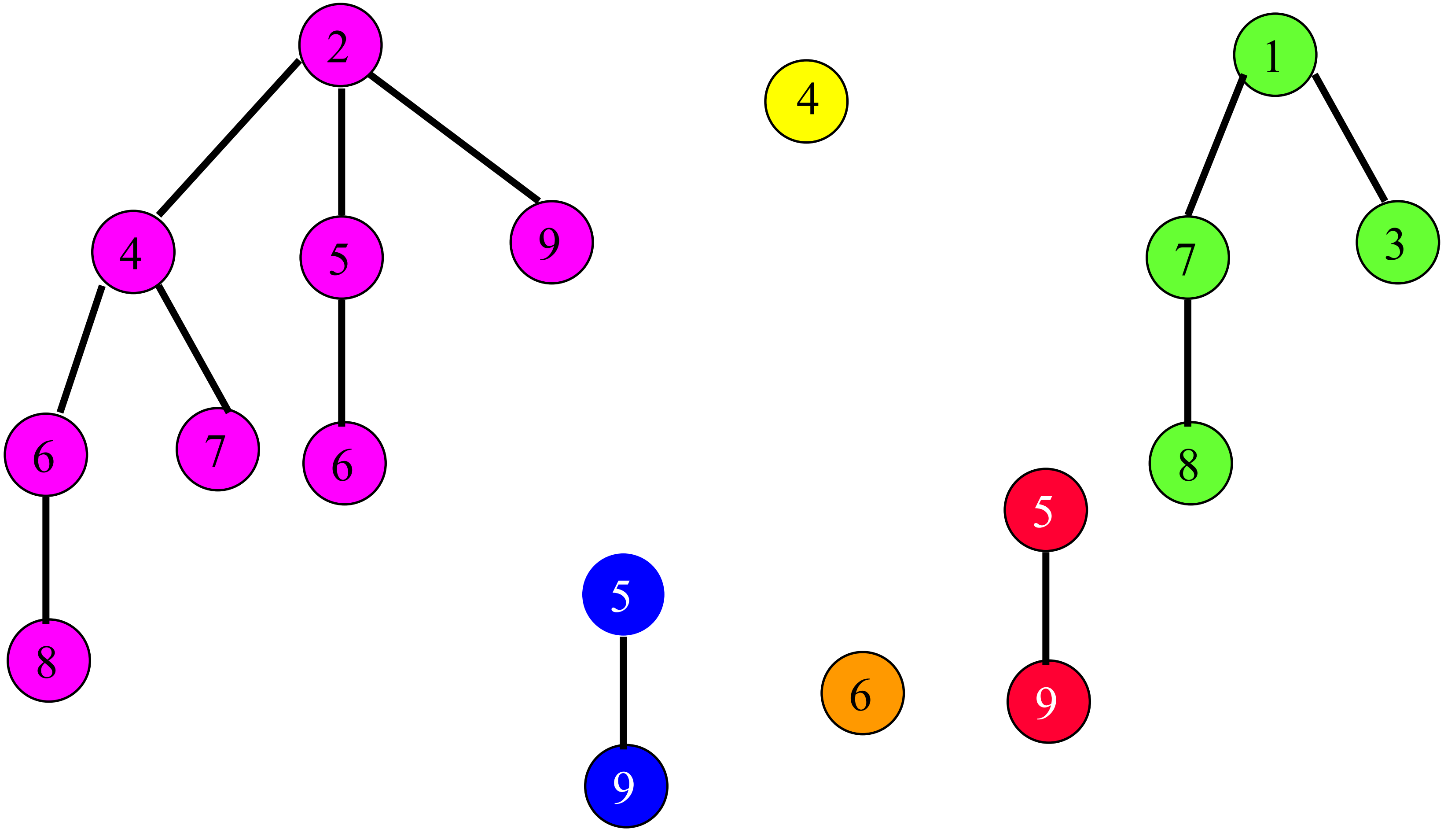
Binomial Heaps

	Leftist trees	Binomial heaps	
		Actual	Amortized
Insert	O(log n)	O(1)	O(1)
Remove min (or max)	O(log n)	O(n)	O(log n)
Meld	O(log n)	O(1)	O(1)

Min Binomial Heap

- Collection of min trees.

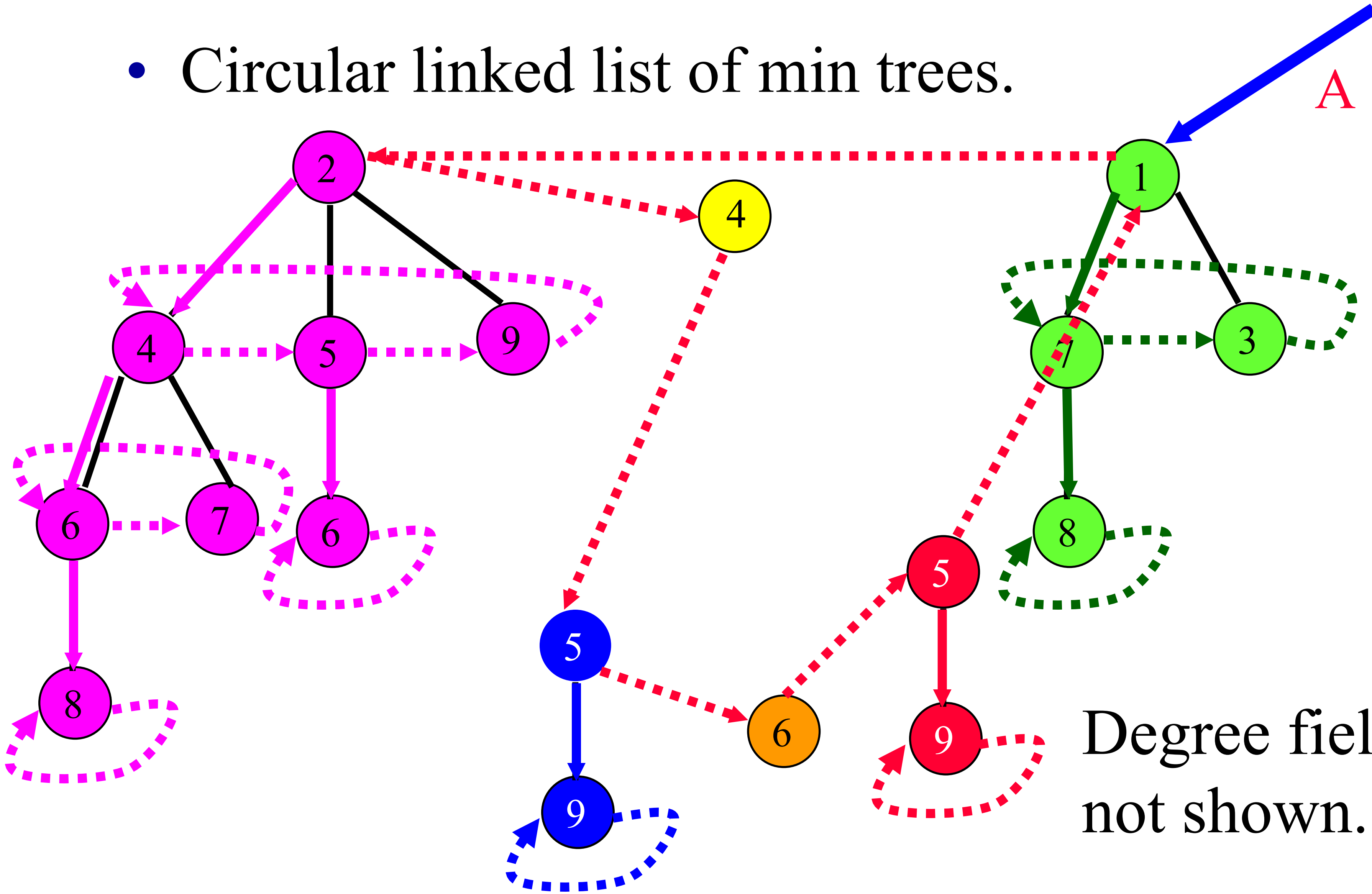


Node Structure

- Degree
  - Number of children.
- Child
  - Pointer to one of the node’s children.
  - Null iff node has no child.
- Sibling
  - Used for circular linked list of siblings.
- Data

Binomial Heap Representation

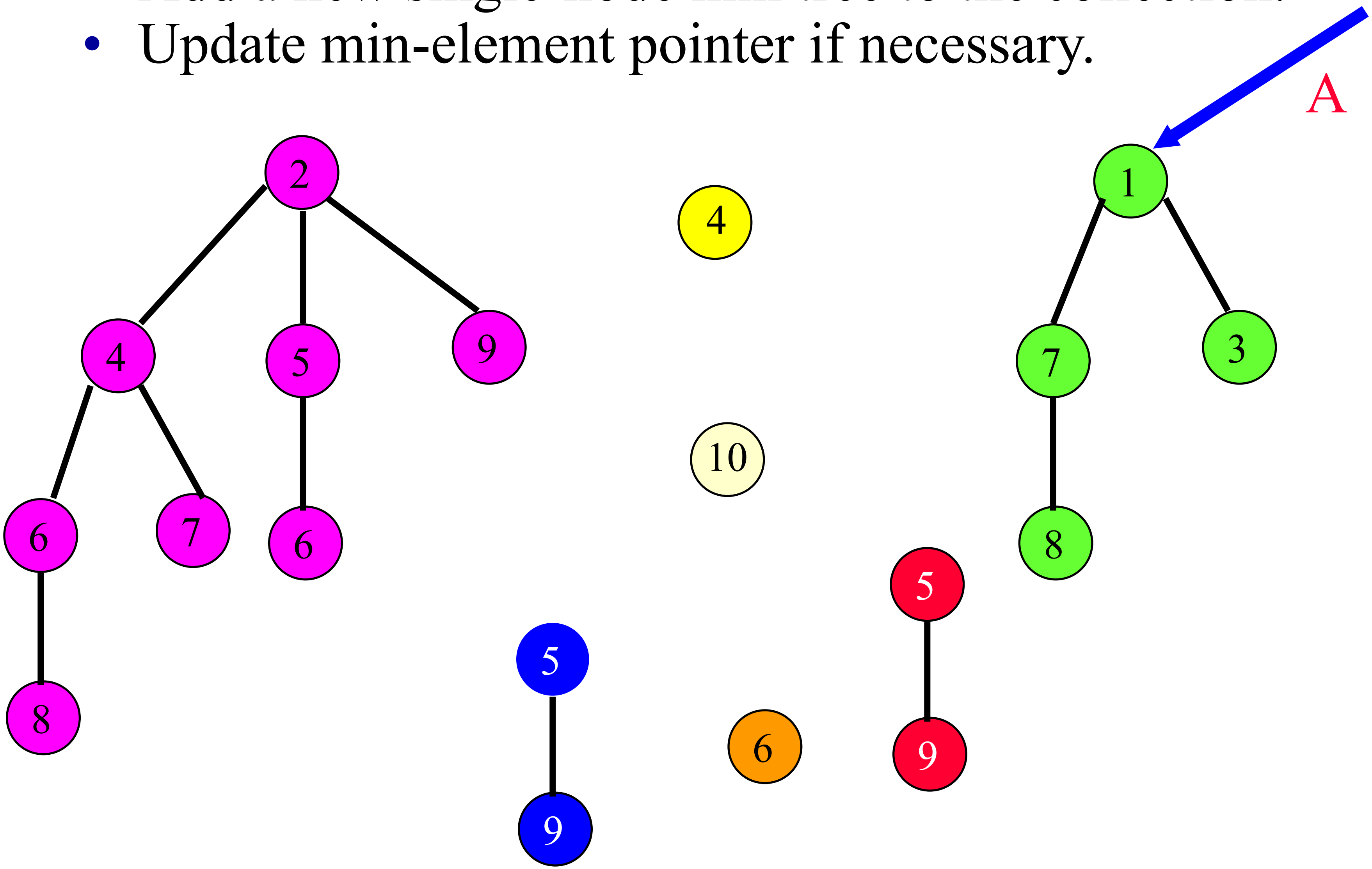
- Circular linked list of min trees.



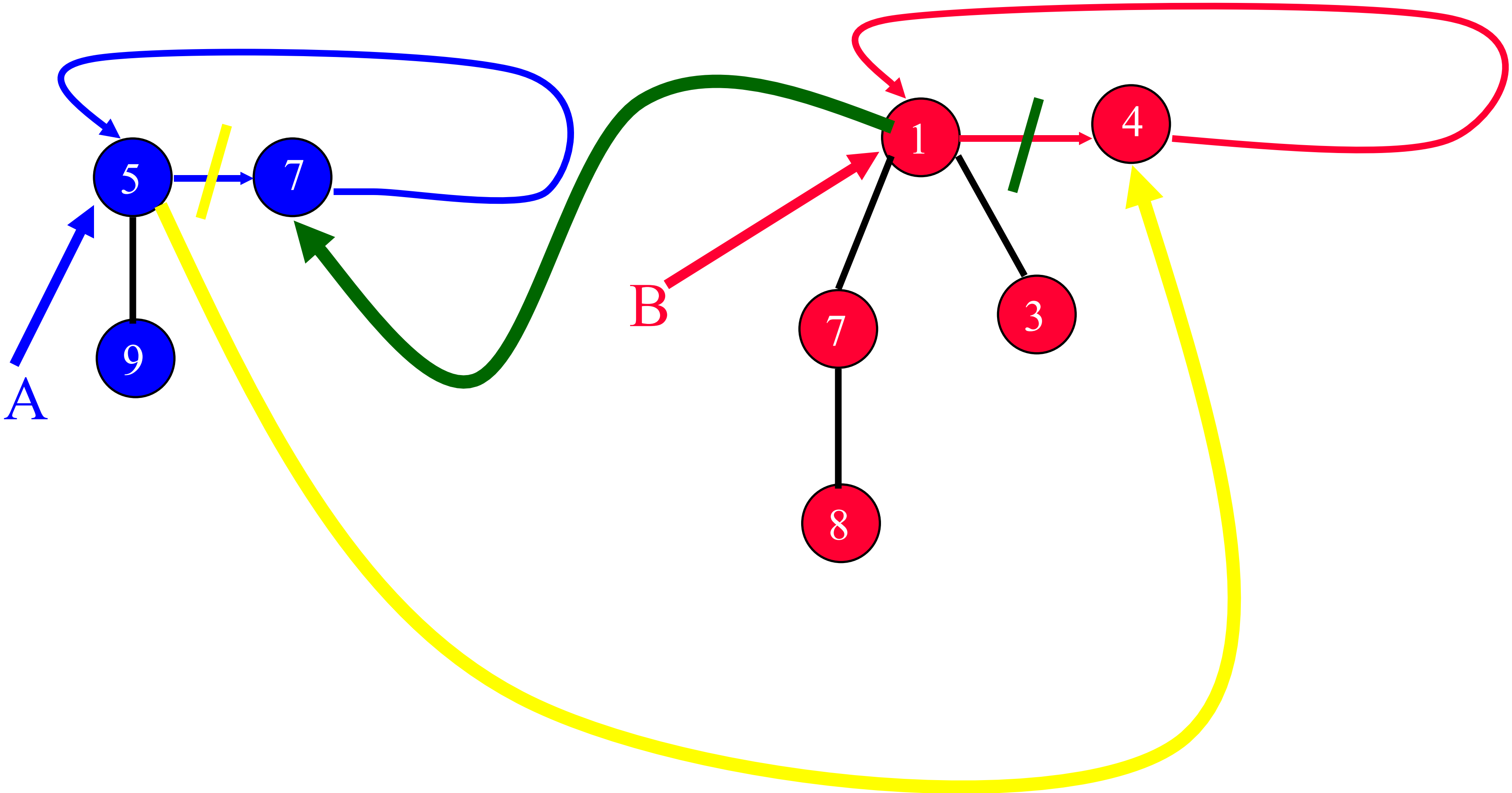
Degree fields not shown.

Insert 10

- Add a new single-node min tree to the collection.
- Update min-element pointer if necessary.

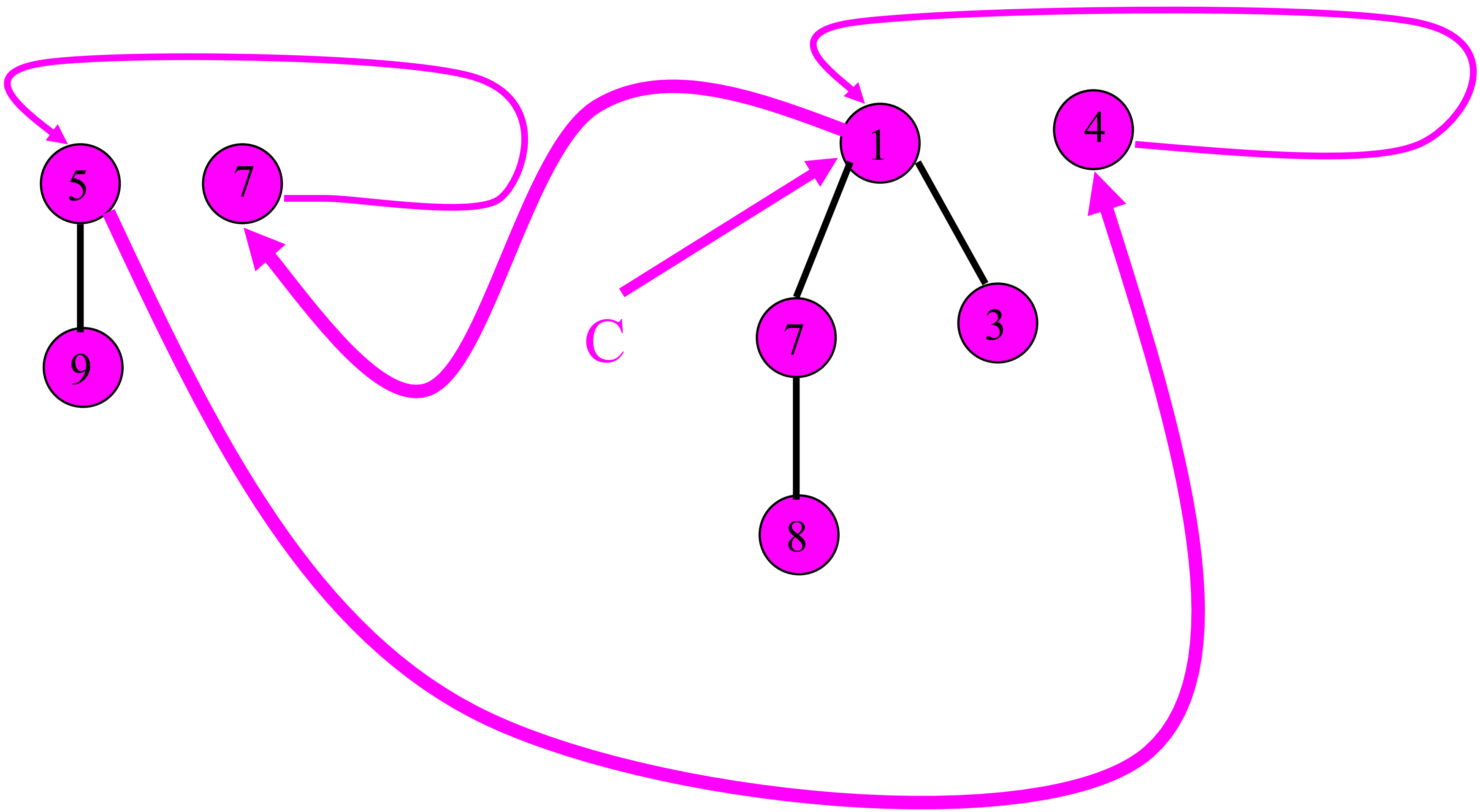


Meld



- Combine the 2 top-level circular lists.
- Set min-element pointer.

Meld

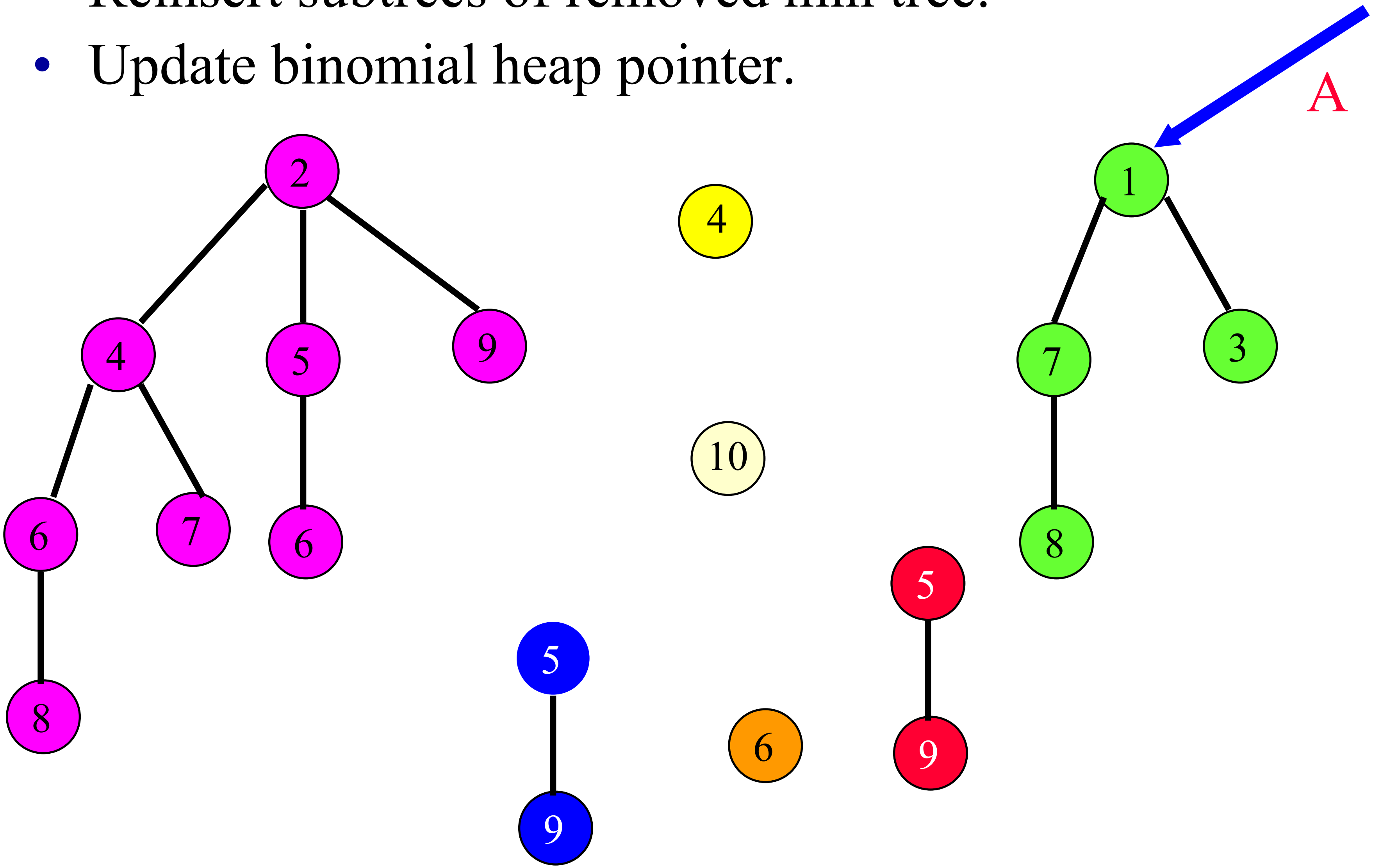


Remove Min

- Empty binomial heap => fail.

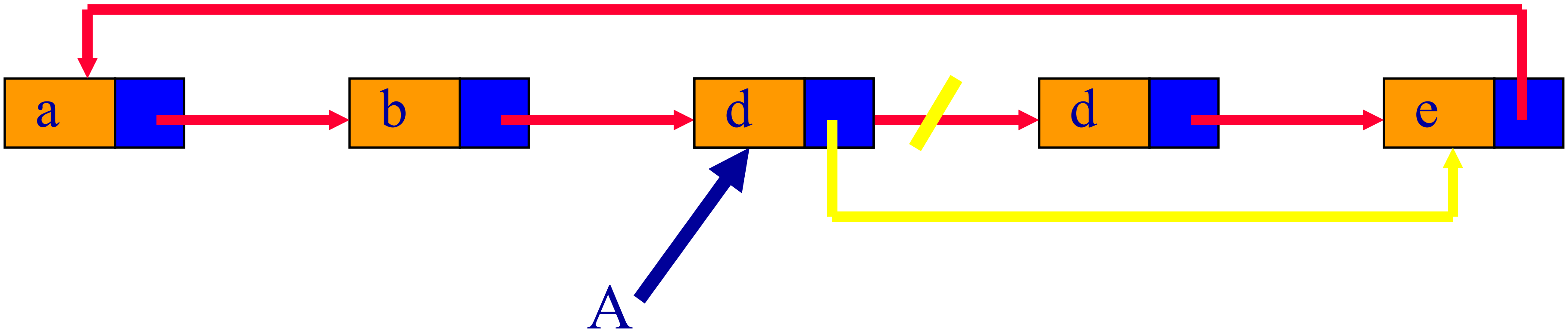
Nonempty Binomial Heap

- Remove a min tree.
- Reinsert subtrees of removed min tree.
- Update binomial heap pointer.



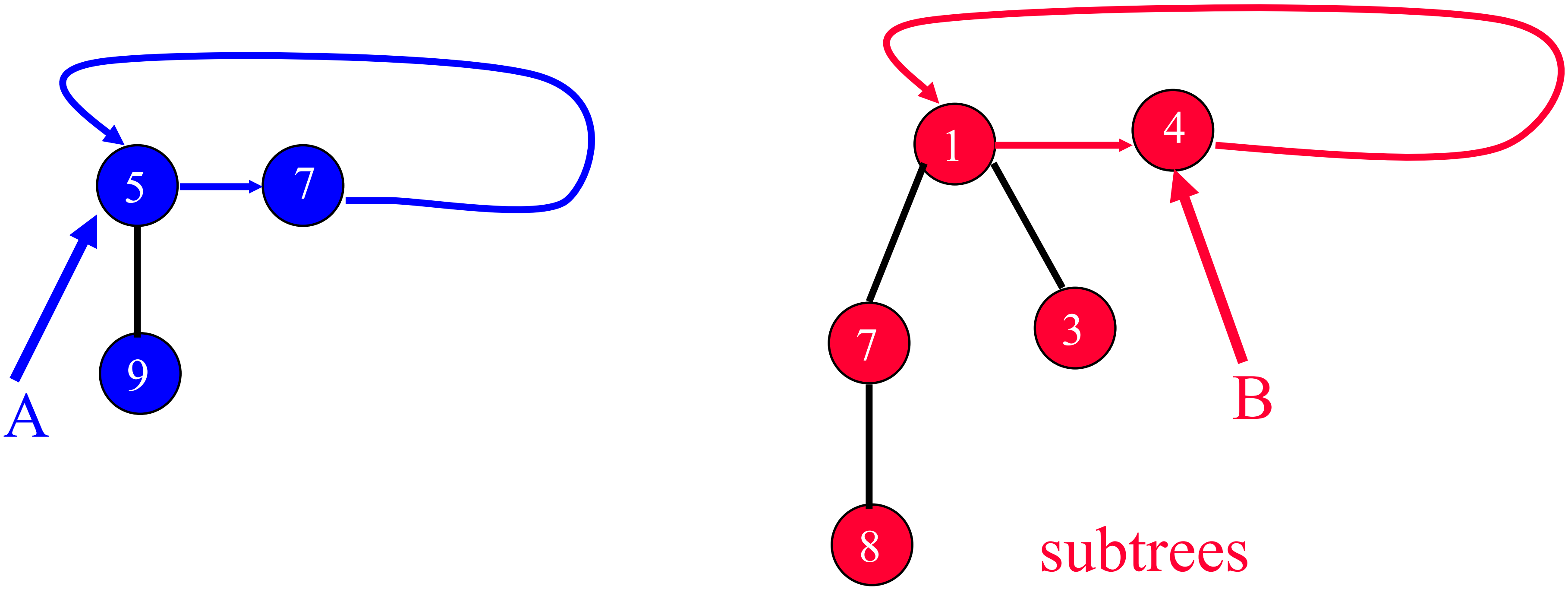
Remove Min Tree

- Same as remove a node from a circular list.



- No next node => empty after remove.
- Otherwise, copy next-node data and remove next node.

Reinsert Subtrees



- Combine the 2 top-level circular lists.
  - Same as in meld operation.

Update Binomial Heap Pointer

- Must examine roots of all min trees to determine the min value.

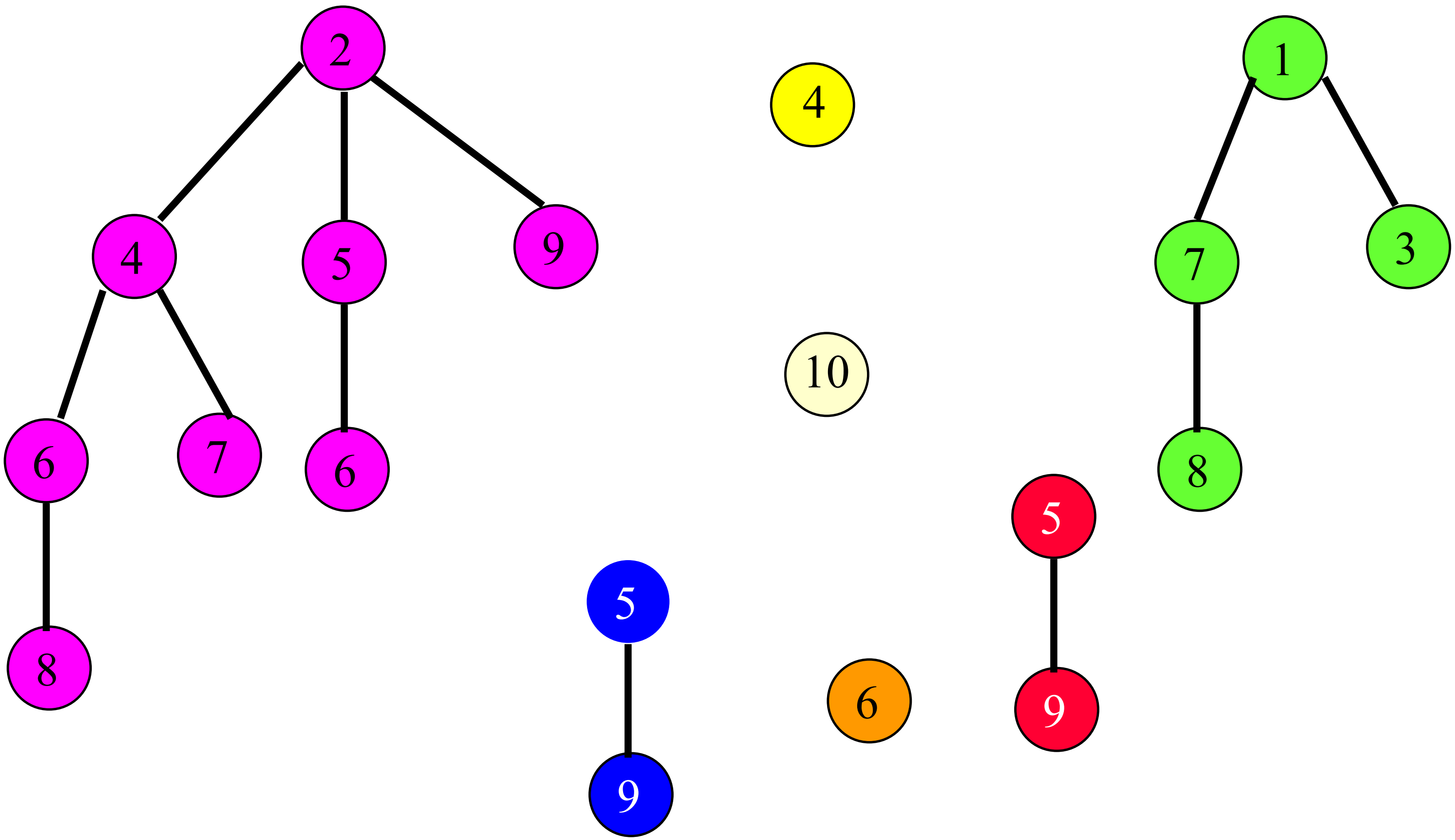
Complexity Of Remove Min

- Remove a min tree.
  - O(1).
- Reinsert subtrees.
  - O(1).
- Update binomial heap pointer.
  - O(s), where s is the number of min trees in final top-level circular list.
  - s = O(n).
- Overall complexity of remove min is O(n).

Enhanced Remove Min

- During reinsert of subtrees, pairwise combine min trees whose roots have equal degree.

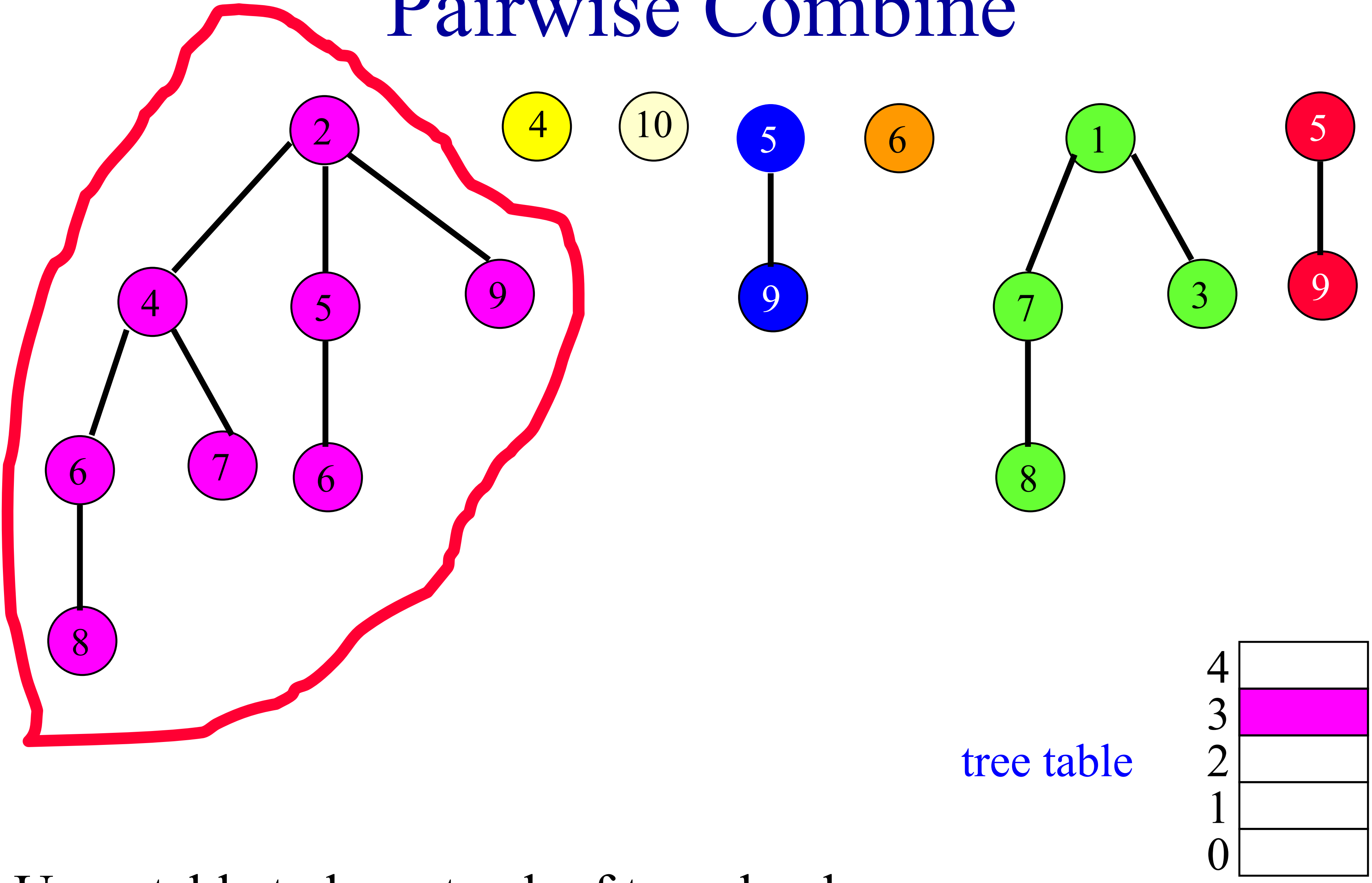
Pairwise Combine



Examine the s = 7 trees in some order.

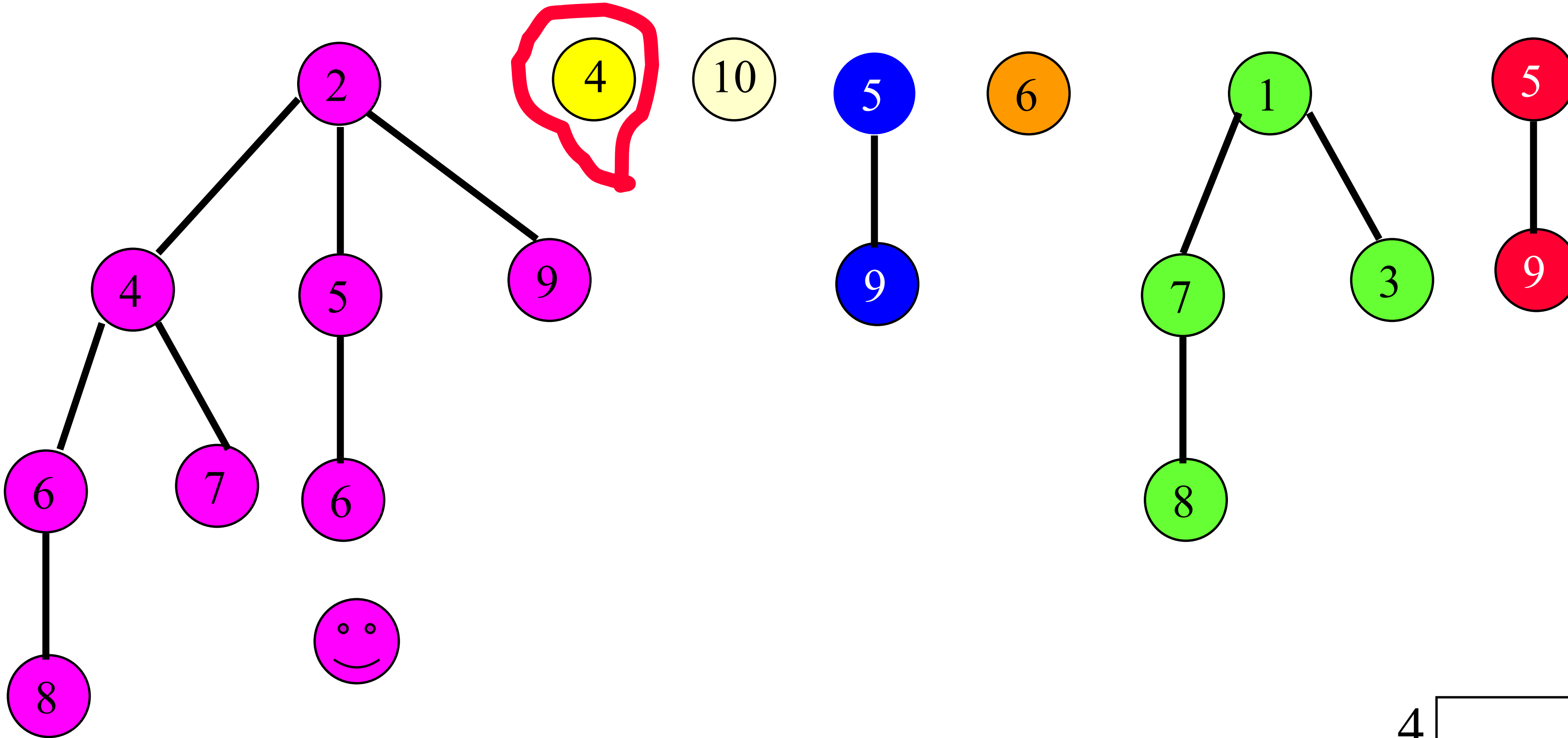
Determined by the 2 top-level circular lists.

Pairwise Combine

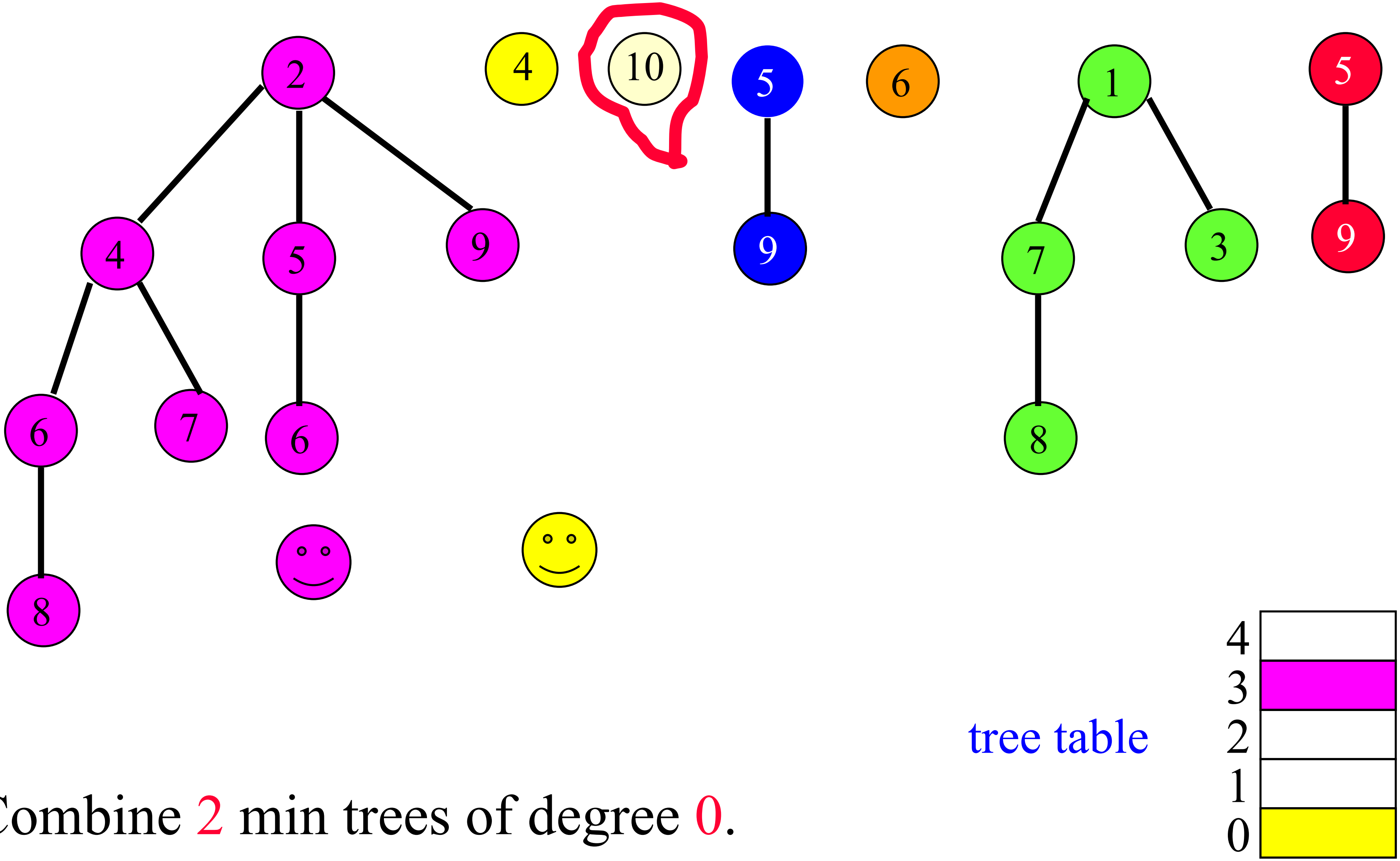


Use a table to keep track of trees by degree.

Pairwise Combine



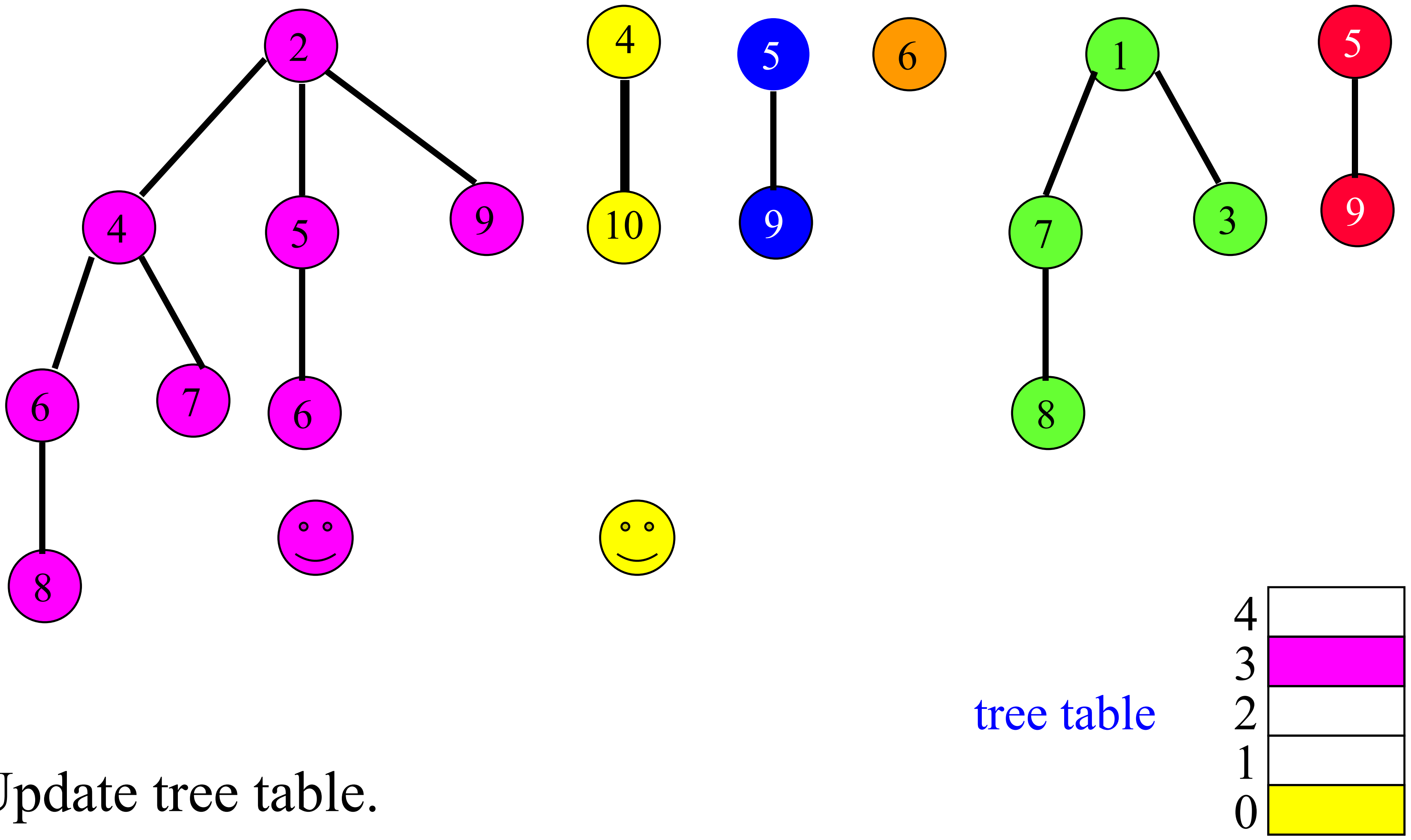
Pairwise Combine



Combine 2 min trees of degree 0.

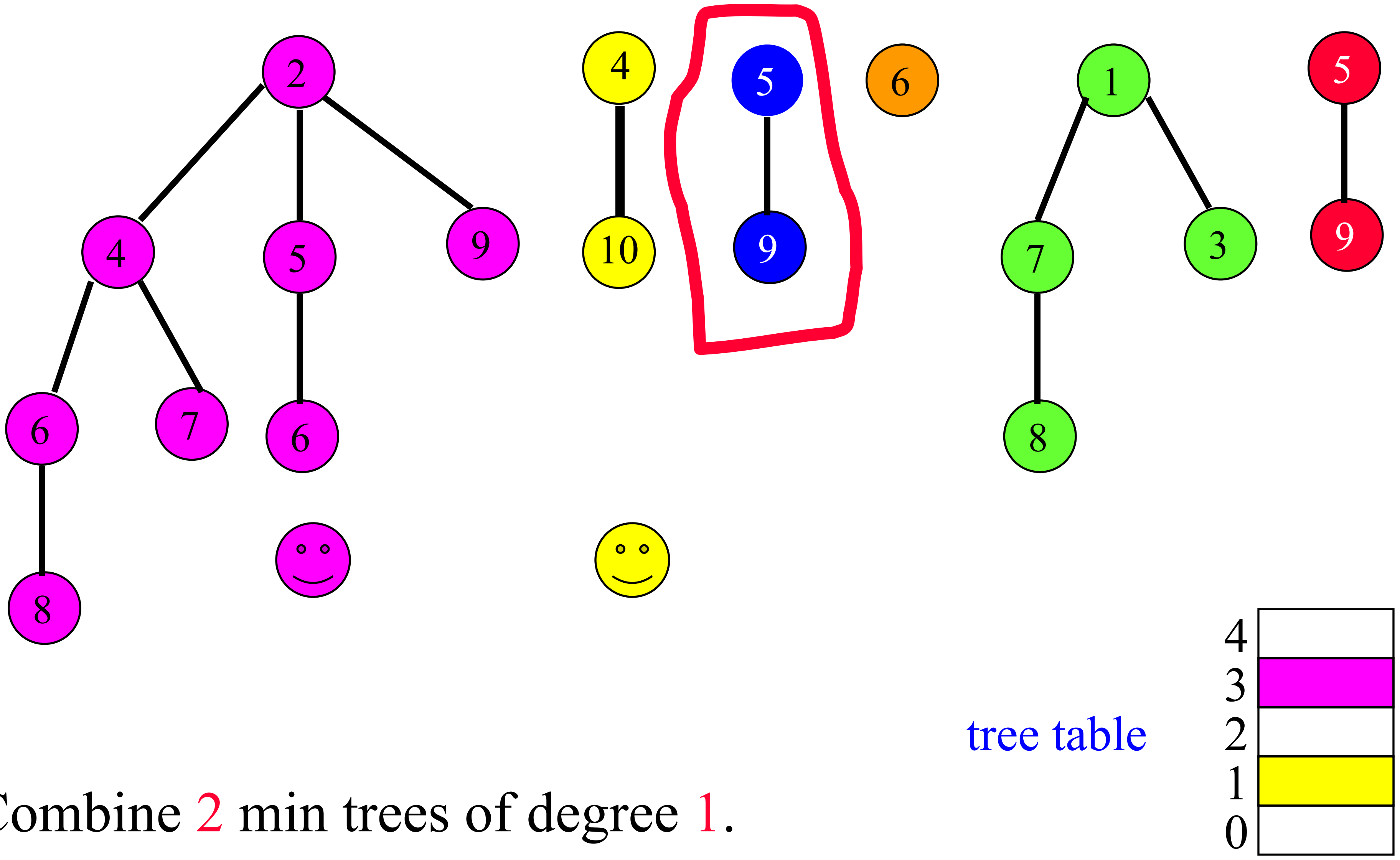
Make the one with larger root a subtree of other.

Pairwise Combine



Update tree table.

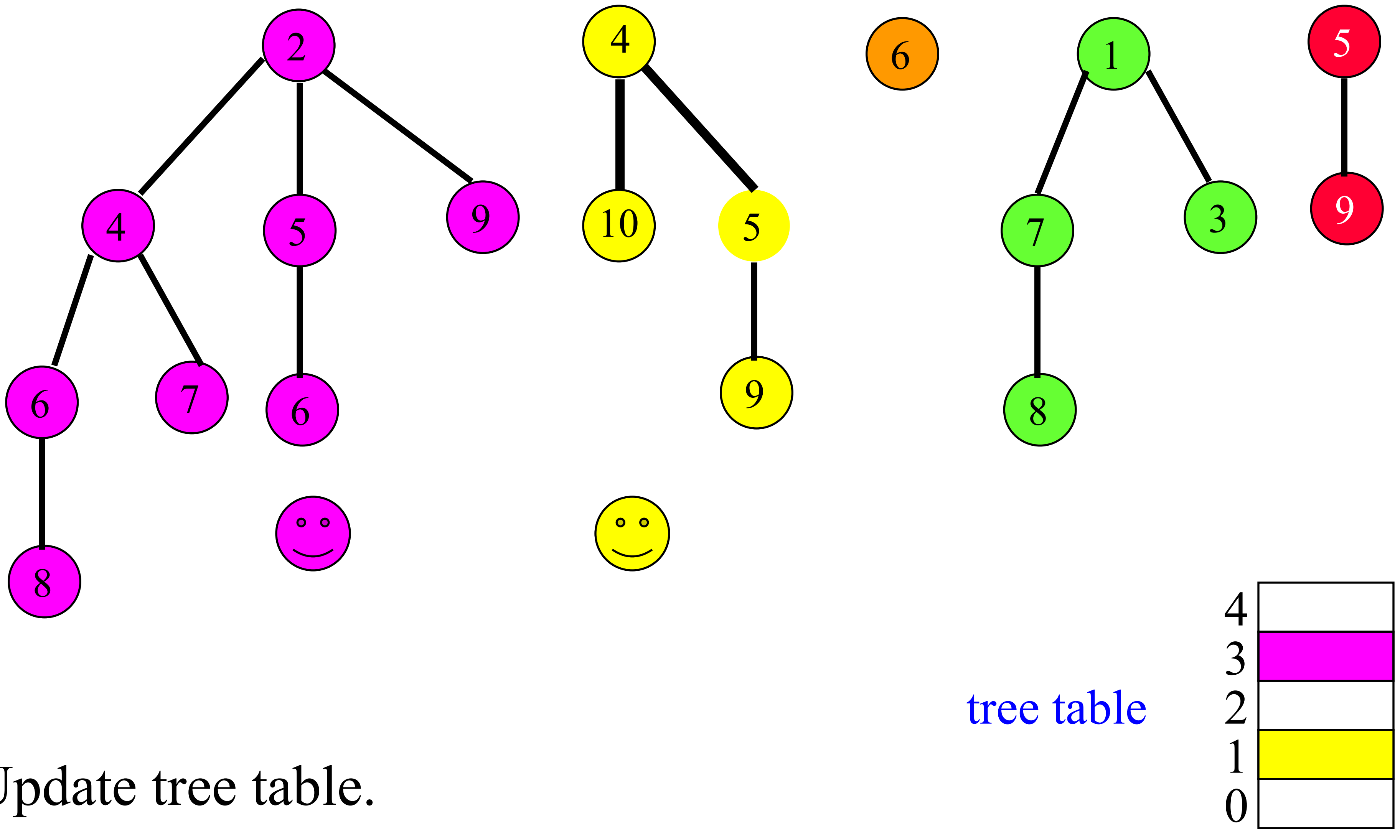
Pairwise Combine



Combine 2 min trees of degree 1.

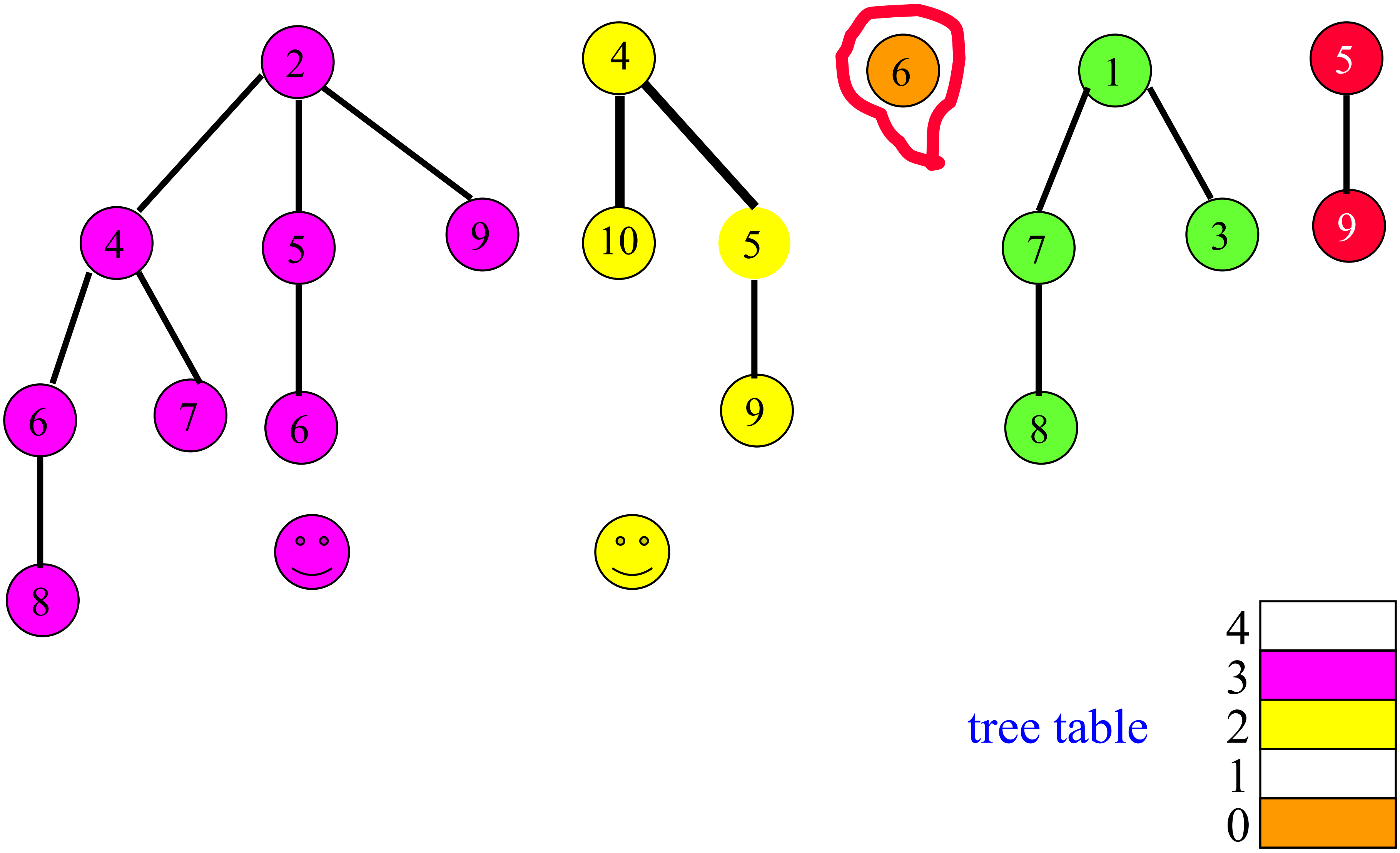
Make the one with larger root a subtree of other.

Pairwise Combine

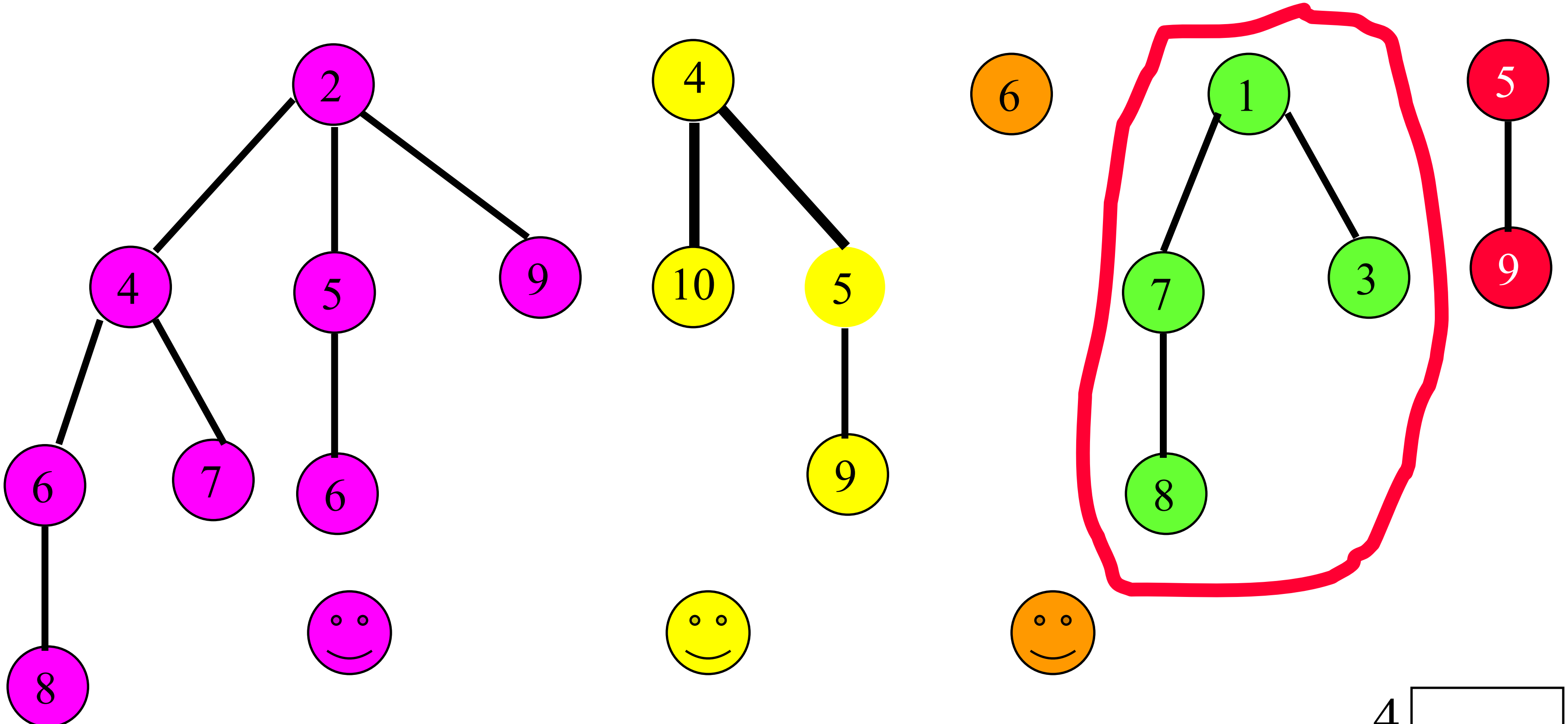


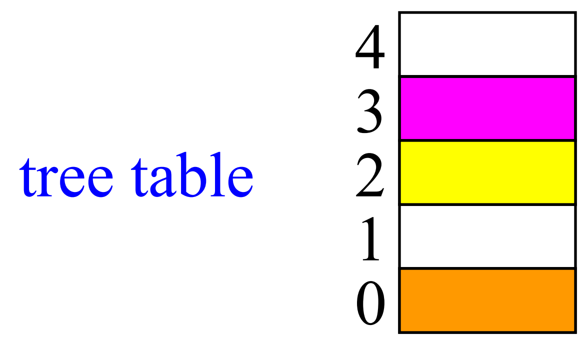
Update tree table.

Pairwise Combine



Pairwise Combine

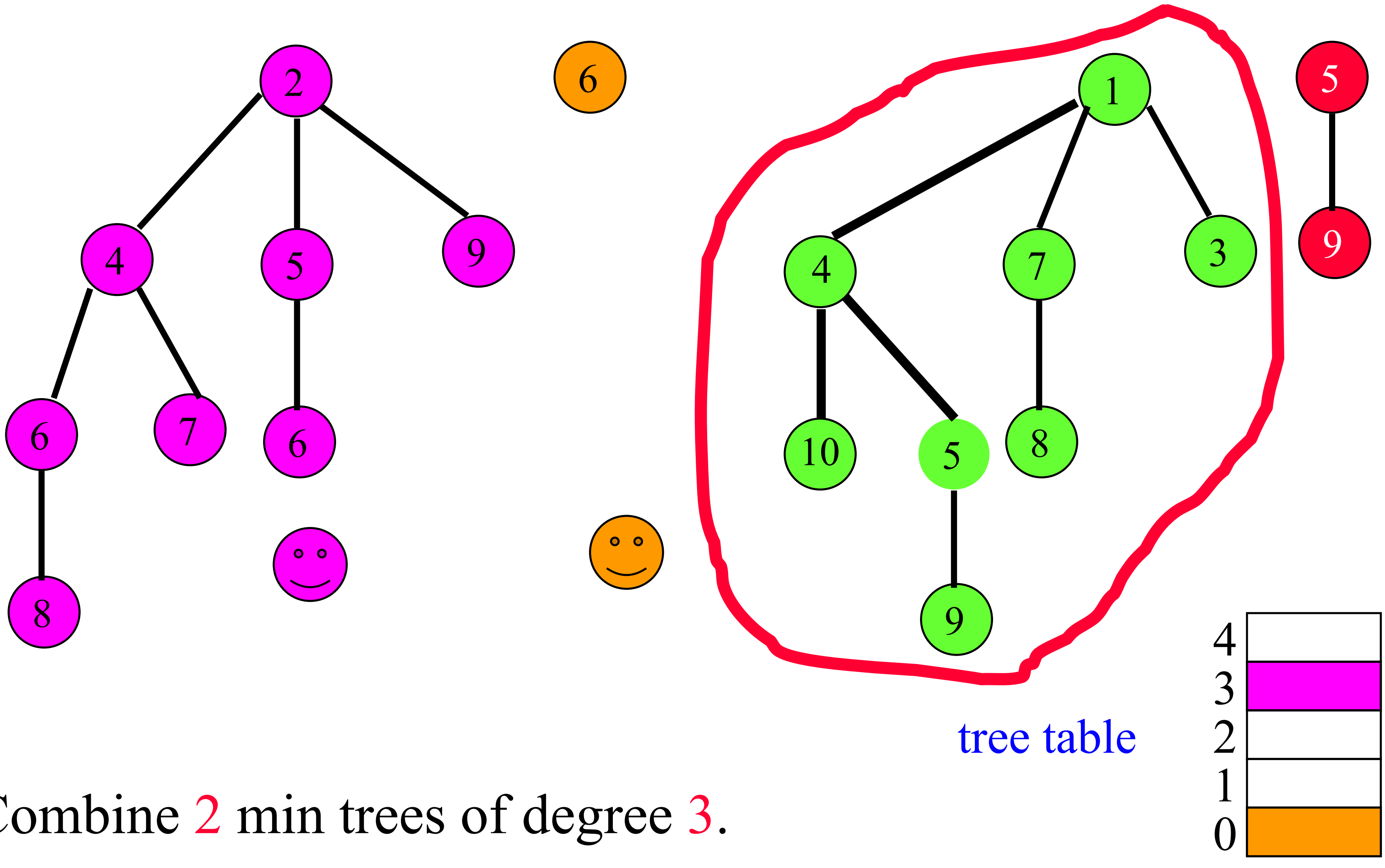




Combine 2 min trees of degree 2.

Make the one with larger root a subtree of other.

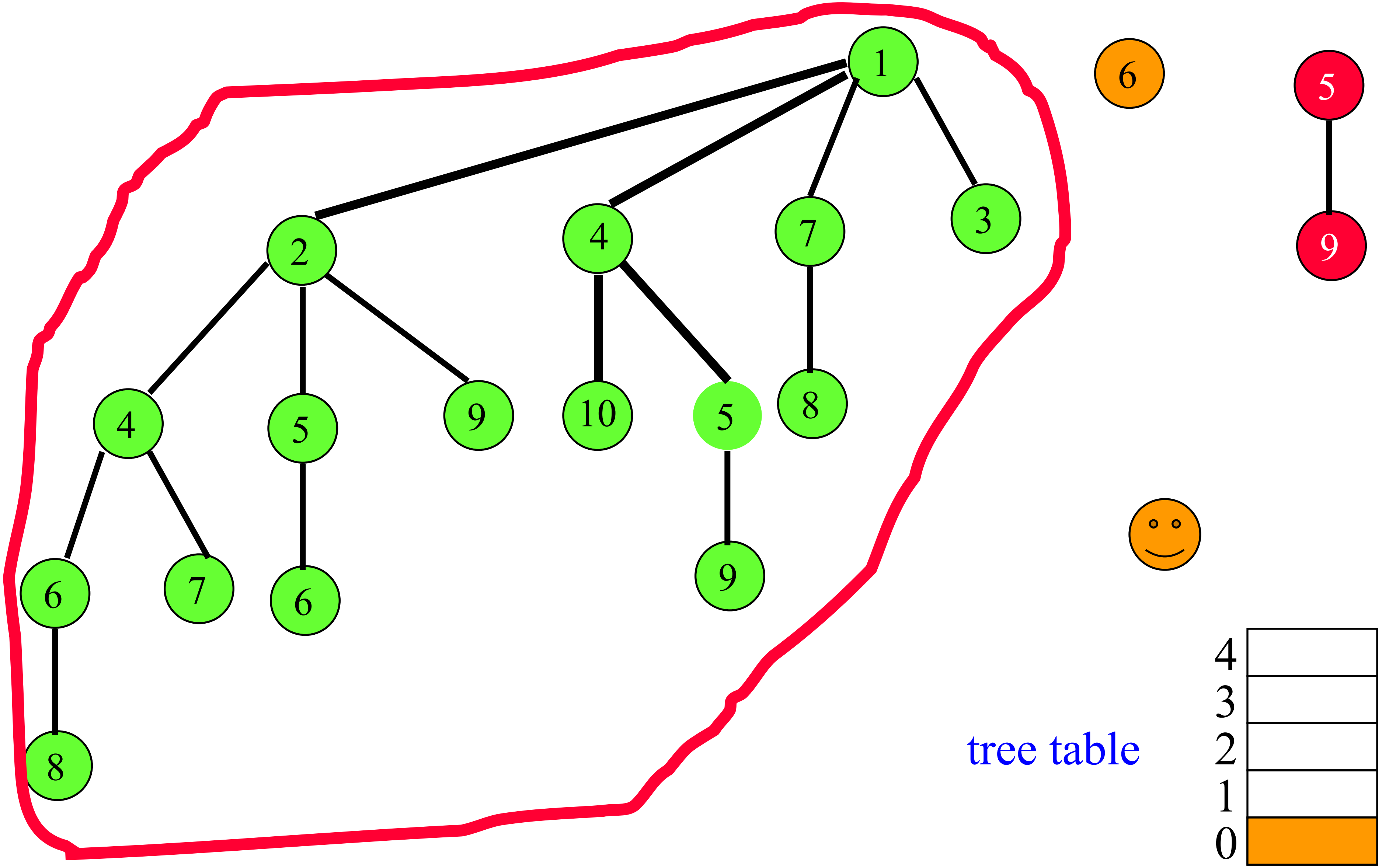
Pairwise Combine



Combine 2 min trees of degree 3.

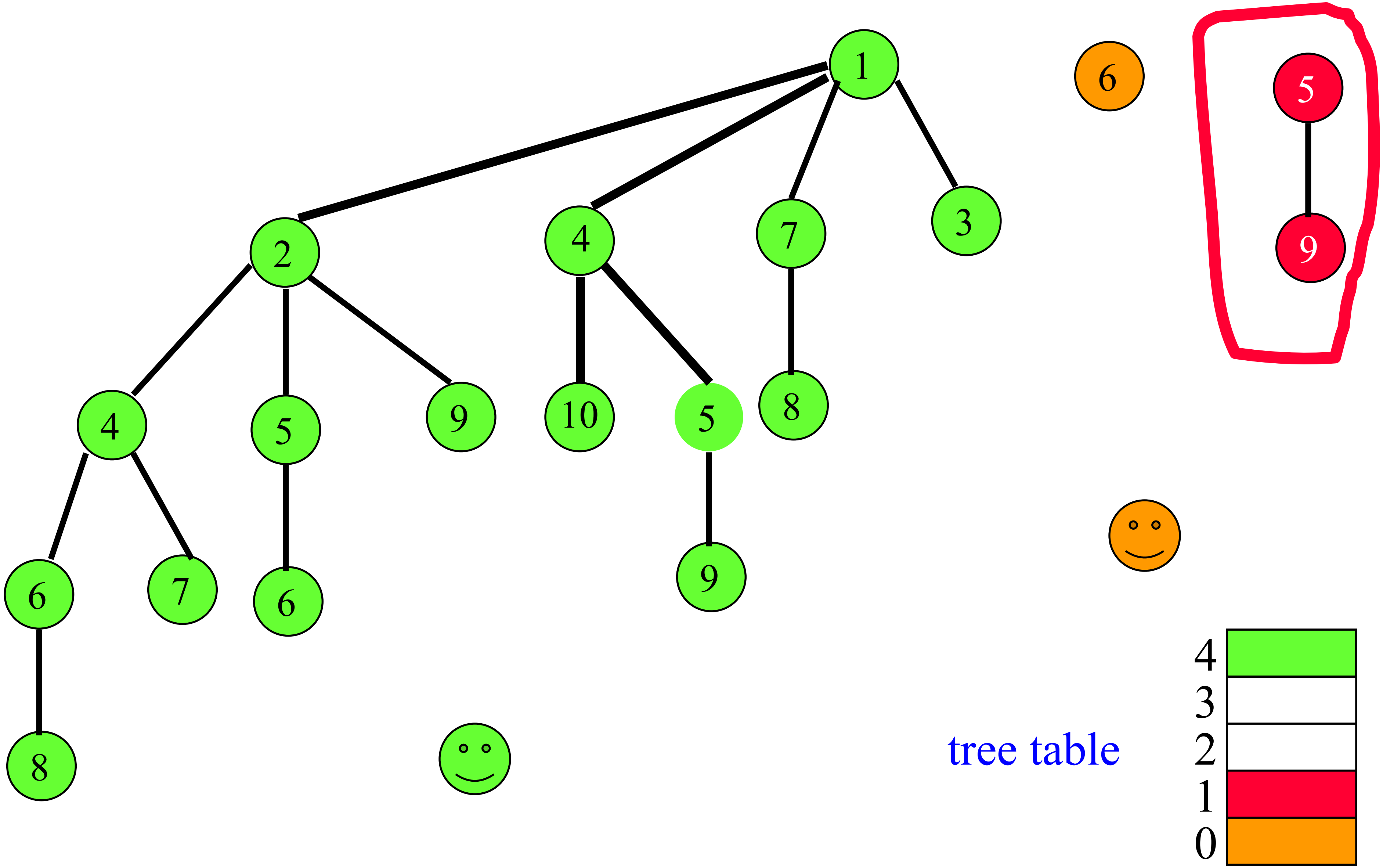
Make the one with larger root a subtree of other.

Pairwise Combine

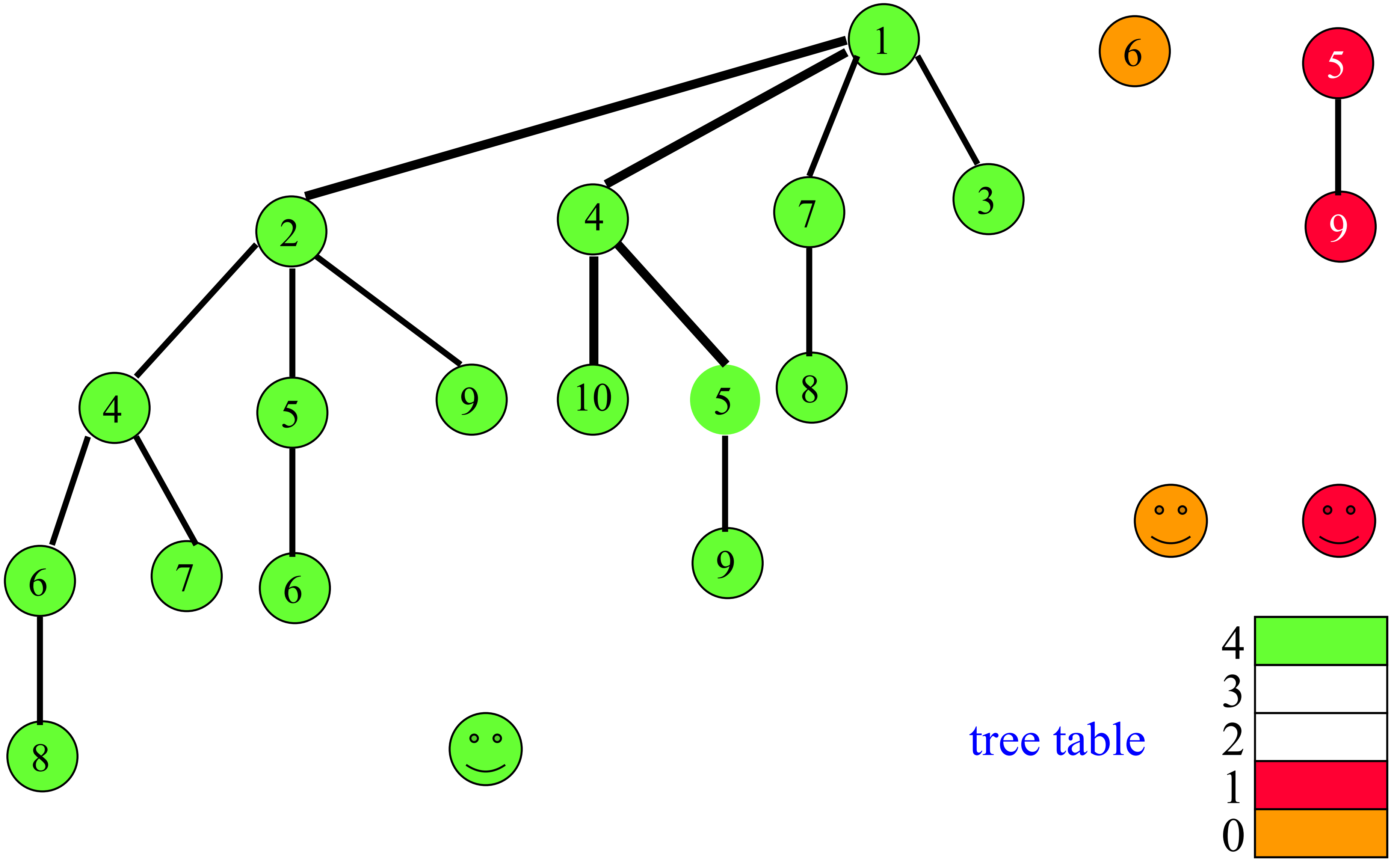


Update tree table.

Pairwise Combine



Pairwise Combine



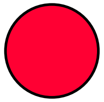
Create circular list of remaining trees.

Complexity Of Remove Min

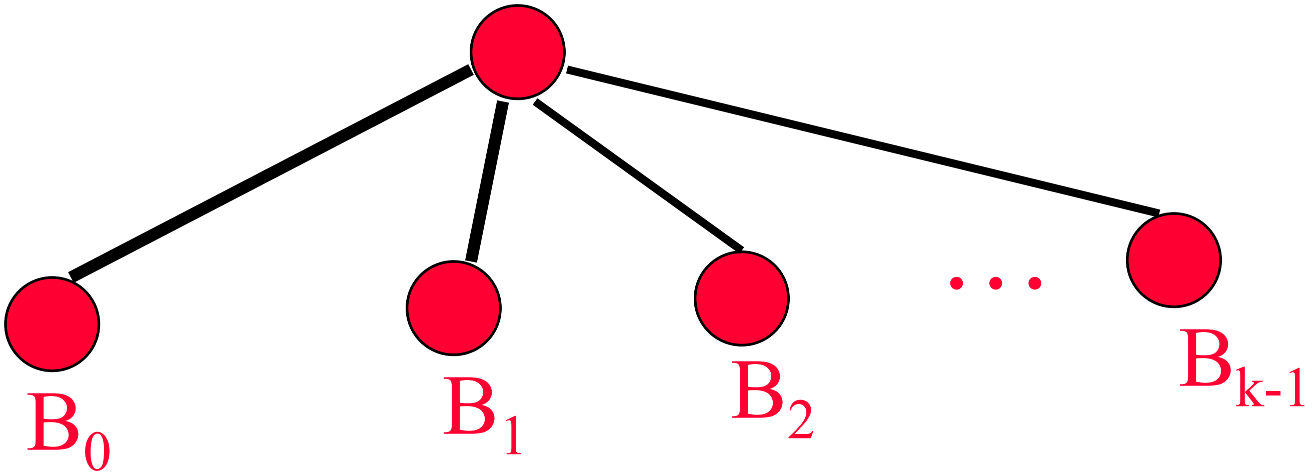
- Create and initialize tree table.
  - O(MaxDegree).
  - Done once only.
- Examine s min trees and pairwise combine.
  - O(s). (WHY?)
- Collect remaining trees from tree table, reset table entries to null, and set binomial heap pointer.
  - O(MaxDegree).
- Overall complexity of remove min.
  - O(MaxDegree + s).

Binomial Trees

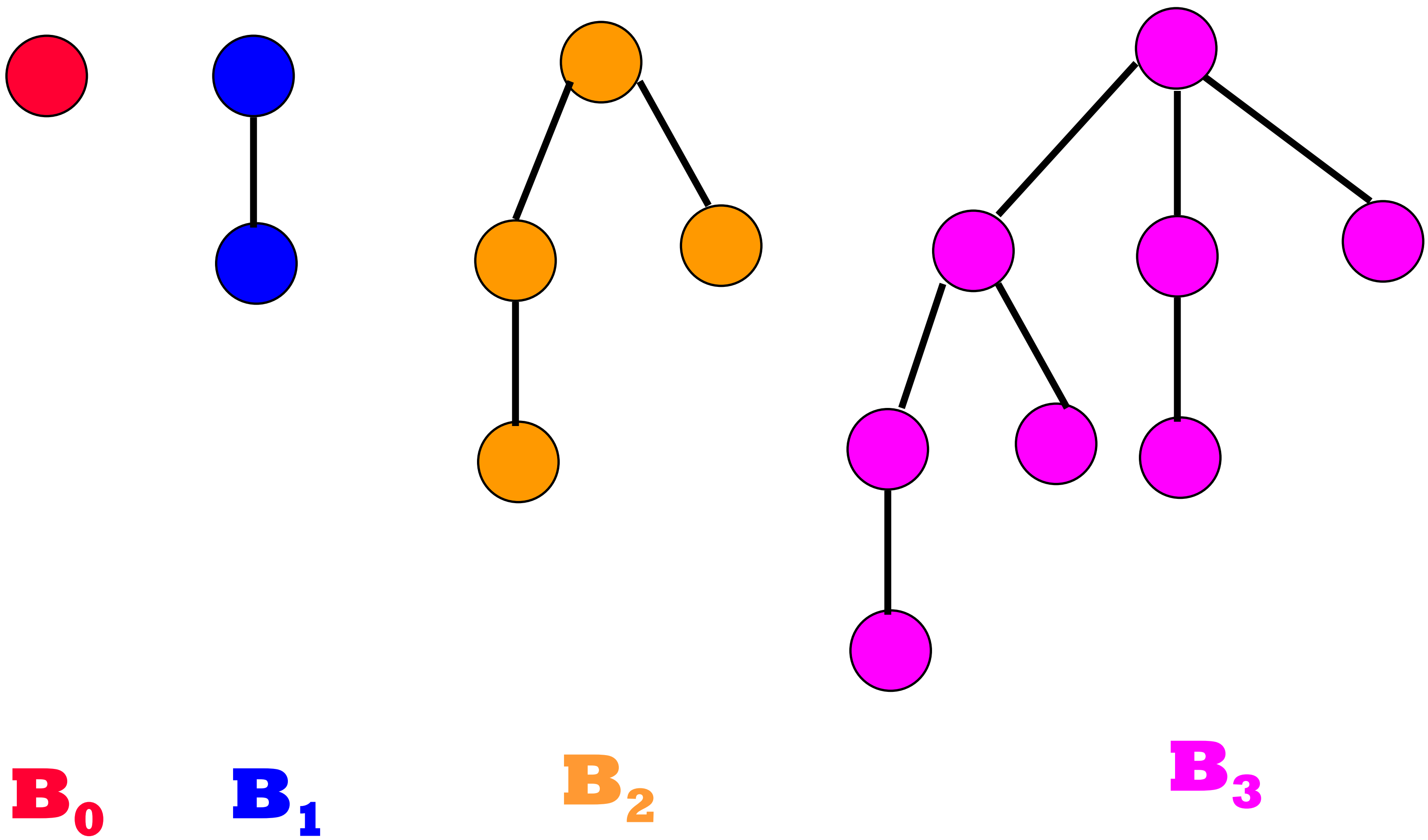
- B_k is degree k binomial tree.

B_0 

- B_k , $k > 0$, is:

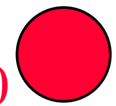


Examples

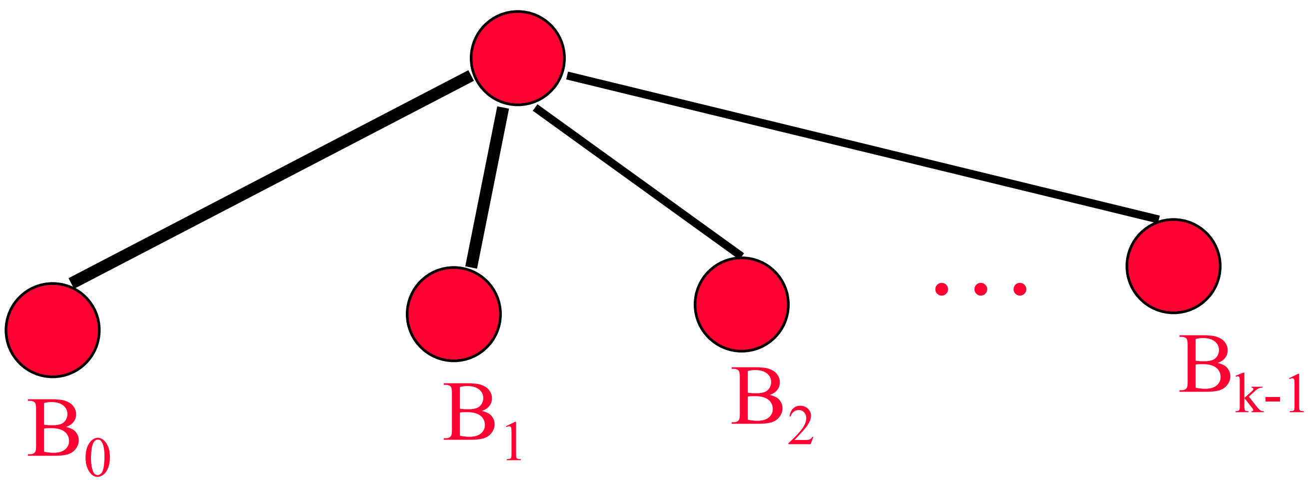


Number Of Nodes In B_k

- $N_k =$ number of nodes in B_k.

B_0  $N_0 = 1$

- B_k , $k > 0$, is:

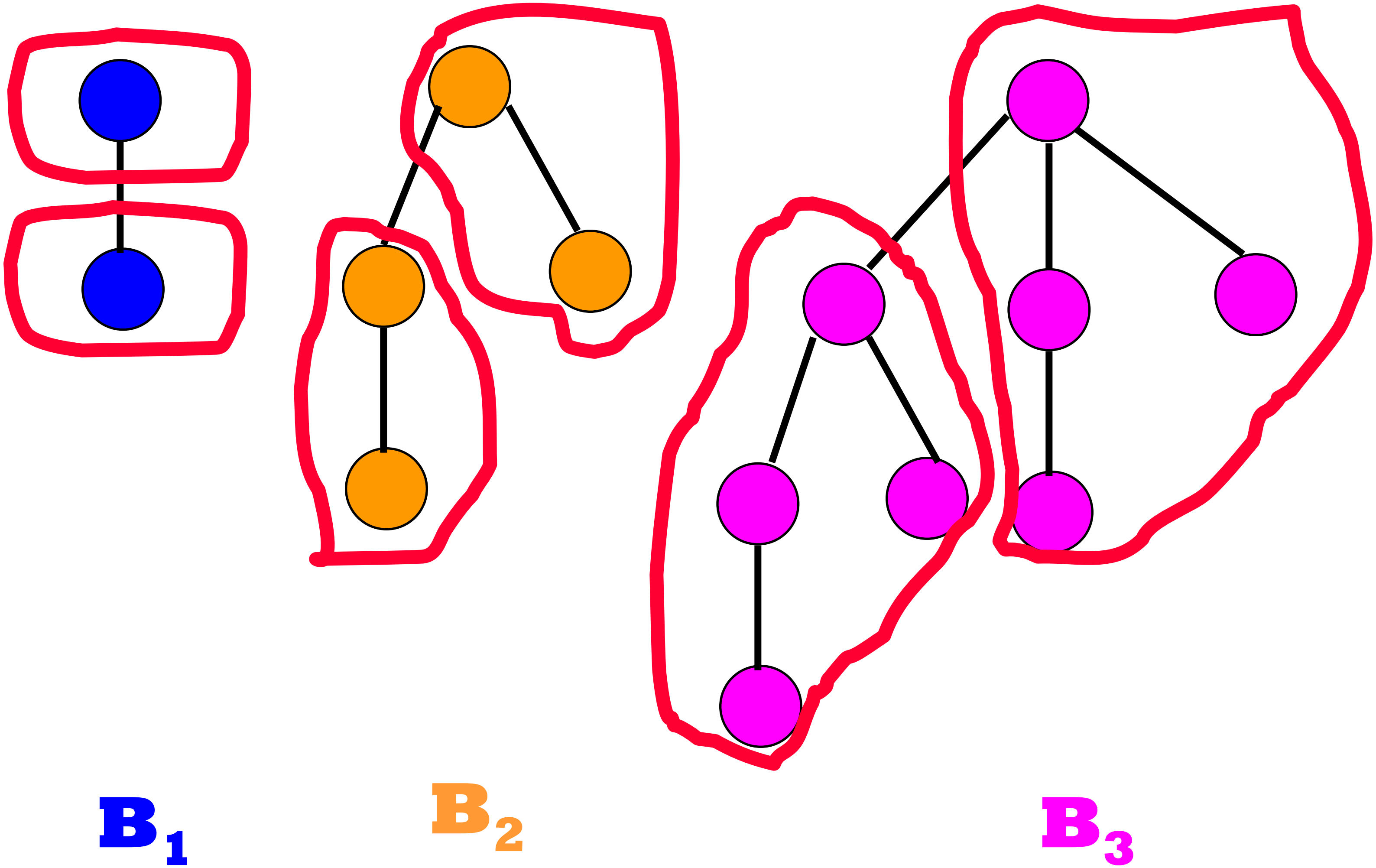


- $N_k = N_0 + N_1 + N_2 + \ldots + N_{k-1} + 1$

  $= 2^k$.

Equivalent Definition

- B_k , $k > 0$, is two B_{k-1}s.
- One of these is a subtree of the other.



N_k And MaxDegree

- $N_0 = 1$
- $N_k = 2N_{k-1}$
  $= 2^k$.
- If we start with zero elements and perform operations as described, then all trees in all binomial heaps are binomial trees.
- So, MaxDegree = O(log n).

Analysis Of Binomial Heaps

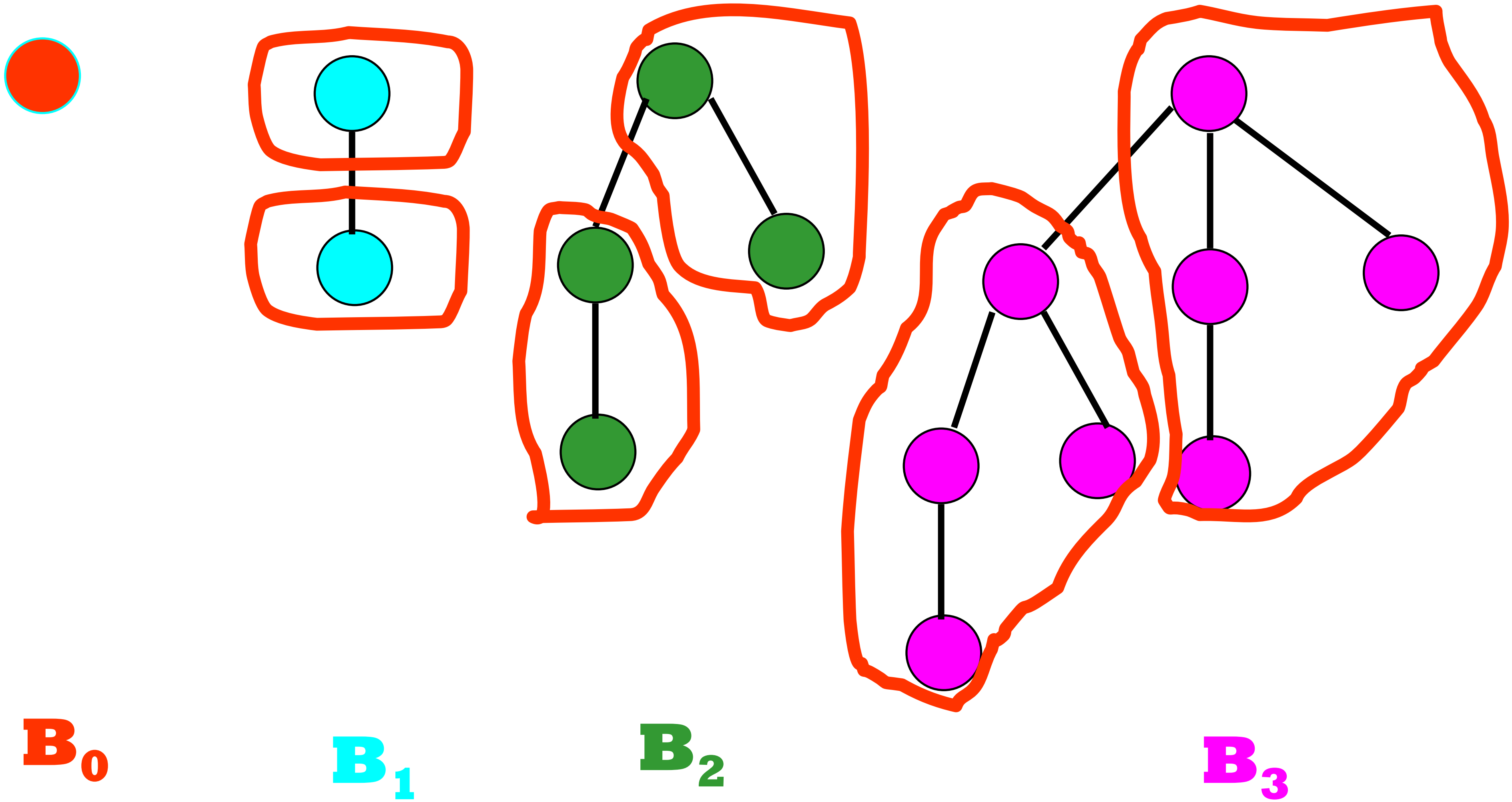
	Leftist trees	Binomial heaps	
		Actual	Amortized
Insert	O(log n)	O(1)	O(1)
Remove min (or max)	O(log n)	O(n)	O(log n)
Meld	O(log n)	O(1)	O(1)

Operations

- Insert
  - Add a new min tree to top-level circular list.
- Meld
  - Combine two circular lists.
- Remove min
  - Pairwise combine min trees whose roots have equal degree.
  - O(MaxDegree + s), where s is number of min trees following removal of min element but before pairwise combining.

Binomial Trees

- B_k, $k > 0$, is two B_{k-1}s.
- One of these is a subtree of the other.



All Trees In Binomial Heap Are Binomial Trees

- Initially, all trees in system are Binomial trees (actually, there are no trees initially).
- Assume true before an operation, show true after the operation.
- Insert creates a B_0.
- Meld does not create new trees.
- Remove Min
  - Reinserted subtrees are binomial trees.
  - Pairwise combine takes two trees of equal degree and makes one a subtree of the other.

Complexity of Remove Min

- Let n be the number of operations performed.
  - Number of inserts is at most n.
  - No binomial tree has more than n elements.
  - MaxDegree <= $\log_2 n$.
  - Complexity of remove min is $O(\log n + s) = O(n)$.

Aggregate Method

- Get a good bound on the cost of every sequence of operations and divide by the number of operations.
- Results in same amortized cost for each operation, regardless of operation type.
- Can’t use this method, because we want to show a different amortized cost for remove mins than for inserts and melds.

Aggregate Method – Alternative

- Get a good bound on the cost of every sequence of remove mins and divide by the number of remove mins.
- Consider the sequence insert, insert, …, insert, remove min.
  - The cost of the remove min is $O(n)$, where n is the number of operations in the sequence.
  - So, amortized cost of a remove min is $O(n/1) = O(n)$.

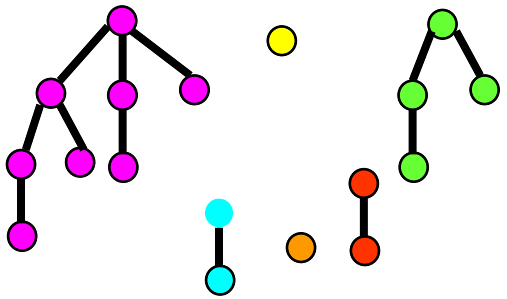
Accounting Method

- Guess the amortized cost.
  - Insert => 2.
  - Meld => 1.
  - Remove min => $3\log_2 n$.
- Show that $P(i) – P(0) >= 0$ for all i.

Potential Function

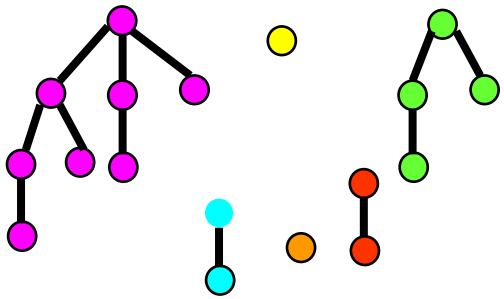
- P(i) = amortizedCost(i) – actualCost(i) + P(i – 1)
- P(i) – P(0) is the amount by which the first i operations have been over charged.
- We shall use a credit scheme to keep track of (some of) the over charge.
- There will be 1 credit on each min tree.
- Initially, #trees = 0 and so total credits and P(0) = 0.
- Since number of trees cannot be <0, the total credits is always >= 0 and hence P(i) >= 0 for all i.

Insert



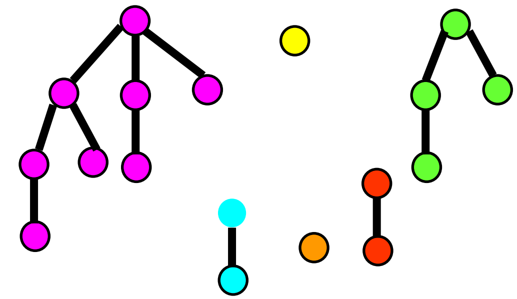
- Guessed amortized cost = 2.
- Use 1 unit to pay for the actual cost of the insert.
- Keep the remaining 1 unit as a credit.
- Keep this credit with the min tree that is created by the insert operation.
- Potential increases by 1, because there is an overcharge of 1.

Meld



- Guessed amortized cost = 1.
- Use 1 unit to pay for the actual cost of the meld.
- Potential is unchanged, because actual and amortized costs are the same.

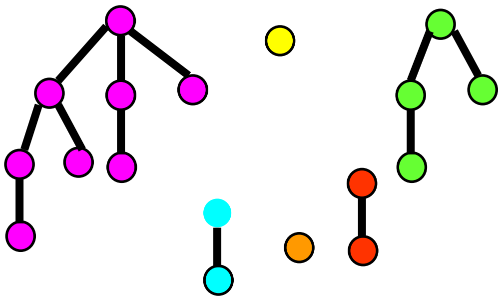
Remove Min



- Let MinTrees be the set of min trees in the binomial heap just before remove min.
- Let u be the degree of min tree whose root is removed.
- Let s be the number of min trees in binomial heap just before pairwise combining.
  - $s = \#MinTrees + u - 1$
- Actual cost of remove min is $\le$ MaxDegree + s

  $\le 2\log_2 n - 1 + \#MinTrees$.

Remove Min



- Guessed amortized cost $= 3\log_2 n$.
- Actual cost $<= 2\log_2 n - 1 + \#MinTrees$.
- Allocation of amortized cost.
  - Use up to $2\log_2 n - 1$ to pay part of actual cost.
  - Keep some or all of the remaining amortized cost as a credit.
  - Put 1 unit of credit on each of the at most $\log_2 n + 1$ min trees left behind by the remove min operation.
  - Discard the remainder (if any).

Paying Actual Cost Of A Remove Min

- Actual cost <= $2\log_2 n - 1 + \#MinTrees$
- How is it paid for?
  - $2\log_2 n - 1$ comes from amortized cost of this remove min operation.
  - #MinTrees comes from the min trees themselves, at the rate of 1 unit per min tree, using up their credits.
  - Potential may increase or decrease but remains nonnegative as each remaining tree has a credit.

Potential Method

- Guess a suitable potential function for which $P(i) - P(0) >= 0$ for all i.
- Derive amortized cost of ith operation using

  $\Delta P = P(i) - P(i-1)$

  $= \text{amortized cost} - \text{actual cost}$
- amortized cost = actual cost + ΔP

Potential Function

- $P(i) = \Sigma\#MinTrees(j)$
  - #MinTrees(j) is #MinTrees for binomial heap j.
  - When binomial heaps A and B are melded, A and B are no longer included in the sum.
- $P(0) = 0$
- $P(i) >= 0$ for all i.
- ith operation is an insert.
  - Actual cost of insert $= 1$
  - $\Delta P = P(i) - P(i - 1) = 1$
  - Amortized cost of insert $=$ actual cost $+ \Delta P$ $= 2$

ith Operation Is A Meld

- Actual cost of meld = 1
- $P(i) = \Sigma \#MinTrees(j)$
- $\Delta P = P(i) - P(i - 1) = 0$
- Amortized cost of meld = actual cost + ΔP

  = 1

ith Operation Is A Remove Min

- old => value just before the remove min
- new => value just after the remove min.
- $\#MinTrees^{old}(j)$ => value of #MinTrees in jth binomial heap just before this remove min.
- Assume remove min is done in kth binomial heap.

ith Operation Is A Remove Min

- Actual cost of remove min from binomial heap k

  $\le 2\log_2 n - 1 + \#MinTrees^{old}(k)$
- $\Delta P = P(i) - P(i-1)$

  $= \Sigma[\#MinTrees^{new}(j) - \#MinTrees^{old}(j)]$

  $= \#MinTrees^{new}(k) - \#MinTrees^{old}(k)$.
- Amortized cost of remove min $=$ actual cost $+ \Delta P$

  $\le 2\log_2 n - 1 + \#MinTrees^{new}(k)$

  $\le 3\log_2 n$.