Chapter 6
Graphs

6.1 The Graph Abstract Data Type

6.1.1 introduction

Graphs have been widely used in:

- **analysis of electrical circuits**
- **finding shortest routes**
- **project planning**

- identification of chemical compounds
- statistical mechanics
- genetics
- cybernetics
- linguistics
- social science
- …

6.1.2 Definitions

- graph G = (V, E)
- vertices V(G) $\neq \varnothing$
- edges E(G)
- undirected graph: (u, v) = (v, u)
- directed graph: <u, v>, u---tail, v---head,

<u,v> $\neq$ <v,u>

Three graphs:

G_1:

$V(G_1)=\{0,1,2,3\}$

$E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

G_2:

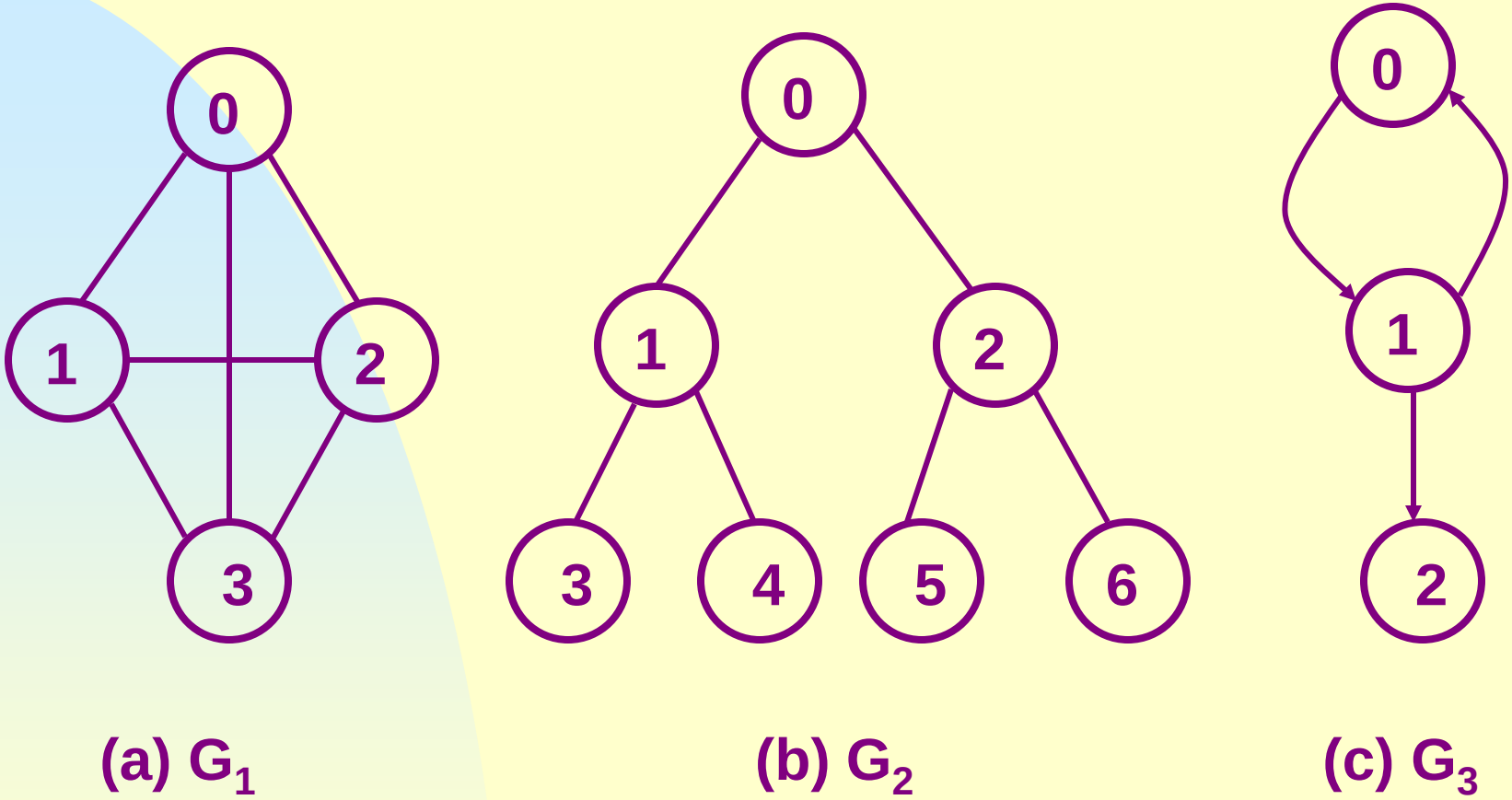
$V(G_2)=\{0,1,2,3,4,5,6\}$

$E(G_2)=\{(0,1),(0,2),,(1,3),(1,4),(2,5),(2,6)\}$

G_3:

$V(G_3) = \{0,1,2\}$

$E(G_3) = \{<0,1>,<1,0>,<1,2>\}$ (directed)



(a) G_1 (b) G_2 (c) G_3

Restrictions:

(1) (v, v) or <v, v> is not legal, such edges are known as self edges.

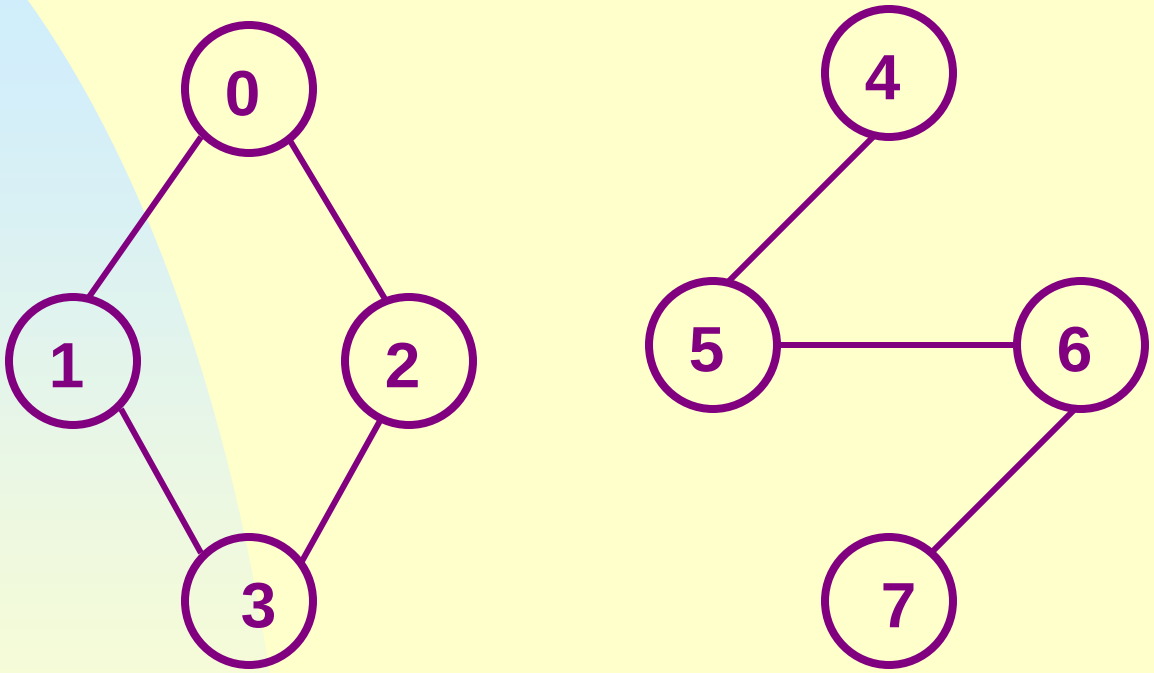
(2) multiple occurrences of the same edges are not allowed. If allowed, we get a multigraph.

- **The maximum number of edges in any n-vertex, undirected graph is n(n-1)/2, and in directed graph is n(n-1).**
- **An n-vertex undirected graph with n(n-1)/2 edges is said to be complete.**

- If $(u, v) \in E(G)$, we say u and v are **adjacent** and edge (u, v) is **incident** on vertices u and v. If <u, v> is a directed edge, then vertex u is **adjacent to** v, and v is **adjacent from** u, <u, v> is **incident to** u and v.
- A **subgraph** of G is a graph G` such that $V(G`) \subseteq V(G)$ and $E(G`) \subseteq E(G)$.
- A **path** from u to v in G is a sequence of vertices $u, i_1, i_2, \ldots, i_k, v$ such that $(u, i_1), (i_1, i_2), \ldots, (i_k, v)$ are edges in E(G). If G` is directed, then $<u, i_1>, <i_1, i_2>, \ldots, <i_k, v>$ are edges in E(G`).
- The **length** of a path is the number of edges on it.

- A **simple path** is a path in which all vertices except possibly the first and last are distinct.
- A **cycle** is a simple path in which the first and last vertices are the same.
- For directed graph, we have directed paths and cycles.
- In an undirected G, u and v are **connected** iff there is a path in G from u to v (also from v to u).
- An undirected G is **connected** iff for every pair of distinct u and v in V(G), there is a path from u to v.

- **A connected component is a maximal connected subgraph.**



G_4 A graph with two connected components

- A **tree** is a connected acyclic graph.
- A directed G is **strongly connected** iff for every pair of distinct u and v in V(G), there is a directed path from u to v and also from v to u.
- A **strongly connected component** is a maximal subgraph that is strongly connected.
- The **degree** of a vertex is the number of edges incident to it.

- For directed G, the **in-degree** of $v \in V(G)$ is the number of edges for which v is the head. The **out-degree** is the number of edges for which v is the tail.
- If d_i is the degree of vertex i in G with n vertices and e edges, then

$$e = \left(\sum_{i=0}^{n-1} di\right)/2$$

- We'll refer to a directed graph as **digraph**, and a undirected graph as **graph**.

ADT 6.1 Graph

```
class Graph
{ // A non empty set of vertices and a set of undirected
  // edges, where each edge is a pair of vertices.
public:
    virtual ~Graph(){ };
      // virtual destructor
    bool IsEmpty() const {return n==0;};
      // return true iff graph has no vertices
    int NumberOfVertices() const {return n;};
      // return the number of vertices in the graph
    int NumberofEdges() const {return e;};
      // return number of edges in the graph
    virtual int Degree(int u) const =0;
      // return number of edges incident to vertex u
```

```
    virtual bool ExisteEdge(int u, int v) const =0;
      // return true iff graph has edge (u, v)
    virtual void InsertVertex (int v) =0;
      // insert vertex v into graph, v has no incident edges
    virtual void InsertEdge (int u, int v) =0;
      // insert edge (u, v) into graph
    virtual void DeleteVertex (int v);
      // delete v and all edges incident to it
    virtual void DeleteEdge (int u, int v) =0;
      // delete edge (u, v) from the graph
private:
     int n;    // number of vertices
     int e;    // number of edges
};
```

6.1.3 Graph Representations

Three most commonly used representations:

(1) Adjacency matrices

(2) Adjacency lists

(3) Adjacency multilists

The actual choice depends on application.

6.1.3.1 Adjacency Matrix

The adjacency matrix of G is a n×n array, say a, such that:

$$a[i,j] = \begin{cases} 1 & \text{iff } (i, j) \in E(G) \text{ (or } \langle i, j\rangle \in E(G) \text{)} \\ 0 & \text{otherwise} \end{cases}$$

The next slide show adjacency matrices of G_1 , G_3 and G_4.

	0	1	2	3
0	0	1	1	1
1	1	0	1	1
2	1	1	0	1
3	1	1	1	0

(a) G_1

	0	1	2
0	0	1	0
1	1	0	1
2	0	0	0

(b) G_3

$$
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

G_4

For an graph, a is symmetric, and

$$d_i = \sum_{j=0}^{n-1} a[i][j]$$

For a digraph, a may not be symmetric, and

$$\text{out-}d_i = \sum_{j=0}^{n-1} a[i][j]$$

$$\text{in-}d_j = \sum_{i=0}^{n-1} a[i][j]$$

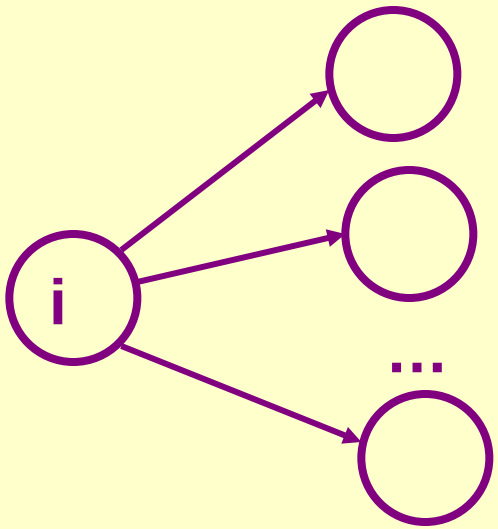
Problem: how many edges are there in G?

Using a, we need $O(n^2)$. When $e << n^2/2$, and if we explicitly represent only edges in G, then we can solve the above problem in $O(e + n)$.

This lead to:

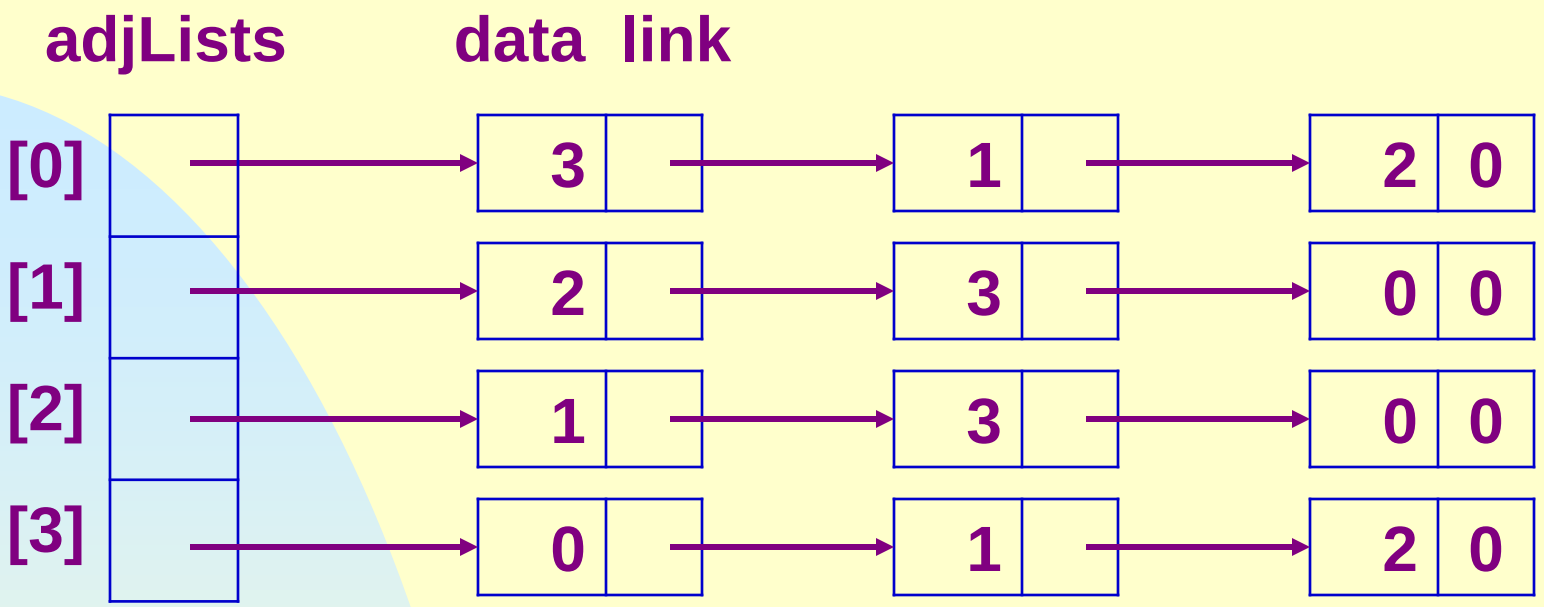
6.1.3.2 Adjacency Lists

- The n rows of the adjacency matrix are represented as n chains.
- The nodes in chain i represent the vertices adjacent from i.

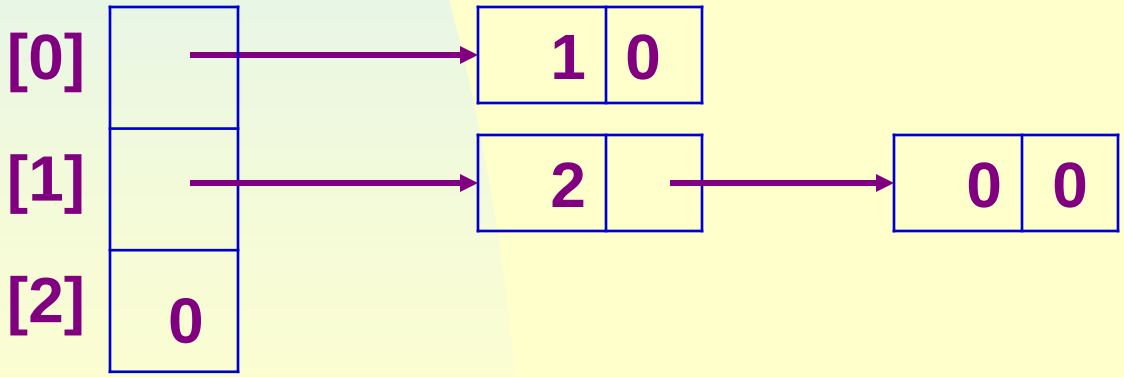


- **The data field of a chain node store the index of an adjacent vertex.**
- **The vertices in each chain are not required to be ordered.**
- **An array adjLists is used for accessing any chain in O(1).**

```
class LinkedGraph
{
public:
    LinkedGraph (const int vertices): e(0)
    {
      if (vertices < 1) throw “Number of vertices must be > 0”;
      n = vertices;
      adjLists = new Chain<int>[n];
    };
private:
    Chain<int>* adjLists;
    int n;
    int e;
};
```

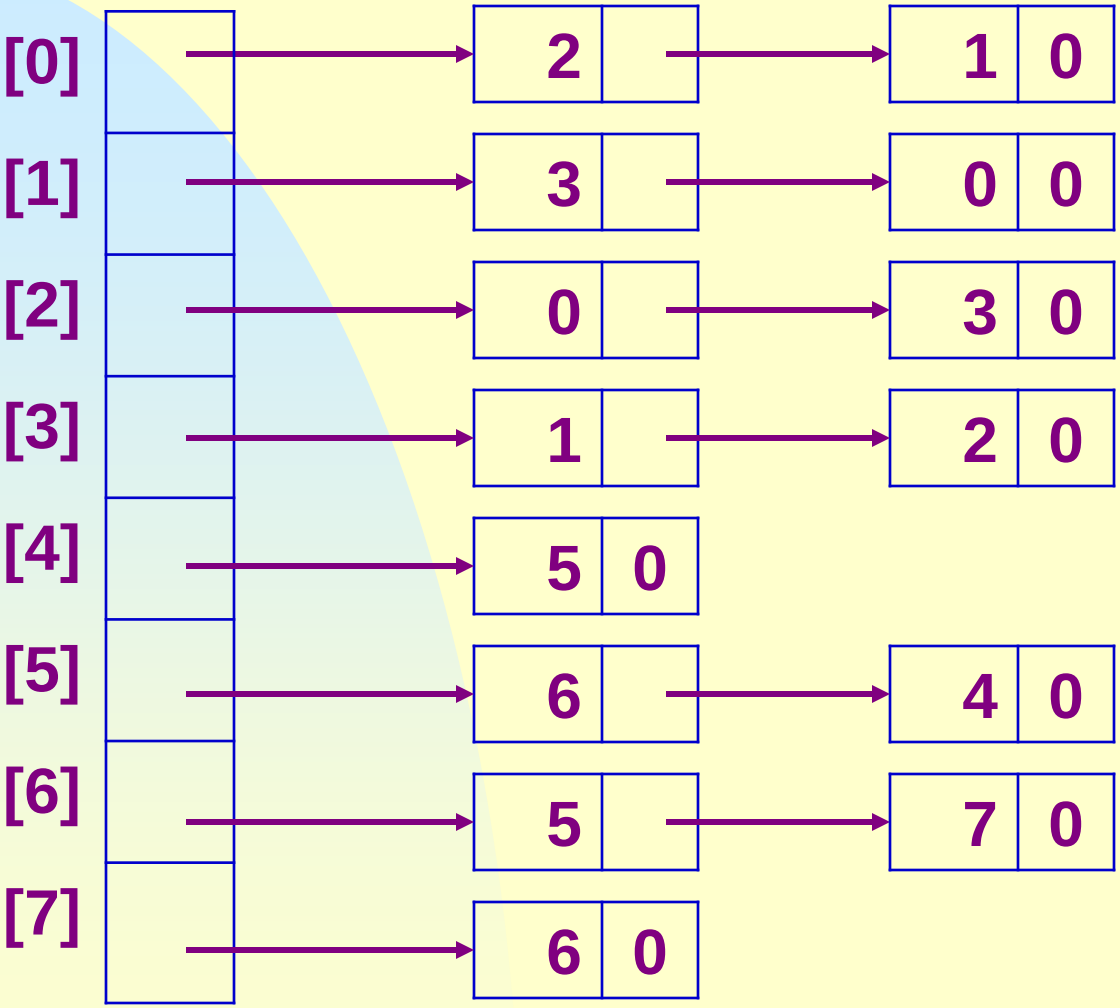


(a) G_1



(b) G_3

adjLists



(c) G_4

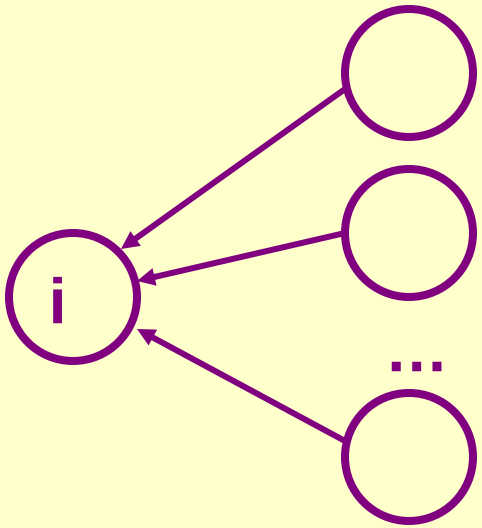
For an undirected graph with n vertices and e edges, this representation requires an array of size n and 2e chain nodes.

Now it is possible to determine the total number of edges in G in O(e+n).

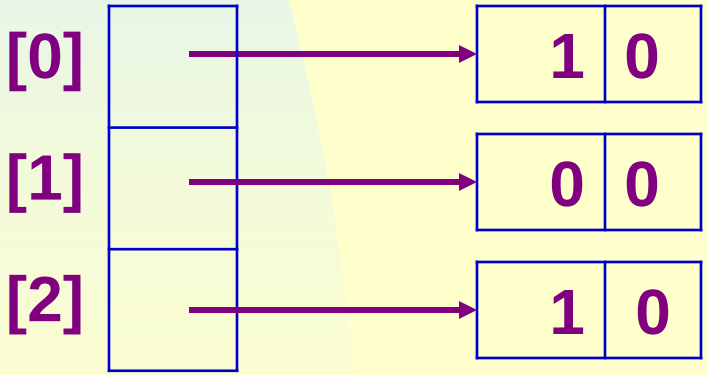
In case of digraph, the in-degree of a vertex is a little more complex to determine --- use a vector c[n], when traverse from i to j, c[j]++.

Inverse adjacency lists:

- one list for each vertex
- the nodes in list i represent the vertices adjacent to vertex i.



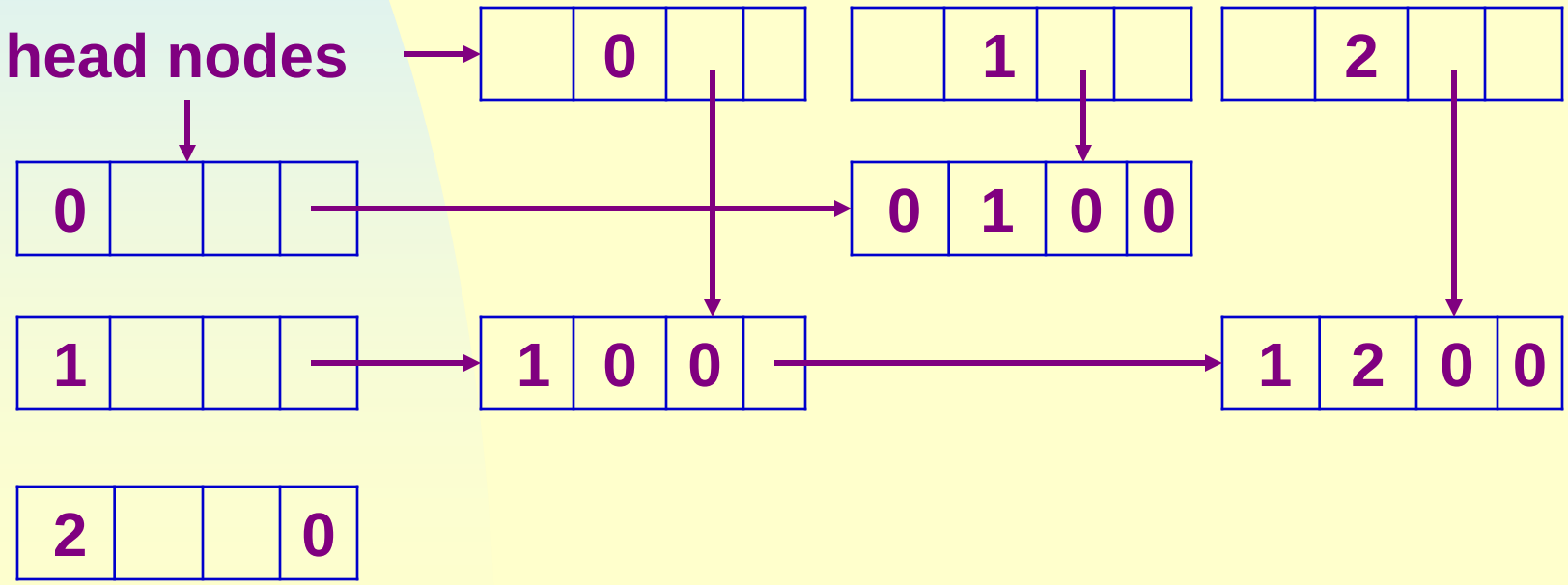
The following is the inverse adjacency lists for G_3 :



Combining nodes for adjacency lists and inverse adjacency lists together forms an orthogonal list node:

tail	head	column link	row link

Here is the orthogonal representation for G_3:



6.1.3.3 Adjacency Multilists

In the adjacency lists of an undirected graph, each (u, v) is represented by 2 entries.

In some situations, it is necessary to be able to determine the second entry for a particular edge and mark that edge as used.

So we need multilists: for each edge there is exactly one node, but it is in two lists.

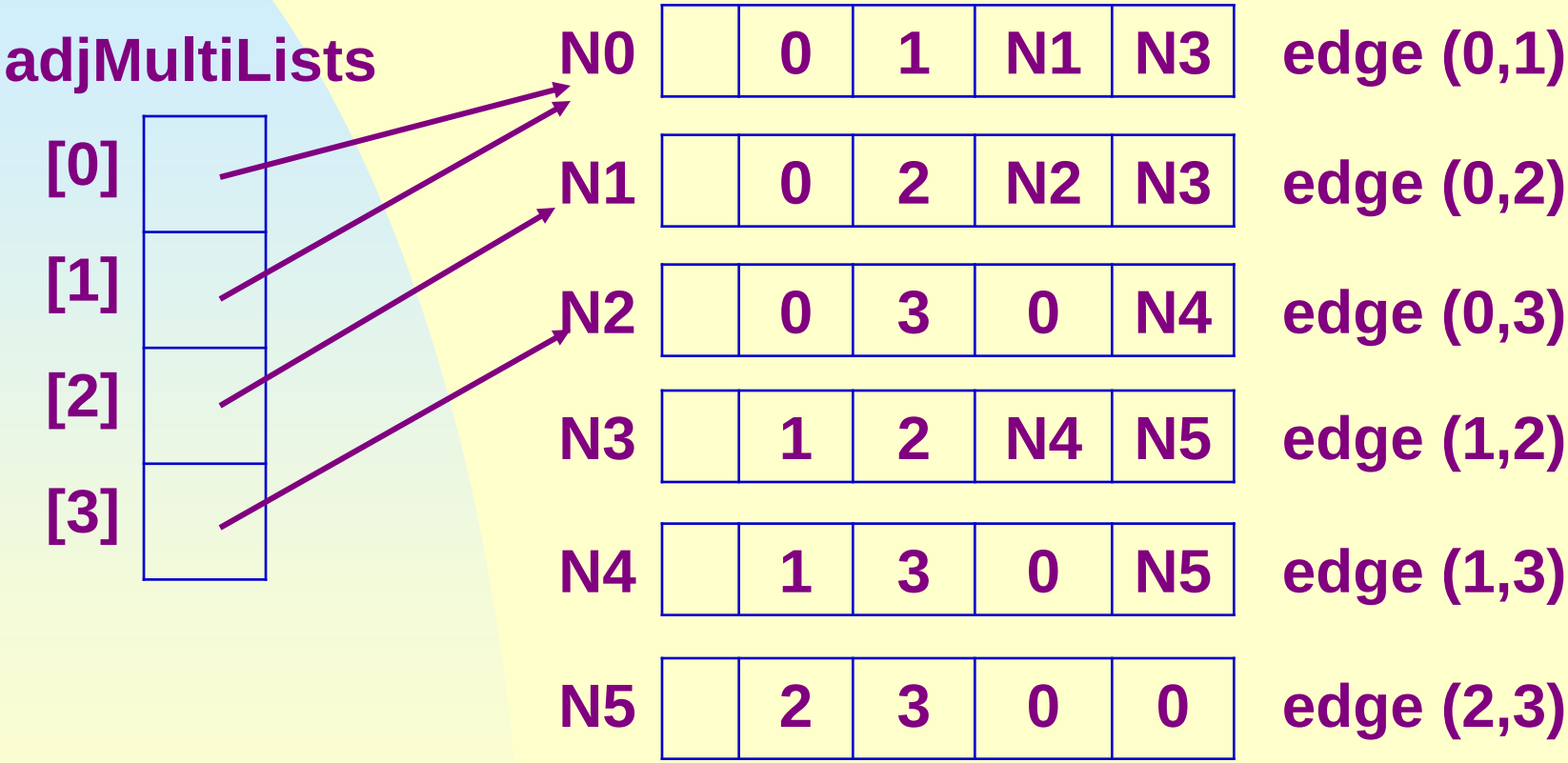
m	vertex1	vertex2	path1	path2

Note an edge can be marked from either path1 or path2.

```
class MGraph;
class MGraphEdge {
friend MGraph;
private:
    bool m;
    int vertex1, vertex2;
    MGraphEdge *path1, *path2;
};
typedef MGraphEdge *EdgePtr ;
class MGraph {
public:
     MGraph(const int);
private:
    EdgePtr *adjMultiLists;
    int n;
    int e;
};
```

```
MGraph::MGraph(const int vertices) : e(0)
{
    if (vertices < 1) throw “Number of vertices must be > 0”;
    n = vertices;
    adjMultiLists = new EdgePtr[n];
    fill(adjMultiLists, adjMultiLists+n,0);
}
```

Here is the adjacency multilists for G_1:



If p points to an MGraphEdge representing (u, v), and given u, to get v we need the following test:

```
if (p→vertex1 == u) v = p→vertex2;else v = p→vertex1;
```

And we can insert an edge in O(1):

```
void MGraph::InsertEdge(int u, int v) {
   MGraphEdge *p = new MGraphEdge;
   p→m = false; p→vertex1 = u; p→vertex2 = v;
   p→path1 = adjMultiLists[u]; p→path2 = adjMultiLists[v];
   adjMultiLists[u] = adhMultiLists[v] = p;
}
```

6.1.3.4 weighted Edges

Edges may have weight.

- **In the case of adjacency matrix, A[i][j] may keep this information.**
- **In the case of adjacency lists, we need a weight field in the list node.**
- **A graph with weighted edges is called a network.**

Exercises: P340-5, 9

6.2 Elementary Graph Operations

Given G = (V, E), and v in V(G), we wish to visit all vertices in G that are reachable from v.

In the following methods, we assume the graphs are undirected, although they work on the directed as well.

6.2.1 Depth-First Search

Idea:

Visit the start v first, next an unvisited w adjacent to v is selected and a depth-first search from w initiated. When u is reached such that all its adjacent vertices has been visited, we back up to the last vertex visited that has an unvisited vertex w adjacent to it and initiate a depth-first search from w. The search terminates when no unvisited vertex can be reached from any of the visited vertices.

```
virtual void Graph::DFS()   // Driver
{
    visited=new bool[n];
     // visited is declared as a bool* data member of Graph
    fill(visited, visted + n, false);
    DFS (0);      // start search at vertex 0.
    delete [ ] visited;
}

virtual void Graph::DFS ( const int v) // workhorse
{//visit all previously unvisited vertices reachable from v
   visited[v] = ture;
   for (each vertex w adjacent to v)  // actual code uses an
                                      // iterator
      if ( !visited[w] ) DFS ( w);
}
```

Example 6.1

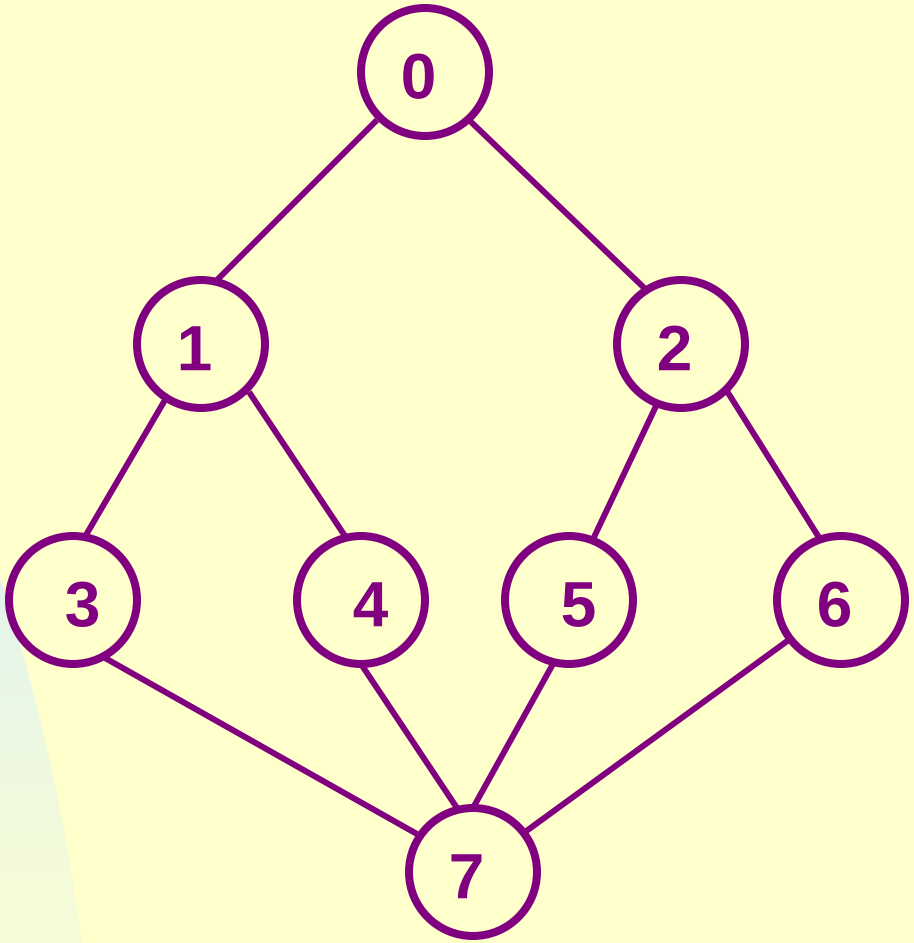


Fig. 6.17 (a) Graph G

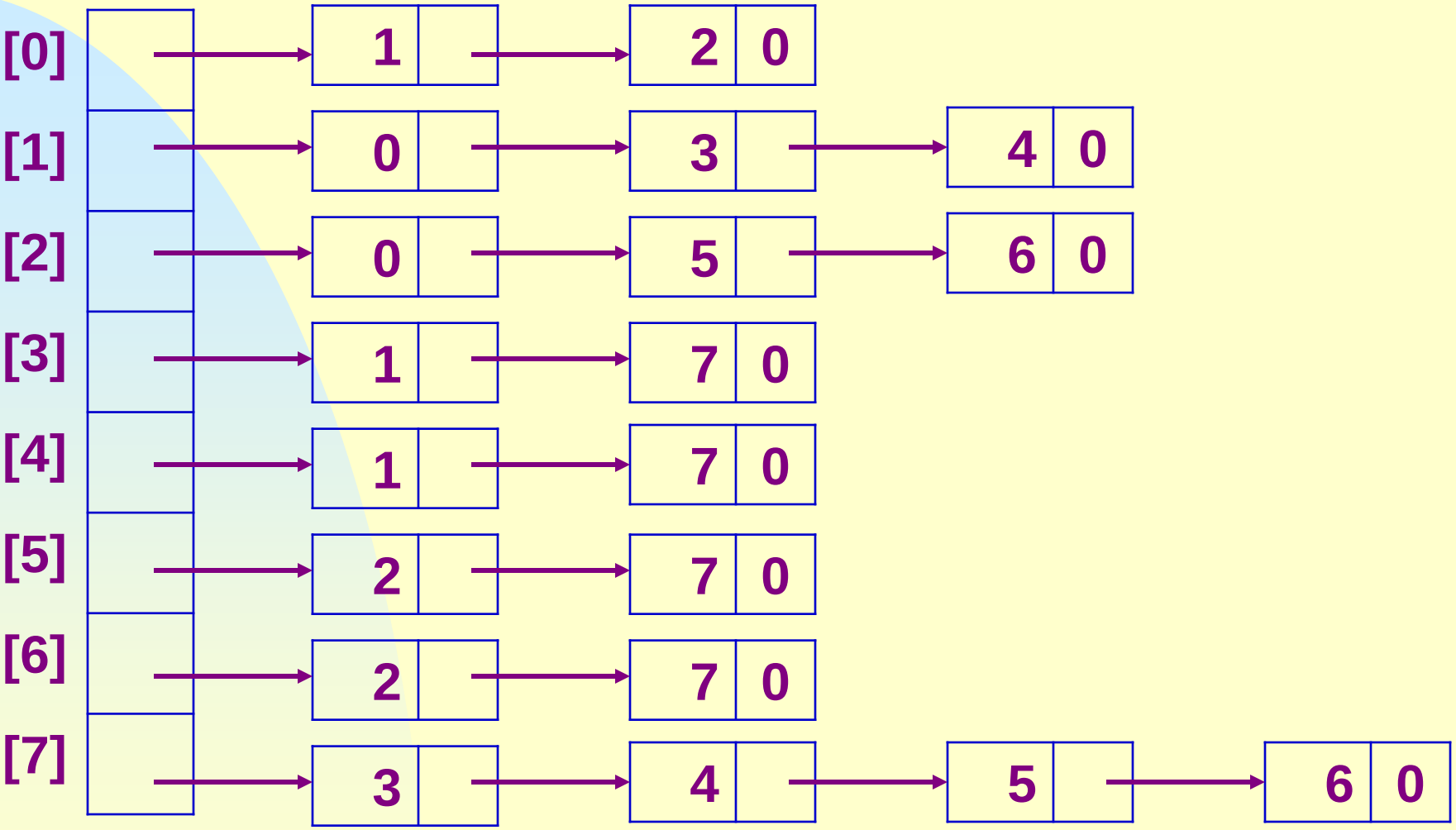


Fig 6.17 (b) Adjacency lists of G

Started from vertex 0, the vertices in G are visited in the order: 0, 1, (0), 3, (1), 7, (3), 4, (1), (7), (back to 7), 5, 2, (0), (5), 6, (2), (7), (back to 2), (back to 5), (7), (back to 7), (6), (back to 3), (back to 1), (4), (back to 0), (2).

Analysis of DFS:

Initiating visited needs O(n). When G is represented by its adjacency lists, DFS examines each node at most once, so the time is O(e+n).

If adjacency matrix is used, the time to determine all vertices adjacent to v is O(n), there are n vertices, the total time is $O(n^2)$.

6.2.2 Breadth-First Search

In a breadth-first search, we begin from the start vertex v. Next, all unvisited vertices adjacent to v are visited. Unvisited vertices adjacent to these newly visited vertices are then visited, and so on.

```
virtual void Graph::BFS (int v)
{ // breadth-first search: begin at v, use a queue.
    visited = new bool[n];
    fill(visited, visited + n, false);
    visited[v] = true;
    Queue<int> q;
    q.Push(v);
```

```
    while ( !q.IsEmpty()) {
        v = q.Front();
        q.Pop();
        for (all vertices w adjacent to v)  // actual code uses an
                                            //  iterator
            if (!visited[w]) {
                q.Push(w);
                visited[w] = true;
            }
    }   // end of while
    delete [ ] visited;
}
```

Example 6.2:

Started from vertex 0, the vertices in G of Fig. 6.17 are visited by breadth-first search in the order:

0, 1, 2, 3, 4, 5, 6, 7.

Analysis of BFS:

Each visited vertex enters the queue exactly once, the while loop is iterated at most n times.

If adjacency matrix is used, the loop takes O(n) for each node visited, the total time is $O(n^2)$.

If adjacency lists are used, the loop has a total cost of $d_0+,\ldots,+ d_{n-1} = O(e)$, the total time is O(n+e).

6.2.3 Connected Components

To obtain all the connected components of a undirected graph, we can make repeated calls to either DFS(v) or BFS(v) for unvisited v.

This leads to function Components.

Function OutputNewComponent output all vertices visited in the most recent invocation of DFS, together with all edges incident on them.

```
virtual void Graph::Components()
{ // Determine the connected components of the graph.
    visited = new bool[n];
    fill(visited, visited+n, false);
    for (int i=0; i<n; i++)
      if (!visited[i]) {
          DFS (i); // find a component
          OutputNewComponent();
      }
    delete [ ] visited;
}
```

Analysis of Components:

If adjacency lists are used, the total time taken by DFS is O(e). The output can be completed in O(e) if DFS keeps a list of all newly visited vertices. The for loops take o(n). The total time is O(e+n).

If adjacency matrix is used, the total time is $O(n^2)$.

6.2.4 Spanning Trees

If G is connected, in DFS or BFS, all vertices are visited, the edges of G are partitioned into 2 sets:

- **T --- Tree edges**
- **N --- Nontree edges**

T may be obtained by inserting "$T = T \cup \{(v, w)\}$" in the if clauses of DFS or BFS.

Any tree consisting solely of edges in G and including all vertices in G is called a spanning tree.

- **Depth-first spanning tree --- the spanning tree resulting from a depth-first search**
- **Breadth-first spanning tree --- the spanning tree resulting from a breadth-first search**

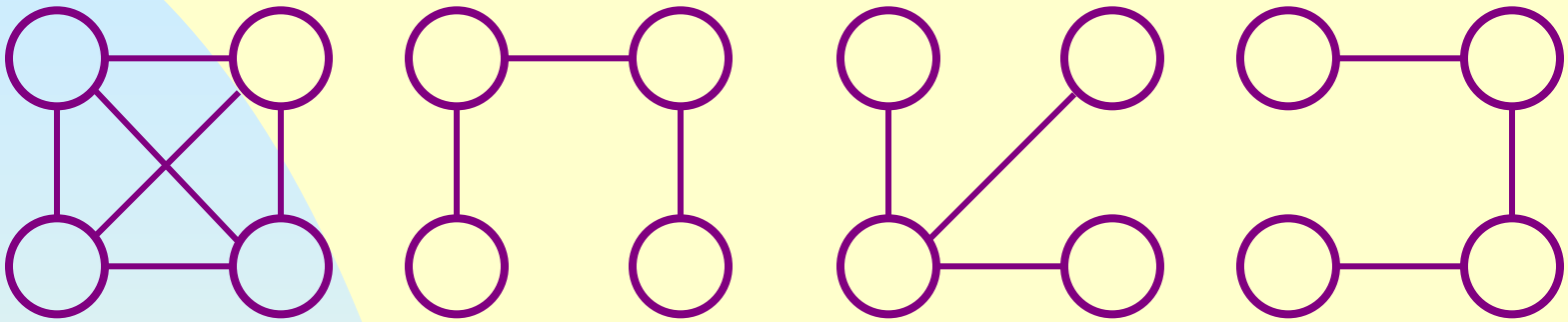


Fig. 6.18 A complete graph and three of its spanning trees

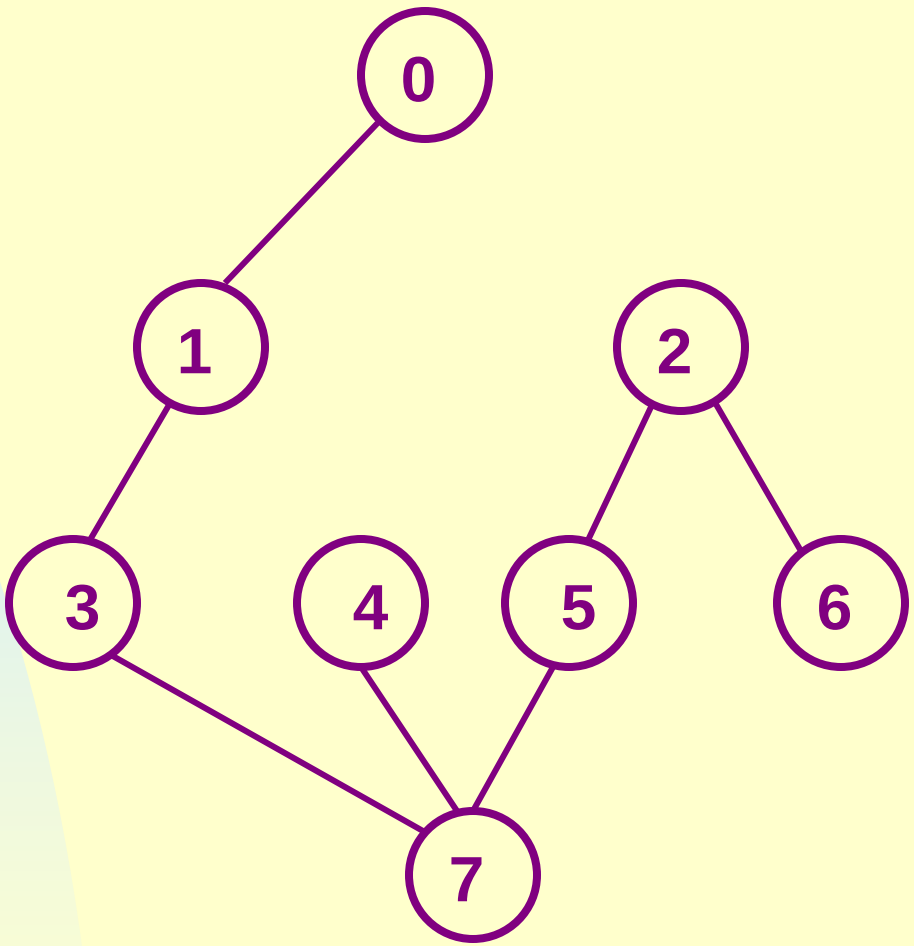


Fig. 6.19 (a) DFS(0) spanning tree for graph of Fig.6.17

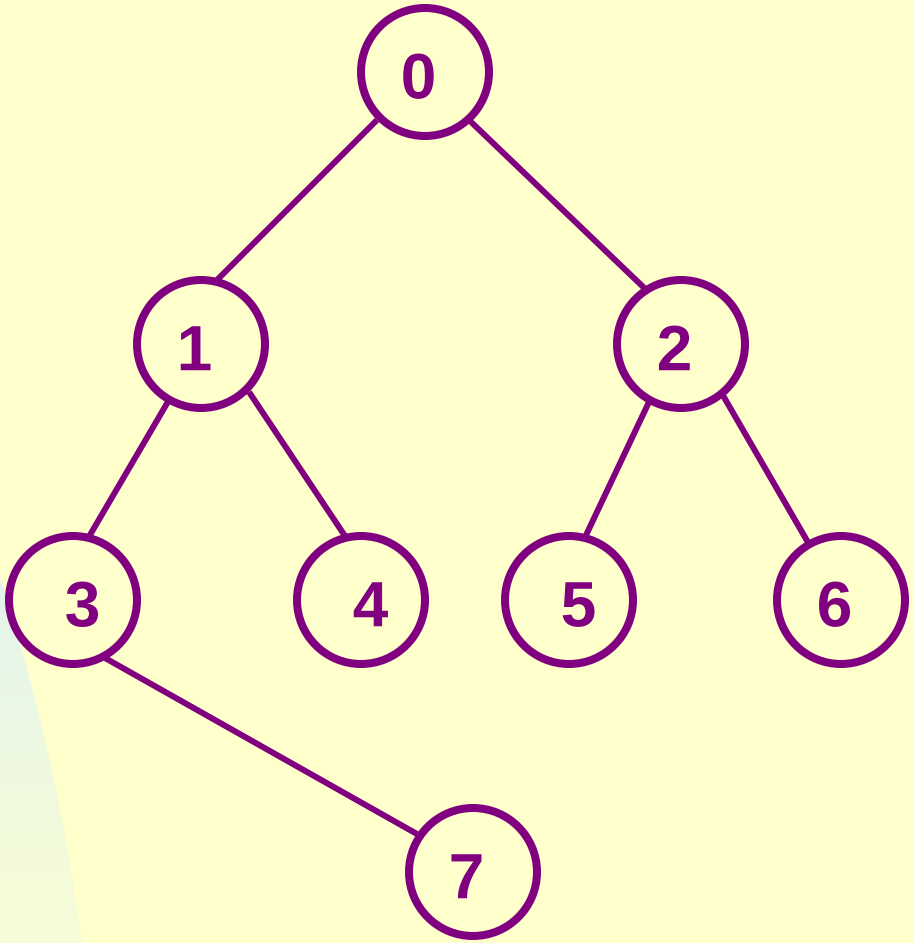


Fig. 6.19 (b) BFS(0) spanning tree for graph of Fig.6.17

If a nontree edge (v, w) is put into any spanning tree T, then a cycle is formed.

Example applications of spanning tree:

(1) Obtain an independent set of circuit equations for an electrical network by introducing nontree edges into the spanning tree one at a time.

(2) Connecting n cities with n-1 communication links.

In practical situation, edges will have weights (or cost) assigned to them. The cost of a spanning tree is the sum of the costs of the edges in that tree.

Exercises: P352-3, 5, 6

6.3 Minimum-Cost Spanning Trees

The cost of a spanning tree of a weighted, undirected graph is the sum of the costs (weights) of the edges in the spanning tree.

A minimum-cost spanning tree is a spanning tree of least cost.

A design strategy called greedy method can be used to obtain a minimum-cost spanning tree.

In the greedy method,

- **construct an optimal solution in stages;**
- **at each stage, make a decision (using some criterion) that appears to be the best (local optimum);**
- **since the decision can't be changed later, make sure it will result in a feasible solution, i.e., satisfying the constraints.**

At the end, if the local optimum is equal to the global optimum, then the algorithm is correct; otherwise, it has produced a suboptimal solution.

Fortunately, in the case of constructing minimum-cost spanning tree, the method is correct.

To construct minimum-cost spanning tree, we use a least-cost criterion and the constraints are:

(1) must use only edges within G.

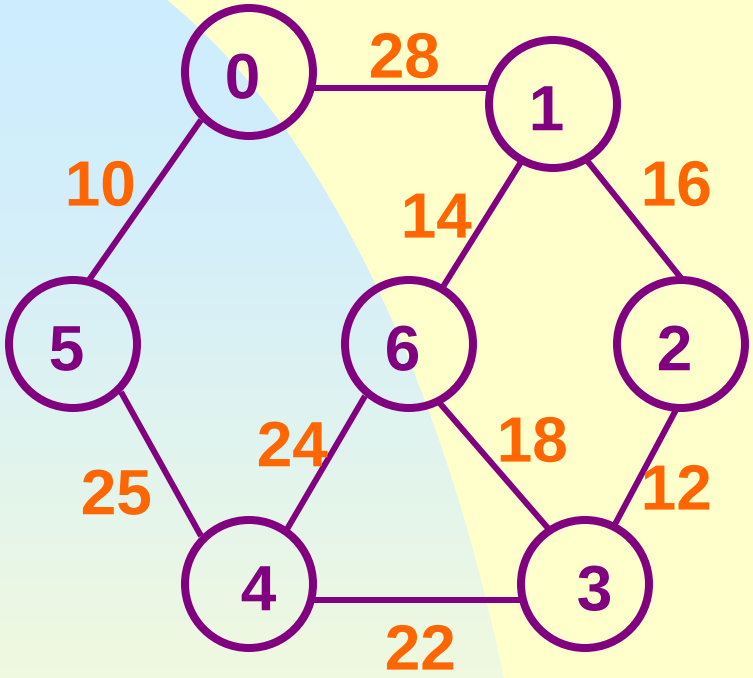
(2) must use exactly n-1 edges.

(3) may not use edges that produce a cycle.

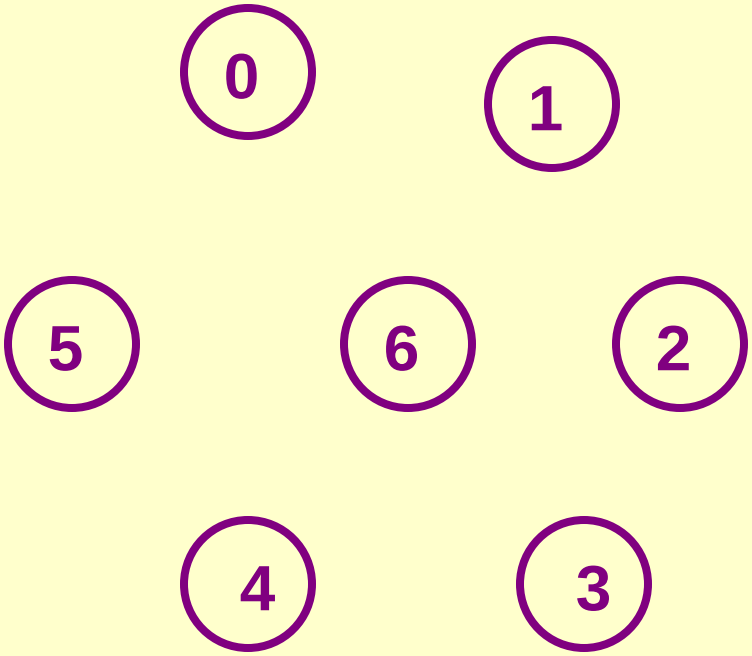
6.3.1 Kruskal Algorithm

- The minimum-cost spanning tree T is built by adding edges to T one at a time.
- Edges are selected for inclusion in T in non-decreasing order of their cost.
- An edge is added to T if it does not form a cycle with the edges already in T.
- Exactly n-1 edges will be selected into T.

Example 6.4:

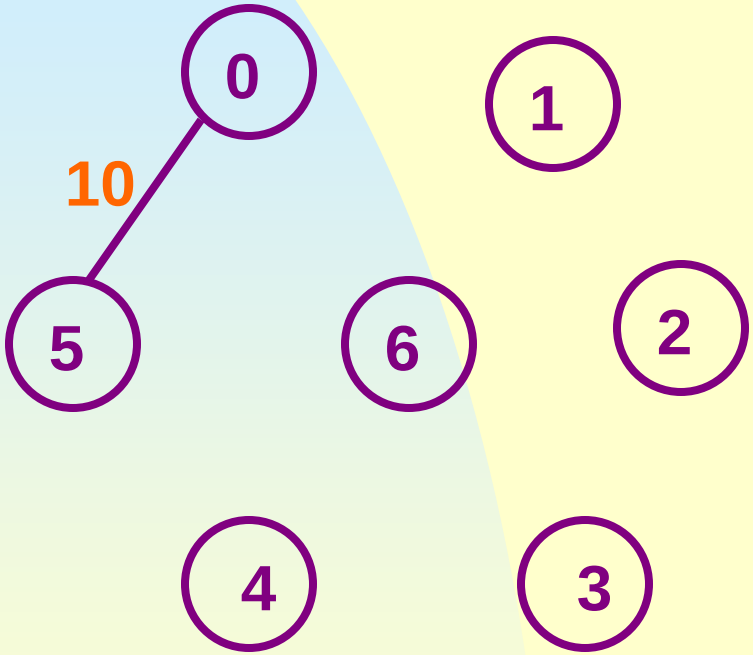


(a) G

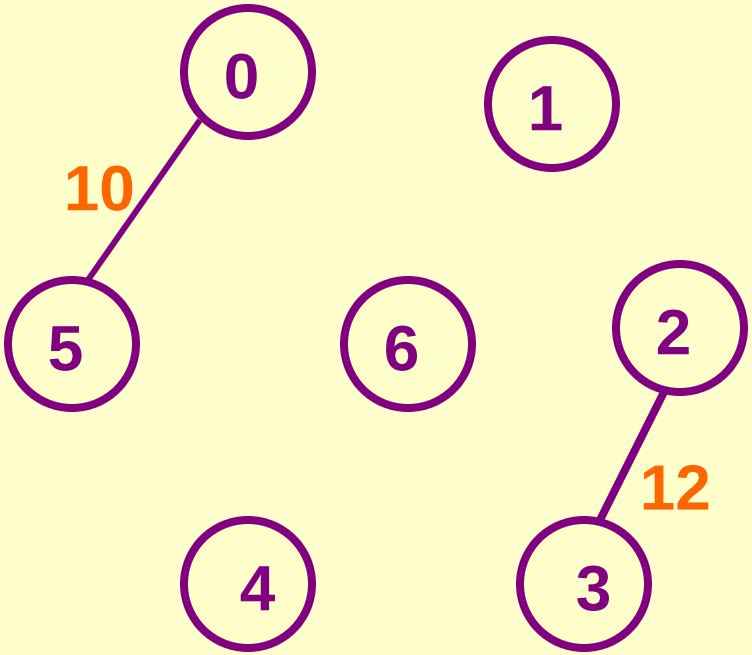


(b)

The cost of remaining edges: 10, 12, 14, 16, 18, 22, 24, 25, 28

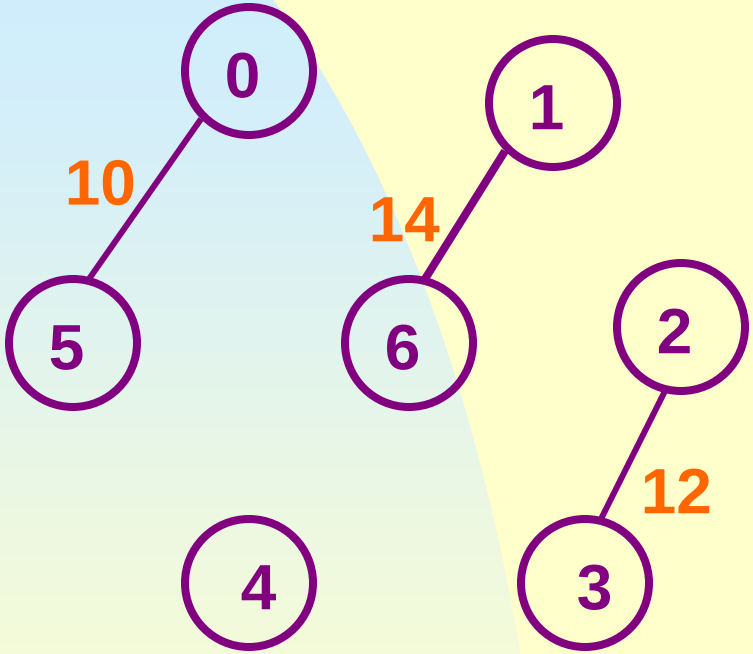


(c)

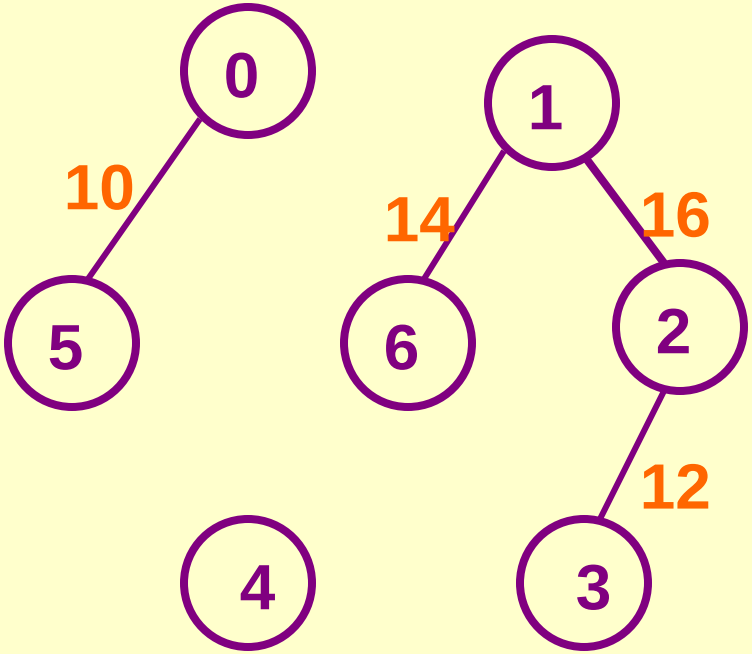


(d)

The cost of remaining edges: 14, 16, 18, 22, 24, 25, 28

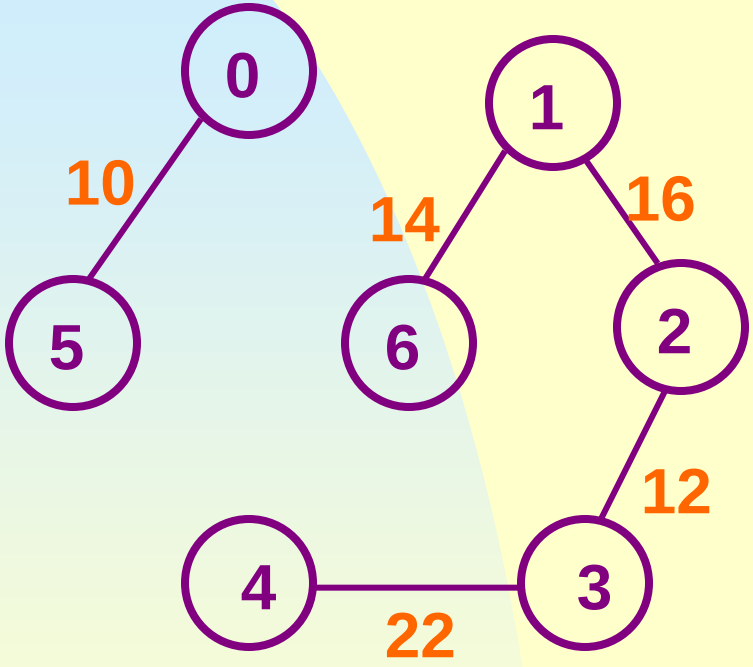


(e)

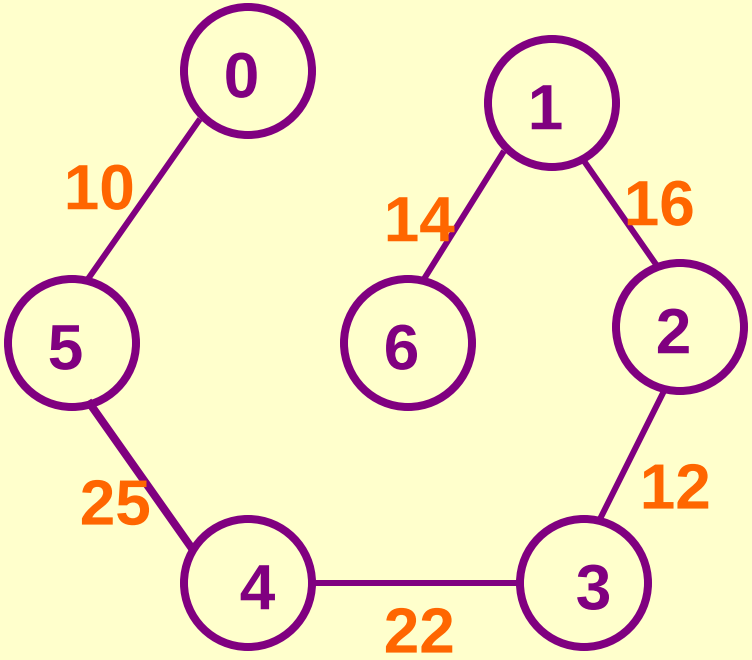


(f)

Remaining edges: 18—(3,6), 22, 24—(4,6), 25, 28



(g) (3,6) discarded



(h) (4, 6) discarded

Given G = (V, E), we have the algorithm:

```
T = ∅;
while ((T contains less than n-1 edges) && (E not Empty)) {
    choose an edge (v, w) from E of lowest cost;
    delete (v, w) from E;
    if ((v, w) does not create a cycle in T) add (v, w) to T;
    else discard (v, w);
}
if (T contains fewer than n-1 edges)
                    cout<<"no spanning tree"<<endl;
```

Analysis:

- **To perform lines 3 and 4, E can be organized as a min heap, so the next edge can be chosen and deleted in O(log e). Initialization of the heap takes O(e) (ref. 7.6).**
- **To perform line 5, vertices in T can be placed into a set using Union-Find. The total cost is O(e. α(e)).**

The total cost: O(e log e).

Correctness:

Theorem 6.1: Let G be any undirected, connected graph, Kruskal's algorithm generates a minimum-cost spanning tree.

The proof is left as a self-study exercise.

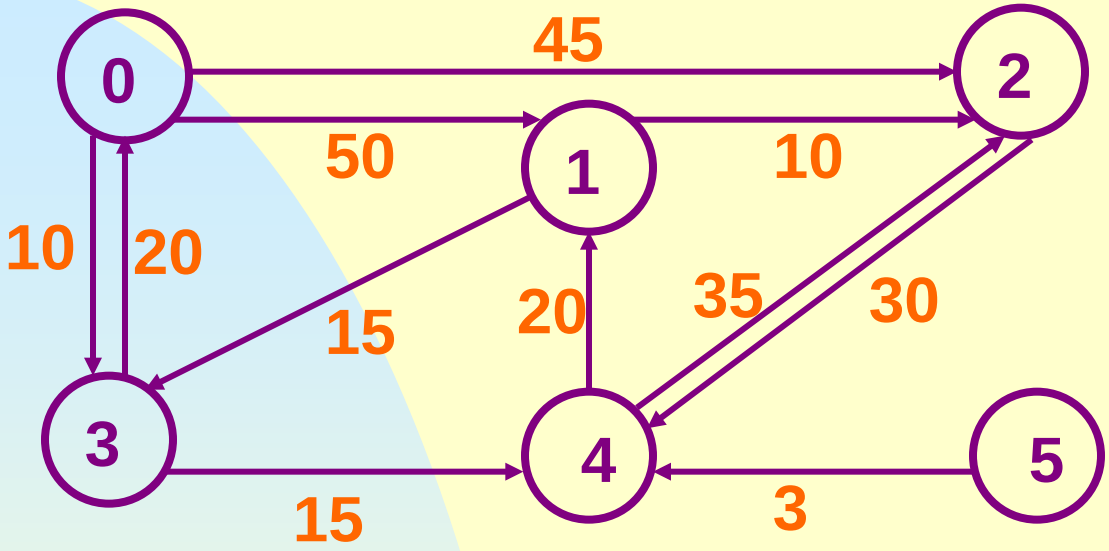
Exercises: P359-1

6.4 Shortest Paths

6.4.1 Single Source/All Destinations: Nonnegative Edge Cost

Problem: given a digraph $G=(V, E)$, a length function $length(i, j) \geq 0$ for $<i, j> \in E(G)$, and a source vertex v, to determine the shortest path from v to all remaining vertices of G.

Example:



(a) G

	path	Length
1)	0, 3	10
2)	0, 3, 4	25
3)	0, 3, 4, 1	45
4)	0, 2	45

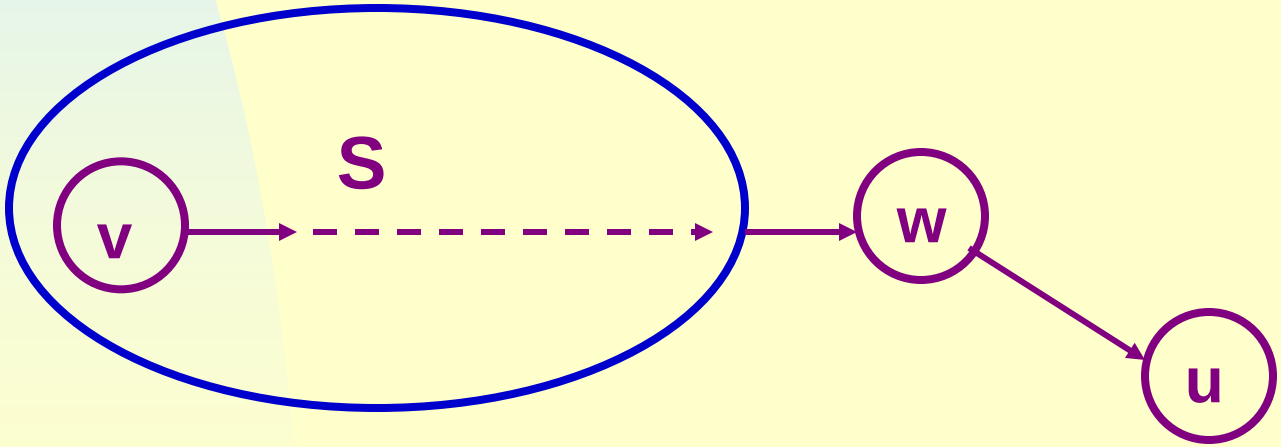
(b) Shortest paths from 0

Let S --- the set of vertices (including v) to which the shortest paths have been found.

For w $\notin$ S, dist[w] --- the length of the shortest path starting from v, going through only the vertices in S, and ending at w.

Assume paths are generated in non-decreasing order of length, observe:

(1) If the next shortest path is to u, then it begins at v, ends at u, and goes through only vertices in S. To prove, assume w on this path is not in S, then the v to u path contains a path from v to w as shown below:

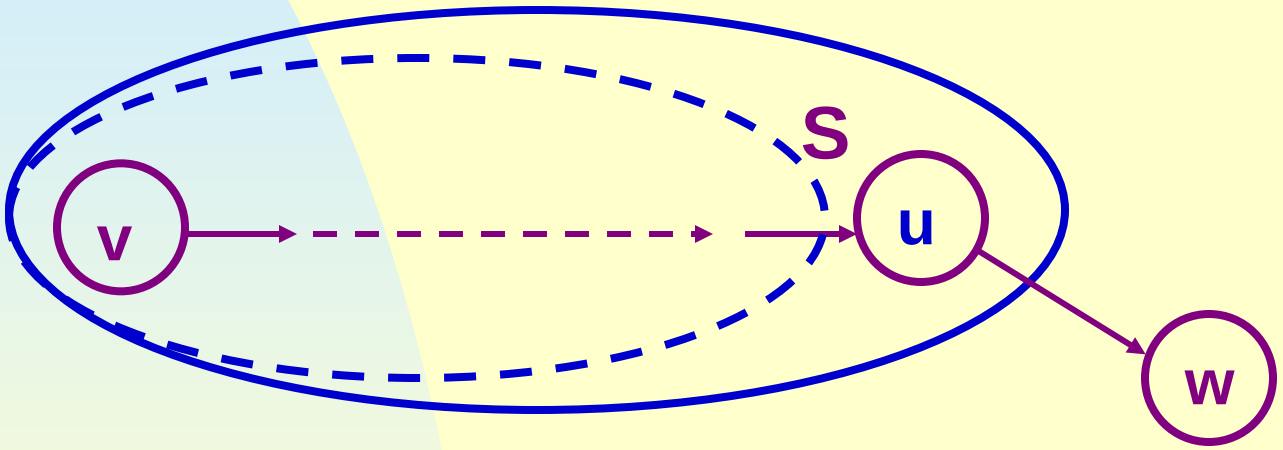


As length(w, u)≥0, length(v---w) must be less than length(v---u), otherwise we don't need to go through w. By assumption, the shorter path(v---w) has been generated already. Hence there is no intermediate vertex that is not in S.

(2) The destination of the next path generated must be u, such that

$$dist[u]=\min_{v \notin S}\{ dist[v]\}$$

(3) Having selected a vertex u as in (2), u becomes a member of S. Now dist[w], w $\notin$S, may change. If it does, it must be due to a shorter path from v to u and then to w.



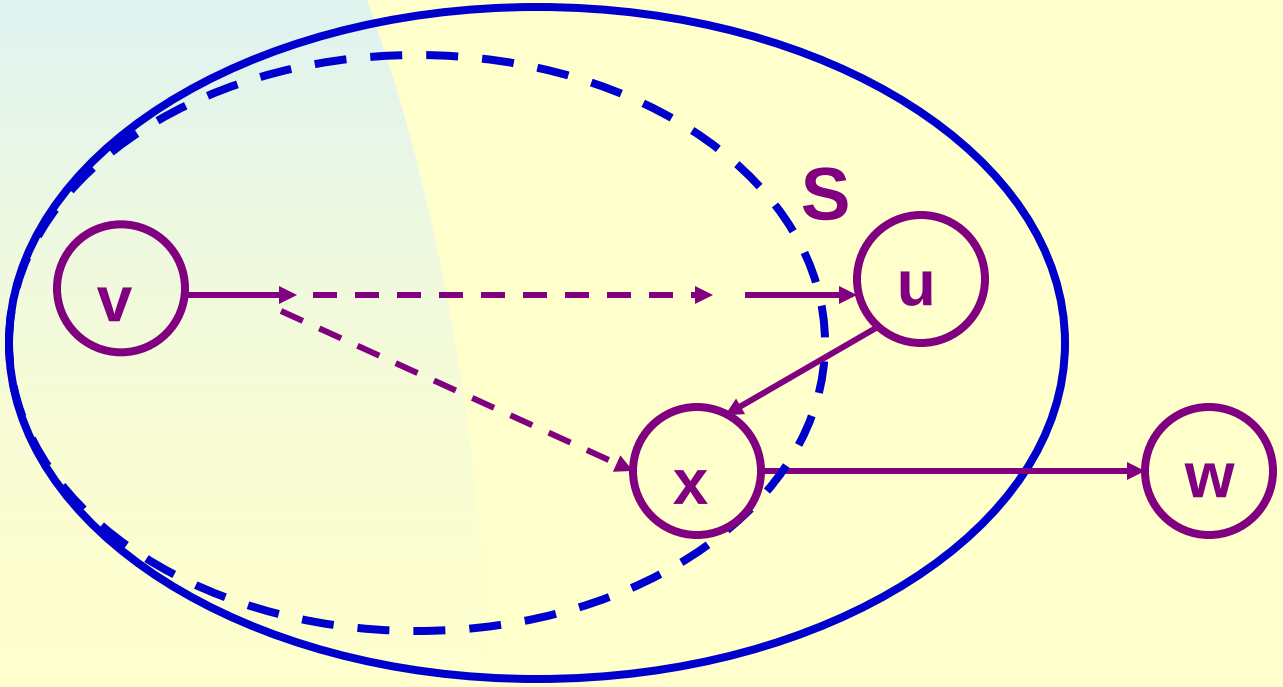
The v to u path must be the shortest v to u path, and the u to w path is just the edge <u, w>. So if dist[w] is to decrease, it is because the current dist[w]>dist[u]+length(u, w).

Note the u to w path with any intermediate vertex $x \in S$ cannot affect dist[w]. Because:

① length(v-u-x)>dist[x]

② length(v-u-x-w)>length(v-x-w)≥dist[w]

As shown below:



Assume:

- **n vertices numbered 0, 1, …, n-1.**
- **s[i]=false if i $\notin$S, s[i]=true if i $\in$S.**
- **length[i][j]. If <i, j> $\notin$E(G) and i$\neq$j, set length[i][j] to some large number.**

```
class MatrixWDigraph {
private:
    double length[NMAX][NMAX];  // NMAX is a constant
    double *dist;
    bool *s;
public:
    void ShortestPath(const int, const int);
    int choose(const int);
};
```

And we assume the constructor will apply space for array dist and s.

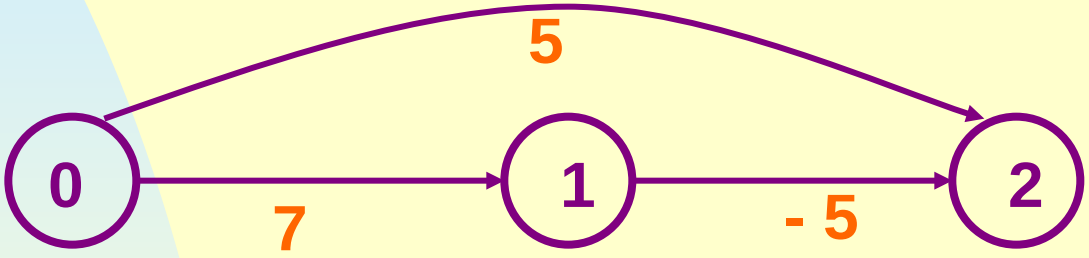
```
void MatrixDigraph::ShortestPath(const int n,  const int v)
{//dist[j],0≤j<n, is set to the length of the shortest path(v-j)
//in a digraph G with n vertices and edge lengths in length[i][j]
    for (int i=0; i<n; i++) { s[i]=false; dist[i]=length[v][i];}
    s[v]=true;
    dist[v]=0;
    for (i=0; i<n-2; i++) { //determine n-1 paths from v
      int u=choose(n); //choose returns a value u such that
                       //dist[u]=minimum dist[w], where s[w]=false
      s[u]= true;
      for ( int w=0; w<n; w++)
        if (!s[w] && dist[u]+length[u][w]<dist[w])
          dist[w]=dist[u]+length[u][w];
    }
}
```

Analysis of ShortestPath:

The time for loop of line 4 is O(n). The for loop of line 7 is executed n-2 times, each requires O(n) at line 8 and at line 11 to 13. So the total time is (n^2).

Even if adjacency lists are used, the overall time for line 11 to 13 can be brought down to O(e), but for line 8 remains On^2).

Note the algorithm does not work when some edges may have negative length, as shown bellow:



6.4.3 All-Pairs Shortest Paths

Problem: find the shortest paths between all pairs of vertices u and v, $u \neq v$.

One solution:

For each vertex in G, takes it as the source, apply ShortestPath, all n times, in $O(n^3)$.

Using the dynamic programming approach, we can obtain a conceptually simpler algorithm that has complexity of $O(n^3)$ and works even when G has edges with negative length so long as G has no cycles with negative length.

$A^k[i][j]$ --- the length of shortest path from i to j going through no intermediate vertex of index greater than k.

$A^{n-1}[i][j]$ --- the length of the shortest i--j path in G.

$A^{-1}[i][j]$ --- length[i][j].

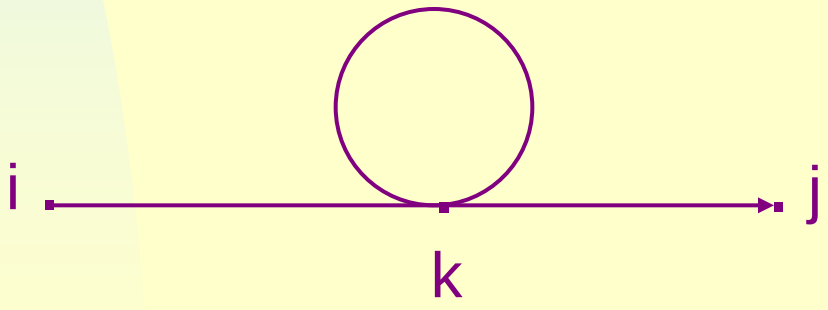
Now the basic idea of the algorithm:

Successively generate A^{-1}, A^{0}, A^{1},…, A^{n-1}. Given A^{k-1}, we can get A^{k} by realizing for every pair of i and j, either of the following 2 cases applies:

(1) The shortest path from i to j going through no vertex with index greater than k does not go through k, so its length is $A^{k-1}[i][j]$.

(2) The shortest path does go through k. The path consists of a subpath from i to k and another one from k to j. These must be the shortest paths from i to k and from k to j going through no vertex with index greater than k, so their lengths are $A^{k-1}[i][k]$ and $A^{k-1}[k][j]$.

Note the above is true only if G has no negative cycle containing k.



From (1) and (2), we have:

$$A^k[i][j]=\min\{A^{k-1}[i][j], A^{k-1}[i][k]+ A^{k-1}[k][j]\}, k\geq 0.$$

Assume:

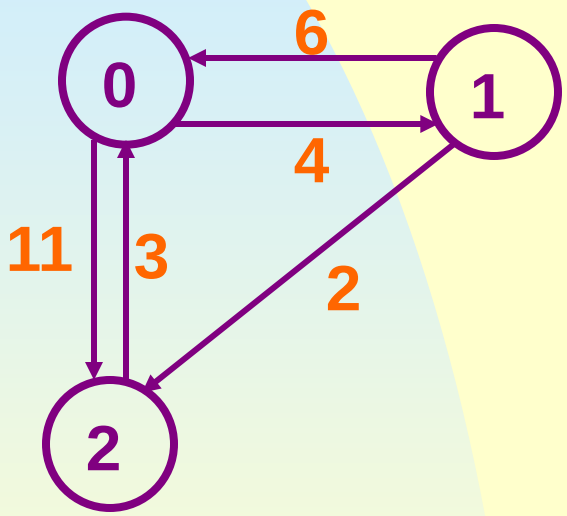
- **G is represented by length-adjacency matrix as for ShortestPath.**
- **double *a is data member of MatrixDigraph and the constructor will apply space for it.**

```
void MatrixDigraph::AllLengths (const int n)
{ //length[n][n] is the adjacency matrix of G with n vertices.
  //a[i][j] is the length of shortest path from i to j
    for (int i=0; i<n; i++)
      for (int j=0; j<n; j++)
            a[i][j] = length[i][j]; // copy length into a
    for ( int k=0; k<n; k++) // for a path with highest index k
      for (i=0; i<n; i++)   // for all pairs
          for (j=0; j<n; j++)
              if (a[i][k]+a[k][j]<a[i][j]) a[i][j]=a[i][k]+a[k][j];
}
```

The computation is done in place using a, because $A^k[i][k]= A^{k-1}[i][k]$ and $A^k[k][j]= A^{k-1}[k][j]$.

The time is obviously $O(n^3)$.

Example 6.7:



A^{-1}	0	1	2
0	0	4	11
1	6	0	2
2	3	∞	0

A^{0}	0	1	2
0	0	4	11
1	6	0	2
2	3	7	0

A^{1}	0	1	2
0	0	4	6
1	6	0	2
2	3	7	0

A^{2}	0	1	2
0	0	4	6
1	5	0	2
2	3	7	0

Exercises: P372-1, P373-2, 5, P375-17

6.5 Activity Networks

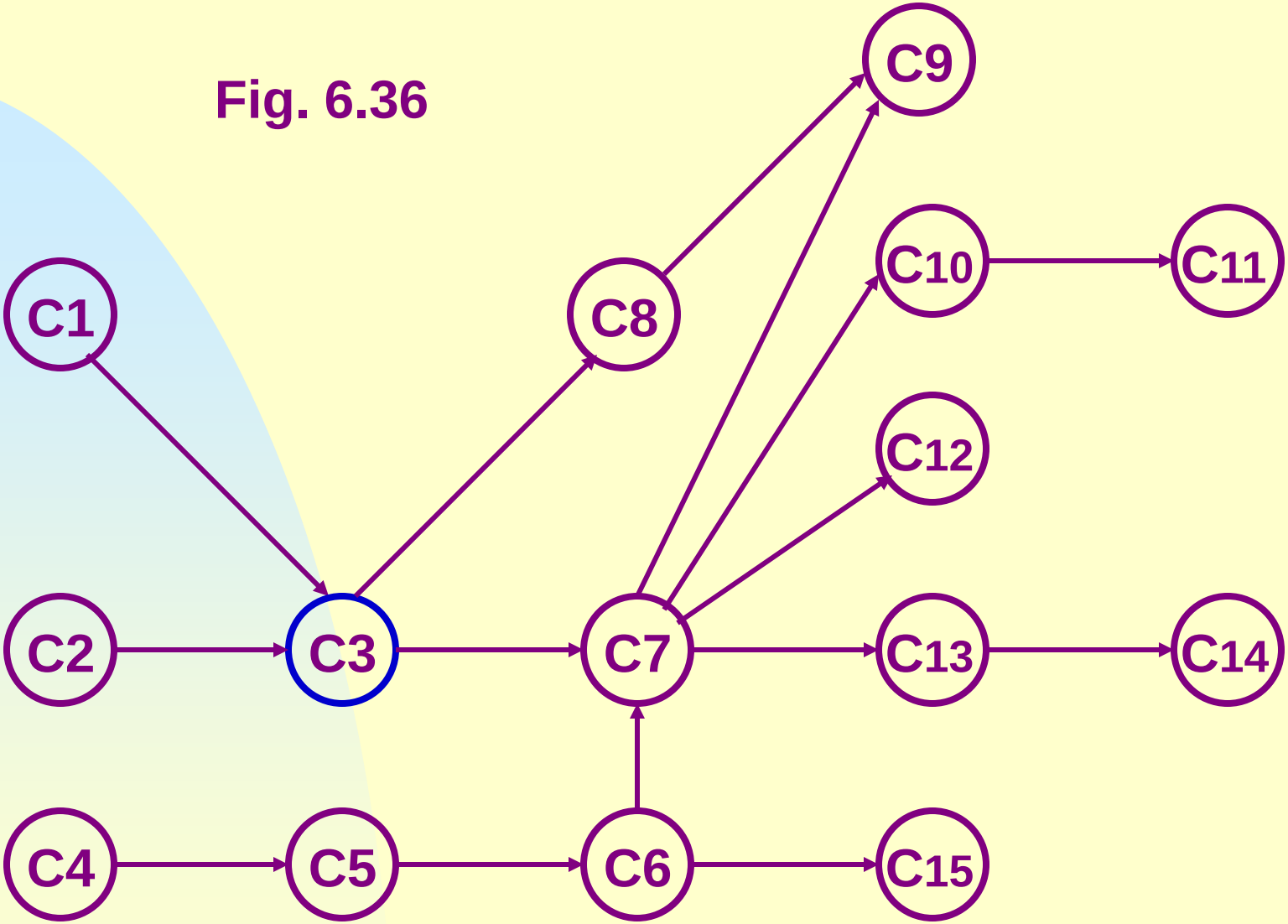
6.5.1 Activity-on-Vertex (AOV) Networks

Definition: A directed graph G in which the vertices represent tasks or activities and the edges represent precedence relations between tasks is an activity-on-vertex network or AOV network.

In the following, we'll see the AOV network corresponding to the courses of the next slide.

Course-No.	Course-Name	Prerequisites
C1	Programming Ⅰ	None
C2	Discrete Mathematics	none
C3	Data Structures	C1, C2
C4	Calculus Ⅰ	none
C5	Calculus Ⅱ	C4
C6	Linear Algebra	C5
C7	Analysis of Algorithms	C3, C6
C8	Assembly Language	C3
C9	Operating System	C7, C8
C10	Programming Languages	C7
C11	Compiler Design	C10
C12	Artificial Intelligence	C7
C13	Computational Theory	C7
C14	Parallel Algorithm	C13
C15	Numerical Analysis	C5

Fig. 6.36



Definition: Vertex i in an AOV network G is a **predecessor** of j iff there is a directed path from i to j. If <i, j> is an edge in G then i is an **immediate predecessor** of j and j **immediate successor** of i.

Definition: A precedence relation that is both transitive and irreflexive is a **partial order.**

A directed graph with no cycle is an **acyclic** graph.

Given a AOV network G, we are concerned with determining whether or not it is irreflexive, i.e., acyclic.

And we need to generate the topological order of vertices in G.

Definition: A topological order is a linear ordering of vertices of a graph such that, for any two vertices i and j, if i is a predecessor of j in the network, then i precedes j in the linear ordering.

Two possible topological orders of Fig. 6.36 are:

C1,C2,C4,C5,C3,C6,C8,C7,C10,C13,C12,C14,C15,C11,C9

and

C4,C5,C2,C1,C6,C3,C8,C15,C7,C9,C10,C11,C12,C13,C14

Topological sorting algorithm

Idea: list a vertex with no predecessor, then delete this vertex together with all edges leading out of it from the network. Repeat the above until all vertices have been listed or all remaining vertices have predecessors.

Formally,

```
Input the AOV network, let n be the number of vertices;
for (int i=0; i<n; i++) // output the vertices
{
   if (every vertex has a predecessor) return;
      // network has a cycle and is infeasible.
   pick a vertex v that has no predecessors;
   cout << v;
   delete v and all edges leading out of v from the network;
}
```

Example:

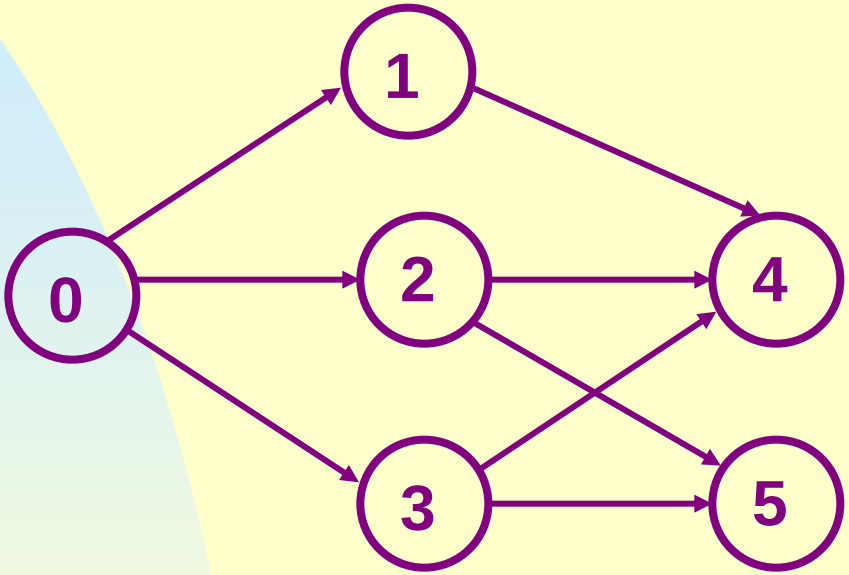
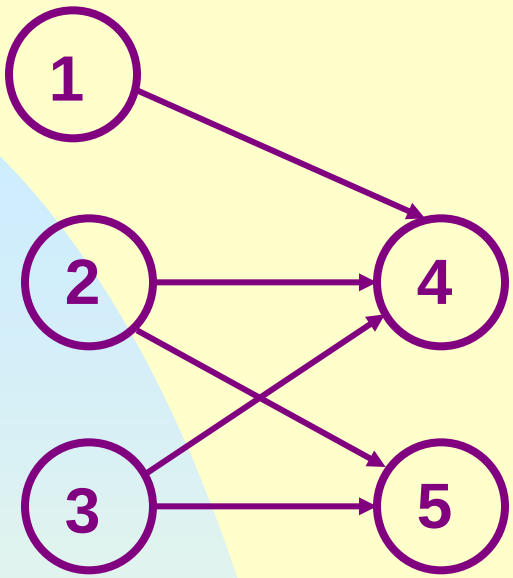
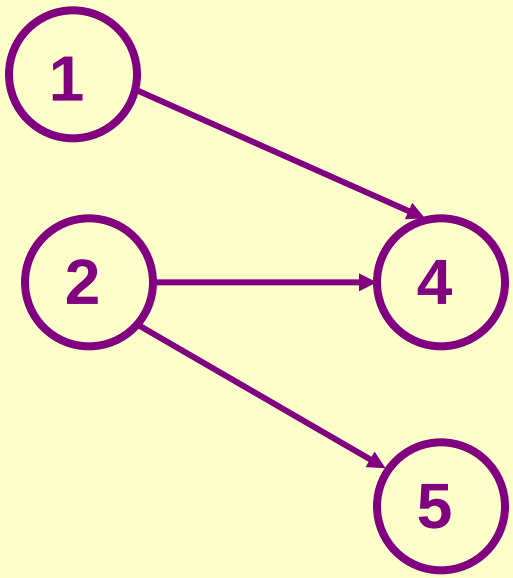


Fig. 6.37

(a) Initial

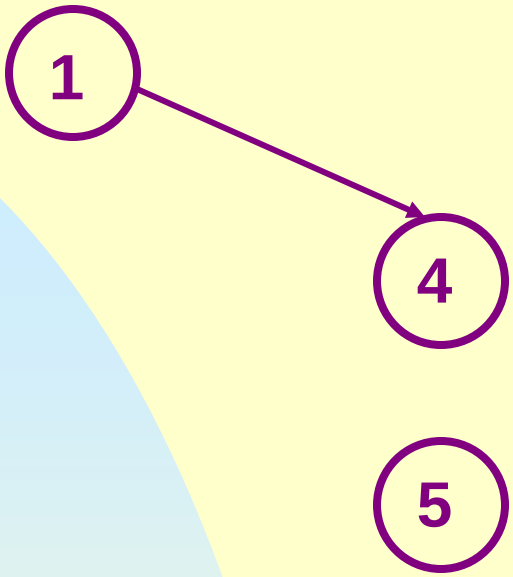


(b) Vertex 0 deleted

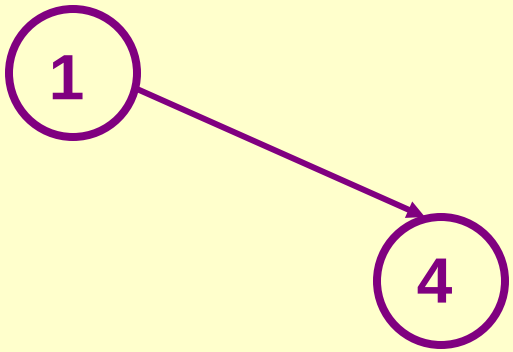


(c) Vertex 3 deleted

Topological order generated: 0, 3,



(d) Vertex 2 deleted



(e) Vertex 5 deleted

(f) Vertex 1 deleted

Topological order generated: 0, 3, 2, 5, 1, 4

The data representation depends on:

(1) Whether a vertex has a predecessor?--- maintain a count of the number of immediate predecessors of each vertex.

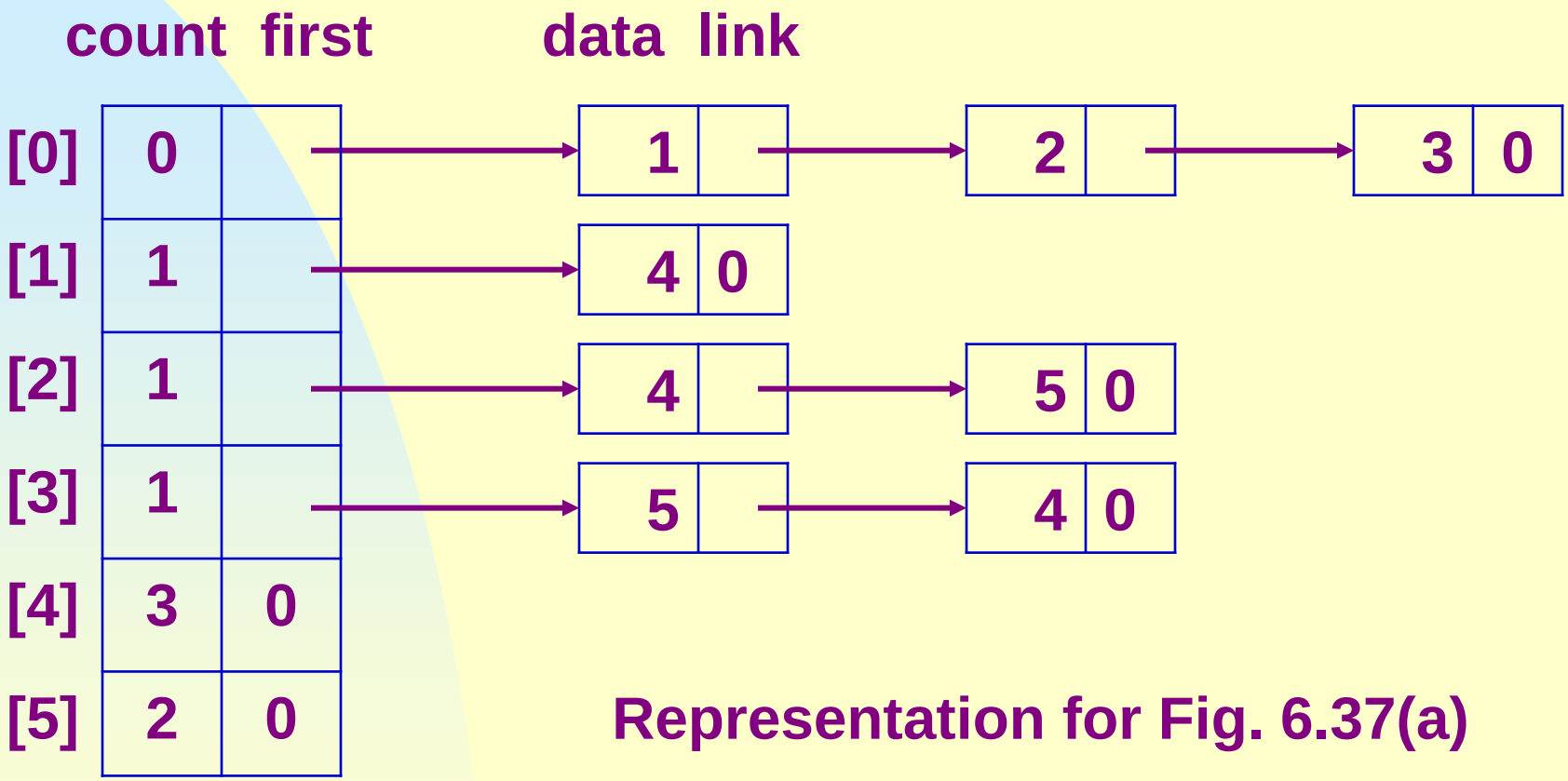
(2) How to delete a vertex together with all its incident edges?--- adjacency lists, and decrease the count of all vertices on its adjacency list.

Whenever the count of a vertex drops to 0, that vertex can be placed onto a list of vertices with 0 count.

Hence, we use adjacency lists representation, and assume:

- **int *count is defined as a data member, and its space is allocated in the constructor.**
- **count[i], 0≤i<n, has been initialized to the in-degree of vertex i. When <i, j> is input, the count of j is increased by 1.**
- **the list of vertices with 0 count is maintained as a custom stack linked through the count field since it is of no use after the count has become 0.**

- **int *t is defined as a data member, and its space is allocated in the constructor. Vertices are stored in t in topological order for future use.**



Representation for Fig. 6.37(a)

```
void LinkedGraph::TopologicalOrder()
{ //The n vertices of a network are listed in topological order
  int top = -1, pos = 0;
  for (int i=0; i<n; i++) //create a linked stack of vertices with
    if (count[i]==0) { count[i]=top;  top=i;} //no predecessors
  for (i=0; i<n; i++)
    if (top==-1) throw “network has a cycle.";
    int j=top;  top=count[top]; //unstack a vertex
    t[pos++] = j;  // store vertex j in topological order
    Chain<int>::ChainIterator ji=adjLists[j].begin();
    while (ji != adjLists[j].end()) { // decrease the count of
        count[*ji]--;                  // the successor vertices of j
        if (count[*ji]==0) {count[*ji]=top; top=*ji;} //add to stack
        ji++; // next successor
     }
}
```

Analysis of TopologicalOrder:

Very efficient because of a judicious choice of data structure.

- Lines 4-5 loop: O(n)
- Lines 6-10 totally O(n)
- The while loop of 11-15 takes $O(d_i)$ for each vertex i, where d_i is the out-degree of i. The total time for this part is $O((\sum_{i=0}^{n-1} d_i)+n)=O(e+n)$.

The total time: O(e+n).

6.5.2 Activity-on-Edge (AOE) Networks

The activity-on-edge (AOE) network:

- **directed edges --- tasks to be performed**
- **vertices --- events, signaling the completion of certain activities.**
- **activities represented by edges leaving a vertex cannot be started until the event at that vertex has occurred.**
- **an event occurs only when all activities entering it have been completed.**

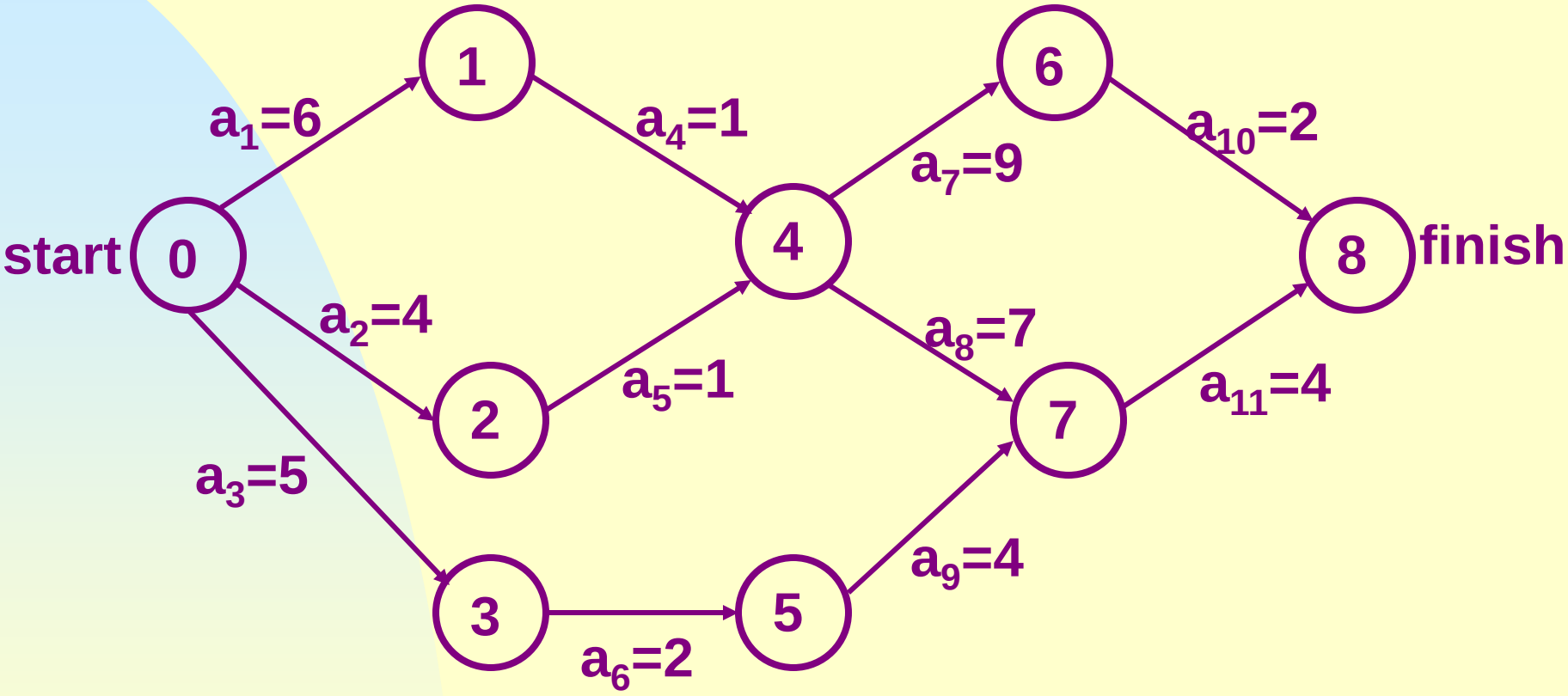

start
0
1
2
3
4
5
6
7
8
finish
$a_1=6$
$a_2=4$
$a_3=5$
$a_4=1$
$a_5=1$
$a_6=2$
$a_7=9$
$a_8=7$
$a_9=4$
$a_{10}=2$
$a_{11}=4$

Fig. 6.39

- **Since activities in an AOE network can be carried out in parallel, the minimum time to complete the project is the length of the longest path from the start to the finish.**
- **A path of longest length is a critical path. e.g., the paths 0, 1, 4, 6, 8 (length 18) and 0, 1, 4, 7, 8.**

- **e(i) --- the earliest start time for activity a_i.**
- **l(i) --- the latest start time for activity a_i without increasing the project duration.**
- **If e(i) = l(i), a_i is called a critical activity. l(i) - e(i) is a measure of the criticality of a_i .**

- **Speed up a critical activity will not necessarily result in a reduced project length unless it is on all critical paths. e.g., speed up a_{11} will not reduce the project length, but a_1 will.**
- **Once we get e(i) and l(i), critical activities can be easily identified.**

6.5.2.1 Calculation of Early Activity Times

How to obtain e(i) and l(i)? It is easy to first get:

ee[j] --- the earliest time for event j.

le[j] --- the latest time for event j.

If a_i is edge <k, l>, then

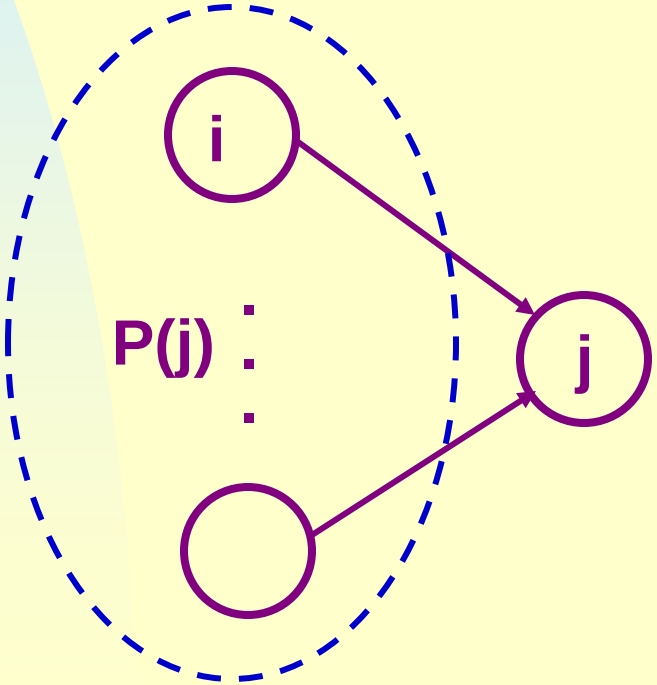
$$\left.\begin{aligned} e(i) &= ee[k] \quad \text{and} \\ l(i) &= le[l] - \text{duration of activity } a_i \end{aligned}\right\} \quad (6.1)$$

ee[j] and le[j] are computed in 2 stages: a forward stage and a backward stage.

The forward stage:

$$\left.\begin{array}{l} ee[0]=0 \qquad \text{(suppose 0 is the start)} \\ ee[j]= \max\limits_{i \in P(j)} \{ee[i]+\text{duration of } \langle i, j\rangle\} \end{array}\right\} \qquad (6.2)$$

where P(j) is the set of all vertices adjacent to j.



To carry the information of duration, we use the structure:

```
struct Pair
{
    int  vertex;
    int  dur;       //activity duration
};

class LInkedGraph {
private:
    Chain<Pair> *adjLists;
    int *count, *t, *ee, *le;
    int n;
```

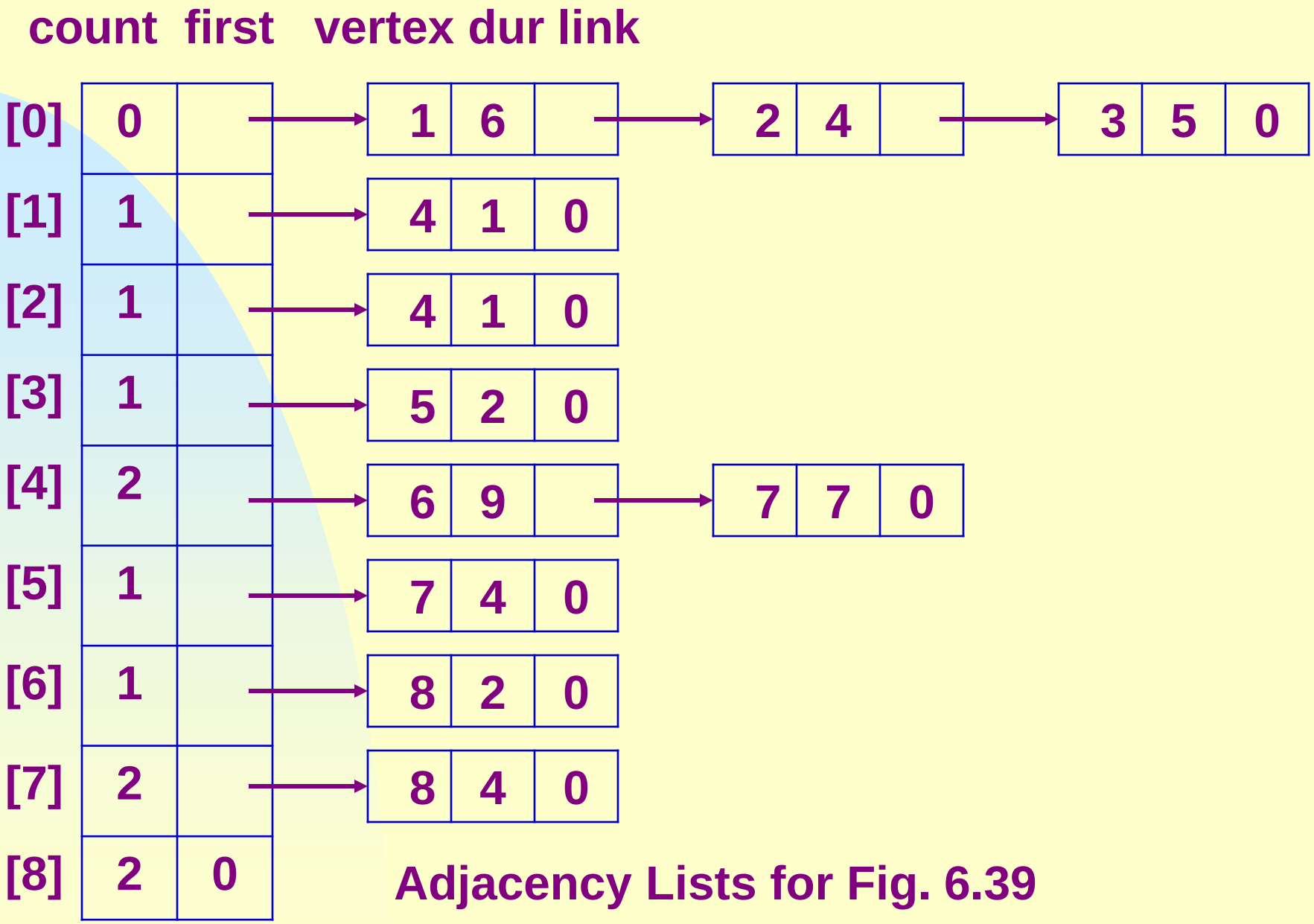
```
public:
    LinkedGraph (const int vertices) : {
      if (vertices < 1) throw “Number of vertices must be > 0”;
      n = vertices;
      adjLists = new Chain<Pair>[n];
      count = new int[n]; t = new int[n];
      ee = new int[n]; le = new int[n];
    };
   void TopologicalOrder();
   void EarliestEventTime();
   void LatestEventTime();
   void CriticalActivities();
};
```

If we initialize array ee to 0, and compute in topological order, then the early start times of all predecessors of j would have been computed prior to the computation of ee[j].

Thus we have:

```
void LinkedGraph::EarliestEventTime()
{ // assume a topological order has already been in t,
  // compute ee[j] according to t
   fill(ee, ee+n, 0);   // initialize ee
   for (i=0; i<n; i++) {
      int j=t[i];
      Chain<Pair>::ChainIterator ji=adjLists[j].begin();
      while (ji!=adjLists[j].end()) {
         int k=ji→vertex; //k is successor of j
         if (ee[k]<ee[j]+ji→dur) ee[k]=ee[j]+ji→dur;
         ji++;
      }
}
```

To illustrate, let's try it out on the network of Fig. 6.39.



Adjacency Lists for Fig. 6.39

ee	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
initial	0	0	0	0	0	0	0	0	0
do 0	0	6	4	5	0	0	0	0	0
do 3	0	6	4	5	0	7	0	0	0
do 5	0	6	4	5	0	7	0	11	0
do 2	0	6	4	5	5	7	0	11	0
do 1	0	6	4	5	7	7	0	11	0
do 4	0	6	4	5	7	7	16	14	0
do 7	0	6	4	5	7	7	16	14	18
do 6	0	6	4	5	7	7	16	14	18
do 8									

Fig. 6.40

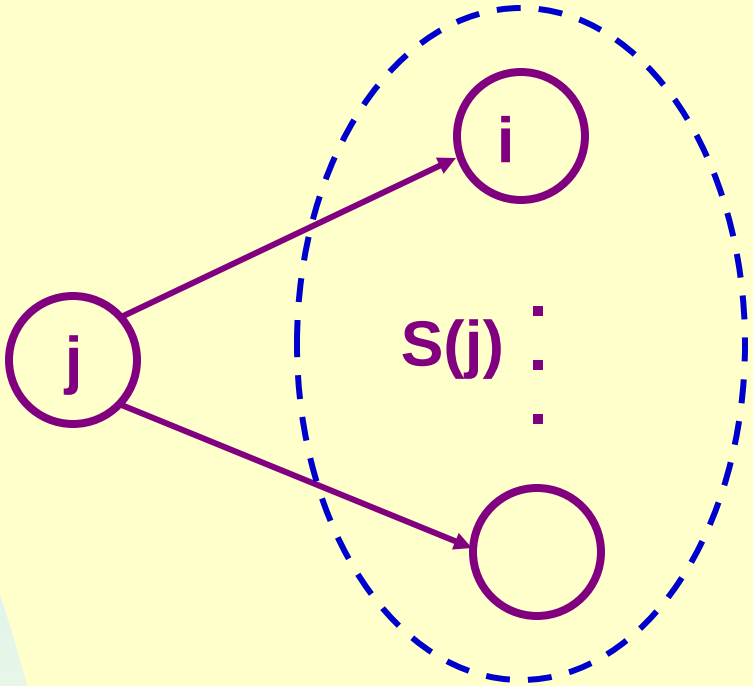
The computing time is obviously O(n+e).

6.5.2.2 Calculation of Late Activity Times

The backward stage:

$$\left.\begin{aligned} &le[n-1]=ee[n-1] \quad \text{(suppose n-1 is the finish)} \\ &le[j]= \min_{i \in S(j)} \{le[i]\text{-duration of } \langle j, i\rangle\} \end{aligned}\right\} \quad (6.3)$$

where S(j) is the set of all vertices adjacent from j.



If the forward stage has already been done, and a topological order get, then le[i], 0≤i<n, can be computed directly, using (6.3), by performing the computations in the reverse topological order.

Hence we have:

```
void LinkedGraph::LatestEventTime()
{ // assume a topological order has already been in t, ee has
  // been computed, compute le[j] in the reverse order of t
   fill(le, le+n, ee[n-1]); // initialize le
   for (i=n-2; i>=0; i--) {
      int j=t[i];
      Chain<Pair>::ChainIterator ji=adjLists[j].begin();
      while (ji!=adjLists[j].end()) {
         int k=ji→vertex; //k is successor of j
         if (le[k]-ji→dur<le[j]) le[j]=le[k]-ji→dur;
         ji++;
      }
}
```

Example:

given the topological order of Fig. 6.39:

0, 3, 5, 2, 1, 4, 7, 6, 8

we may compute le[i] for i=8, 6, 7, 4, 1, 2, 5, 3, 0.

le[8] = ee[8] = 18

le[6] = min {le[8] – 2} = 16

le[7] = min {le[8] – 4} = 14

le[4] = min {le[6] – 9, le[7] – 7} = 7

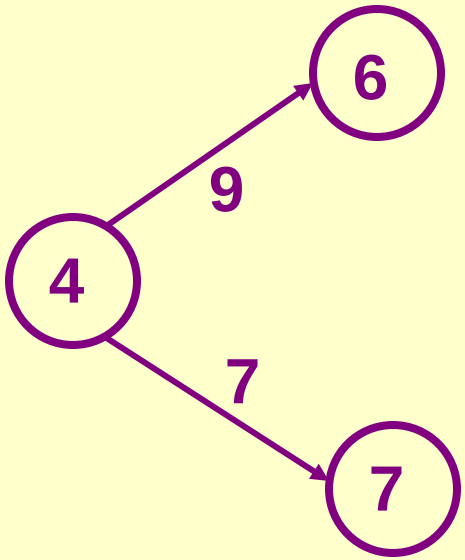
le[1] = min {le[4] – 1} = 6

le[2] = min {le[4] – 1} = 6

le[5] = min {le[7] – 4} = 10

le[3] = min {le[5] – 2} = 8

le[0] = min {le[1]–6, le[2]–4, le[3]–6} = 0



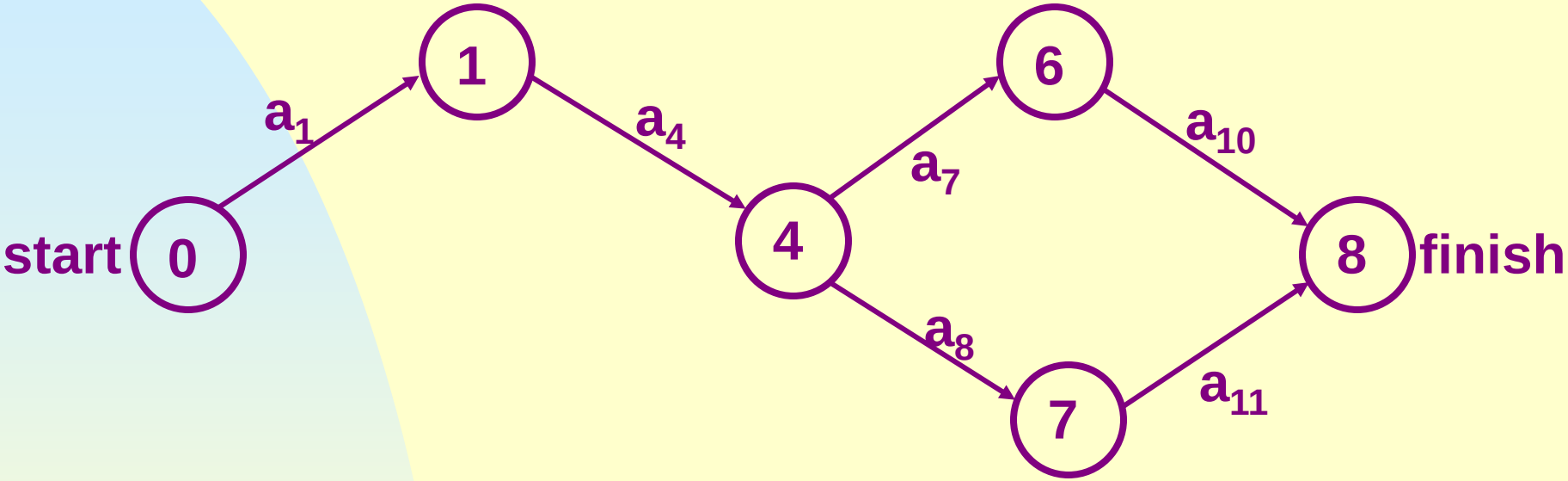
Now we can use (6.1) to output critical activities:

```
void LinkedGraph::CriticalActivities()
{ // compute e[i] and l[i], output critical activities
    int i=1; // the numbering counter for activities
    int u, v, e, l; // e, l are the earliest, latest start time of <u, v>
    for (u=0; u<n; u++) {  // scan the adjacency lists.
      Chain<Pair>::ChainIterator ui=adjLists[u].begin();
      while (ui!=adjLists[u].end()) {
        int v=ui→vertex; // <u, v> is an edge numbered i
        e=ee[u]; l=le[v]-ui→dur;
        if (l==e) cout <<"a"<<i<<"<"<<u<<","<<v<<">"
                          <<"is a critical activity"<<endl;
        ui++; i++;
      }
    }
}
```

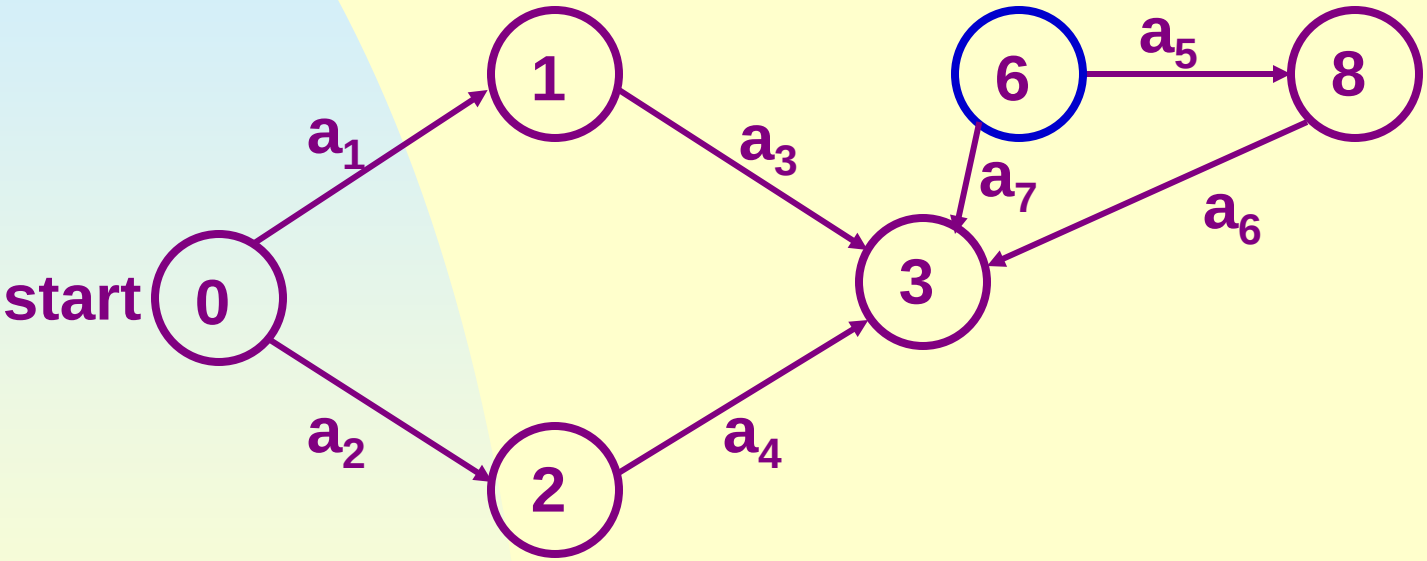
For Fig. 6.39, the e(i) and l(i) are:

activity	e	l	l-e	l-e=0
a_1 <0, 1>	0	0	0	Y
a_2 <0, 2>	0	2	2	N
a_3 <0, 3>	0	3	3	N
a_4 <1, 4>	6	6	0	Y
a_5 <2, 4>	4	6	2	N
a_6 <3, 5>	5	8	3	N
a_7 <4, 6>	7	7	0	Y
a_8 <4, 7>	7	7	0	Y
a_9 <5, 7>	7	10	3	N
a_{10} <6, 8>	16	16	0	Y
a_{11} <7, 8>	14	14	0	Y

The critical activities: a_1, a_4, a_7, a_8, a_{10}, and a_{11}. Deleting non-critical ones we get:



Suppose vertex 0 is the start, since all activity times are assumed >0, if finally ee[i]==0 (i≠0) then we can determine that vertex i is not reachable from the start.



Exercises: P389-2, p390-5