Arrays

2.2 The Array as an Abstract Data Type

Array:

A set of pairs: <index, value> (correspondence or mapping)

Two operations: retrieve, store

Now we will use the C++ class to define an ADT.

GeneralArray

class GeneralArray **{**

// a set of pairs <index, value> where for each value of

// index in IndexSet there is a value of type **float**. IndexSet is

// a finite ordered set of one or more dimensions.

public:

GeneralArray(**int** j, RangeList list, **float** initValue = defaultValue)**;**

// This constructor creates a j dimensional array of floats;

// the range of the kth dimension is given by the kth element of list.

// For all $i \in$ IndexSet, insert <i, initValue> into the array.

float Retrieve(index i)**;**

// **if** (i$\in$IndexSet) **return** the float associated with i in the

// array;**else** throw an exception.

void Store(index i, **float** x)**;**

// **if** (i$\in$IndexSet) replace the old value associated with i

// by x**;** **else** throw an exception.

}; //end of GeneralArray

Note:

Not necessarily implemented using consecutive memory

Index can be coded any way

GeneralArray is more general than C++ array as it is more flexible about the composition of the index set

To be simple, we will hereafter use the C++ array

2.3The polynomial abstract data type

Array can be used to implement other abstract data types. The simplest one might be:

Ordered or linear list.

Example:

(Sun, Mon, Tue, Wed, Thu, Fri, Sat)

(2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A)

( )   // empty list

More generally, An ordered list is either empty or $(a_0, a_1, .., a_{n-1})$. // index important

Main operations:

(1) Find the length, n, of the list.

(2) Read the list from left to right (or right to left)

(3) Retrieve the ith element, $0 \le i < n$.

(4) Store a new value into the ith position, $0 \le i < n$.

(5) Insert a new element at position i, $0 \le i < n$, causing elements numbered i, i+1,…n-1 to become numbered i+1, i+2,…n.

(6) Delete the element at position i, $0 \le i < n$, causing elements numbered i+1, i+2,…n-1 to become numbered i, i+1,…n-2.

How to represent ordered list efficiently?

- **Sequential mapping**
  - **Use array: $a_i \leftrightarrow$ index i**
- **Complexity**
  - **Random access any element in**
    - **O(1).**
  - **Operations (5) and (6)?**
    - **Data movement**
    - **O(n)**

Now let us look at a problem requiring ordered list.

Problem:

Build an ADT for the representation and manipulation of symbolic polynomials in a single variable (say x).

$A(x)=3x^2+ 2x+4$

$B(x)=x^4+ 10x^3+ 3x^2+1$

Degree: the largest exponent

ADT Polynomial

class Polynomial **{**

// $p(x)=a_0x^{e0}+,\ldots,+ a_nx^{en}$; a set of ordered pairs of $<e_i, a_i>$,

// where a_i is a nonzero **float** coefficient and e_i is a

// non-negative exponent

public:

Polynomial ( )**;**

// Construct the polynomial $p(x)=0$

void AddTerm (Exponent e, Coefficient c)**;**

// add the term <e,c> to ***this**, so that it can be initialized

Polynomial Add (Polynomial poly)**;**

// return the sum of the polynomials ***this** and poly

Polynomial Mult (Polynomial poly)**;**

// return the product of the polynomials ***this** and poly

float Eval (**float** f)**;**

// evaluate polynomial ***this** at f and return the result

}

Polynomial Representation

Let a be $A(x)=a_n x^n + a_{n-1} x^{n-1} + ,\ldots, + a_1 x + a_0$

Representation 1

private:

```
int degree;  // degree ≤ MaxDegree
float coef[MaxDegree+1];
```

a.degree= ?

n;

a.coef[i] = ?

a_{n-i}, $0 \le i \le n$

Simple algorithms for many operations.

Representation 2

When a.degree << MaxDegree, representation 1 is very poor in memory use. To improve, define variable sized data member as:

```
private:
    int degree;
    float *coef;
Polynomial::Polynomial(int d)
 {
    int degree=d;
    coef= new float[degree+1];
 }
```

Representation 2 is still not desirable.

For instance, $x^{1000}+1$

makes 999 entries of the coef be zero.

So, we store only the none zero terms:

Representation 3

$$A(x) = b_m x^{em} + b_{m-1} x^{em-1} +,\ldots,+ b_0 x^{e0}$$

Where $b_i \neq 0, \quad e_m > e_{m-1} >,\ldots, e_0 \geq 0$

```
class Polynomial; // forward declaration
class Term {
friend Polynomial;
private:
    float coef; // coefficient
    int exp;    // exponent
};

class Polynomial {
public:
  ……
private:
  Term *termArray;
  int capacity; // size of termArray
  int terms; // number of nonzero terms
}
```

For $A(x) = 2x^{1000} + 1$

A.termArray looks like:

coef	2	1		
exp	1000	0		

Many zero --- good

Few zero --- ?

not very good

may use twice as much space as in presentation 2.

Polynomial Addition

Use presentation 3 to obtain C = A +B.

Idea:

Because the exponents are in descending order, we can adds A(x) and B(x) term by term to produce C(x).

The terms of C are entered into its termArray by calling function NewTerm.

If the space in termArray is not enough, its capacity is doubled.

```
Polynomial  Polynomial::Add (Polynomial b)
{   // return the sum of the polynomials *this and b.
  Polynomial c;
  int aPos=0, bPos=0;
  while (( aPos < terms) && (b < b.terms))
     if (termArray[aPos].exp==b.termArray[bPos].exp) {
        float t = termArray[aPos].coef + termArray[bPos].coef
        if ( t )
            c.NewTerm (t, termArray[aPos].exp);
        aPos++;  bPos++;
     }
     else if (termArray[aPos].exp < b.termArray[bPos].exp) {
            c.NewTerm (b.termArray[bPos].coef,
                            b.termArray[bPos].exp);
        bPos++;
     }
```

```
    else {
        c.NewTerm (termArray[aPos].coef, termArray[aPos].exp);
        aPos++;
    }
  // add in the remaining terms of *this
  for ( ; aPos < terms; aPos++ )
      c.NewTerm(termArray[aPos].coef, termArray[aPos].exp );
  // add in the remaining terms of b
  for ( ; bPos < b.terms; bPos++ )
  c.NewTerm(b.termArray[bPos].coef, b.termArray[bPos].exp);
  return c;
}
```

```
void Polynomial::NewTerm(const float theCoeff,
                          const int theExp)
{ // add a new term to the end of termArray.
   if (terms == capacity)
   { // double capacity of termArray
      capacity *= 2;
      term *temp = new term[capacity]; // new array
      copy(termArray, termAarry + terms, temp);
      delete [ ] termArray;  // deallocate old memory
      termArray = temp;
   }
  termArray[terms].coef = theCoeff;
  termArray[terms++].exp = theExp;
}
```

Analysis of Add:

Let m, n be the number of nonzero terms in a and b respectively.

- **line 3 and 4---O(1)**
- **in each iteration of the while loop, aPos or bPos or both increase by 1, while loop terminates when either aPos, equals a.terms or b.Pos equals b.terms, the number of iterations of this loop ≤ m+n-1**
- **if ignore the time for doubling the capacity, each iteration takes O(1)**
- **line 20--- O(m), line 23--- O(n)**

Asymptotic computing time for Add: O(m+n+time spent in array doubling)

Analysis of doubling capacity:

- **the time for doubling is linear in the size of new array**
- **initially, c.capacity is 1**
- **suppose when Add terminates, c.capacity is 2^k**
- **the total time spent over all array doubling is**

$$O\left(\sum_{i=1}^{k} 2^i \right) = O(2^{k+1}) = O(2^k)$$

- **since c.terms > 2^{k-1} and m + n ≥ c.terms, the total time for array doubling is**

$$O(c.terms) = O(m + n)$$

- **so, even consider array doubling, the total run time of Add is O(m + n).**
- **experiments show that array doubling is responsible for very small fraction of the total run time of Add.**

Exercises: P93-2,6, P94-9

Sparse Matrices

Introduction

A general matrix consists of m rows and n columns (m × n)of numbers, as:

$$
\begin{array}{c|ccc|}
 & 0 & 1 & 2 \\
\hline
0 & -27 & 3 & 4 \\
1 & 6 & 82 & -2 \\
2 & 109 & -64 & 11 \\
3 & 12 & 8 & 9 \\
4 & 48 & 27 & 47
\end{array}
$$

Fig.2.2(a) 5×3

$$
\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 15 & 0 & 0 & 22 & 0 & -15 \\
1 & 0 & 11 & 3 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & -6 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 91 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 28 & 0 & 0 & 0
\end{array}
$$

Fig. 2.2(b) 6×6

A matrix of m × m is called a square.

A matrix with many zero entries is called sparse.

Representation:

- **A natural way ---**
  - **a[m][n]**
  - **access element by a[i][j], easy operations. But**
  - **for sparse matrix, wasteful of both memory and time.**
- **Alternative way ---**
  - **store nonzero elements explicitly. 0 as default.**

SparseMatrix

```
class SparseMatrix
 { // a set of <row, column, value>, where row, column are
  // non-negative integers and form a unique combination;
  //  value is also an integer.
 public:
     SparseMatrix ( int r, int c, int t);
     // creates a r×c SparseMatrix with a capacity of t nonzero
     // terms
     SparseMatrix Transpose ( );
     // return the SparseMatrix obtained by transposing *this
     SparseMatrix  Add ( SparseMatrix b);
     SparseMatrix  Multiply ( SparseMatrix b);
 };
```

Sparse Matrix Representation

Use triple <row, col, value>, sorted in ascending order by <row, col>.

```
class  SparseMatrix;
class  MatrixTerm {
friend class SparseMatrix;
Private:
    int row, col, value;
};
```

We need also

the number of rows

the number of columns

the number of nonzero elements

And in class SparseMatrix:

private:

```
Int rows, cols, terms, capacity;
MatrixTerm *smArray;
```

Now we can store the matrix of Fig.2.2 (b) as Fig.2.3 (a).

smArray	row	col	value
[0]	0	0	15
[1]	0	3	22
[2]	0	5	-15
[3]	1	1	11
[4]	1	2	3
[5]	2	3	-6
[6]	4	0	91
[7]	5	2	28

Fig.2.3 (a)

Transposing a Matrix

Transpose:

If an element is at position [i][j] in the original matrix, then it is at position [j][i] in the transposed matrix.

Fig.2.3(b) shows the transpose of Fig2.3(a).

First try:

For (each row i)

✓**store element (i, j, value) of the original matrix**

✓**as (j, i, value) of the transpose;**

Difficulty:

NOT knowing where to put (j, i, value) until all other elements preceding it have been processed.

smArray	row	col	value
✓[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

Improvement:

For (all elements in col j)

✓store (i, j, value) of the original matrix

✓as (j, i, value) of the transpose;

Since the rows are in order, we will locate elements in the correct column order.

smArray	row	col	value
✓[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

```
SparseMatrix SparseMatrix::Transpose ( )
{ // return the transpose of *this
    SparseMatrix b(cols, rows, terms);
    if (terms > 0)
    {   //nonzero matrix
      int currentB = 0;
```

```
    for ( int c=0; c<cols; c++ ) // transpose by columns
      for ( int i=0; i<terms; i++ )
      // find and move terms in column c
        if ( smArray[i].col == c )
        {
          b.smArray[CurrentB].row = c;
          b.smArray[CurrentB].col = smArray[i].row;
          b.smArray[CurrentB++].value= smArray[i].value;
        }
   } // end of if (terms > 0)
  return b;
}
```

Time complexity of Transpose:

- **line 7-15 loop--- cols times**
- **line 10 loop--- terms times**
- **other line--- O(1)**

Total time: O(cols* terms)

Additional space: O(1)

Think:

O(cols* terms) is not always good. If terms = O(cols* rows) then it becomes O($cols^2$* rows)---too bad!

Since with 2-dimensional representation, we can get an easy O(cols* rows) algorithm as:

```
for (int j=0;j < columns;j++)
    for (int i=0; i < rows; i++)  B[j][i] = A[i][j];
```

Further improvement:

If we use some more space to store *some knowledge* about the matrix, we can do much better: doing it in O(cols + terms).

- **get the number of elements in each column of**

  ***this (= the number of elements in each row of B);**
- **obtain the starting point of each row of B;**
- **move the elements of *this one by one into their right position in B.**

Now the algorithm FastTranspose.

```
SparseMatrix SparseMatrix::FastTranspose ( )
{ // return the transpose of *this in O(terms+cols) time.
   SparseMatrix b(cols, rows, terms);
   if (terms > 0)
   { // nonzero matrix
      int *rowSize = new int[cols];
      int *rowStart = new int[cols];
      // compute rowSize[i] = number of terms in row i of b (col i of *this)
      fill(rowSize, rowSize + cols, 0);  // initialze
      for (i=0; i<terms; i++ )  rowSize[smArray[i].col]++;
```

```
// rowStart[i] = starting position of row i in b
rowStart[0] = 0;
for (i=1;i<cols;i++) rowStart[i]=rowStart[i-1]+rowSize[i-1];
for (i=0; i<terms; i++)
{     // copy from *this to b
      int j = rowStart[smArray[i].col];
      b.smArray[j].row = smArray[i].col;
      b.smArray[j].col = smArray[i].row;
      b.smArray[j].value = smArray[i].value;
      rowStart[smArray[i].col]++;
 }   // end of for
```

```
      delete [ ] rowSize;
      delete [ ] rowStart;
   }  // end of if
   return b;
}
```

smArray	row	col	value
[0]	0	0	15
[1]	0	3	22
[2]	0	5	-15
[3]	1	1	11
[4]	1	2	3
[5]	2	3	-6
[6]	4	0	91
[7]	5	2	28

smArray	row	col	value
✓[0]	0	0	15
[1]	0	4	91
[2]	1	1	11
[3]	2	1	3
[4]	2	5	28
[5]	3	0	22
[6]	3	2	-6
[7]	5	0	-15

After line 13, we get :

	[0]	[1]	[2]	[3]	[4]	[5]
RowSize=	**2**	**1**	**2**	**2**	**0**	**1**
RowStart=	**0**	**2**	**3**	**5**	**7**	**7**

Note the error in P101 of the text book!

Analysis:

3 loops:

- **line 10--- O(terms)**
- **line 13--- O(cols)**
- **line 14 – 21--- O(terms)**

and line 9--- O(cols), other lines--- O(1)

Total: O(cols+terms)

This is a typical example for trading space for time.

Exercises: P107-1, 2, 4

The String Abstract Data Type

A string S = $s_0, s_1, \ldots, s_{n-1}$,

where $s_i \in$ char, $0 \le i < n$, n is the length.

ADT 2.5 String

class String

{ // a finite set of zero or more characters**;**

public:

String (**char** *init, **int** m)**;**

// initialize ***this** to string init of length m

bool operator == (String t)**;**

// if ***this** equals t, return **true** else **false**.

bool operator ! ( )**;**

// if ***this** is empty return **true** else **false**.

int Length ( )**;**

// return the number of chars in ***this**

String Concat (String t)**;**

String Substr (**int** i, **int** j)**;**

int Find (String pat)**;**

// return an index i such that pat matches the substring of ***this**

// that begins at position i. Return –1 if pat is either empty or not

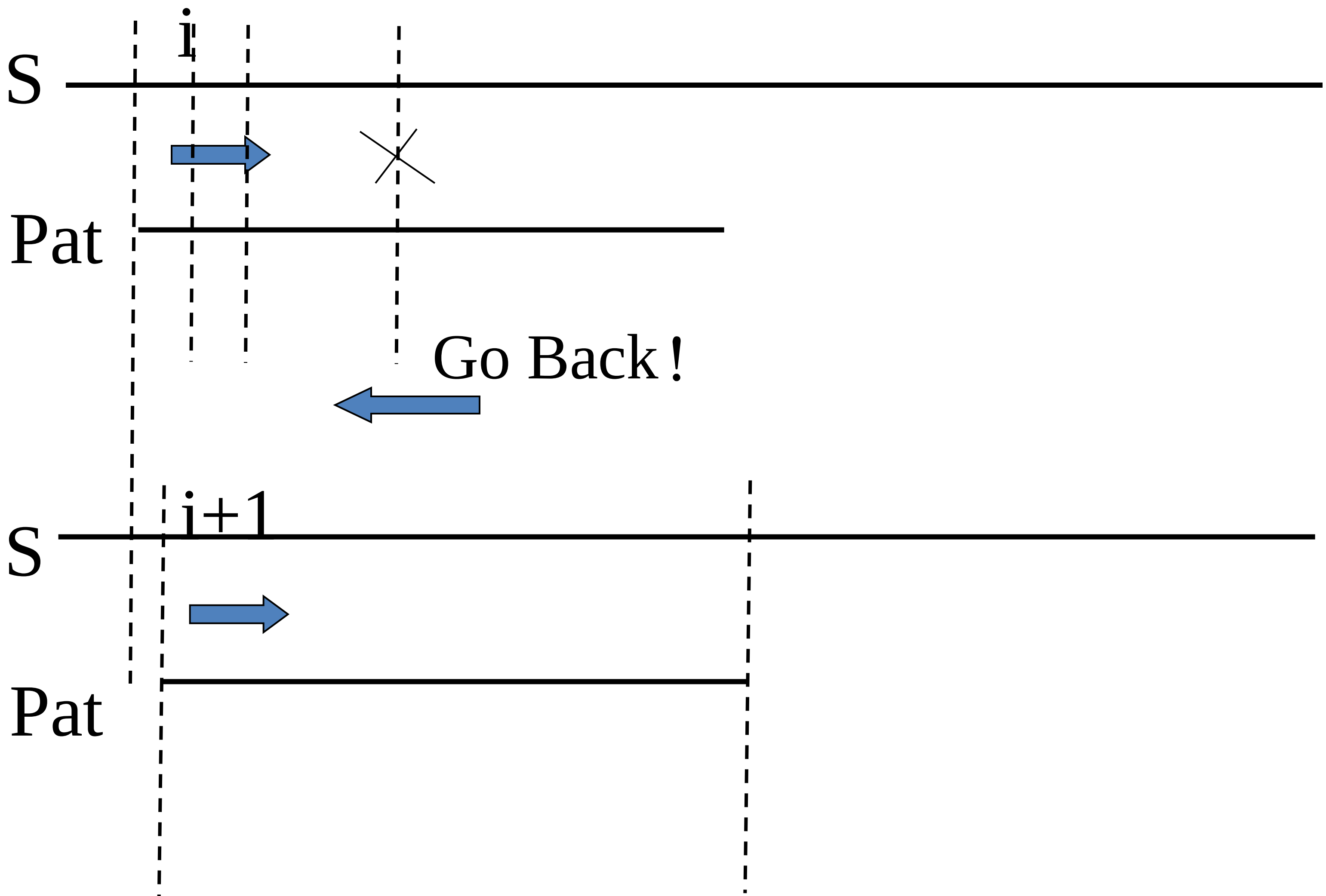
// a substring of ***this**.

};

Assume strings are represented by:

```
private:
    char* str;
```

String Pattern Matching: A Simple Algorithm



```
int String::Find ( String pat )
{ // Return –1 if pat does not occur in *this; otherwise
  // return the first position in *this, where  pat begins.
     if (pat.Length( ) == 0) return -1; // pat is empty
     for (int start=0; start<=Length( ) - pat.Length( ); start++)
     { //  check  for match beginning at str[start]
     for (int j=0; j<pat.Length( )&&str[start+j]==pat.str[j];j++)
             if (j== pat.Length( )) return start; // match found
        // no match at position start
      }
      return -1;     // pat does not occur in s
}
```

The complexity of it is O(LengthP * LengthS).

Problem:

rescanning.

String Pattern Matching: The Knuth-Morris-Pratt Algorithm

Can we get an algorithm which *avoid rescanning* the strings and works in O(LengthP + LengthS)?

This is optimal for this problem, as in the worst it is necessary to look at characters in the pattern and string at least once.

Basic Ideas:

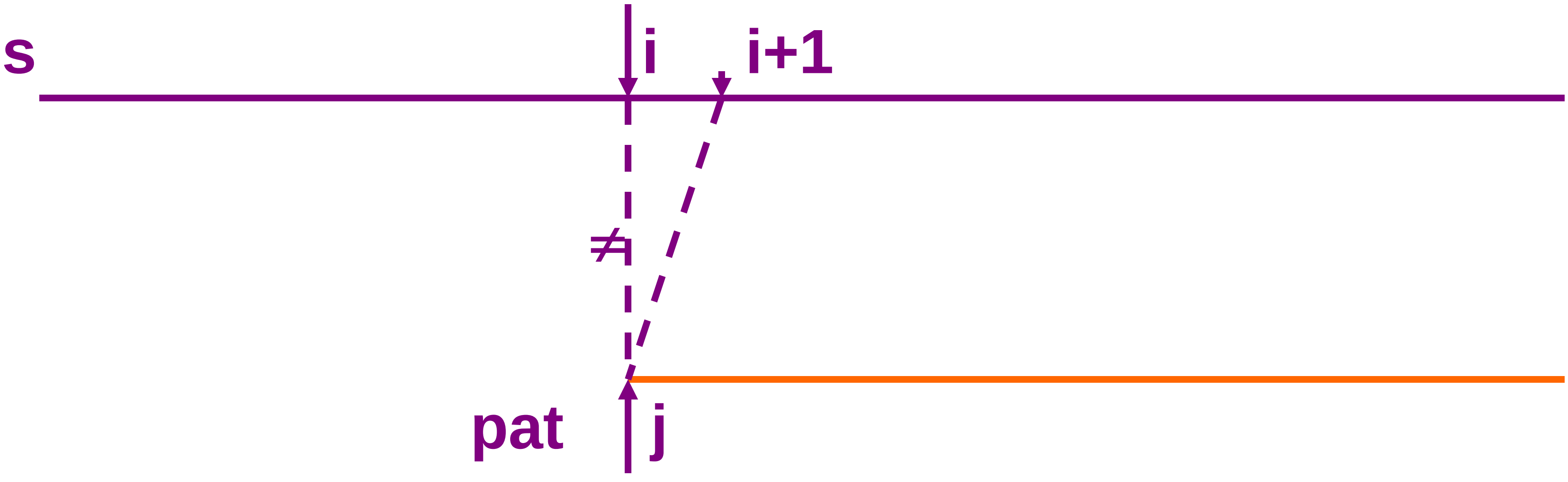
Rescanning to avoid missing the target ---

too conservative

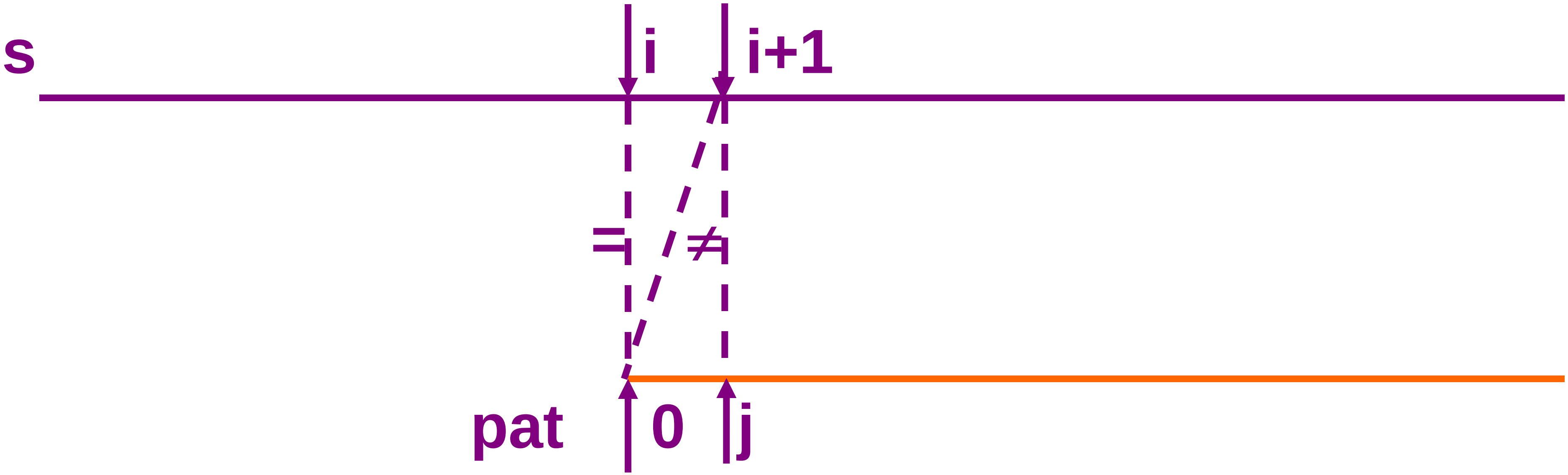
If we can go without rescanning, it is likely to do the job in O(LengthP + LengthS).

Preprocess the pattern, to get some knowledge of the characters in it and the position in it, so that if a mismatch occurs we can determine where to continue the search and avoid moving backwards in the string.

Now we show details about the idea.



case: j = 0



case: j =1

An concrete example:

i

↓

s = . . . a b d a b ? . . . .

≠

pat = a b d a b c a c a b

↑

j=5

An concrete example:

i

↓

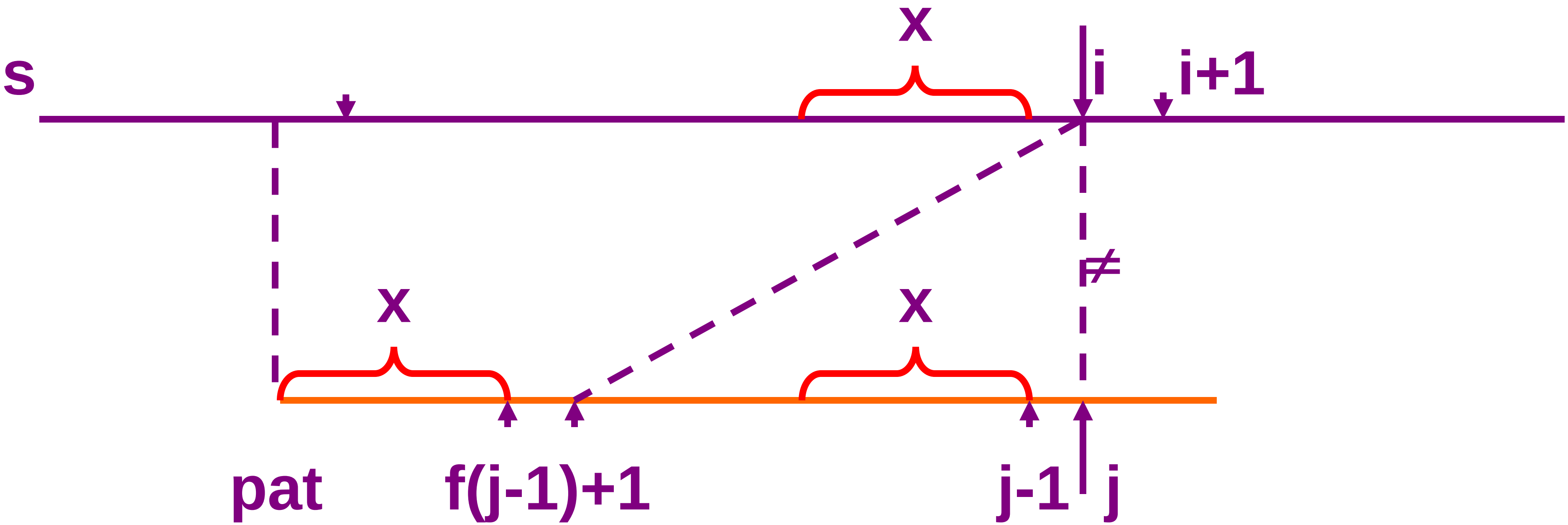
s = . . . a b d a b ? . . . .

≠

pat = a b d a b c a c a b

↑

j=5



case: $j \neq 0$

To formalize the above idea:

Definition: if $p=p_0p_1\ldots p_{n-1}$ is a pattern, then its failure function f, is defined as:

$$f(j) = \begin{cases} \text{largest } k < j \text{, such that } p_0p_1\ldots p_k = p_{j-k}p_{j-k+1}\ldots p_j & \text{if such } k \geq 0 \text{ exists} \\ -1 & \text{otherwise} \end{cases}$$

For example, pat = a b c a b c a c a b, we have

j	0	1	2	3	4	5	6	7	8	9
pat	a	b	c	a	b	c	a	c	a	b
f	-1	-1	-1	0	1	2	3	-1	0	1

Note:

- largest : jump farthest but no match be missed
- k < j : avoid dead loop

From the definition of f, we have the following rule for pattern matching:

If a partial match is found such that $s_{i-j}\ldots s_{i-1} = p_0p_1\ldots p_{j-1}$ and $s_i \neq p_j$ then matching may be resumed by comparing s_i and $p_{f(j-1)+1}$ if $j \neq 0$.

If $j=0$, then we may continue by comparing s_{i+1} and p_0.

The failure function is represented by an array of integers f, which is a private data member of String.

Now the algorithm FastFind.

```
int String::FastFind (String pat)
{ // Determine if pat is a substring of s
   int PosP = 0,  PosS = 0;
   int LengthP= pat.Length( ), LengthS= Length( );
   while ((PosP  < LengthP) && (PosS  < LengthS))
      if ( pat.str[PosP] == str[PosS] ) { // characters match
      PosP ++; PosS ++;
      }
      else
      if ( PosP==0)   // characters mismatch at position 0 in the pattern
          PosS++;
        else PosP= pat.f [PosP-1] + 1;
   if ((PosP < LengthP) || LengthP==0)) return –1;
   else return PosS  - LengthP ;
}
```

Analysis of FastFind:

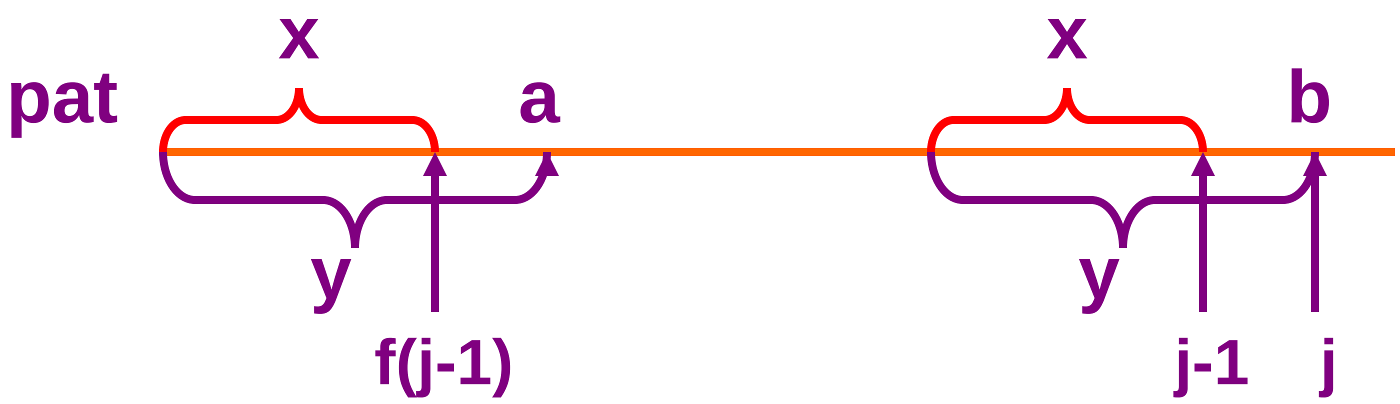
Line 7 and 11 --- at most LengthS times, since PosS is increased but never decreased. So PosP can move right on pat at most LengthS times (line 7).

Line 12 moves PosP left, it can be executed at most LengthS times. Note that $0 \leq f(posP-1)+1 < posP$.

Consequently, the computing time is O(LengthS).

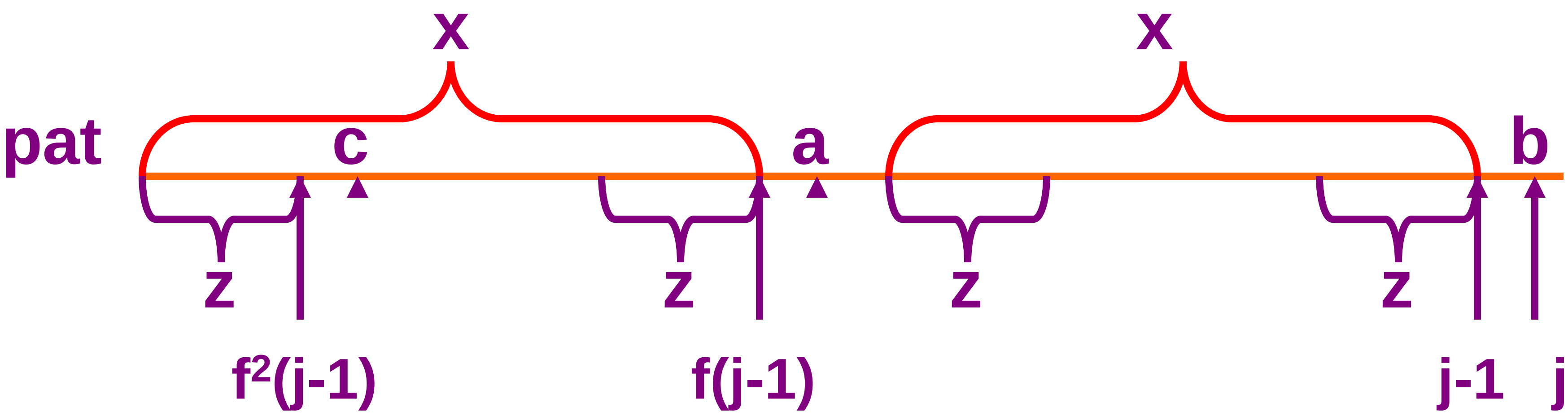
How about the computing of the f for the pattern? By similar idea, we can do it in O(LengthP).

f(0)=-1, now if we have f(j-1), we can compute f(j) from it by the following observation:



If a=b, then f(j)=f(j-1)+1

Else:



If c=b, Then $f(j)=f(f(j-1))+1=f^2(j-1)+1$

Else ……

In general, we have the following restatement of the failure function:

$$
f(j) = \begin{cases} -1 & \text{if } j=0 \\ f^m(j-1)+1 & \text{where } m \text{ is the least } k \text{ for which} \\ & p_{f^k(j-1)+1} = p_j \\ -1 & \text{if there is no } k \text{ satisfying the above} \end{cases}
$$

Now we get the algorithm to compute f.

```
void String::Failurefunction( )
{ // compute the failure function of the pattern *this.
   int LengthP= Length( );
   f [0]= -1;
   for (int j=1; j< LengthP; j++) // compute  f[j]
  {
       int i=f [j-1];
       while ((*(str+j)!=*(str+i+1)) && (i>=0)) i=f[i]; // try for m
        if ( *(str+j)==*(str+i+1))
            f[j]=i+1;
       else f[j]= -1;
   }
}
```

Analysis of FailureFunction:

In each iteration of the while i decreases (line 8, and $f(j)<j$)

i is reset (line 7) to –1 (when the previous iteration went through line 11), or to a value 1 greater than its value on the previous iteration (when through line 10). **i is always not less than -1.**

There are only LengthP –1 executions of line 7, the value of i has a total increment of at most LengthP –1.

i cannot be decremented more than LengthP –1 times, the while loop is iterated at most LengthP –1 times over the whole algorithm.

Consequently, the computing time of FailureFunction is O(LengthP).

Now we can see, when the failure function is not known in advance, pattern matching can be carried out in time O(LengthP + LengthS) by first computing the failure function and then performing a pattern match using the FastFind.

Exercises: P118-1, P119-7, 9

Experiment 1: P123-8