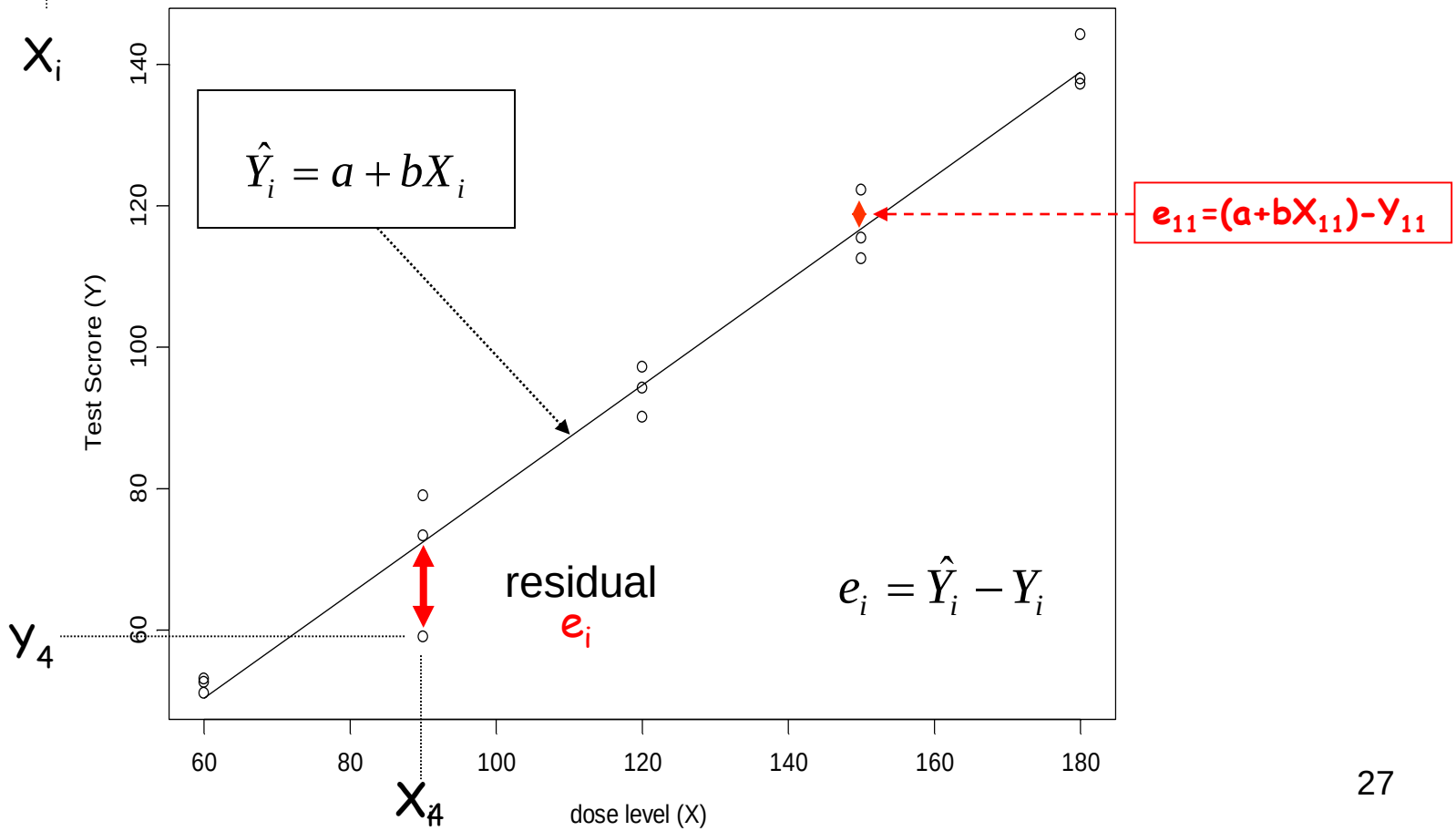
….BUT WE DO NOT EXPECT A PERFECT FIT…



The residuals



The difference between the observed response and the predicted response.



Estimation (II): The Least Squares Criterion

- How to estimate the intercept and slope?
- We want that the fitted model (the line which describes the relationship between Y and X) will be "close" to the data.
- The residual sum of squares = sum (residual)^2.
- The least squares criterion: choose intercept and slope which minimize the residual sum of squares

$$\boxed{RSS = \sum_{i=1}^{n} \left(\hat{Y}_i - Y_i\right)^2}$$

Parameter estimates

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i\,, \quad \varepsilon_i \sim N(0, \sigma^2)$$

Unknown parameters in the model

$\bar{x}$ & $\bar{y}$

$$\hat{\beta}_1 = \frac{\sum_{i=i}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

OLS estimates for β_0 and β_1 (for a and b)

Parameter estimate for the variance

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i\,, \quad \varepsilon_i \sim N(0, \sigma^2)$$

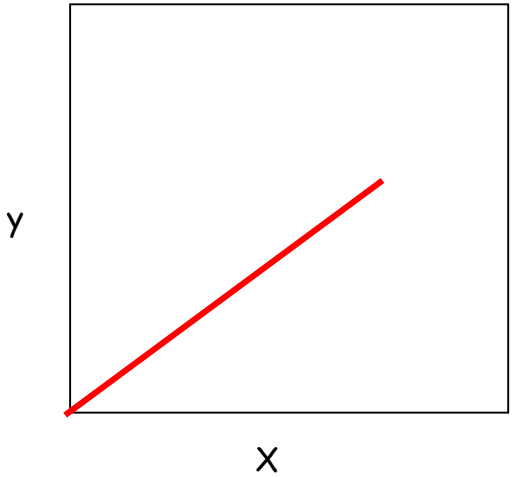
$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n} r_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - (b_0 + b_1 x_i))^2$$

$\hat{\beta}_0$ & $\hat{\beta}_1$

Inference

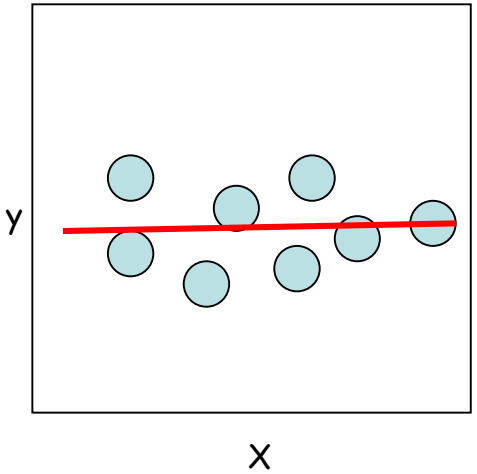
$$H_0 : \beta_0 = 0$$
$$H_1 : \beta_0 \neq 0$$

Intercept equals to zero



$$H_0 : \beta_1 = 0$$
$$H_1 : \beta_1 \neq 0$$

Slope equals to zero



Test statistics

- For the simple linear regression model:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad i = 1,\ldots n$$

$$\varepsilon_i \sim N(0, \sigma^2)$$

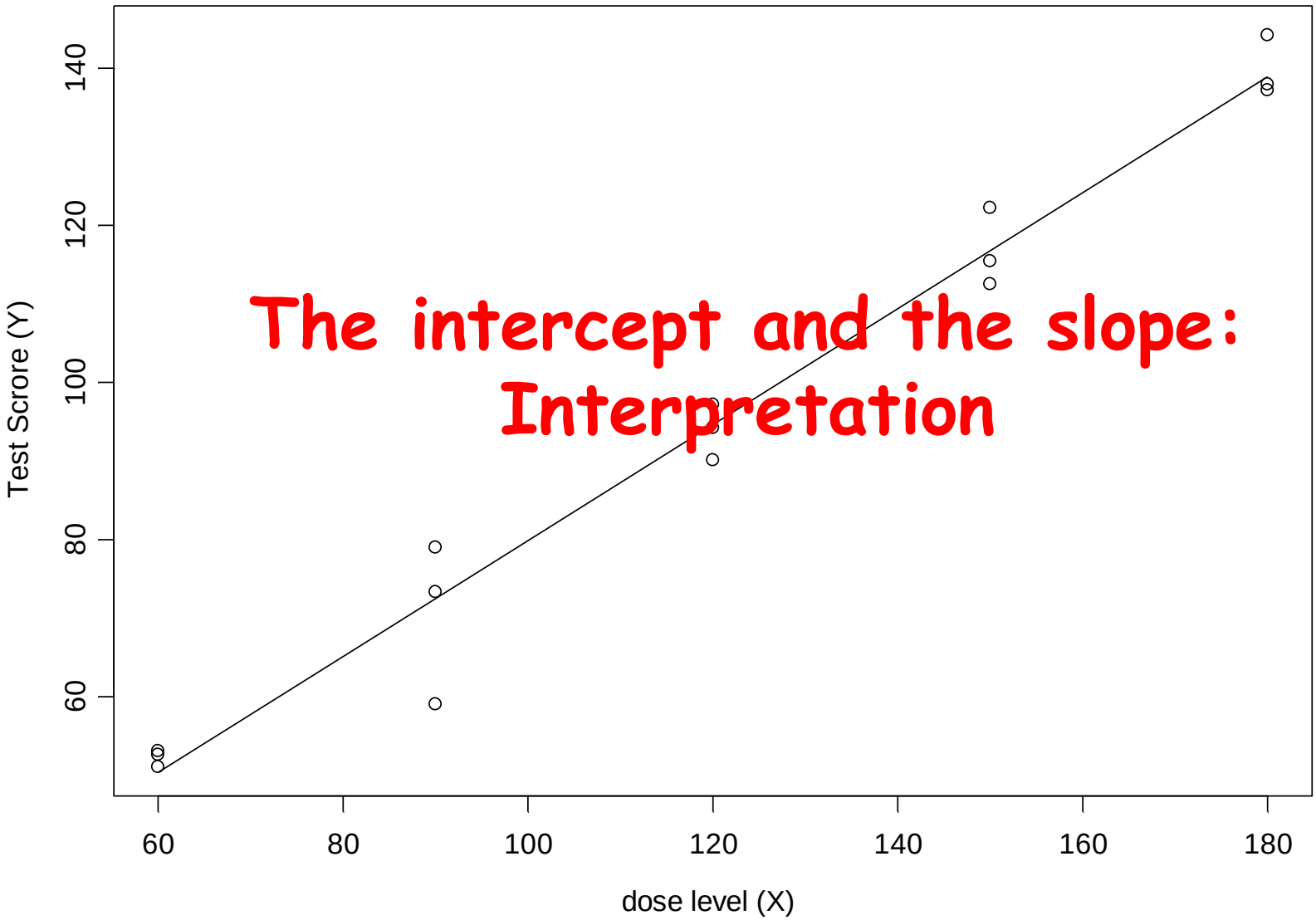
Test statistic for β_1

$$\frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \sim t_{n-2}$$

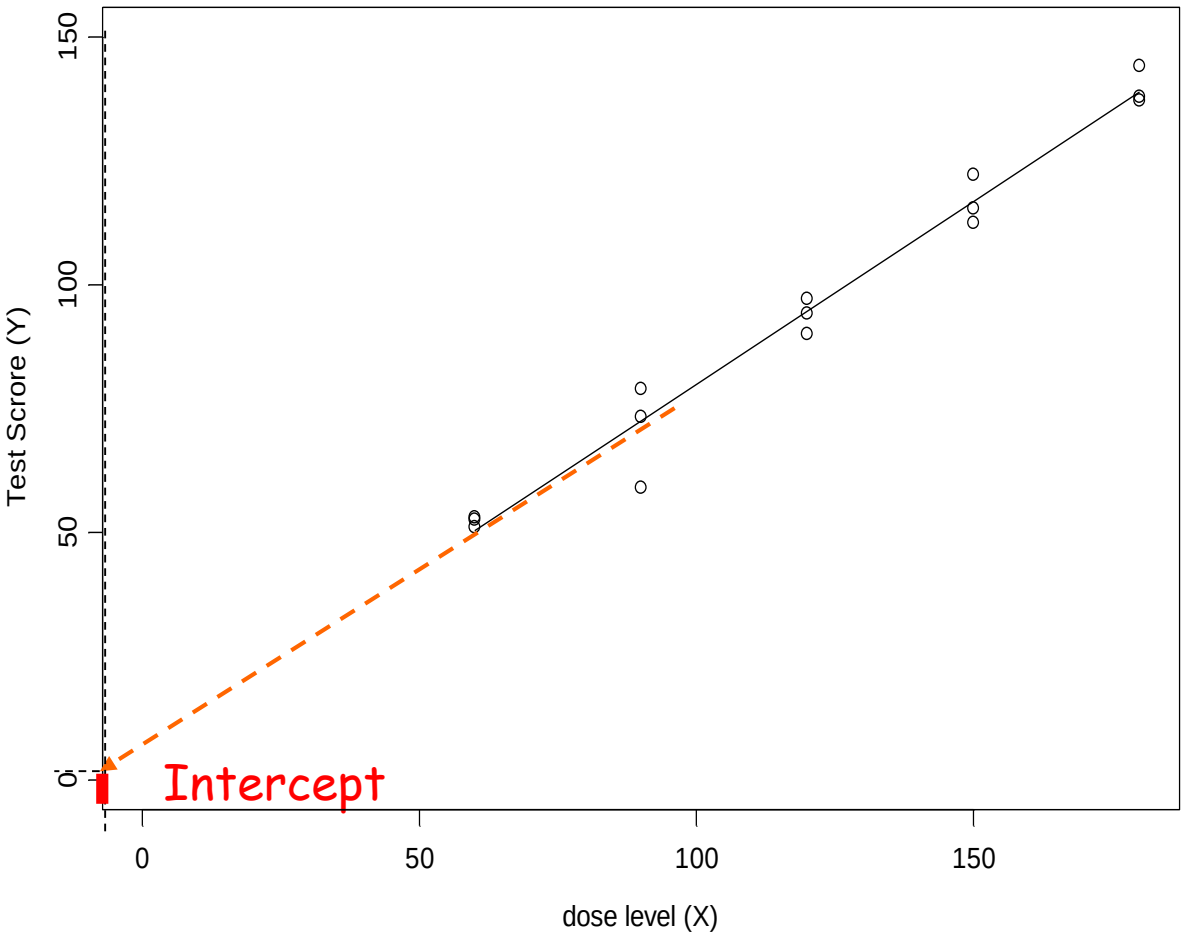
Test statistic for β_0

$$\frac{(\hat{\beta}_0 - \beta_0)}{\hat{\sigma}\sqrt{\dfrac{1}{n} + \dfrac{\bar{x}}{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}}} \sim t_{n-2}$$

The intercept and the slope: Interpretation



Interpretation: the Intercept



The intercept is the predicted test score for dose level zero:

For X_i=0 we have:

Predicted test score=12.09+0.

Interpretation: the slope

Suppose that we have two rats: the first received a dose of 100 and the second dose of 101.

WHAT IS THE DIFFERENCE BETWEEN THE PREDICTED VALUES OF THE TWO RATS ?

- Dose level 100:

Predicted value=12.09+0.69*100

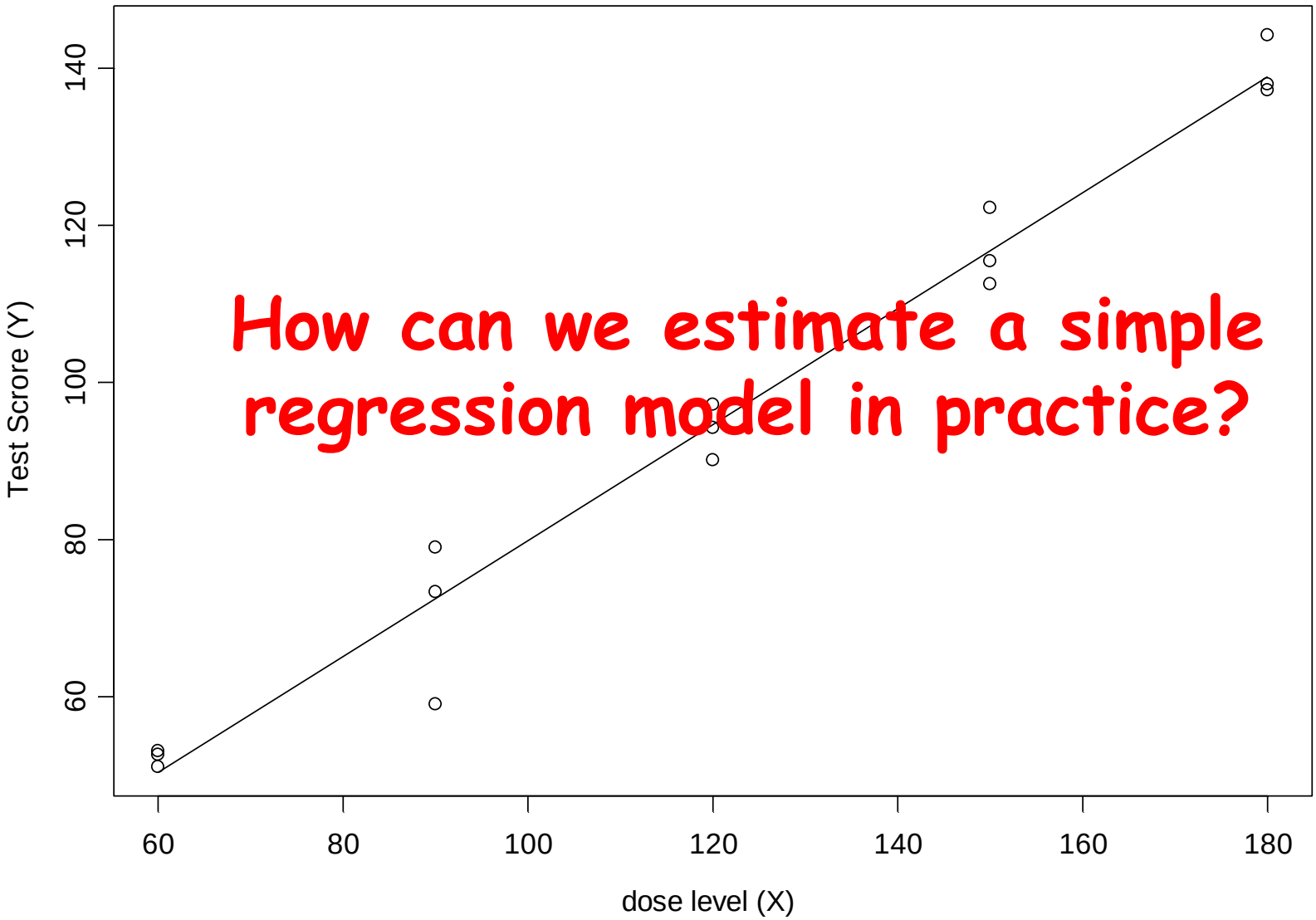
- Dose level 101:

Predicted value=12.09+0.69*101

(12.09+0.69*101)-(12.09+0.69*100)=0.69

The difference is equal to 0.69 which is the value of the slope

The slope is the change in the (expected) response level for a unit change in the predictor



How can we estimate a simple regression model in practice?

Part 2
Fitting a simple linear regression in R using the lm() function

The Data in R

```
Dose <- c(60,60,60,90,90,90, 120,120,120,150,150,150,180,180,180)
Score <- (56.07362,49.45516,56.07840,74.18539,73.13873,77.35170,95.37789,93.03198,
        92.46663,117.61100,123.56117,119.12260,130.81847,137.31600,139.09742)
dose.data <- cbind(Dose, Score)
print(dose.data)
     Dose    Score
 [1,]   60 56.07362
 [2,]   60 49.45516
 [3,]   60 56.07840
 [4,]   90 74.18539
 [5,]   90 73.13873
 [6,]   90 77.35170
 [7,]  120 95.37789
 [8,]  120 93.03198
 [9,]  120 92.46663
[10,]  150 117.61100
[11,]  150 123.56117
[12,]  150 119.12260
[13,]  180 130.81847
[14,]  180 137.31600
[15,]  180 139.09742
```

Name of the data

Dependent variable

Predictor

The function lm( ) in R

- Simple linear regression model can be fitted in R using the function lm( ).

- The model statement:

Score ~ Dose

- Example of R script for function lm( )

```
fit.dose <- lm(Score ~ Dose,data = dose.data)
```

Dependent variable

Predictor

y ~ x

Fitting the model in R

```
Dose <- c(60,60,60,90,90,90, 120,120,120,150,150,150,180,180,180)
Score <- (56.07362,49.45516,56.07840,74.18539,73.13873,77.35170,95.37789,93.03198,
        92.46663,117.61100,123.56117,119.12260,130.81847,137.31600,139.09742)
dose.data <- cbind(Dose, Score)
print(dose.data)
     Dose    Score
 [1,]   60  56.07362
 [2,]   60  49.45516
 [3,]   60  56.07840
 [4,]   90  74.18539
 [5,]   90  73.13873
 [6,]   90  77.35170
 [7,]  120  95.37789
 [8,]  120  93.03198
 [9,]  120  92.46663
[10,]  150 117.61100
[11,]  150 123.56117
[12,]  150 119.12260
[13,]  180 130.81847
[14,]  180 137.31600
[15,]  180 139.09742
```

Dependent variable

Predictor

```
> fit.dose <- lm(Score ~ Dose)

> summary(fit.dose)
```

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

lm(y ~ x)

Output

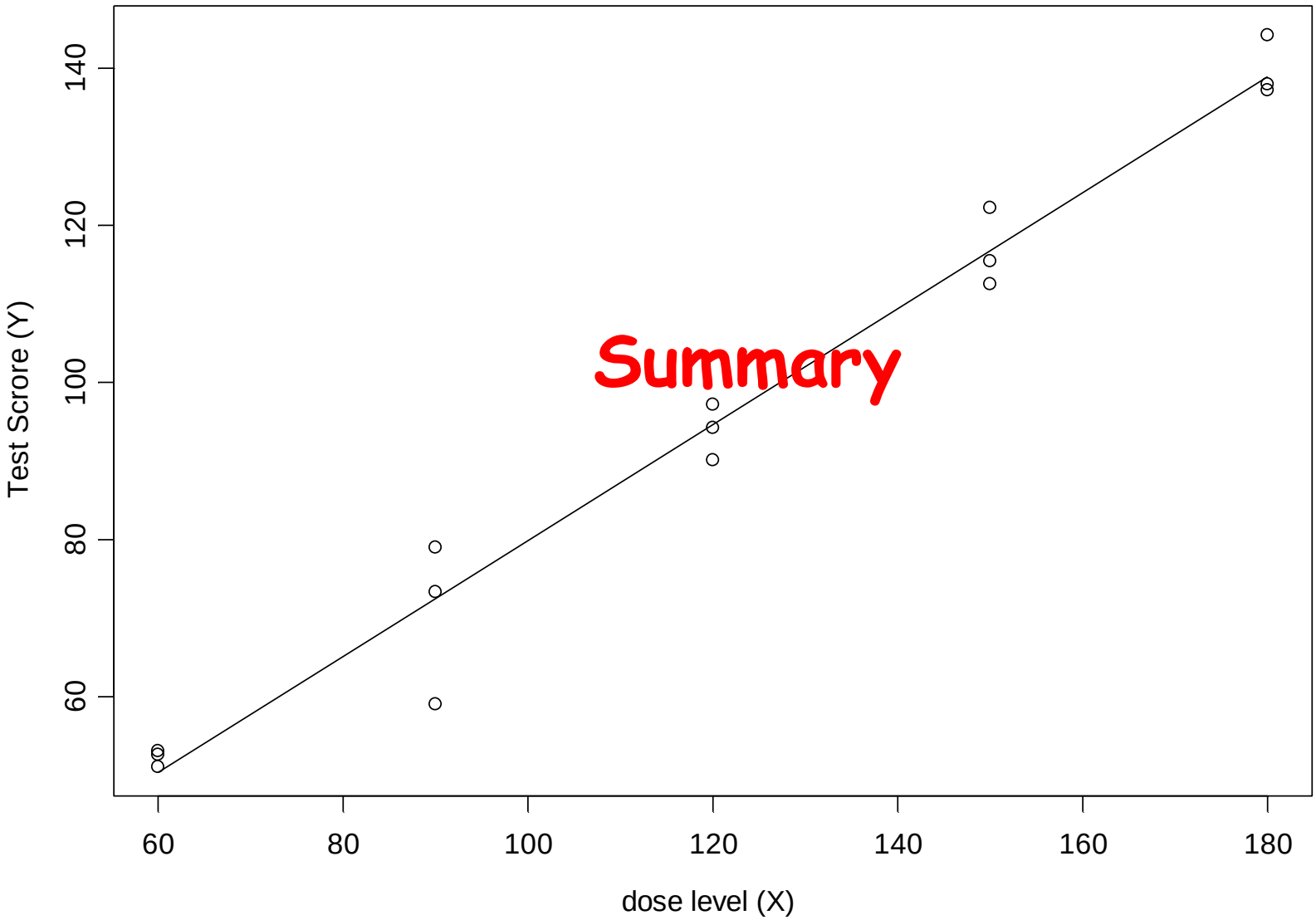
ESTIMATION

INFERENCE

```
Coefficients:
            Estimate   Std. Error t value Pr(>|t|)
(Intercept) 12.06329    0.71389   4.445 0.000661 ***
Dose         0.69652    0.02132  32.666 7.28e-14 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.504 on 13 degrees of freedom
Multiple R-squared: 0.988,     Adjusted R-squared: 0.987
F-statistic:  1067 on 1 and 13 DF,  p-value: 7.279e-14
```

The intercept: what is the test score for dose=0

<u>The slope:</u> how much the response change for a unit change in the predictor


Summary
Test Scrore (Y)
60
80
100
120
140
60
80
100
120
140
160
180
dose level (X)

Technical details (Estimation)

- A simple linear regression model has the form:

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

- α (β_0) and β (β_1) are the parameters in the model and ε is the random error.

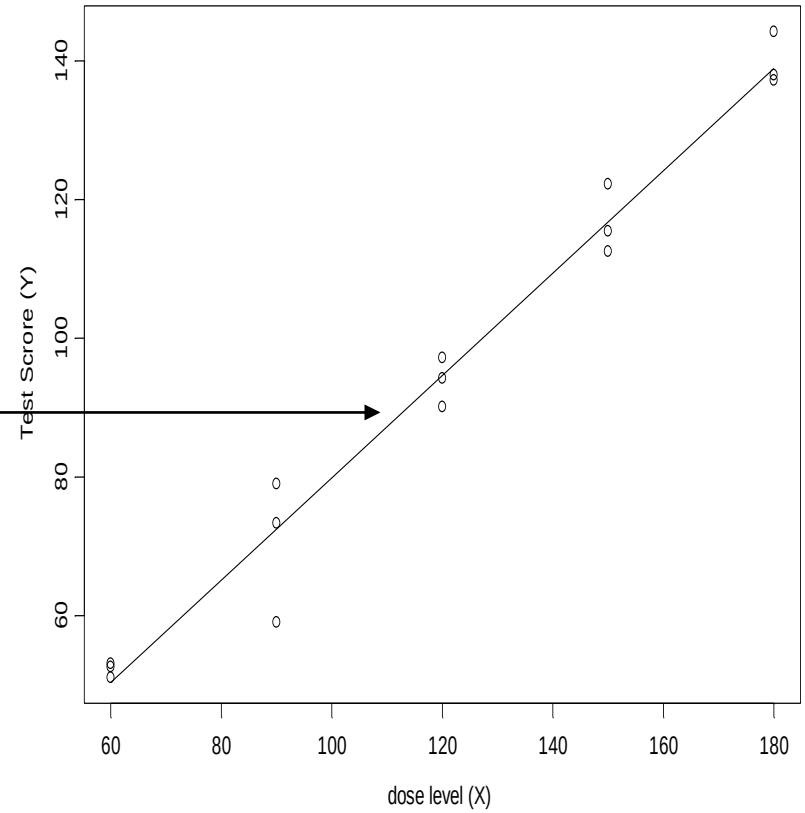
- We can estimate α and β by minimizing the residual sum of squares

$$RSS = \sum_{i=1}^{n} \left(\alpha - \beta X_i - Y_i\right)^2$$

Technical details (Estimation)

- The estimated model

$$\hat{Y}_i = a + bX_i$$



- The residual

$$e_i = \hat{Y}_i - Y_i$$

Technical Details (Estimation)

We assume that the relationship between Y_i and X_i can be described with a statistical model

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

We assume that the random error ε is normally distributed.	$\varepsilon \sim N(0, \sigma^2)$
The mean of ε is equal to zero	$E(\varepsilon_i) = 0$
The conditional mean of Y_i (given the value of X_i)	$E(Y_i \mid X_i) = \alpha + \beta X_i$
The estimator for the conditional mean of Y_i (the fitted model=the regression line)	$\hat{E}(Y_i \mid X_i) = a + bX_i = \hat{Y}_i$
The residual: the estimator for ε	$e_i = \hat{Y}_i - Y_i$
Least square criterion: choose a and b that minimize the residuals sum of squares	$RSS = \sum_{i=1}^{n} (\alpha - \beta X_i - Y_i)^2$

Part 3
Model diagnostic

Simple regression model: assumptions

We consider the following linear regression model

$$Y_i = \alpha + \beta \times X_i + \varepsilon_i$$

The random error is assumed to be normal distributed:

$$\varepsilon_i \sim N(0, \sigma^2)$$

We also assume that the variance is constant, i.e., $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are outcome of a normal distribution with mean zero and equal variances.

How to check the model assumptions? (1)

- The random error, ε_i, is unknown but we can estimate ε_i with the residuals

$$e_i = Y_i - \hat{Y}_i$$

Observed Predicted

- The residuals can be used in order to check the model asaumptions.

- We focus on:

  1) the distribution of e_i

  2) the variability of e_i

How to check the model assumptions? (2)

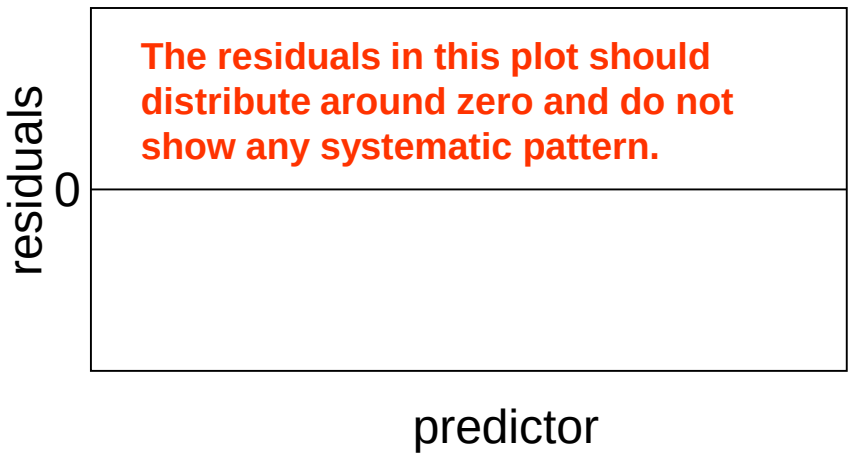
- We assume that the mean of Y_i is linear with respect to X:

$$E(Y)_i = \alpha + \beta \times X_i$$

- This is true only if

$$E(\varepsilon_i) = 0$$

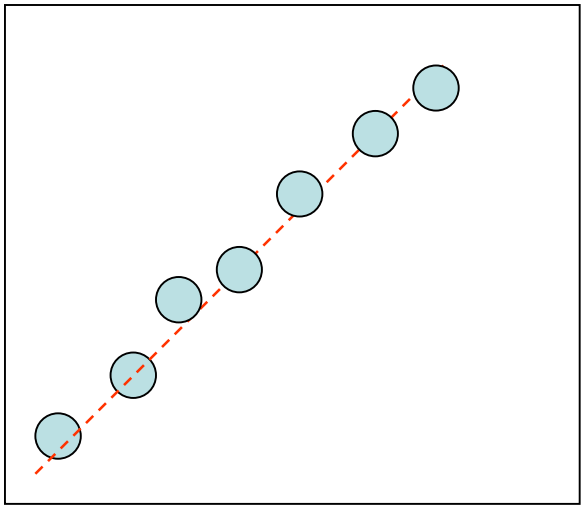
- The residuals can be used in order to check the linearity assumption .



Assumption 1: The distribution of e_i

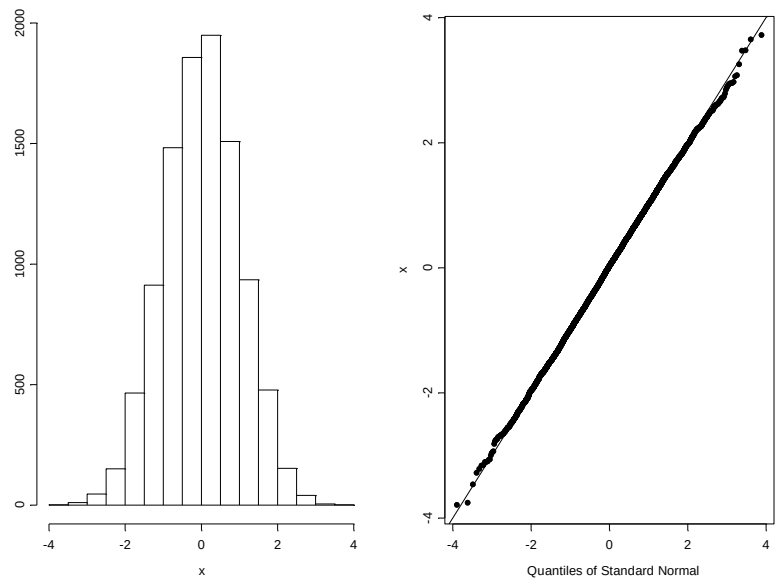
- The distribution of e_i is expected to be normal with mean zero and variance σ^2.
- qq-normal plot (or normal probability plot) is a graphical tool that can be used in order to asses the normality assumption.

- If the normalty assumption holds we expect qq-normal plot will be a srtight line.



An example of qq-normal plot form N(0,1)

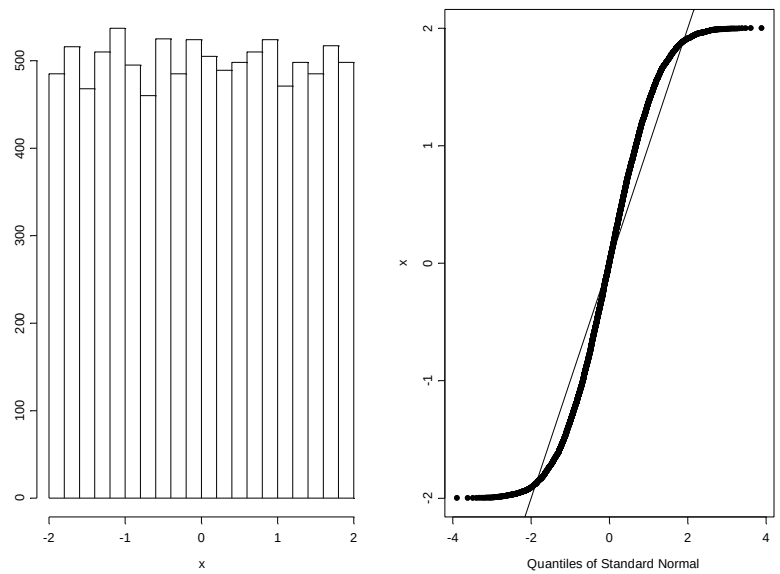
- Sample of 10000 observations from N(0,1)
- The qqnormal plot is a stright line.
- If the random error ε_i is normal distributed, the qqnormal plot of the residuals should be a stright line.



An example of qq-normal plot of a heavy tail distribution

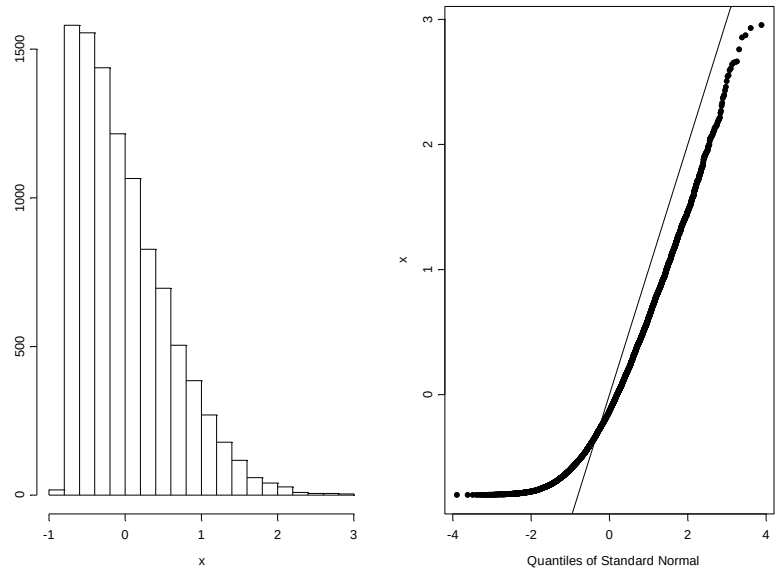
- Sample of 10000 observations from

  U(-2,2).
- S shape of the qqnormal plot.
- This is an example of a symatric distribution with more observations (relativly to the normal distribution) at the tails.



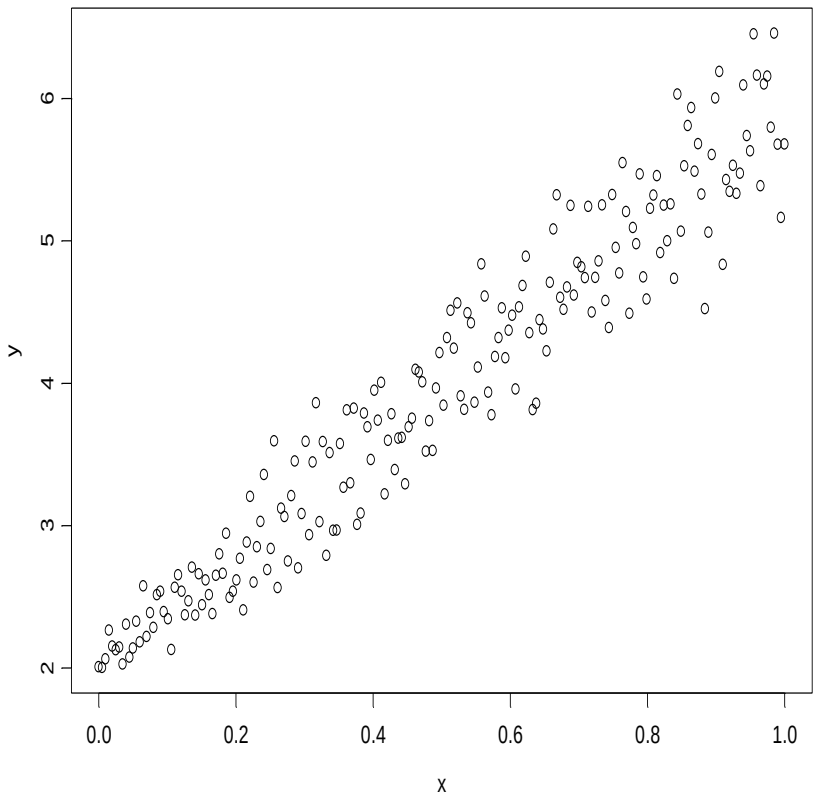
An example of qq-normal plot of a skwed distribution

- Sample of 10000 observations from a skewed distribution.
- The distribution is skewed to the right and the points in the qqplot are not follow the stright line.



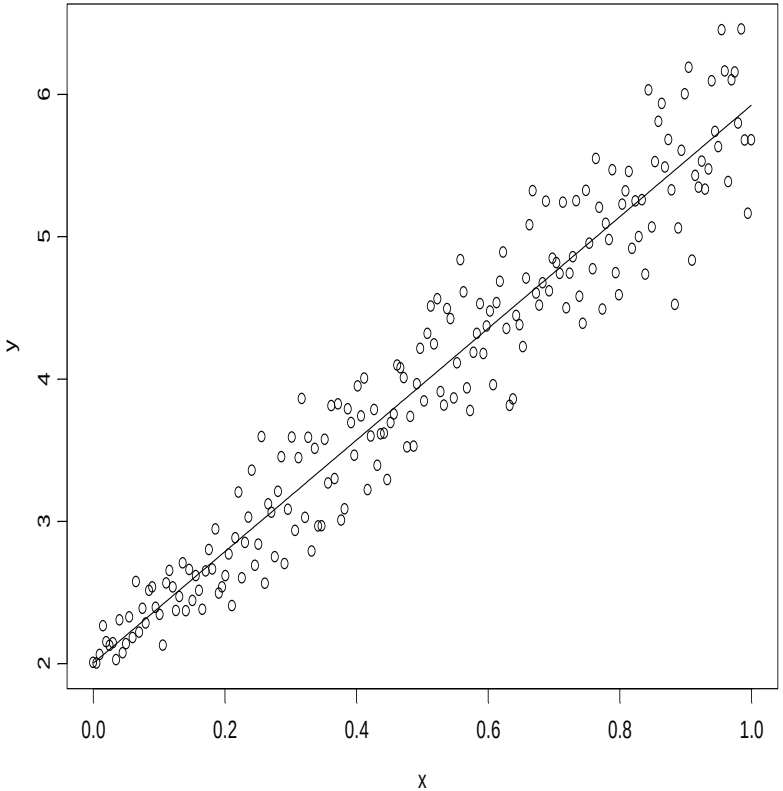
Assumption 2: Constant variance

- This is an example of a dataset in which the variance is not constant.
- The variance increases when the value of X increases.
- However, there is a linear relatioship between the predictor and the response.



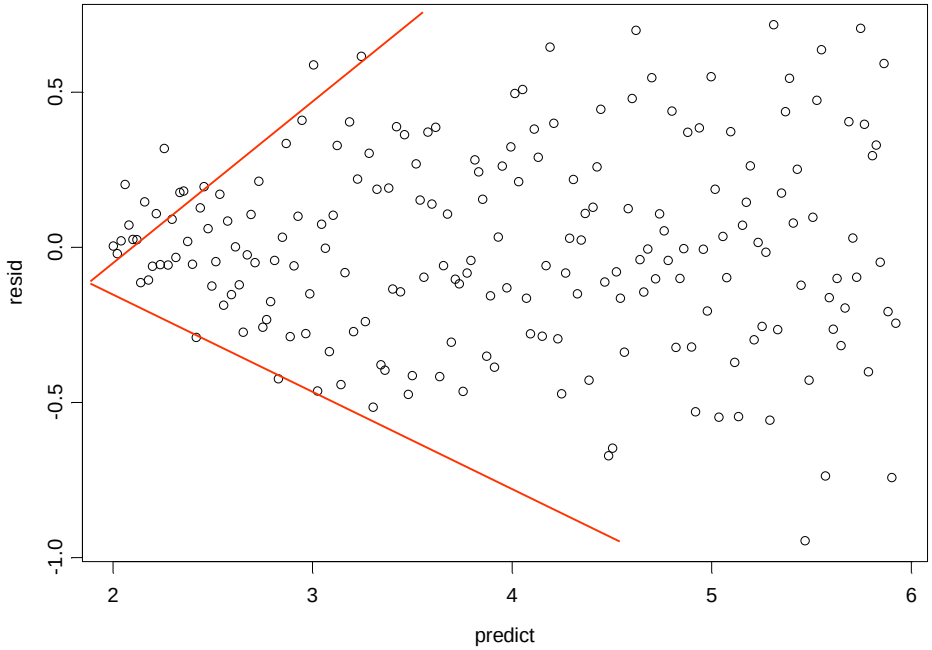
The data and the resrssion line

- The model seems to fit the data well, it captures the structure of the mean.

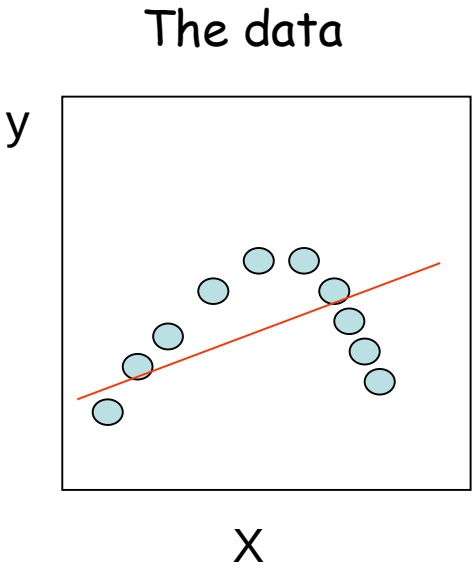


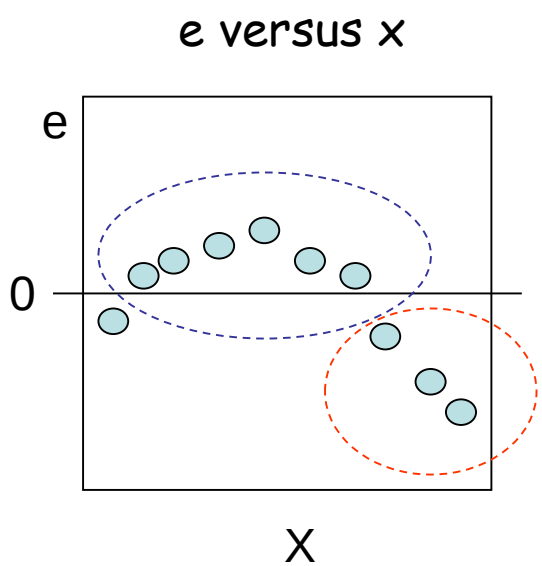
Residuals plot: residuals versus the predicted values

- A clear pattern.
- As the predicted values increase the variability among the residuals increase (a “megaphone” shape).



Assumption 3: Linearity

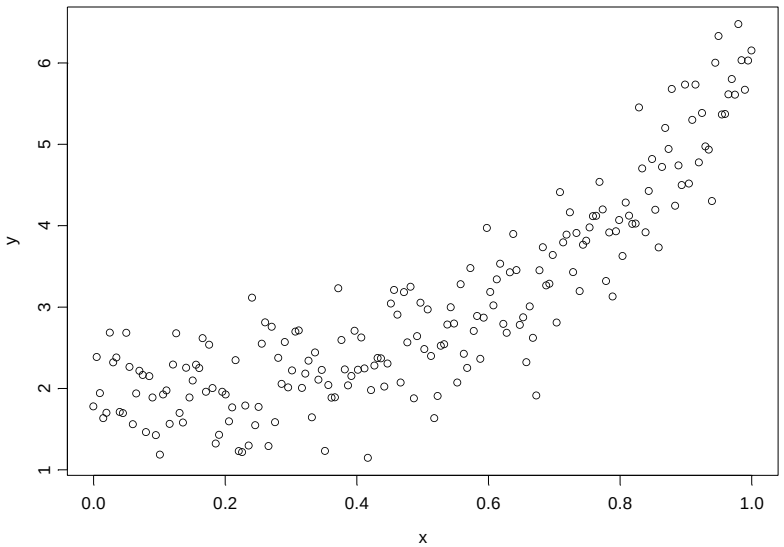




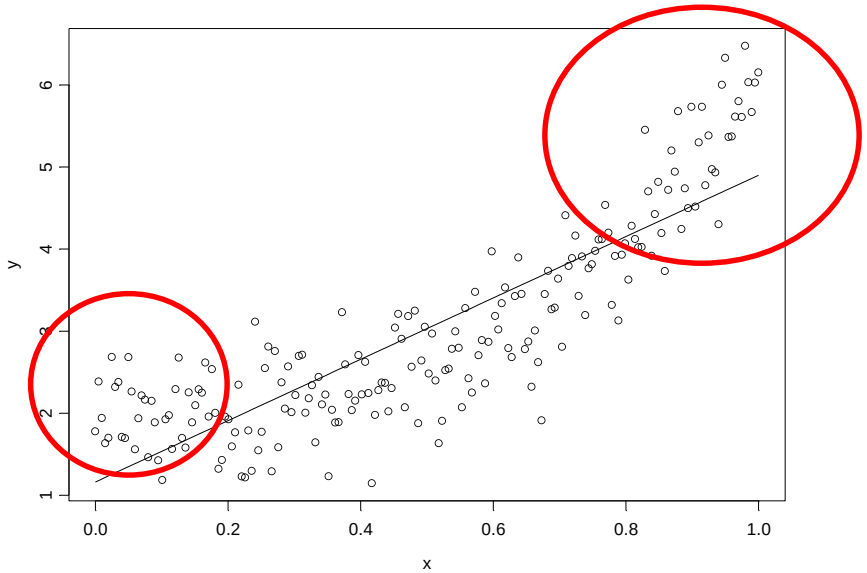
Example of systematic pattern in the residual plot.

- The scatterplot of the data reveals that the association between the response and the predictor is not linear.
- The residuals plot (in the right) reveals a clear pattern among the residuals which depends on the value of X.

Systematic patterns



data

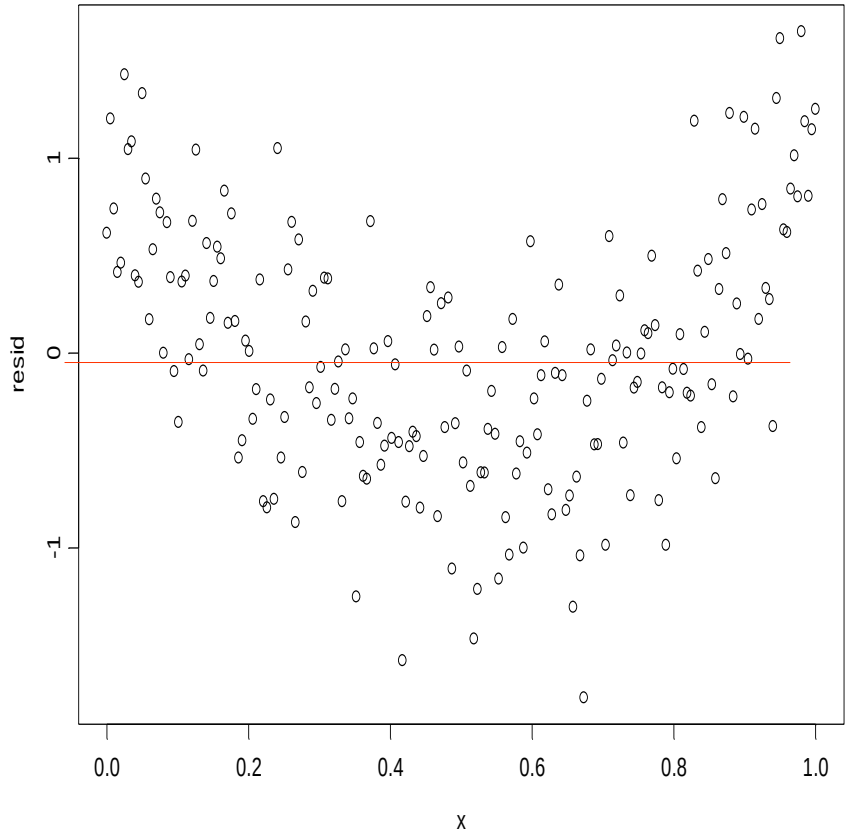


data and fitted model

- The model underestimates the value of Y when the value of X is relatively small or large.

Linearity: residuals plot

- Clear systematic pattern among the residuals.
- The residuals are positive for small and large value of X and negative in the middle.
- This means that there is structure in the data that the linear regression model did not capture.



Bottom line about model diagnostic

- We let the residuals to tell us the story.
- Departure from model assumptions (constant variance, normality and linearity) can be investigated using qq-plot and residuals plots.

Part 4
Model diagnostic using R

Fitting the model in R

```
Dose <- c(60,60,60,90,90,90, 120,120,120,150,150,150,180,180,180)
Score <- (56.07362,49.45516,56.07840,74.18539,73.13873,77.35170,95.37789,93.03198,
        92.46663,117.61100,123.56117,119.12260,130.81847,137.31600,139.09742)
dose.data <- cbind(Dose, Score)
print(dose.data)
      Dose     Score
 [1,]   60  56.07362
 [2,]   60  49.45516
 [3,]   60  56.07840
 [4,]   90  74.18539
 [5,]   90  73.13873
 [6,]   90  77.35170
 [7,]  120  95.37789
 [8,]  120  93.03198
 [9,]  120  92.46663
[10,]  150 117.61100
[11,]  150 123.56117
[12,]  150 119.12260
[13,]  180 130.81847
[14,]  180 137.31600
[15,]  180 139.09742
```

$$Y_i = \alpha + \beta X_i + \varepsilon_i$$

```
> fit.dose <- lm(Score ~ Dose)
```

The output

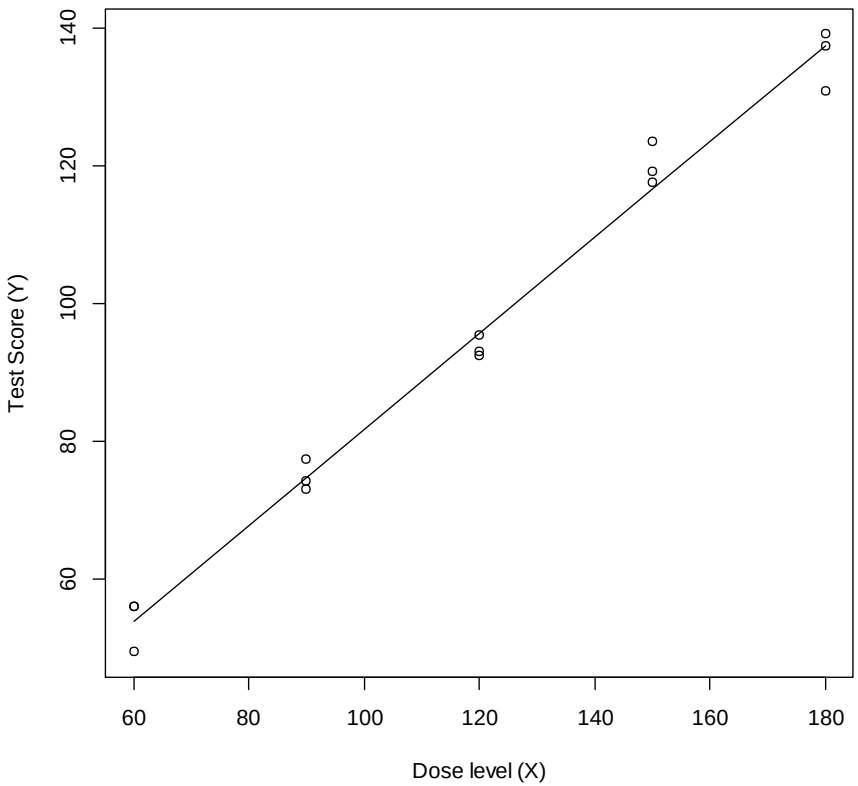
```
>summary(fit.dose)
Call:
lm(formula = Test_score ~ Dose_level, data = dose)

Residuals:
   Min      1Q Median      3Q     Max
-6.619 -2.113 -0.121  2.221  7.020

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 12.06329    2.71389   4.445 0.000661 ***
Dose_level   0.69652    0.02132  32.666 7.28e-14 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.504 on 13 degrees of freedom
Multiple R-squared: 0.988,      Adjusted R-squared: 0.987
F-statistic:  1067 on 1 and 13 DF,  p-value: 7.279e-14
```

INFERENCE

ESTIMATION

Data and predicted model

```
>plot(Dose,Score,
    ylab = "TestScore (Y)",
    xlab = "Doselevel (X)")
>x <- Dose
>y <- fit.dose$fit
>lines(x,y)
```



The output

ANOVA Table:

```
> aov(fit.dose)

Call:
   aov(formula = fit.dose)

Terms:
                    Dose_level Residuals
Sum of Squares   13098.798   159.579
Deg. of Freedom          1        13

Residual standard error: 3.503618
Estimated effects may be unbalanced
```

Regression Sum of Squares

RSS=Residual Sum Squares

Graphical output

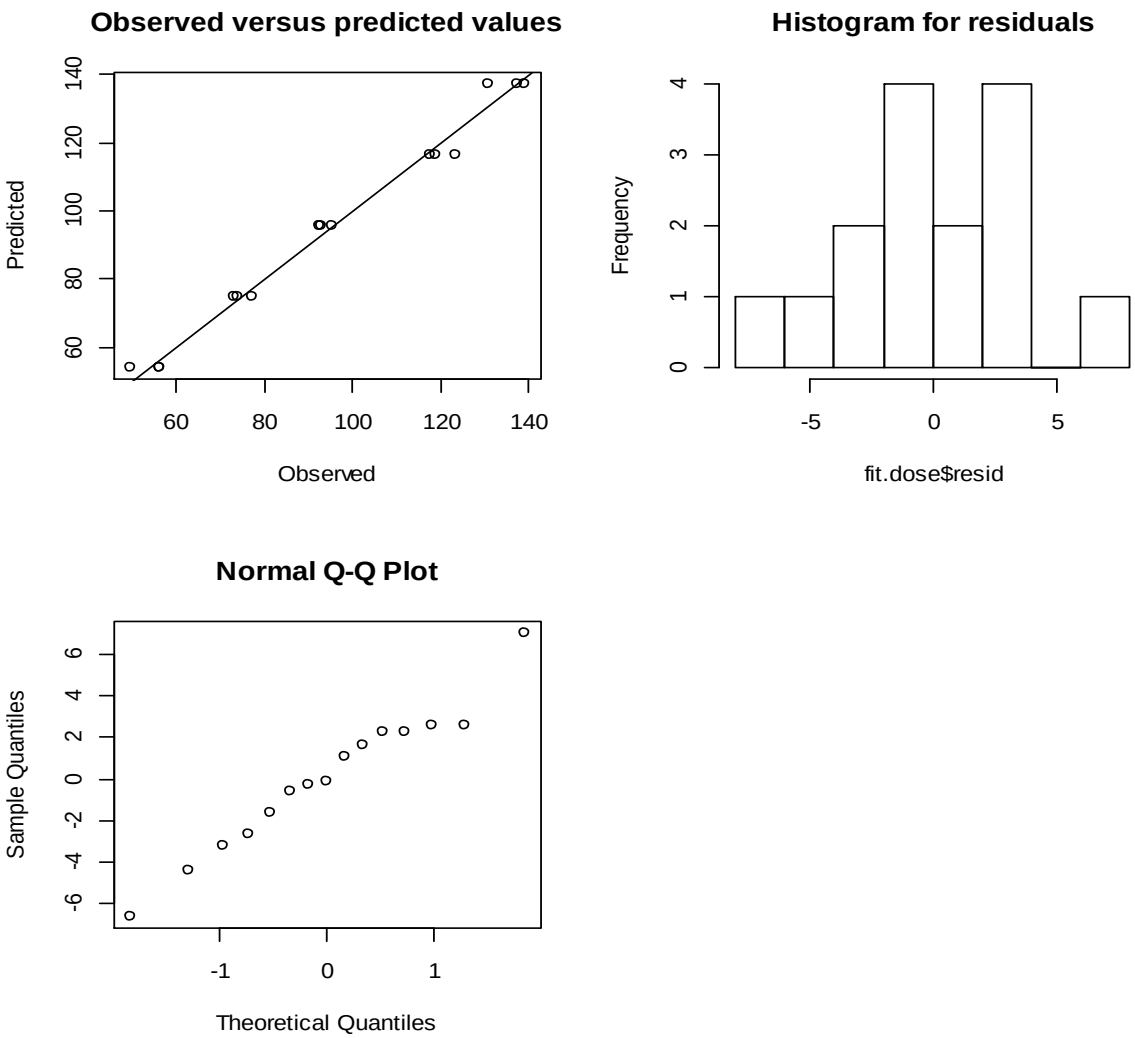
```
> par(mfrow=c(2,2))
> plot(fit.dose$fit,xlab="Observed",
  ylab="Predicted", main = "Observed versus,
  predicted values")
> abline(0,1)
> hist(fit.dose$resid,col=0,main="Histogram for
+ residuals")
> qqnorm(fit.dose$resid)
```

This statemnt produces the plot of the observed versus the predicted values (to check if the variance is constant)

This statemnt produces the Histogram of residuals (to check normality)

This statemnt produces the qqnormal plot (to check normality)

Graphical output



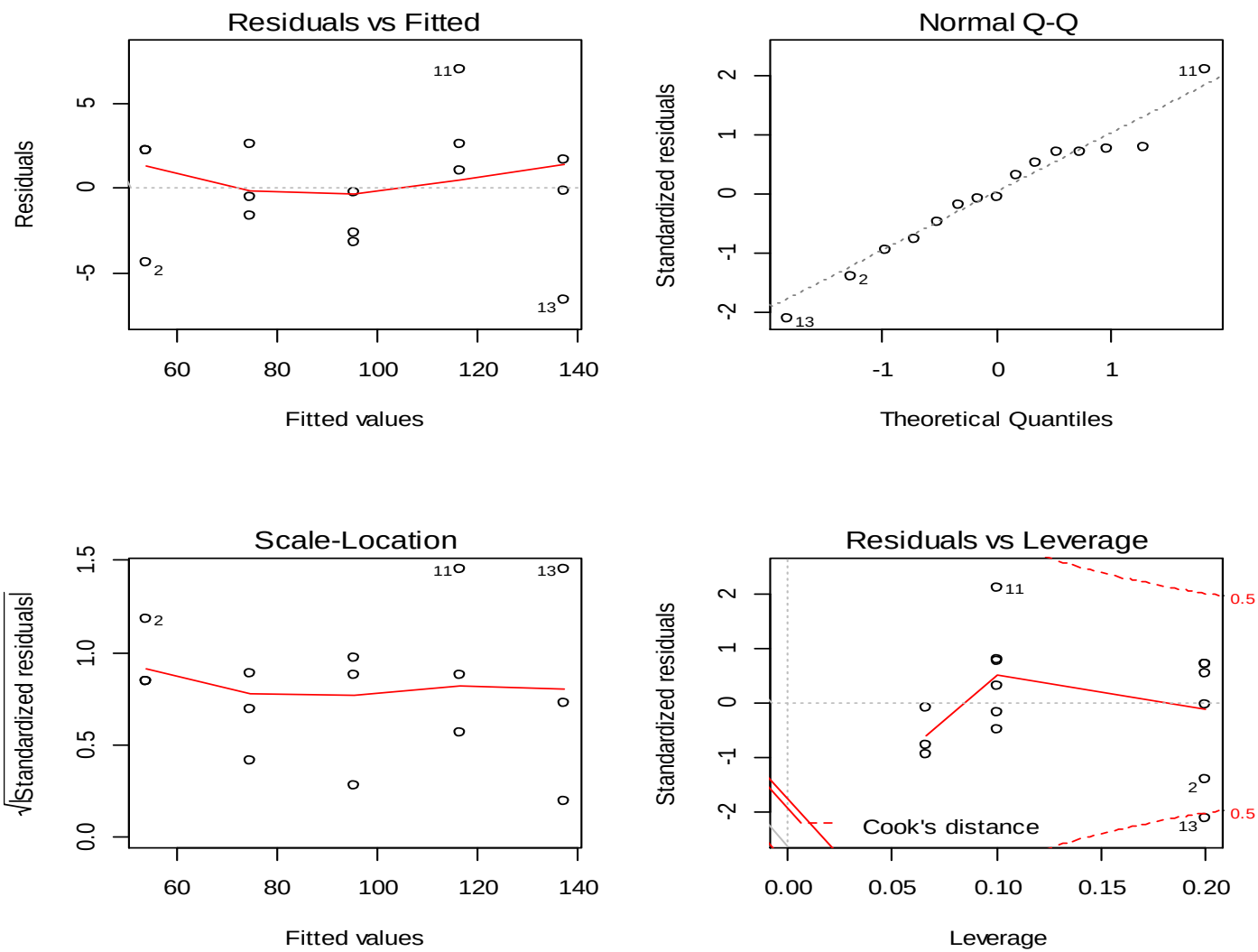
Diagnostic plots

```
> par(mfrow = c(2,2))
> plot(fit.dose)
```

This statement produces the following figures:

- A plot of the residuals versus the predicted values (to check if the variance is constant).
- A qqnormal plot (to check normality).
- A scale-location plot (to check if the variance is constant),
- A plot of residuals versus leverage (to check if there are influential observations).

Diagnostic plots



Residual analyses

```
## Normality test ##
> shapiro.test(residuals(fit.dose))
```

```
Shapiro-Wilk normality test

data:  residuals(fit.dose)
W = 0.9723, p-value = 0.8907
```

```
##Constant variance test ##
> library(lmtest)
> bptest(fit.dose)
```

```
studentized Breusch-Pagan test

data:  fit.dose
BP = 1.0129, df = 1, p-value = 0.3142
```

```
#Testing the Independence Assumption #
library(lmtest)
dwtest(fit.dose, alternative =
+ "two.sided")
```

```
Durbin-Watson test

data:  fit.dose
DW = 2.0775, p-value = 0.8863
alternative hypothesis: true autocorrelation is not 0
```

Part 5

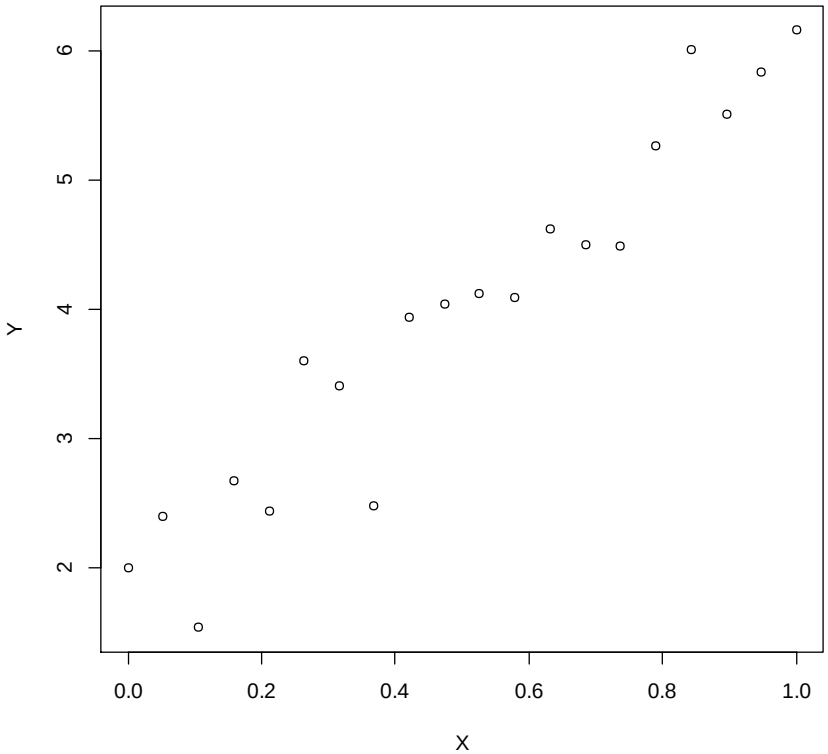
Four examples for model diagnostic

Four examples

- Example 1: all model asgsumptions hold.
- Example 2: the variance is not constant.
- Example 3: structure in the residuals.
- Example 4: the distribution of the residuals is not a Normal distribution.

Example 1: the data

- The sample size is equal to 20.
- The observation unit (x_i, y_i), $i=1,\ldots,20$.
- The relatioship between X and Y seems to be linear.



Example 1 External data: example1.txt

Formulation of the model

We consider a linear regression model of the form

$$Y_i = \alpha + \beta \times X_i + \varepsilon_i$$

It is further assumed that the random error is normal distributed with mean 0 and constant variance σ^2.

$$\varepsilon_i \sim N(0, \sigma^2)$$

The model in R:

```
fit.example1 <- lm(y ~ x, data = example1)
summary(fit.example1)
aov(fit.example1)
```

ANOVA table and parameter estimates

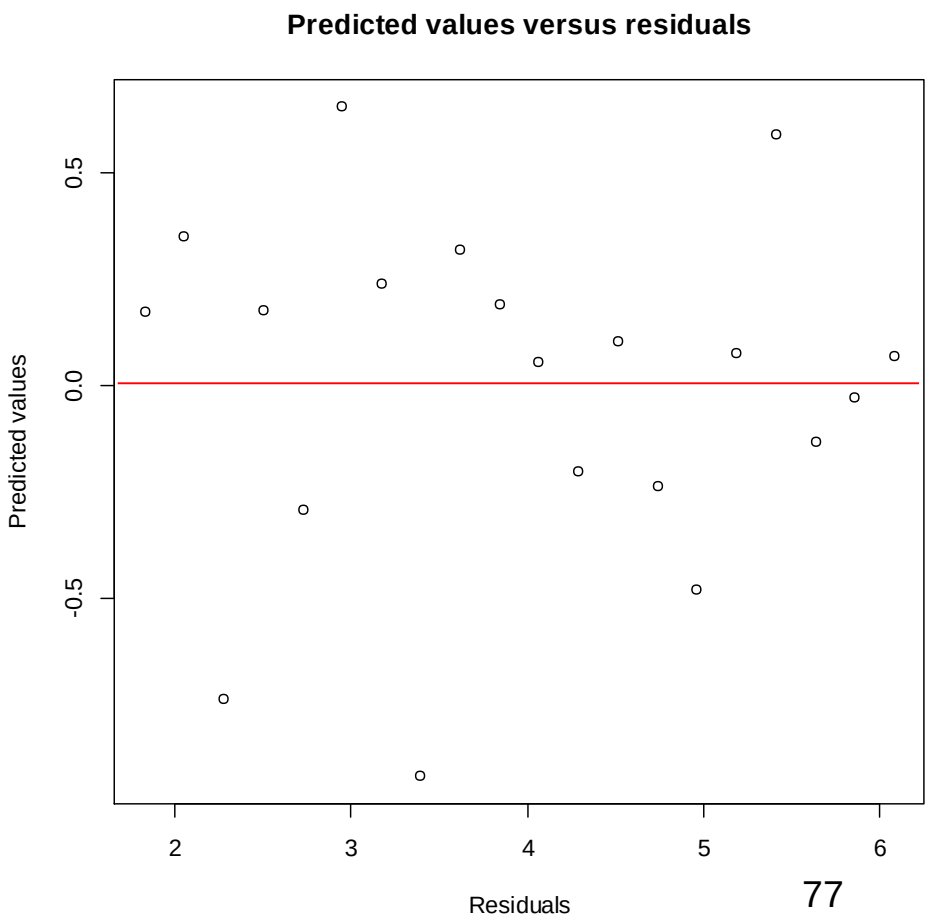
```
Call:
  aov(formula = fit.example1)

Terms:
                    x Residuals
Sum of Squares  33.38747   2.99696
Deg. of Freedom       1        18

Residual standard error: 0.4080414
Estimated effects may be unbalanced
```

ANOVA table and parameter estimates

```
Call:
lm(formula = y ~ x, data = example1)

Residuals:
    Min      1Q   Median      3Q     Max
-0.91702 -0.21027  0.07406  0.20531  0.65608

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.8260     0.1759   10.38 4.99e-09 ***
x             4.2582     0.3007   14.16 3.36e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.408 on 18 degrees of freedom
Multiple R-squared: 0.9176,     Adjusted R-squared: 0.9131
F-statistic: 200.5 on 1 and 18 DF,  p-value: 3.364e-11
```

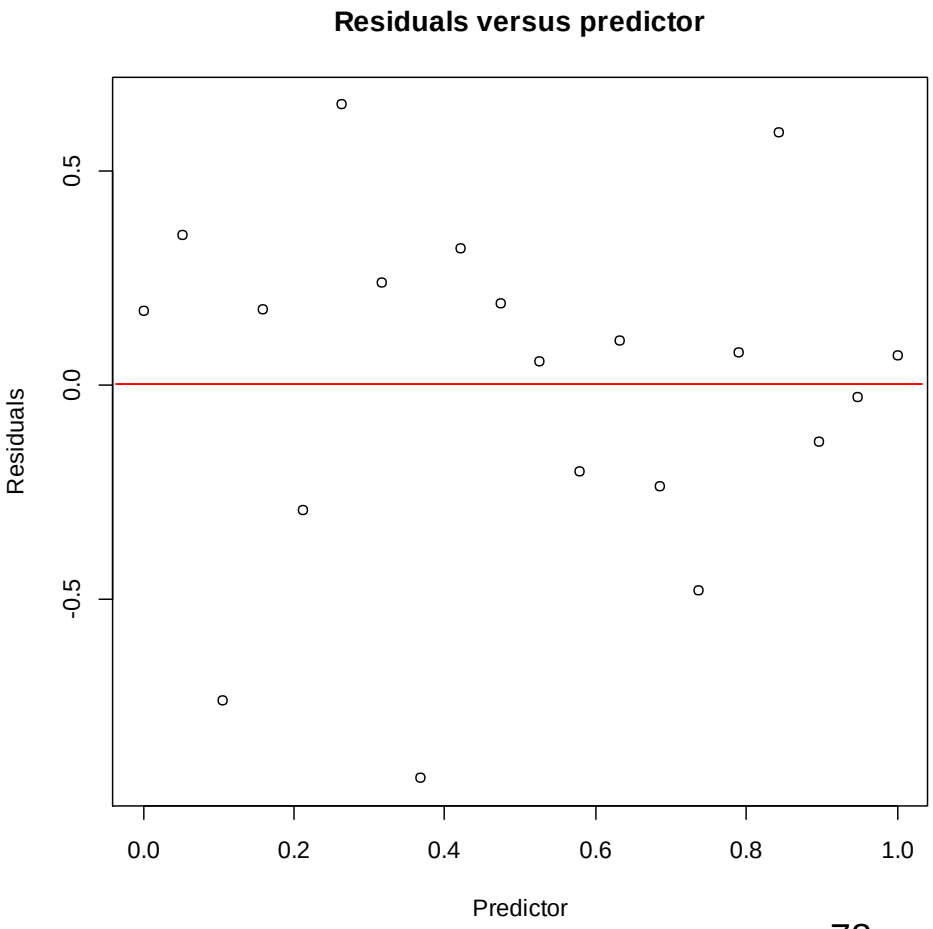
Constant variance: residuals versus predicted values

- We focus in this plot on the variabilty, if it constant we do not expect to patterns in this plot.



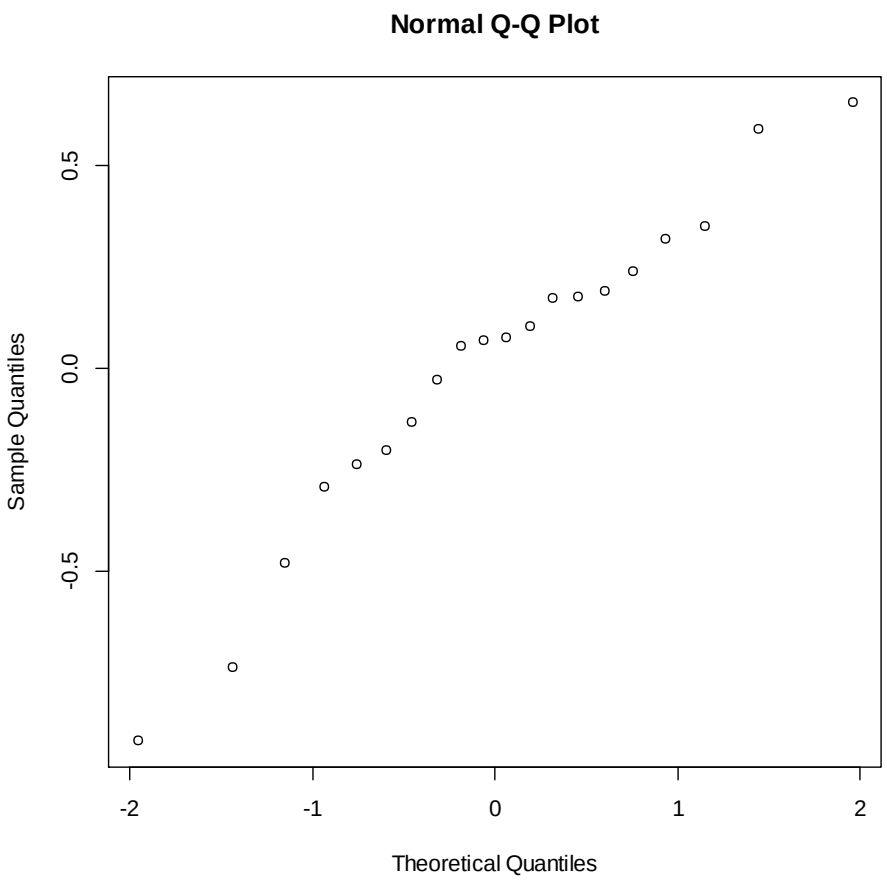
Linearity: residuals versus the predictor

- If the linear model is a "good model" (this means that the assumption that the mean of Y is linear with respect to X) we do not expect to patterns in this plot.

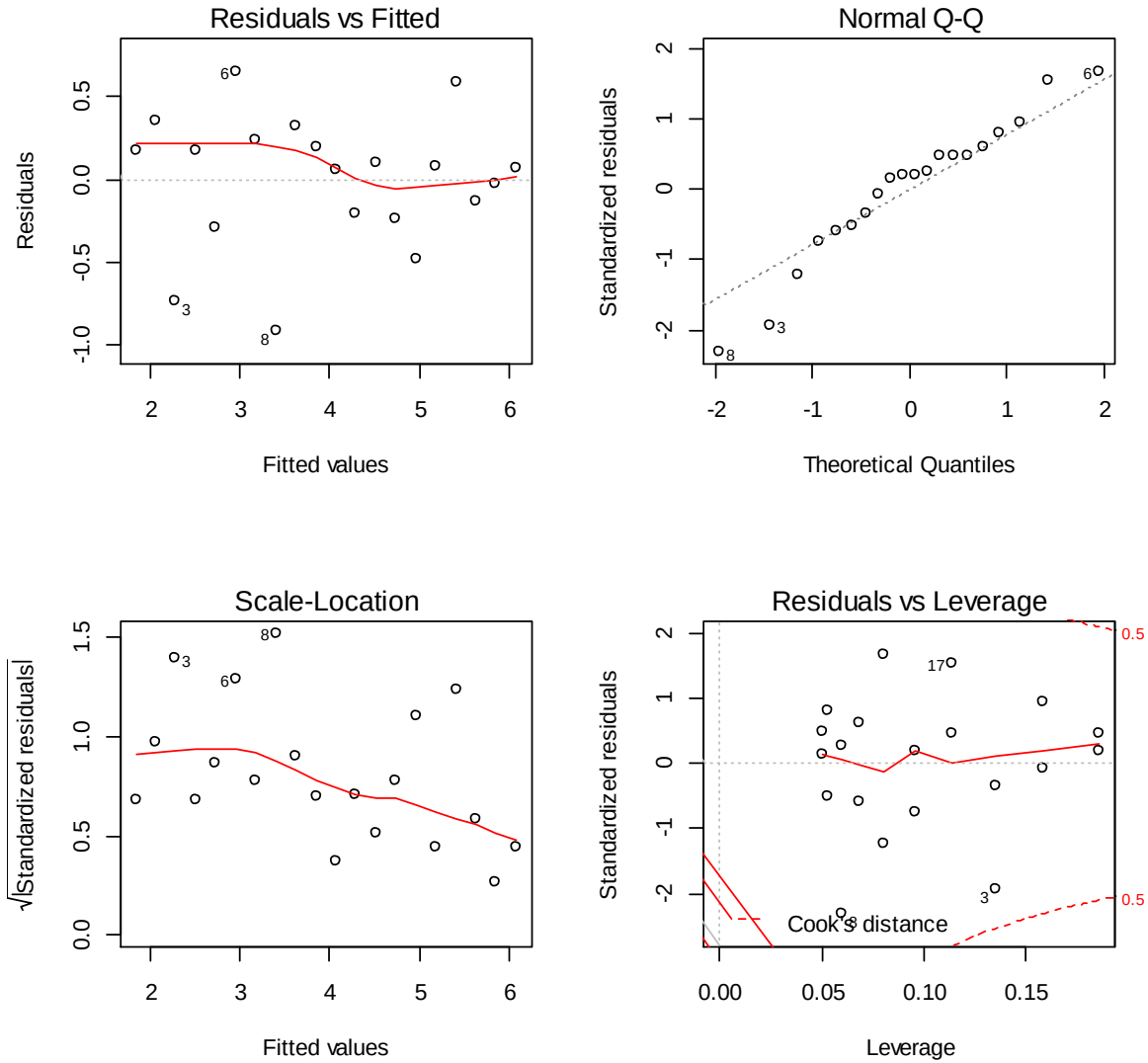


Normality: qqpolt for the residuals

- If the random error is normal distributed the points in the qqnormal plot should follow a stright line pattern.



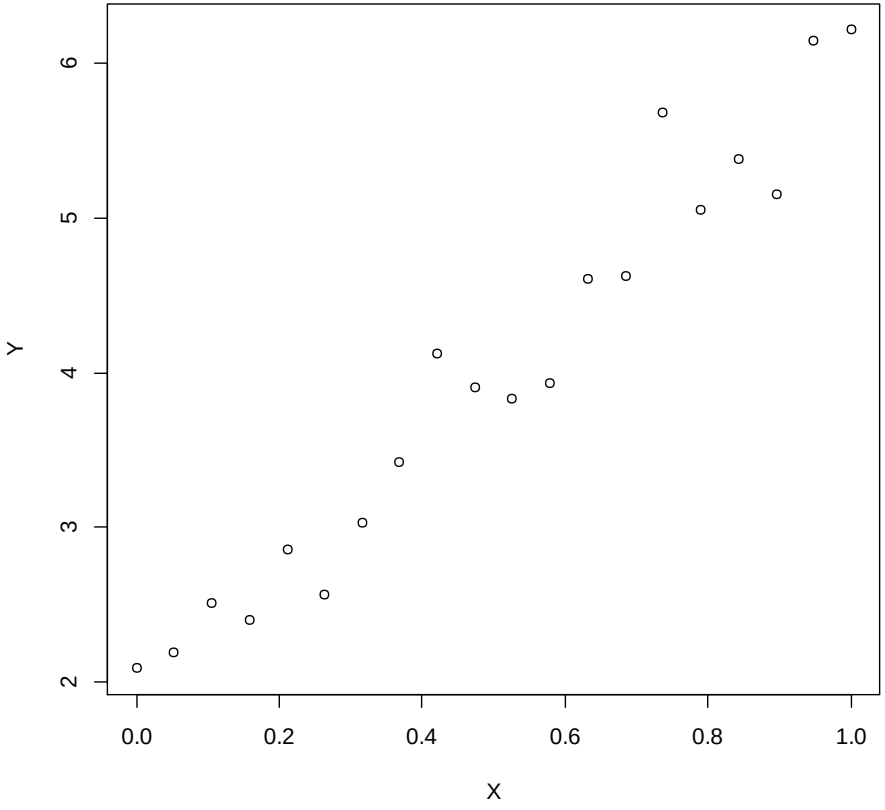
Diagnostic plots



Example 2: the data

- Sample size is 20.
- The relatioship seems to be linear. So the regression model should fit the data.
- Model in R:

```
####  Scatter plot of the data  ####
plot(y ~ x, data = example2, ylab = "Y", xlab = "X")
####  Fitting the model     ####
fit.example2 <- lm(y ~ x, data = example2)
summary(fit.example2)
aov(fit.example2)
```



Example 2 External data: example2.txt

ANOVA table and parameter estimates

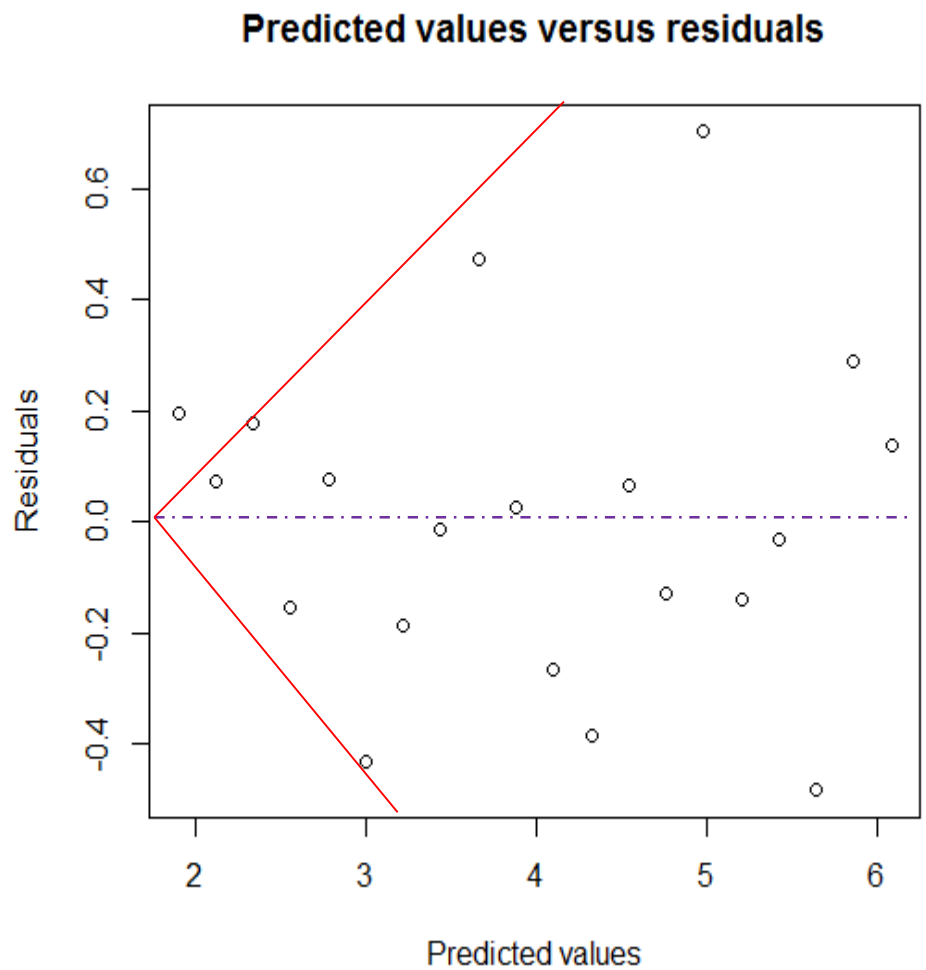
```
> aov(fit.example2)

Call:
   aov(formula = fit.example2)

Terms:
                    x Residuals
Sum of Squares  32.28117   1.64056
Deg. of Freedom       1        18

Residual standard error: 0.3018981
Estimated effects may be unbalanced
```

ANOVA table and parameter estimates

```
> summary(fit.example2)

Call:
lm(formula = y ~ x, data = example2)

Residuals:
    Min      1Q   Median      3Q     Max
-0.48422 -0.16228  0.00692  0.14724  0.70333

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.8938     0.1301   14.55 2.13e-11 ***
x             4.1870     0.2225   18.82 2.75e-13 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3019 on 18 degrees of freedom
Multiple R-squared: 0.9516,     Adjusted R-squared: 0.9489
F-statistic: 354.2 on 1 and 18 DF,  p-value: 2.745e-13
```

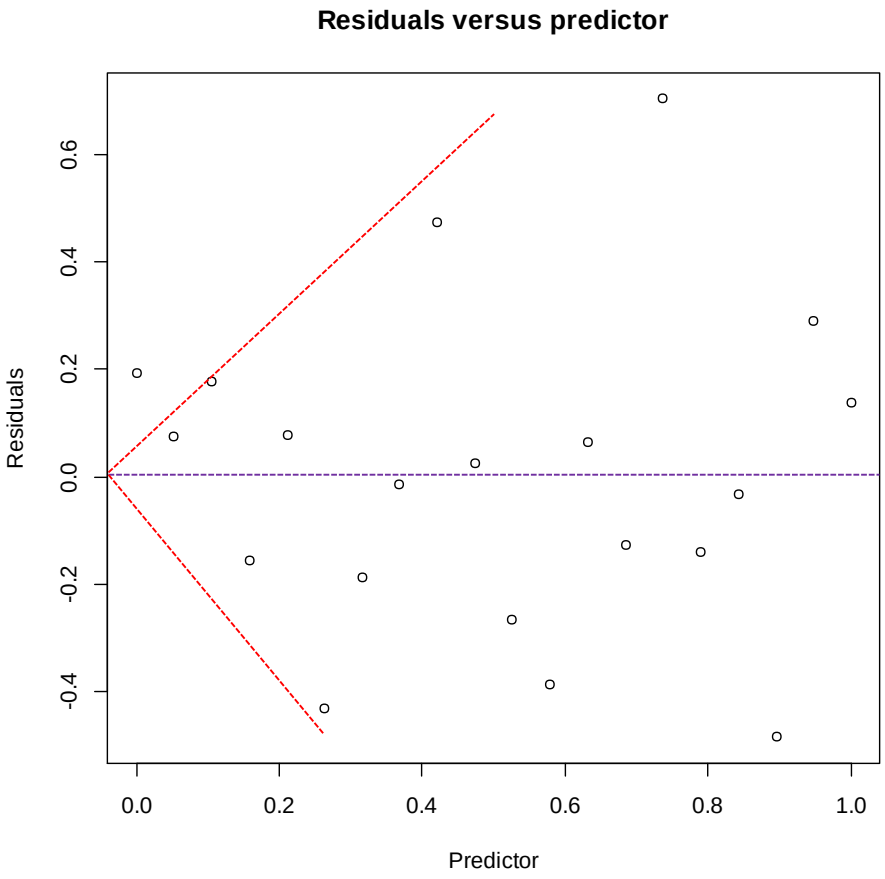
Constant variability

- A “megaphon” shape.
- The variability is not constant.
- The variability increase as the predicted values increase.



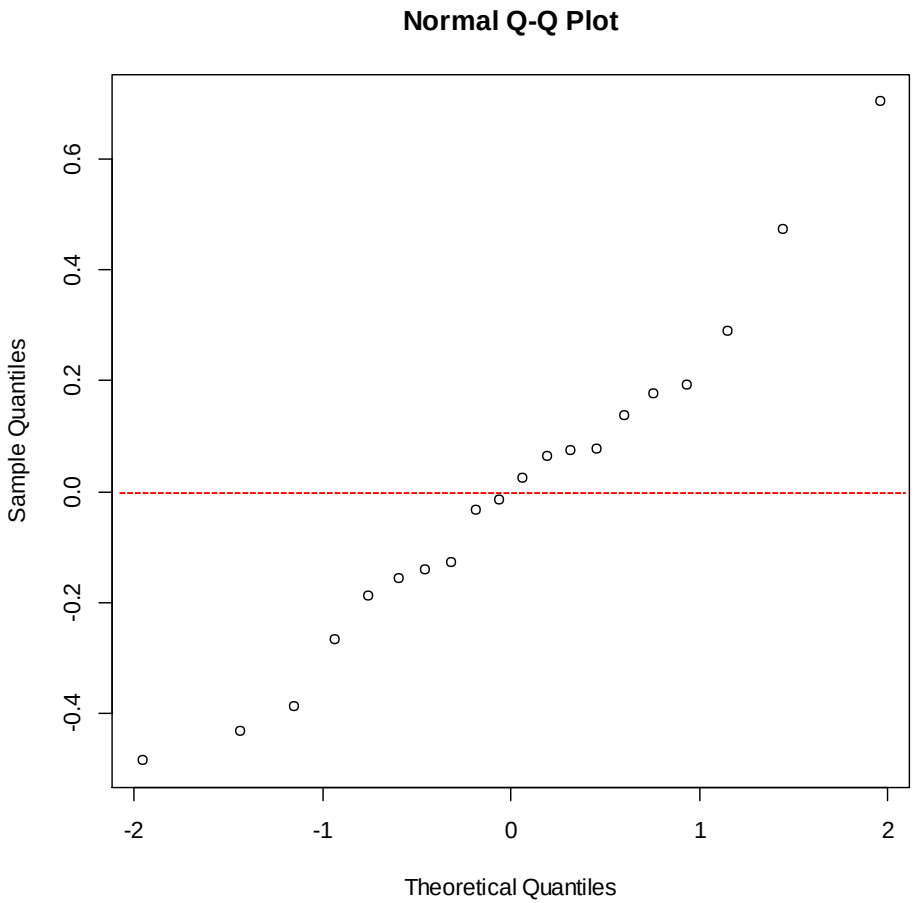
Linearity and constant variability: residuals versus the predictor

- Residuals distributed around zero.
- This means that the linear regression model captures the main pattern in the data.
- BUT it is clear that the variability is not constant.

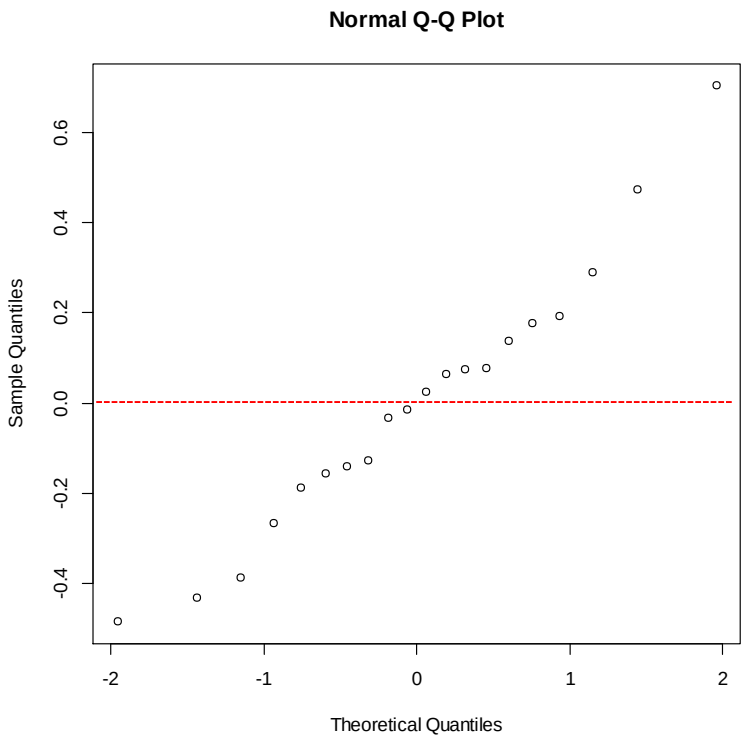


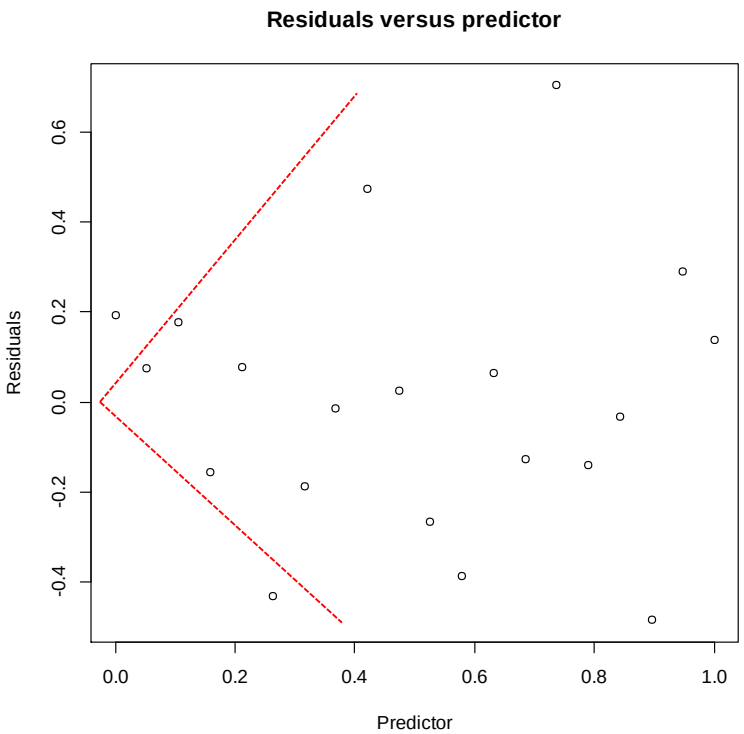
A qq-normal plot

- No pattern is detected so we conclude that the random error is normal distributed.



Do not use only one residuals plot for model diagnostic

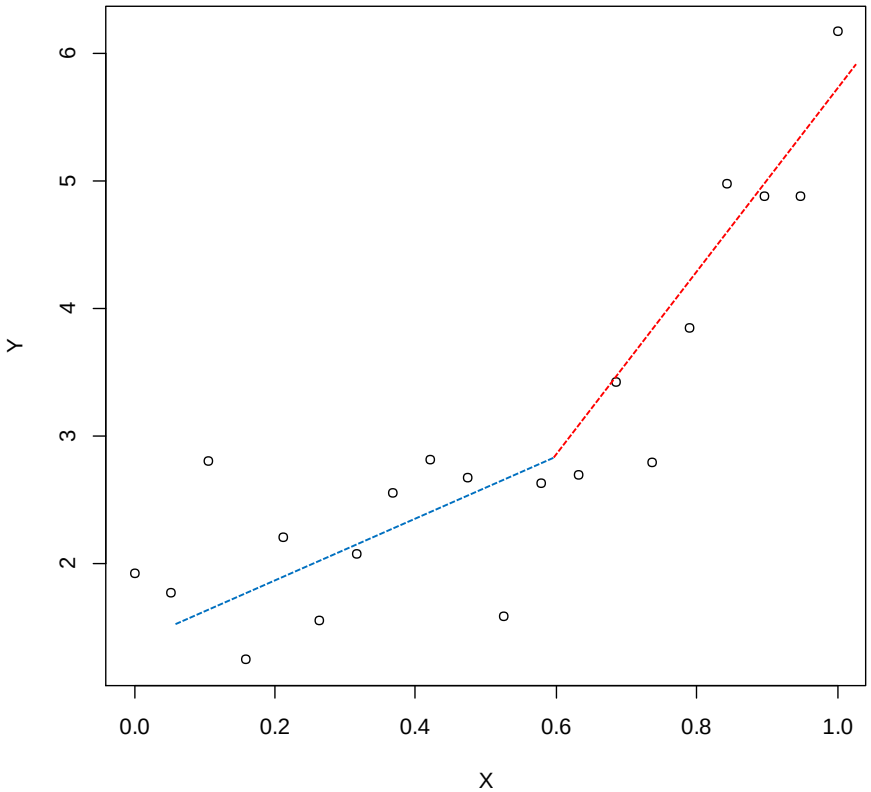




Another example in which the qqnormal plot indicate that the random error is normal distributed and the the plot with the residuals versus the predictor indicates on non constant variance.

Example 3: the data

- Sample size is 20.
- The relatioship seems to be linear BUT NOT A STRIGHT LINE.
- This means that a simple linear regression model will not be able to capture all structure of the data.



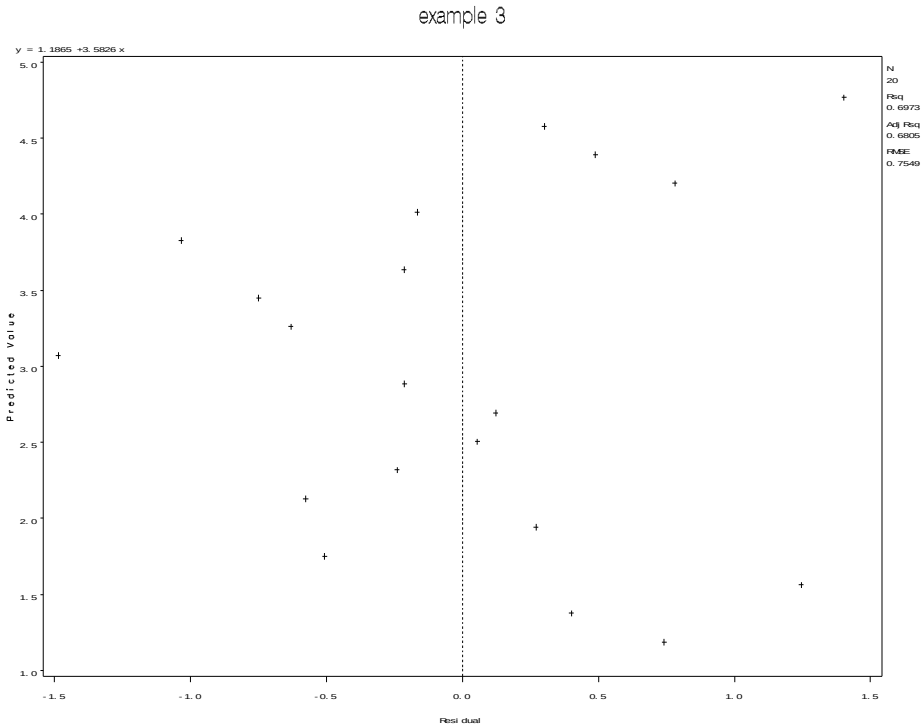
Example 3 External data: example3.txt

ANOVA table and parameter estimates

```
> aov(fit.example3)

Call:
   aov(formula = fit.example3)

Terms:
                    x Residuals
Sum of Squares  23.63339  10.25865
Deg. of Freedom       1        18

Residual standard error: 0.7549339
Estimated effects may be unbalanced
```

Constant variability

- The residuals plot do not reavel any pattern which indicates that the variance is not constant.

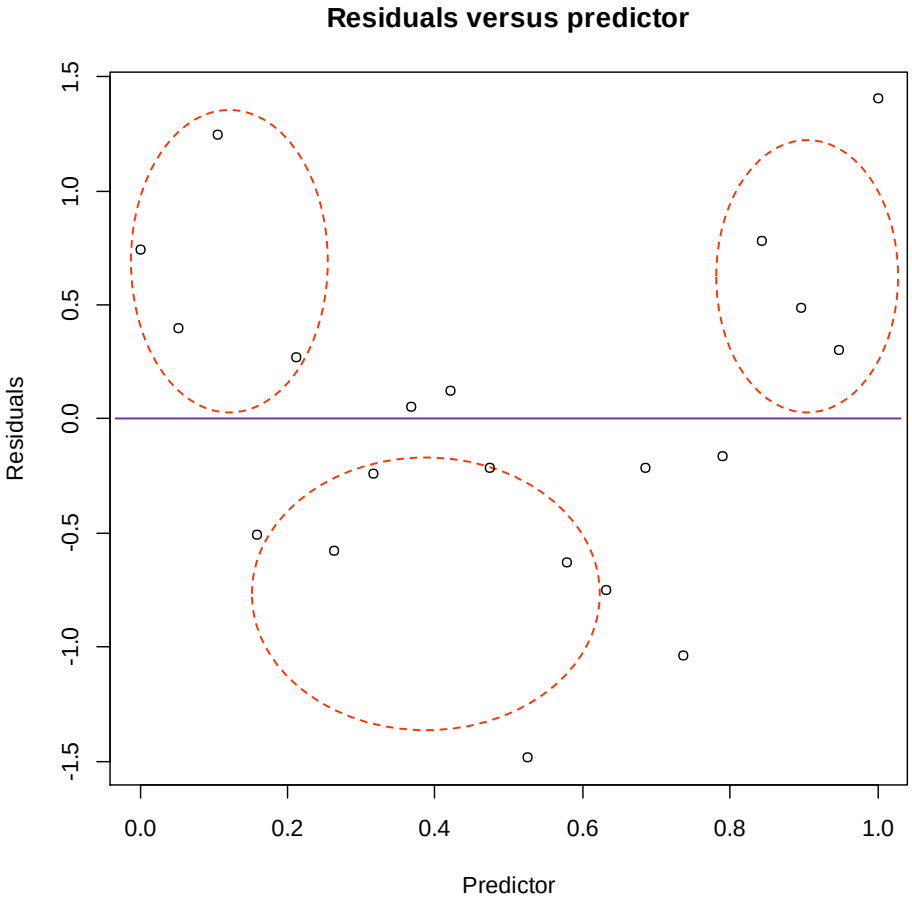


ANOVA table and parameter estimates

```
> summary(fit.example3)

Call:
lm(formula = y ~ x, data = example3)

Residuals:
    Min      1Q   Median      3Q     Max
-1.48495 -0.52386 -0.05503  0.42272  1.40292

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.1865     0.3254   3.647  0.00185 **
x             3.5826     0.5563   6.440 4.64e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.7549 on 18 degrees of freedom
Multiple R-squared: 0.6973,     Adjusted R-squared: 0.6805
F-statistic: 41.47 on 1 and 18 DF,  p-value: 4.64e-06
```

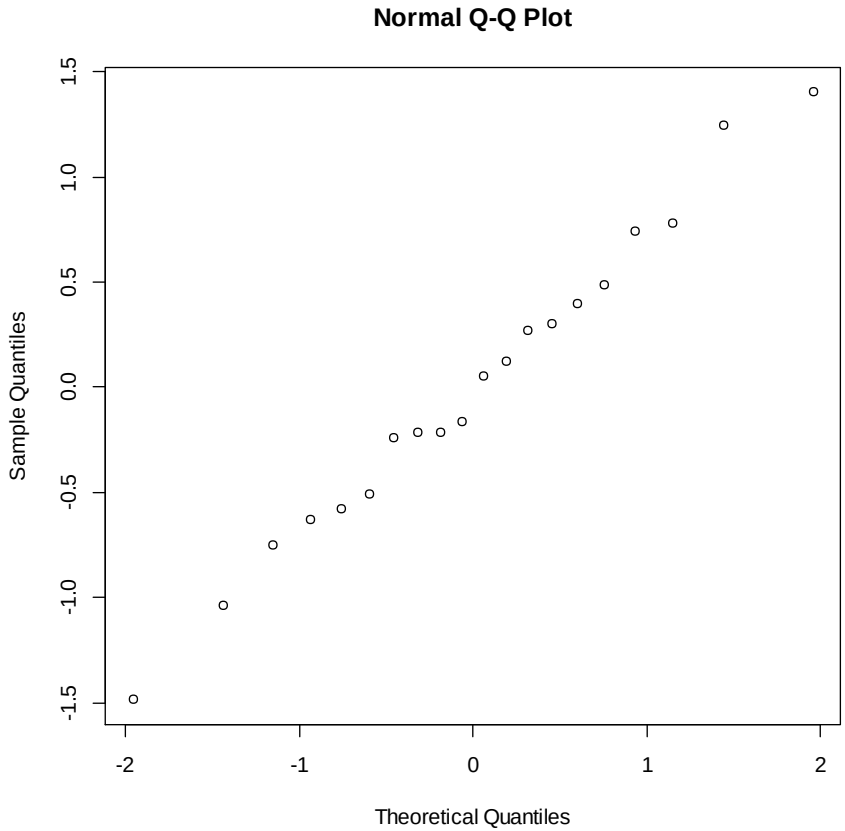
Linearity

- Pattern in the residual plot.
- We observed groups with positive and negative residuals.
- This means that the model does not capture all structure in the data.



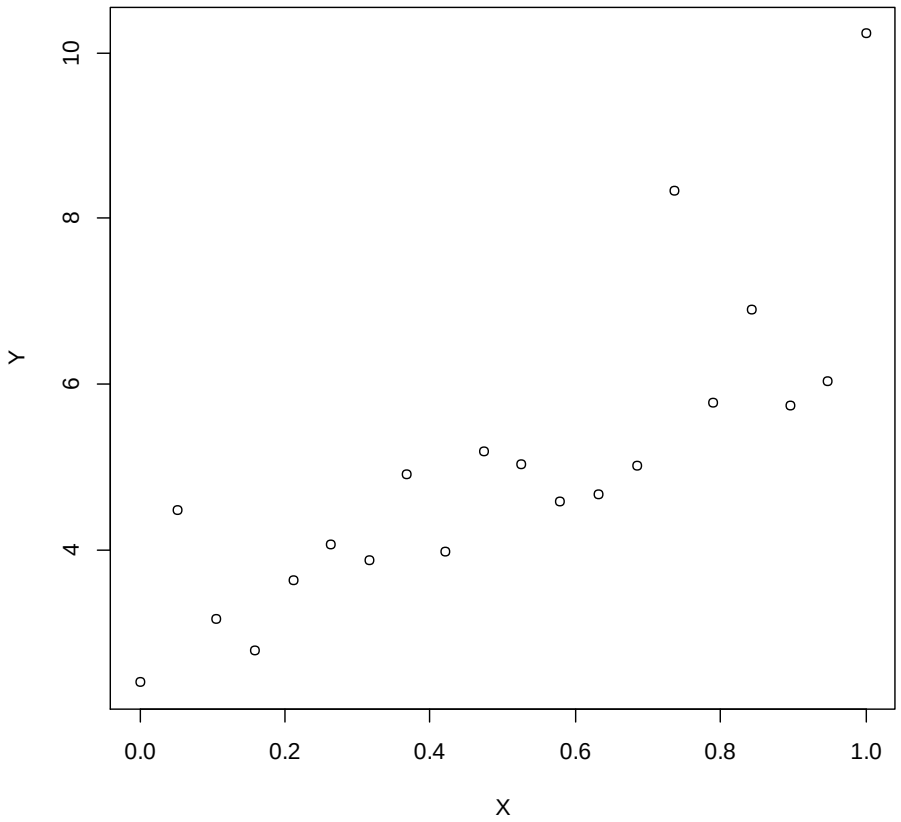
Normality

- A pattern of a sgtight line in the qqnormal plot.
- This indicate that the residuals follow a normal distribution.



Example 4: the data

- This is an example in which the three residuals plots reveal the same problem of the model which is not related to linearity and constant variability.



Example 4 External data: example4.txt

ANOVA table and parameter estimates

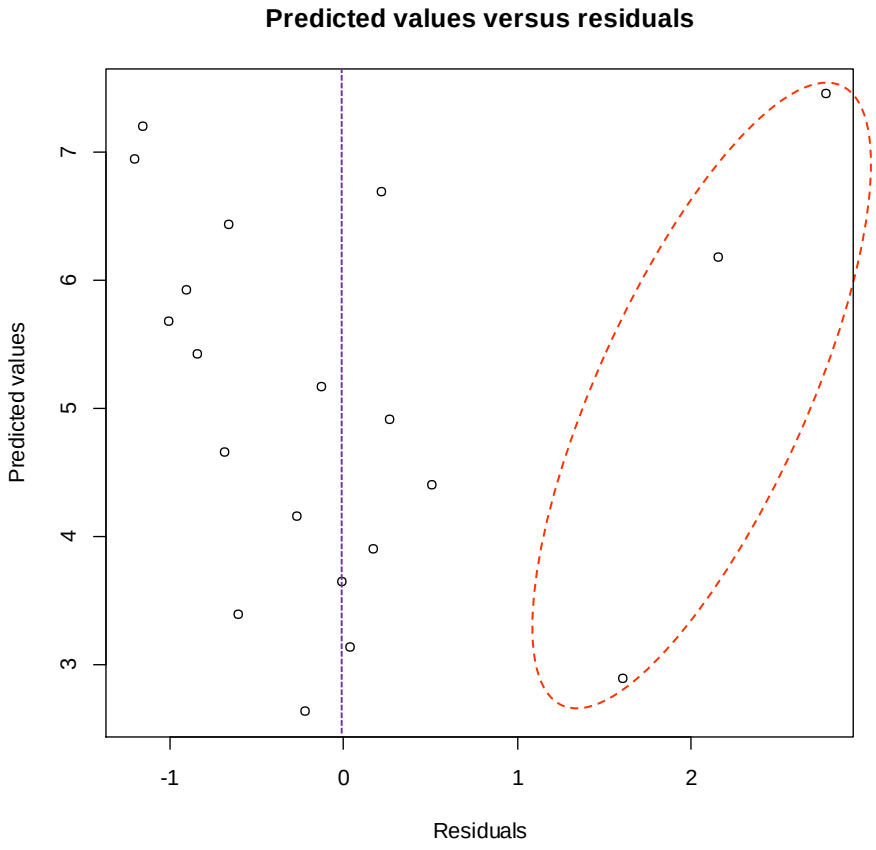
```
> aov(fit.example4)

Call:
   aov(formula = fit.example4)

Terms:
                    x Residuals
Sum of Squares  42.91607  22.07751
Deg. of Freedom       1        18

Residual standard error: 1.107487
Estimated effects may be unbalanced
```

ANOVA table and parameter estimates

```
> summary(fit.example4)

Call:
lm(formula = y ~ x, data = example4)

Residuals:
   Min      1Q  Median      3Q     Max
-1.2074 -0.7238 -0.1791  0.2265  2.7738

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.6287     0.4773   5.507 3.14e-05 ***
x             4.8277     0.8161   5.915 1.34e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.107 on 18 degrees of freedom
Multiple R-squared: 0.6603,     Adjusted R-squared: 0.6414
F-statistic: 34.99 on 1 and 18 DF,  p-value: 1.341e-05
```

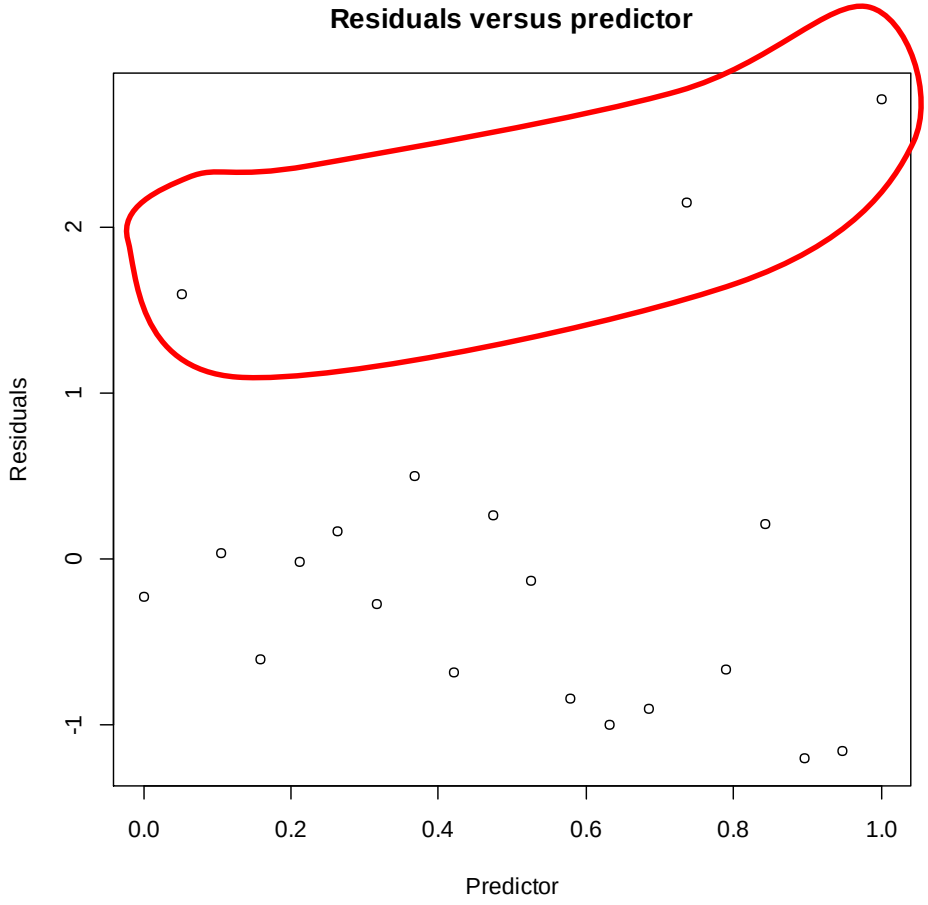
Residuals versus predicted values

- Three positive outliers.



Residuals versus the predictor

- There are more negative residuals than positive residuals and three positive outliers.



Normality

- The pattern in the qqnormal plot indicates on departure from normality.
- Mind the three outliers.