TUTORAGGIO 5

* INTEGRALI

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RIPASSO INTEGRALI

Usando il teorema dei residui posso calcolare vari integrali:

I) Data $f(z)$ con solo singolarità isolate e γ curva chiusa:

$$I = \int_\gamma dz\, f(z) = 2\pi i \sum \text{Residui}$$

II) Integrale reale di funzioni trigonometriche nel periodo.

Esempio: $I = \int_0^{2\pi} dx \, \frac{1}{2+\cos x}$

Faccio la sostituzione: $e^{ix} = z$ per cui $\int_0^{2\pi} \to \int_{|z|=1}$

Diventa del tipo I !

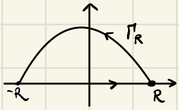
Attenzione: devo avere $f = f(e^{ix})$!

III) Integrali su asse reale di funzioni con sole singolarità isolate

Esempi:
$$I_1 = \int_{-\infty}^{+\infty} dx \, \frac{1}{x^2+1}$$
$$I_2 = \int_{-\infty}^{+\infty} dx \, \frac{x \operatorname{sen}(\pi x)}{(x^2+1)^2}$$

Posso usare lemmi di Jordan. L'idea è che se guardo:

$\tilde{I}_R = \int_{\gamma_R} dz\, f(z)$



: $\lim_{R\to+\infty} \tilde{I}_R = 2\pi i \sum \text{Residui}$

$$\tilde{I}_R = \left(\int_{-R}^{R} + \int_{\Gamma_R} \right) dz\, f(z)$$

- $\lim_{R\to\infty} \int_{-R}^{R} dz\, f(z) = \int_{-\infty}^{+\infty} dz\, f(z) = I$
- $\lim_{R\to+\infty} \int_{\Gamma_R} dz\, f(z) = 0$ per lemma di Jordan

Quindi: $\lim_{R\to\infty} \tilde{I}_R = I = 2\pi i \sum \text{Residui}$

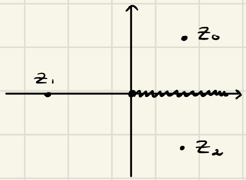
Attenzione alle fregature!
- Se $f(z)$ è dispari $I = 0$
- Se $\int_0^{+\infty} dz\, f(z)$ e $f(z)$ è pari allora $= \frac{1}{2}\int_{-\infty}^{+\infty} dz\, f(z)$

Ma cosa succede se:
- $f(z)$ ha singolarità non isolate
- Dominio $\neq (-\infty, +\infty)$

ESERCIZI

Ⓐ $$I = \int_0^{+\infty} dx \, \frac{x^{1/3}}{x^3+8}$$ Non è pari!

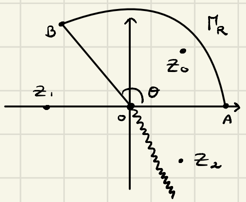
Studio la funzione $f(z) = \dfrac{z^{1/3}}{z^3+8}$



z_k soluzioni di $z^3 = -8$

$$z_k = 2e^{i\pi/3 + 2\pi i k/3} = $$
$$= 2e^{i\pi/3}, \quad -2, \quad 2e^{i\pi 5/3}$$

Nota che il taglio posso metterlo nella direzione che voglio!
Consideriamo la seguente curva γ:



$\theta = \dfrac{2\pi}{3}$

Il taglio è posizionato fuori dalla curva γ

$$\tilde{I} = \int_\gamma dz \, \frac{z^{1/3}}{z^3+8}$$

Studiamo $\lim_{R\to+\infty} \tilde{I}$:

- $\lim_{R\to+\infty} \int_{\Gamma_R} dz\, f(z) = 0$ per I° Jordan : $z f(z) \to 0$ per $|z| \to +\infty$
- $\lim_{R\to+\infty} \int_{OA} dz\, f(z) = I$
- $\lim_{R\to+\infty} \int_{BO} dz\, f(z) = -\int_0^{+\infty} d(e^{2i\pi/3} z)\, f(e^{2i\pi/3} z) =$

$$= -e^{2\pi i/3} \int_0^{+\infty} dz \; e^{2\pi i/9} \frac{z^{1/3}}{z^3 \underbrace{e^{2\pi i}}_{=1} + 8}$$

ecco perché ho scelto $\theta = 2\pi/3$!

$$= -e^{i\pi 8/9} I$$

Però anche:

$$\lim_{R\to+\infty} \tilde{I} = 2\pi i \,\mathrm{Res}[f, z_0] = 2\pi i \,(z-z_0) \frac{z_0^{1/3}}{\underbrace{8+z_0^3}_{=0} + 3z_0^2(z-z_0) + \dots} =$$

$$= 2\pi i \frac{z_0^{1/3} \cdot z_0}{3 z_0^3} = 2\pi i \frac{z_0^{4/3}}{-3\cdot 8}$$

$$= -\frac{i\pi}{12}\, 2^{4/3}\, e^{i\pi 4/3}$$

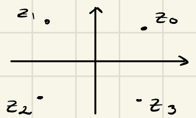
Quindi: $$\tilde{I} = \left(1-e^{i\pi 8/9}\right) I = -\frac{i\pi}{12}\, 2^{4/3}\, e^{i\pi 4/9}$$

$$I = -\frac{i\pi}{12}\, 2^{4/3}\, \frac{1}{e^{-i\pi 4/9} - e^{i\pi 4/9}} =$$

$$= \frac{2^{1/3}\pi}{12 \,\mathrm{sen}\,(4\pi/9)}$$

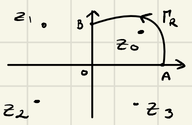
(B) $$I = \int_0^{+\infty} dx\, \frac{x^3 \,\mathrm{sen}\, x^2}{x^4+1}$$

Consideriamo $$f(z) = \frac{z^3 e^{iz^2}}{z^4+1} :$$



Con poli: $z^4+1=0 \;\rightarrow\; z_k = e^{i\pi/4 + 2\pi i k/4}$

Prendiamo:



$$\int_\gamma = \int_{OA} + \int_{\Gamma_R} + \int_{BO}$$

per $R \to +\infty$ fa 0 per II° lemma di Jordan

Quindi per $R \to +\infty$:

- $$\int_{OA} dz\, f(z) = \int_0^{+\infty} dz\, \frac{z^3 e^{iz^2}}{z^4+1}$$

- $$\int_{BO} dz\, f(z) = -e^{i\pi/2} \int_0^{+\infty} dz\, \frac{e^{i\pi 3/2} z^3 e^{iz^2 e^{i\pi}}}{z^4 e^{2\pi i}+1}$$

!! $\quad$ $=1$ ecco perché $\theta = \pi/2$

$$= -\int_0^{+\infty} dz\, \frac{z^3 e^{-iz^2}}{z^4+1}$$

Quindi: $$\tilde{I} = \int_0^{+\infty} dz\, \frac{z^3}{z^4+1}\left(e^{iz^2} - e^{-iz^2}\right) =$$

$$= \int_0^{+\infty} dz\, \frac{z^3}{z^4+1}\, 2i \,\mathrm{sen}\, z^2 = 2i I$$

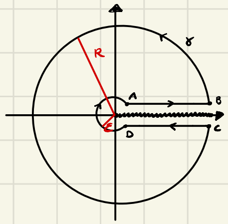
Ma anche $$\tilde{I} = 2\pi i \,\mathrm{Res}\left[\frac{z^3 e^{iz^2}}{z^4+1},\, z = e^{i\pi/4}\right] = 2\pi i \frac{e^{i\pi 3/4}\, e^{i e^{i\pi/2}} \left(z - e^{i\pi/4}\right)}{1 - 1 + 4e^{i\pi 3/4}\left(z - e^{i\pi/4}\right) + \dots}$$

$$= 2\pi i \,\frac{e^{-1}}{4}$$

Quindi: $\quad I = 2\pi i \dfrac{e^{-1}}{4} \cdot \dfrac{1}{2i} = \dfrac{\pi}{4e}$

ⓒ $\quad I = \displaystyle\int_0^{+\infty} dx \, \frac{\log x}{x^{1/2}(x^2+1)}$

Consideriamo: $\quad \displaystyle\int_\gamma dz \, \frac{\log z}{z^{1/2}(z^2+1)}$



$$\int_\gamma = \int_{AB} + \underbrace{\int_{\Gamma_R}}_{=0 \text{ per I° Jordan}} + \int_{CD} + \int_{\Gamma_\varepsilon}$$

Studio $\quad \displaystyle\int_{\Gamma_\varepsilon} dz \, f(z) \overset{\text{per } \varepsilon \to 0}{=} \int_0^{2\pi} \varepsilon \, d(e^{i\theta}) \, \frac{\log \varepsilon + i\theta}{\varepsilon^{1/2} e^{i\theta/2} (\varepsilon^2 e^{2i\theta} + 1)}$

$$\simeq i \varepsilon^{1/2} \int_0^{2\pi} d\theta \, (\log \varepsilon + i\theta) \, e^{i\theta/2}$$

Quindi: $\quad \left| \int_{\Gamma_\varepsilon} dz \, f(z) \right| < \varepsilon^{1/2} \left| 2\pi \log \varepsilon + \underbrace{\sup \left| \theta e^{i\theta/2} \right|}_{=2\pi} \right| = 2\pi \varepsilon^{1/2} \left| \log \varepsilon + 1 \right|$

Per $\varepsilon \to 0$ $\quad \left| \int_{\Gamma_\varepsilon} dz \, f(z) \right| < 0 \quad \longrightarrow \quad \int_{\Gamma_\varepsilon} dz \, f(z) = 0$!

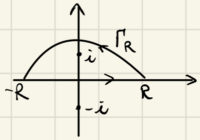
Allora:

$$\tilde{I} = \lim_{\substack{R \to +\infty \\ \varepsilon \to 0}} \left(\int_{AB} + \int_{CD} \right) dz \, f(z) = \int_0^{+\infty} dz \, \frac{\log z}{z^{1/2}(z^2+1)} - \int_0^{+\infty} dz \, \frac{\log z + 2\pi i}{e^{i\pi} z^{1/2}(z^2+1)} =$$

$$= 2 \int_0^{+\infty} dz \, \frac{\log z}{z^{1/2}(z^2+1)} - 2\pi i \underbrace{\int_0^{+\infty} dz \, \frac{1}{z^{1/2}(z^2+1)}}_{I'}$$

Quindi $\quad \tilde{I} = 2I - 2\pi i I'$

Risolvo con $\displaystyle\int_\gamma dz \, \frac{1}{z^{1/2}(z^2+1)}$



Per compito!

D $\quad I = \int_0^{+\infty} dx \, \frac{1}{1+x^n} \qquad$ per $n \geq 2$

I poli sono dati da:

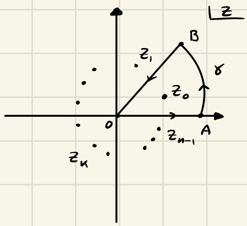
$$z = \sqrt[n]{-1} = e^{\frac{i\pi}{n} + \frac{2\pi i k}{n}} \qquad \text{per } k = 0, \dots, n-1$$

I poli sono quindi in: $\quad z_k = e^{2\pi i \frac{k + 1/2}{n}} = e^{\frac{i\pi}{n}}, e^{\frac{3i\pi}{n}}, e^{\frac{5i\pi}{n}}, \dots$

Nota che non è mai possibile avere $z_k \in \mathbb{R}^+$!

Abbiamo due possibili strategie:

I) Prendo la curva γ:



Dove: $\ast$ $OA = [0, R]$

$\ast$ $AB =$ arco circonferenza raggio R e angolo θ

$\ast$ $BO = [Re^{i\theta}, 0]$

Se prendiamo $\theta = \frac{2\pi i}{n}$ allora $\operatorname{Arg} z_0 < \theta < \operatorname{Arg} z_1 < \operatorname{Arg} z_2 < \dots$

Abbiamo che:

$$\int_{AB} dz\, f(z) = 0 \qquad \text{perché} \quad z f(z) \to 0 \quad \text{se} \quad |z| \to \infty$$

Poi abbiamo:

$$\ast \quad \int_{OA} dz \, \frac{1}{1+z^n} = \int_0^{+\infty} dz \, \frac{1}{1+z^n} = I$$

$$\ast \quad \int_{BO} dz \, \frac{1}{1+z^n} = -\int_0^{+\infty} dz \, e^{\frac{2\pi i}{n}} \frac{1}{1+z^n e^{2\pi i}} =$$

$$= -e^{\frac{2\pi i}{n}} \int_0^{+\infty} dz \, \frac{1}{1+z^n} = e^{\frac{2\pi i}{n}} I$$

Quindi: $\quad \tilde{I} = \int_\gamma dz \, \frac{1}{1+z^n} = \left(1 - e^{\frac{2\pi i}{n}}\right) I$

$$= 2\pi i \operatorname{Res}[f, z_0]$$

Calcolo :

$$\mathrm{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z) =$$

$$= \lim_{z \to z_0} \frac{z - z_0}{1 + z^n} =$$

$$= \lim_{z \to z_0} \frac{z - z_0}{1 + z_0^n + n z_0^{n-1}(z - z_0) + \dots} =$$

$$= \frac{1}{n z_0^{n-1}} =$$

$$= \frac{z_0}{n z_0^n} = - \frac{e^{\frac{i\pi}{n}}}{n}$$

Per cui:

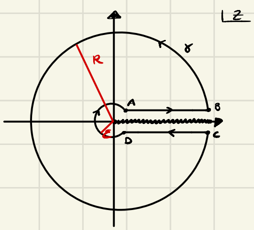
$$I = \frac{2\pi i}{1 - e^{2\pi i/n}} (-) \frac{e^{\frac{i\pi}{n}}}{n} =$$

$$= \frac{2\pi i}{n} \left(e^{i\pi/n} - e^{-i\pi/n} \right) =$$

$$= \frac{\pi}{n \,\mathrm{sen}(\pi/n)}$$

#) Guardiamo l'integrale : $\tilde{I} = \int_\gamma dx \, \frac{\log x}{1 + x^n}$

Consideriamo il dominio di integrazione dato da :



$$\int_\gamma = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} =$$

$$= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4$$

Dove: BC = circonferenza raggio $R \to \infty$

DA = circonferenza raggio $\varepsilon \to 0$

Ma nota che:

* $\tilde{I}_2 = \int_{BC} dz \, \frac{\log z}{1 + z^n} = 0$ perché $z f(z) \to 0$ se $|z| \to \infty$

* $\tilde{I}_4 = \int_{DA} dz \, \frac{\log z}{1 + z^n} = 0$

Poi posso scrivere:

$$\tilde{I}_1 + \tilde{I}_3 = \int_0^{+\infty} dz \, \frac{\log z}{1+z^n} + \int_{+\infty}^{0} dz \, \frac{\log z + 2\pi i}{1+z^n} =$$

$$= \int_0^{+\infty} dz \, \frac{\log z - \log z - 2\pi i}{1+z^n} =$$

$$= -2\pi i \int_0^{+\infty} dz \, \frac{1}{1+z^n} =$$

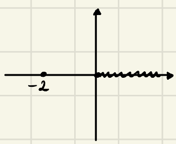
$$= -2\pi i \, I$$

Inoltre: $\tilde{I} = \sum_k \text{Res}[f, z_k]$

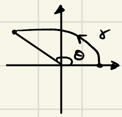
Potrei risolverlo così ma servono n generici residui!

Ⓔ $\quad I = \displaystyle\int_0^{+\infty} dx \, \frac{1}{(x+2)\, x^{1/3}}$

La funzione $f(z) = \dfrac{1}{(z+2)\, z^{1/3}}$ ha un solo polo:



Vorrei prendere una curva γ del tipo:



in questo modo posso spostare il taglio fuori da γ ed ignorarlo.

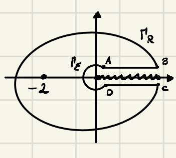
Però θ deve essere scelto affinché $\quad f(e^{i\theta} z) = (\text{prefattore})\, f(z)$

$$= \frac{1}{(z e^{i\theta} + 2)\, z^{1/3} e^{i\theta/3}}$$

Vedo però che deve essere $\quad e^{i\theta} = 1 \rightarrow \theta = 2\pi$.

Ma quindi faccio un giro completo! Non posso ignorare il taglio!

Considero:



$$\tilde{I} = \lim_{\substack{R \to +\infty \\ \varepsilon \to 0}} \left(\int_{AB} + \underbrace{\int_{\Gamma_R}}_{=0 \text{ per I° Jordan}} + \int_{CD} + \int_{\Gamma_\varepsilon} \right) dz \, f(z)$$

$$= 2\pi i \, \text{Res}[f, z=-2]$$

Studio i vari pezzi:

$$\lim_{\varepsilon \to 0} \left| \int_{\Gamma_\varepsilon} dz \, f(z) \right| = \lim_{\varepsilon \to 0} \left| \int_0^{2\pi} \varepsilon i \, d\theta \, \frac{1}{(\varepsilon e^{i\theta} + 2)\, \varepsilon^{1/3} e^{i\theta/3}} \right| \leq \lim_{\varepsilon \to 0} \varepsilon^{2/3} \underbrace{\sup |e^{i\theta/3}|}_{1}$$

$$\leq 0$$

Quindi: $\displaystyle\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} dz \, f(z) = 0$

$$\lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \int_{AB} dz\, f(z) = \int_0^{+\infty} dz\, f(z) = I$$

$$\lim_{\substack{R\to+\infty\\ \varepsilon\to 0}} \int_{CD} dz\, f(z) = -\int_0^{+\infty} dz\, \frac{1}{(z e^{2\pi i}+2)\, z^{1/3}\, e^{2\pi i/3}} = -e^{-2\pi i/3}\, I$$

Quindi:

$$\left(1-e^{-2\pi i/3}\right) I = 2\pi i \,\operatorname{Res}[f, -2]$$

$$= 2\pi i\, (-2)^{1/3} = 2\pi i\, 2^{1/3}\, e^{-i\pi/3}$$

$$I = 2\pi i\, 2^{1/3}\, \frac{e^{-i\pi/3}}{1-e^{-2\pi i/3}} = 2\pi i\, 2^{1/3}\, \frac{1}{e^{i\pi/3}-e^{-i\pi/3}}$$

$$= \frac{2^{1/3}\,\pi}{\operatorname{sen} \pi/3}$$