Notes of Math 719: Partial Differential Equation
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Contents

1 Laplace's equation

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Elliptic PDEs:

Example 1.1. Laplacian:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial}{\partial x_n^2}$$
$$= div\nabla$$

Example 1.2. n = 3: Newtonian gravity

$$\text{magnitutde} = \frac{\kappa m m_1}{\|a - a_1\|} \quad \text{(inverse square)}$$
$$m\ddot{a} = \frac{\kappa m m_1}{\|a - a_1\|} \frac{a_1 - a}{\|a_1 - a\|}$$
$$= -m\nabla u(a)$$
$$\text{where} \quad u(x) = \frac{-k m_1}{\|x - a_1\|}$$

N masses $m_1, \ldots, m_N \geqslant 0$, location $a_1, \ldots, a_N \in \mathbb{R}^3$

$$u(x) = -\kappa \sum_{k=1}^{N} \frac{m_k}{\|x - a_k\|}$$

Example 1.3. continuous distribution of mass

$$\varrho(x) \geqslant 0$$

$$u(x) = -\kappa \int_{\mathbb{R}^3} \frac{\varrho(y)}{\|x - y\|} \, dy$$

suppose supp $\varrho \subseteq \omega$ bdd open set

Remark. $supp\varrho := \overline{\{x \in \mathbb{R}^3 : S(x) \neq 0\}}$

Laplace

$$\Delta u(x) = 0 \, for x \in \mathbb{R}^3 \bar{\omega}$$

$$\Delta_x u(x) = -\kappa \int_\omega \varrho(y) \Delta_x \frac{1}{\|x - y\|} \, dy$$

$$\Delta \frac{1}{\|x\|} = div\nabla \frac{1}{\|x\|}$$
$$= div\left(-\frac{x}{\|x\|^3}\right)$$
$$= \left(-\frac{3}{\|x\|^3} + 3x\frac{x^2}{\|x\|^5}\right)$$

Example 1.4.

$$\varrho = \text{const on } B_\mathbb{R}$$
$$= \frac{m}{4\pi\mathbb{R}^3}$$
$$and u = -\kappa m \int_{B_\mathbb{R}} \frac{1}{\|x - y\|} \, dy$$

Claim 1.1.

$$u(x) = u(Ox) \text{ where} O \in SO(3)$$
$$u(Ox) = -\kappa m \int_{B_\mathbb{R}} \frac{1}{\|Ox - Oz\|} \, dz$$
$$= -\kappa m \int_{B_\mathbb{R}} \frac{1}{\|x - z\|} \, dz$$

$$u(x) = u(r)$$
$$u : \mathbb{R}^3 \to \mathbb{R}$$

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$$u(x) = f(r)$$

[**YW:** TODO: Complete notes between Sep 4 and Sep 13]

[**Date:** Sep 13, 2024]

Last time: Interior estimate

Suppose $R > 0$ and $u \in C^2(B_{2R})$ is harmonic, then

$$\|\nabla^k u(x)\| \leqslant C_k \frac{1}{R^{k+3}} \int_{B_{2R} \setminus B_R} \|u\| \, dy \quad \forall x \in B_R \tag{1}$$

Question: Why estimate of this form?

Symmetries of $\Delta u = 0$.

1. homeoneity/scaler mult

$$u \mapsto \mu \quad (m \in \mathbb{R})$$

2. scaling symmetry

$$u_{\lambda(x)} := u(\lambda x)$$
$$u \mapsto u_\lambda \, (u(\lambda \cdot))$$

Here $\Delta u_\lambda = \Delta[u(\lambda x)] = \lambda^2 \Delta u(\lambda u) = 0$

3. translation:

$$u \mapsto u(\cdot - x_0) \quad (x_0 \in \mathbb{R}^3)$$

4. rotation and reflection

$$u \mapsto u(\mathcal{O}^{-1}) \quad (\mathcal{O} \in O(n))$$

1. $k = 0, R = 1$, What if

$$\|mu\|(x) \leqslant C \int_{B_2 \setminus B_1} \|mu\|^2 \, dy \quad \text{in } B_1$$
$$\implies \|u(x)\| \leqslant Cm \int_{B_2 \setminus B_1} \|u\|^2 \, dy$$
$$\implies u = 0$$

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Then this is nonsense. We need the power in the integral to be 1, i.e.,

$$\|mu\|(x) \leqslant C \int_{B_2 \setminus B_1} \|mu\| \, dy \quad \text{in } B_1$$

To prove (1), it suffices to prove it for

$$\int_{B_{2R} \setminus B_R} \|u\| = 1$$

Suppose (1) holds for all harmonic function u s.t.

$$\int_{B_{2R} \setminus B_R} \|u\| = 1$$

Given arbitrary harmonic function v, we define

$$u := \frac{v}{\int_{B_{2R} \setminus B_R} \|v\| \, dy}$$

Then u is still harmonic

$$\implies \frac{\|v\|}{\int_{B_{2R} \setminus B_R} \|v\| \, dy} = \|u\| \leqslant C \frac{1}{R^3}$$

To prove (1), it suffices to prove it for $R = 1$

Given harmonic function v on B_{2R}

$$u(x) := v(Rx)$$

Here $u(x)$ is a harmonic function on B_2.

Then

$$\|\nabla^k u(x)\| \leqslant C \int_{B_2 \setminus B_1} u(y) \, dy$$

$$= \int_{B_2 \setminus B_1} \|v(Ry)\| \, dy$$

(Change of Variable: $z = Ry$)

$$= \frac{C}{R^3} \int_{B_{2R} \setminus B_R} \int \|v(z)\| \, dz$$

where $\|\nabla^k (v(Rx))\| := R^k \|(\nabla^k v)(Rx)\| = R^k \|(\nabla^k v)(q)\|$ where $q = Rx \in B_R$

To prove (1), it's enough to do it for $R = 1$ and $\int_{B_2 \setminus B_1} \|u\| \, dy = 1$

1.1

Given

$$\phi u = 2\Delta * (\Delta\phi u) + G * (\Delta\phi u) \tag{2}$$

where $u \in C^2$ and $\Delta u = 0$ and $\phi \in C_0^\infty$

One option $u \in L_{loc}^1$ ($u \in L^1(K)$) where K is compact , it's called locally integrable, is harmonic if (2) holds $\forall\phi \in C_0^\infty$

Another option $u \in L_{loc}^1$ is harmonic if

$$\nabla(u * \phi_\epsilon) = 0 \qquad \text{for all } \epsilon$$

Example 1.5.

$\frac{1}{|x|}$ is $L_{loc}^1(\mathbb{R}^3)$

$(1 + |x|)^{-3-\epsilon} \in L^1(\mathbb{R}^3)$

Definition 1.1. $\Omega \subseteq R^3$ open, $u \in L_{loc}^1(\Omega)$ is weakly harmonic if

$$\int u\nabla\phi \, dy = 0 \quad \forall\phi \in C_0^\infty(\Omega)$$

Remark.

$$\int \Delta u \cdot \phi = \int (div\nabla u)\phi$$
$$= -\int \nabla u \nabla\phi = \int u\nabla\phi$$

This is integration by parts.

If you need a generalization, you need to make it easy to check and easy to work with.

Lemma 1.2 (Weyl's lemma)**.**

If u is weakly harmonic in Ω, then u is smooth and $\Delta u = 0$ in Ω

To prove this, we need

1. If u is C^2 and $\Delta u = 0$, then u is weakly harmonic

2. If u is C^2 and weakly harmonic, them $\Delta u = 0$

Proof of 2. above. Suppose not.

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$$\int u\Delta\phi = 0 \quad \forall\phi$$

But $\exists x_0$ s.t. $\Delta u(x_0) \neq 0$

$$\int \Delta u\phi = 0 \quad \phi$$

Choose ϕ s.t.

$$\int \Delta u\phi \neq 0$$

Countradiction. $\qquad\qquad\square$

proof of Wely's lemma.

    (3) $u \in L^1_{loc}$ is weakly harmonic, then $\phi_\epsilon * u$ is also weakly harmonic.

$$\implies \phi_\epsilon * u \text{ is strongly harmonic}$$

We need to check

$$\int (u * \phi_\epsilon)\Delta\psi \, dy = 0 \quad \forall\psi$$

$\qquad\qquad\square$