Notes of Math 719: Partial Differential Equation
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Contents

1 Laplace's equation

[**Date:** Sep 4, 2024]

Elliptic PDEs:

Example 1.1. Laplacian:

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial}{\partial x_n^2} \\ &= div \nabla \end{aligned}$$

Example 1.2. n = 3: Newtonian gravity

$$\begin{aligned} \text{magnitutde} &= \frac{\kappa m m_1}{\|a - a_1\|} \quad \text{(inverse square)} \\ m\ddot{a} &= \frac{\kappa m m_1}{\|a - a_1\|} \frac{a_1 - a}{\|a_1 - a\|} \\ &= -m \nabla u(a) \\ \text{where} \quad u(x) &= \frac{-k m_1}{\|x - a_1\|} \end{aligned}$$

N masses $m_1, \ldots, m_N \geqslant 0$, location $a_1, \ldots, a_N \in \mathbb{R}^3$

$$u(x) = -\kappa \sum_{k=1}^{N} \frac{m_k}{\|x - a_k\|}$$

Example 1.3. continuous distribution of mass

$$\begin{aligned} \varrho(x) &\geqslant 0 \\ u(x) &= -\kappa \int_{\mathbb{R}^3} \frac{\varrho(y)}{\|x-y\|}\, dy \end{aligned}$$

suppose supp $\varrho \subseteq \omega$ bdd open set

Remark. *supp*$\varrho := \overline{\{x \in \mathbb{R}^3 : S(x) \neq 0\}}$

Laplace

$$\Delta u(x) = 0\, for x \in \mathbb{R}^3 \bar{\omega}$$

$$\Delta_x u(x) = -\kappa \int_{\omega} \varrho(y) \Delta_x \frac{1}{\|x-y\|}\, dy$$

$$\begin{aligned} \Delta \frac{1}{\|x\|} &= div \nabla \frac{1}{\|x\|} \\ &= div(-\frac{x}{\|x\|^3}) \\ = \big(-\frac{3}{\|x\|^3} &+ 3x\frac{x^2}{\|x\|^5}) \end{aligned}$$

Example 1.4.

$$\begin{aligned} \varrho &= \text{const on } B_{\mathbb{R}} \\ &= \frac{m}{4\pi\mathbb{R}^3} \\ \text{and} u &= -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|x-y\|}\, dy \end{aligned}$$

Claim 1.1.

$$\begin{aligned} u(x) &= u(Ox)\, \text{where} O \in SO(3) \\ u(Ox) &= -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|Ox - Oz\|}\, dz \\ &= -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|x - z\|}\, dz \end{aligned}$$

$$u(x) = u(r)$$
$$u : \mathbb{R}^3 \to \mathbb{R}$$
$$u(x) = f(r)$$

[**YW:** TODO: Complete notes between Sep 4 and Sep 13]

[**Date:** Sep 13, 2024]

Last time: Interior estimate

Suppose $R > 0$ and $u \in C^2(B_{2R})$ is harmonic, then

$$\|\nabla^k u(x)\| \leqslant C_k \frac{1}{R^{k+3}} \int_{B_{2R} \setminus B_R} \|u\| \, dy \quad \forall x \in B_R \tag{1}$$

Question: Why estimate of this form?

Symmetries of $\Delta u = 0$.

1. homeoneity/scaler mult

$$u \mapsto \mu \quad (m \in \mathbb{R})$$

2. scaling symmetry

$$u_{\lambda(x)} := u(\lambda x)$$
$$u \mapsto u_\lambda \; (u(\lambda \cdot))$$

   Here $\Delta u_\lambda = \Delta[u(\lambda x)] = \lambda^2 \Delta u(\lambda u) = 0$

3. translation:

$$u \mapsto u(\cdot - x_0) \quad (x_0 \in \mathbb{R}^3)$$

4. rotation and reflection

$$u \mapsto u(\mathcal{O}^{-1}) \quad (\mathcal{O} \in O(n))$$

1. $k = 0, R = 1$, What if

$$\|mu\|(x) \leqslant C \int_{B_2 \setminus B_1} \|mu\|^2 \, dy \quad \text{in } B_1$$

$$\begin{aligned}\implies \|u(x)\| &\leqslant Cm \int_{B_2\backslash B_1} \|u\|^2 \, dy \\ &\implies u = 0\end{aligned}$$

Then this is nonsense. We need the power in the integral to be 1, i.e.,

$$\|mu\|(x) \leqslant C \int_{B_2\backslash B_1} \|mu\| \, dy \quad \text{in } B_1$$

To prove (1), it suffices to prove it for

$$\int_{B_{2R}\backslash B_R} \|u\| = 1$$

Suppose (1) holds for all harmonic function u s.t.

$$\int_{B_{2R}\backslash B_R} \|u\| = 1$$

Given arbitrary harmonic function v, we define

$$u := \frac{v}{\int_{B_{2R}\backslash B_R} \|v\| \, dy}$$

Then u is still harmonic

$$\implies \frac{\|v\|}{\int_{B_{2R}\backslash B_R} \|v\| \, dy} = \|u\| \leqslant C \frac{1}{R^3}$$

To prove (1), it suffices to prove it for $R = 1$

Given harmonic function v on B_{2R}

$$u(x) := v(Rx)$$

Here $u(x)$ is a harmonic function on B_2.

Then

$$\begin{aligned}\|\nabla^k u(x)\| &\leqslant C \int_{B_2\backslash B_1} u(y) \, dy \\ &= \int_{B_2\backslash B_1} \|v(Ry)\| \, dy \\ \text{(Change of Variable: } z = Ry) \\ &= \frac{C}{R^3} \int_{B_{2R}\backslash B_R} \int \|v(z)\| \, dz\end{aligned}$$

where $\|\nabla^k (v(Rx))\| := R^k \|(\nabla^k v)(Rx)\| = R^k \|(\nabla^k v)(q)\|$ where $q = Rx \in B_R$

To prove (1), it's enough to do it for $R = 1$ and $\int_{B_2 \setminus B_1} \|u\| \, dy = 1$

1.1

Given

$$\phi u = 2\Delta * (\Delta \phi u) + G * (\Delta \phi u) \tag{2}$$

where $u \in C^2$ and $\Delta u = 0$ and $\phi \in C_0^\infty$

One option $u \in L^1_{loc}$ ($u \in L^1(K)$) where K is compact , it's called locally integrable, is harmonic if (2) holds $\forall \phi \in C_0^\infty$

Another option $u \in L^1_{loc}$ is harmonic if

$$\nabla(u * \phi_\epsilon) = 0 \qquad \text{for all } \epsilon$$

Example 1.5.

$\frac{1}{|x|}$ is $L^1_{loc}(\mathbb{R}^3)$

$(1 + |x|)^{-3-\epsilon} \in L^1(\mathbb{R}^3)$

Definition 1.1. $\Omega \subseteq R^3$ open, $u \in L^1_{loc}(\Omega)$ is weakly harmonic if

$$\int u \nabla \phi \, dy = 0 \quad \forall \phi \in C_0^\infty(\Omega)$$

Remark.

$$\int \Delta u \cdot \phi = \int (div \nabla u) \phi$$
$$= - \int \nabla u \nabla \phi = \int u \nabla \phi$$

This is integration by parts.

If you need a generalization, you need to make it easy to check and easy to work with.

Lemma 1.2 (Weyl's lemma)**.**

If u is weakly harmonic in Ω, then u is smooth and $\Delta u = 0$ in Ω

To prove this, we need the following claim:

Claim 1.3.

1. If u is C^2 and $\Delta u = 0$, then u is weakly harmonic
2. If u is C^2 and weakly harmonic, them $\Delta u = 0$

Proof of Claim 1.3. Suppose not.

$$\int u\Delta\phi = 0 \quad \forall\phi$$

But $\exists x_0$ s.t. $\Delta u(x_0) \neq 0$

$$\int \Delta u\phi = 0 \quad \phi$$

Choose ϕ s.t.

$$\int \Delta u\phi \neq 0$$

Countradiction. $\square$

proof of Wely's lemma.

(3) $u \in L^1_{loc}$ is weakly harmonic, then $\phi_\epsilon * u$ is also weakly harmonic.

$$\implies \phi_\epsilon * u \text{ is strongly harmonic}$$

We need to check

$$\int (u * \phi_\epsilon)\Delta\psi \, dy = 0 \quad \forall\psi$$

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Remark. $\Delta(f * \phi_\epsilon) = \Delta f * \phi_\epsilon \implies$ Mollify harmonic function , get a harmonic function.

Enough to work with balls. Enough to work in B_3 and prove smoothness in B_1.
Because of translation and scalling symmetry.
$u \in L^1(B_3)$ weakly harmonic. We define

$$u_\epsilon(x) = u * \phi_\epsilon(x) \quad \text{for } x \in B_2 \text{ and } 0 < \epsilon \leqslant \frac{1}{2}$$

Want to check u_ϵ is weakly harmonic in B_2.

$$\begin{aligned}
\forall\psi \in C_0^\infty(B_2) : \int_{B_2} u_\epsilon(x)\Delta\psi(x)\,dx &= \int_{B_2}\int_{\mathbb{R}^3} \phi_\epsilon(x-y)u(y)\Delta\psi(x)\,dy\,dx \\
&= \int u(y)(\phi_\epsilon(-\cdot) * \Delta\psi)(y)\,dy \\
&= \int_{\mathbb{R}^2} u(y)\Delta(\phi_\epsilon(-\cdot)\psi)(y)\,dy \\
&= 0 \text{ (by definition of weakly harmonic)}
\end{aligned}$$

$\square$

Use interior estimates on u_ϵ

$$|\nabla^k u_\epsilon| \leqslant C_k \int_{B_2 \setminus B_1} \int |u_\epsilon| \, dy \leqslant C_k \int_{B_3} |u| \, dy$$

Use Arzela-Ascoli: For all k, $\nabla^k u_\epsilon \longrightarrow \nabla^k u$ uniformly in B_1

Weak version of $\Delta u = f$?

Definition 1.2. $\Omega \subseteq \mathbb{R}^3$ open, $u, f \in L^1_{loc}(\Omega)$. Then we say that

$$\Delta u = f \quad \text{weakly harmonic in } \Omega$$

if

$$\int u \Delta \psi = \int f \psi \quad \forall \psi \in C_0^\infty(\Omega)$$

Previously: $f \in C^2$ and compactly supported

$$\Delta(G * f) = f$$

General f holds?

Example 1.6. $G * f$ makes sense for $f \in L^1$

$$-\frac{1}{4\pi} \int \frac{f(y)}{|x-y|} \, dy$$

Notice that $\frac{1}{|x|}$ is not integrable.

Now let $-\frac{1}{4\pi|x|} = \text{G}$. And $G_1 = G\mathbf{1}_{B_1} \in L^1 \bigcap L^{3-}$, $G_2 = G\mathbf{1}_{\mathbb{R}^3 \setminus B_1} \in L^\infty \bigcap L^{3+}$, where L^3 means that $L^{3-\epsilon} \quad \forall \epsilon > 0$.

$$\int_{B_1} \frac{1}{|x|^3} = c \int_{r=0}^1 r^{-3} r^2 \mathrm{d}x = \infty$$
$$\int_{\mathbb{R}^3 \setminus B_1} = c \int_{r=1}^\infty r^{-3} r^2 \mathrm{d}x = \infty$$

And $G * f = \underbrace{G_1 * f}_{\in L^1 \bigcap L^{3-}} + \underbrace{G_2 * f}_{\in L^{3+} \bigcap L^\infty} \in L^1_{loc}$

Exercise 1.1. $f \in L^1 + L^p$ if $p < \frac{n}{2}$

Claim 1.4. $\Delta(G * f) = f$ is weakly harmonic in $\mathbb{R}^3$

Proof. To check:

$$\int (G*f)\Delta\varphi = \int f\varphi \forall\varphi \in C_0^\infty(\mathbb{R}^3)$$

$$\begin{aligned}\int (G*f)\Delta\varphi = \lim_{\epsilon\to 0^+}\int (K_\epsilon * f)\Delta\varphi = \lim_{\epsilon\to 0^+}\int (\Delta K_\epsilon) * f\varphi \\ = \int f\varphi\end{aligned}$$

□

Proposition 1.5. Suppose $u_1, u_2,\ f \in L^1_{loc}$ and $\Delta u_1 = \Delta u_2 = f$ weakly harmonic in $\mathbb{R}^3$ Then

1. $u_1 - u_2$ is smooth and harmonic
2. If u_1, u_2 are bounded, $u_1 - u_2$ is constant.
3. If $|u_1|, |u_2| \longrightarrow 0$ as $|x| \longrightarrow \infty$, (say $u_1, u_2 \in L^1 + L^p, p < \infty$) then $u_1 \equiv u_2$.

   Exercise 1.2. show $u_1 - u_2$ is bounded

[**Date:** Sep 18, 2024]
Mean value formula/property

Proposition 1.6. Suppose u is harmonic on $B_{\mathbb{R}}(x_0)$. Then

1. $u(x_0) = \fint_{\partial B_r(x_0)} u(y)\,dS \quad \forall r \in (0,R)\ u(x_0) =$ avg of u over $\partial B_r(x_0)$
2. $u(x_0) = \fint_{B_r(x_0)} = u(y)\,dy \quad \forall r \in (0,R)\ u(x_0) =$ avg of u over $B_r(x_0)$

Proof. 1. proof of (1).

$$\fint_{\partial B_r(x_0)} u(y)\,dS = q(r) \longrightarrow u(x_0) \text{ as } r \to 0^+$$

To show: $\frac{dq}{dr} = 0 \quad (q \equiv$ const $)$

$$\begin{aligned} x_0 &= 0 \\ &= \frac{1}{|\partial B_1| r^{n-1}} \int_{\partial B_r} u(y)\,dS \\ (r = rz \in \partial B_1) &= \frac{1}{|\partial B_1|}\int_{\partial B_1} u(rz)\,d\tilde{S} \end{aligned}$$

We know $dS = r^{n-1} d\tilde{S}$.

$$\begin{aligned}\frac{dq}{dr} &= \frac{1}{|\partial B_1|}\int_{\partial B_1} (\nabla u)(rz)\cdot z\,d\tilde{S} \\ &= -\frac{1}{|\partial B_1|}\int_{B_1} div[(\nabla u)(rz)]\,dz = 0\end{aligned}$$

$$\implies \Delta u(rz)r = 0$$

The second proof:

$\varphi = G * (\Delta\varphi u + 2\nabla\varphi\nabla u) := G * f$ where $f := \Delta\varphi u + 2\nabla\varphi\nabla u$

Choose $\varphi \equiv 1$ on $B_{r+\epsilon}$.

$$\begin{aligned} u(0) &= (G * f)(0) \\ &= \int G(0-y)f(y)\,dy \end{aligned}$$

$$\begin{aligned} \fint_{B_r} u(x)\,dx &= \fint_{B_r}\int_y G(x-y)f(y)\,dydx \\ &= \int_y \fint_{B_r} G(x-y)\,dxf(y)\,dy \end{aligned}$$

Here we know $G(0-y)$ gravitational point of point mass at 0 measured at $-y$. $\fint_{B_r} G(x-y)\,dx$ gravitational point of body with mass distributed over B_r measured at $-y$.

□

1.2 Maximum principle

Lemma 1.7. Suppose u is harmonic in $B_R(x_0)$ and suppose $u(x_0) = \sup_{B_R(x_0)} u(x)$, then

$$u \equiv \text{ const}$$

Example 1.7. n = 2.

1. 1, x_1, x_2
2. $x_1^2 - x_2^2$. Suppose $u(0) = 0$, $\nabla u(0) = 0$

$$\nabla^2 = \begin{bmatrix} ,2 \\ 1,2 \end{bmatrix} = []$$

row avB $u = \frac{\lambda_1}{2}y_1^2 + \frac{\lambda_2}{2}y_2^2$ and $\text{tr}\nabla^2 u = \Delta u = \lambda_1 + \lambda_2$

Proof.

$$\begin{aligned} u(x_0) &\geqslant u(x) \quad \forall x \in b_R(x_0) \\ u(x_0) \geqslant \fint_{B_R(x_0)} u(x)\,dx &= u(x_0) \quad \text{mean val property} \end{aligned}$$

$\square$

Remark. $\sup_\Omega u = \sup_{\partial\Omega} u$

Corollary 1.8 (strong maximum principle)**.** $\Omega \in \mathbb{R}^3$ bounded domain

Suppose u harmonic in Ω and $u(x_0) = \sup_\Omega u \quad$ for some $x_0 \in \Omega$

Then

$$u \equiv \text{ const}$$

Remark. $S = \{x \in \Omega : u(x) = u(x_0)\}$

If S is non-empty, open and closed (in Ω). "Relatively closed". Since Ω is connected:

$$S = \Omega$$

Remark. When you have a maximum principle, and then you have a Minimum principle. And you have a Comparison principle

$$u_1, u_2 \text{ on } \Omega$$
$$u_1 \geqslant u_2 \text{ on } \partial\Omega$$

Then we have

$$u_1 > u_2 \text{ on } \Omega (\text{ or } u_1 = u_2)$$

And we prove this by $u_1 - u_2$ also harmonic function. $u_1 - u_2 \geqslant 0$ on $\partial\Omega$. Then we apply min principle.

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Minimum Principle:

Given $\Omega \subseteq \mathbb{R}^3$ bounded domain. Suppose u is harmonic in Ω and $u(x_0) = \inf_\Omega u$ for some $x_0 \in \Omega$. Then $u \equiv$ const.

Proposition 1.9 (Harneck's principle/inequality)**.** Suppose u is harmonic and $u > 0$ in Ω. Let $K \subseteq \Omega$ compact set. Then $\exists C = C(\Omega, K) > 0$ s.t.

$$\sup_K u \leqslant C \inf_K u$$

Remark. This implies minimum principle: let $v = u - \inf_K u + \epsilon$

We have $\inf_K v = \epsilon \implies \sup_K v \leqslant C \inf_K v = C\epsilon$

Proof. (In Evans, he proved with mean value formula. $u(x) = \fint_{B_R(x)} u \, dy$. Relate the value between two balls.)

We used to use scaling argument to prove things. Now we use compactness argument.

Harneck: $\forall \Omega, \forall K \subseteq \Omega$ compact, $\exists C$ s.t.

$\forall u > 0$ on Ω harmonic,

$$\sup_K u \leqslant C \inf_K u$$

Negsta: $\exists \Omega$, $\exists K \subseteq \Omega$ compact s.t. $\forall C > 0$:

$\exists u_C > 0$ harmonic on Ω s.t.

$$\sup_K u > C \inf_K u$$

Suppose $C = N \in \mathbb{N}$:

$$\sup_K u > C \inf_K u$$

Define $v_N := \frac{u_N}{\sup_K u_N}$. We have

$$1 = \sup_K v_N > N \inf_K v_N \implies \inf_K v_N < \frac{1}{N}$$

Let $a \in K$. Find $B_{r_a} \in \Omega$

$$1 \geqslant v_N(a) = \fint_{B_r(a)} v_N(y)\, dy$$

$$\underbrace{|\nabla v_N^k(x)|}_{x \in B_{\frac{r_a}{2}}(a)} \leqslant C(k) r_a^{-k} \fint_{B_r(a)} v_N(y)\, dy$$

Interior estimates theorem: $\Omega_1 = \bigcup_{i=1}^M B_{r_{a_1}}(a_i)$ finitely many balls. Here Ω_1 covers K.

$$|\nabla^k v_N| \leqslant C(k) \quad x \in \Omega_1$$

And we have $C(k) = C \max_{i=1,\dots,M} r_{a_1}^{-k}$.

By Arzela-Ascoli $\implies \exists$subsequence (not relabeling) $v_N \to v_\infty$ uniformly in Ω, v_∞ is harmonic in Ω

$$\sup_K v_\infty = 1,\ \inf_K v_\infty = 0,\ v_\infty \geqslant 0$$

Countradiction! (Since this implies that v is constant.) $\square$

Remark. It's called compactness-contradiction argument cf.Terry Tao.

1.3 Boundary value problems (BVPs)

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{Dirichlet BCs} \end{cases} \tag{3}$$

Example 1.8. electrostatics: density of temperature or concentration of chemical... We have <u>diffusion</u>.

Given $c(x,t)$. (Fick's law)

$$\begin{aligned} \frac{d}{dt}\int_O c(x,t)\,dx &= \kappa\int_{\partial O} \nabla c\cdot\vec{n}dS \\ &= \kappa\int_O div\nabla c\,dx = \kappa\int_O \Delta c\,dx \quad \forall O \end{aligned}$$

This is Heat equation

$$\partial_t c = \kappa\Delta c \quad in\Omega$$

If there is no flux:

$$\begin{aligned} \nabla c\cdot\vec{n}|_{\partial\Omega} = 0 \quad \text{Neumann BCs} \\ \frac{\partial c}{\partial n} = 0 \end{aligned}$$

[**Date:** Sep 23, 2024]

Def: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ Dirichlet problem and Neumann problem.

$$\begin{cases} \Delta u = f, & \text{in}\Omega \\ u|_{\partial\Omega} = g \end{cases} \tag{4}$$

Now consider

$$\begin{cases} \Delta u = f, & \text{in } \mathbb{R}^3_+ \\ u|_{\partial\mathbb{R}^3_+} = g \end{cases} \tag{5}$$

$R\vec{x} = (x_1, x_2, -x_3)$ and $\Delta[u(R\vec{x})] = (\Delta u)(R\vec{x})$

By Odd-in-x_3, we mean

$$\begin{aligned} u(\vec{x}) &= -u(R\vec{x}) \\ u(x_1,x_2,x_3) &= -u(x_1,x_2,-x_3) \end{aligned}$$

(If constant, then $u|_{\partial\mathbb{R}^3_+} = 0$)

$$\tilde{f}(x) = \begin{cases} f(\vec{x}), & x_3 > 0 \\ -f(R\vec{x}), & x_3 < 0 \end{cases} \tag{6}$$

$$\tilde{(u)}(x) \text{ sol of } \Delta\tilde{u} = \tilde{u} = \tilde{f} \text{ on } \mathbb{R}^3 \tag{7}$$

$$\tilde{(u)}(x) = (G * \tilde{f})(x) \tag{8}$$

where $\tilde{u}$ is odd.

$$\begin{aligned} \Delta\tilde{u} &= \tilde{f} \\ \underbrace{(\Delta\tilde{u}(R\vec{x}))}_{=\Delta[\tilde{u}(R\vec{x})]} &= \tilde{f}(R\vec{x}) \\ &= -\tilde{f}(x) \\ \implies \Delta[-\tilde{u}(R\vec{x})] = \tilde{f}(x) \end{aligned}$$

sol unique means:

$$\begin{aligned} \implies \tilde{(u)}(\vec{x}) &= -\tilde{u}(R\vec{x}) \\ \implies \tilde{u} &\text{ is odd} \end{aligned}$$

Now

$$\begin{aligned} \tilde{u}(x) &= G * \tilde{f} \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\tilde{f}(y)}{|x-y|}\,dy \\ = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\Big[\mathbf{1}_{\{y_3>0\}} f + \mathbf{1}_{\{y_3<0\}}(-f)(R\vec{y})\Big]\,dy \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3_+} \frac{1}{|x-y|} f(y) - \frac{1}{|x-R\vec{y}|} f(y)\,dy \\ &\overset{y*=Ry}{=} -\frac{1}{4\pi}\int_{\mathbb{R}^3_+} \Big[\frac{1}{|x-y|} - \frac{1}{|x-y*|}\Big] f(y)\,dy \end{aligned}$$

Then we get:

$$G_{\mathbb{R}^3_+}(x,y) = -\frac{1}{4\pi}\Big(\frac{1}{|x-y|} - \frac{1}{|x-y*|}\Big) \quad \text{Green's function for half space}$$

Thus the Boundary value problem is:

$$\begin{cases} \Delta u = f, & \text{in } \mathbb{R}^3_+ \\ u|_{\partial \mathbb{R}^3_+} = g \end{cases}$$

$$\begin{aligned} \implies u(x) &= \int_{R^3_+} G_{\mathbb{R}^3_+}(x-y) f(y)\, dy \\ &= \int_{\{x_3>0\}} \int_{R^2} \quad \text{"Horizontal Convolution"} \end{aligned} \tag{9}$$

Remark. This is not a convolution.

1.3.1 General props of Green's function

$$\begin{cases} \Delta u = f, \\ u|_{\partial\Omega} = 0, \end{cases}$$

$$\begin{aligned} \iff u(x) &= \int_\Omega G_\Omega(x,y) f(y)\, dy \\ &= \int_\Omega \delta_x(y) f(y)\, dy \\ &= f(x) \end{aligned} \tag{10}$$

0 Boundary

$$G(\cdot, y)|_{\partial\Omega} = 0$$
$$G(x,y) = 0 \quad \text{when } x \in \partial\Omega$$

Physically Green's function: Heat but the boundary keeps temperature 0.

Fix $y \in \Omega$. Look for sol to

$$\Delta_x G_\Omega(x,y) = \delta(y-x)$$
$$G_\Omega(\cdot,y)|_{\partial\Omega} = 0$$

$$G_\Omega(x,y) = K_0(x,y) + H^{(y)}(x) \quad \text{where } K_0 = -\frac{1}{4\pi|x|} \text{ and } K_0(x,y) = -\frac{1}{4\pi|x-y|} \text{ where space Green's func} \tag{11}$$

where $H^{(y)}$ should be harmonic in x.

$$\Delta_x(K_0(x,y) + H^{(y)}(x)) = \delta(y-x) + 0$$

And

$$K_0(x,y) + H^{(y)}(x) = 0 \quad \text{when } x \in \partial\Omega$$

Then

$$\begin{cases} \Delta_x H^{(y)}(x) = 0, \\ H^{(y)}(x)|_{\partial\Omega} = K_0(x,y), \end{cases} \tag{12}$$

Philosophy:

1. ~~To solve the problem in Ω, find G_Ω.~~

1* Solve problem using a different method

2. To find G_Ω, need to solve problem for $H^{(y)}$.

3. Read infrastion off of G_Ω, like on HW4

$$G_\Omega(x,y) = G_\Omega(y,x)$$

Green's function is symmetric

$$\begin{cases} u, v \in C^2(\bar{\Omega}), \\ u, v|_{\partial\Omega} = 0, \end{cases} \tag{13}$$

$$\int_\Omega \Delta uv = \int_\Omega u\delta v$$
$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, x, y \in \mathbb{R}^n$$
$$\iff A \text{ symmetric}$$

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$$\begin{aligned} \Delta u &= f \\ u|_{\partial\Omega} &= 0 \\ \iff \int_\Omega G(x,y)f(y)\,dy &\quad \text{where you put the pole in } y \\ \Delta_x G(x,y) &= \delta(x-y) \\ G(x,y) &= 0 \quad x \in \partial\Omega \end{aligned} \tag{14}$$

Question:

$$\begin{aligned} \Delta u = 0 &\quad \text{in } \Omega \\ u|_{\partial\Omega} &= g \end{aligned} \tag{15}$$

Definition 1.3 (Green's formula)**.**

$$\int_\Omega div\vec{v}\,dx = \int_{\partial\Omega} \vec{v}n\,dS \tag{16}$$

$$
\begin{aligned}
\vec{v} &= \vec{u}f \\
div(\vec{u}f) &= (div\vec{u})f + \nabla f \cdot \vec{u} \\
\int_\Omega (div\vec{u})f + \nabla f \cdot \vec{u}\,dx &= \int_{\partial\Omega} (\vec{u}n)f\,dS \\
\int_\Omega (div\vec{u})f\,dx \\
= \int_{\partial\Omega} (\vec{u}n)f\,dS - \int_\Omega \nabla
\end{aligned}
$$

We have bounded terms

$$\int (fy)' = \int f'g + fg'$$

Suppose $u, v \in C^2(\bar{\Omega})$ scaler fns

$$
\begin{aligned}
\int_\Omega \Delta uv\,dx &= \int_{\partial\Omega} (\nabla u\vec{n})v\,dS - \int_\Omega \nabla u \nabla v\,dx \\
= \int_\Omega (\nabla u\vec{n})v\,dS &- \int_\Omega u(\nabla v\vec{n})\,dS + \int_\Omega u\Delta v\,dx
\end{aligned}
$$

Suppose u is harmonic, suppose $v(y) = G_\Omega(x, y)$. We substitute this into the above formula. If u is harmonic function in Ω, then

$$u(x) = \int_\Omega u \frac{\partial G}{\partial n_y}(x, y)\,dS(y)$$

This is the formula to solve

$$\begin{cases} \Delta u = 0, \\ u|_{\partial\Omega} = g, \end{cases} \tag{17}$$

Problem 1.1. Suppose define

$$u(x) = \int_{\partial\Omega} g(y) \frac{\partial G}{\partial n_y}(x, y)\,dS$$

Then $u \longrightarrow g$ as $x \longrightarrow \partial\Omega$

Here we call

$$P_\Omega(x, y) = \frac{\partial G}{\partial n_y}(x, y)$$

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial y_3}$$

$$\frac{1}{4\pi}\frac{\partial}{\partial y_3}\Big(\frac{1}{2}\frac{2(x_3-y_3)}{|x_y|^3}+\frac{1}{2}\frac{2(x_3+y_3)}{|x-y|^3}\Big)$$
$$\frac{1}{2\pi}\frac{x_3}{|x-y|^3}$$

$$u(x)=\frac{1}{2\pi}\int_{\underbrace{\mathbb{R}^3}_{\partial\mathbb{R}^3_+}}\frac{x_3}{|x'-y'|^2+x_3^2}g(y)\,dy$$

$$\frac{1}{2\pi}\frac{\epsilon}{(|y|^2+\epsilon^2)^{\frac{3}{2}}}=\phi_\epsilon$$
$$(\text{Check })\phi=\frac{1}{2\pi}\frac{1}{(|y|^2+1)^{\frac{3}{2}}}$$
$$\phi_\epsilon=\frac{1}{2\pi}\frac{1}{\epsilon^2}\frac{1}{(|\frac{y}{\epsilon}|^2)^{\frac{3}{2}}}$$

And we know $\int\phi\,dy=1$

Conclusion:

$$P((y,\epsilon),y)\text{ is a mollifier.}$$
$$u(x_1,x_3,\epsilon)=\int_{\mathbb{R}^3}P(x_1-y_1,x_2-y_2,\epsilon)g(y_1,y_2)\,dy$$

Proposition 1.10. If g is bdd and continuous, then $u\longrightarrow g$ as $x_3\longrightarrow 0^+$ locally uniformly.

$$\begin{aligned}\Delta u&=0in\mathbb{R}^3_+\\ u|_{\partial\mathbb{R}^3_+}&=g\end{aligned}\tag{18}$$

Question: uniqueness?

We want something like Liouville theorem in half space.

Theorem 1.11 (comp principle)**.** $u,v\in C^2(\bar\Omega)$, if $u|_{\partial\Omega}=v|_{\partial\Omega}$, then

$$u\equiv v$$

Theorem 1.12 (Uniqueness)**.** Suppose you have u,v solve

$$\begin{aligned}\Delta u_k&=f\text{in }\mathbb{R}^3_+\\ u_k|_{\partial\mathbb{R}^3_+}&=g,\quad k=1,2.\end{aligned}$$

Suppose $u, v \longrightarrow 0$ as $|x| \longrightarrow +\infty$, then

$$u \equiv v$$

[**Date:** Sep 27, 2024]

Heat equation:

$$\partial_s u - \kappa \Delta u = 0 \quad \text{where } u = u(x,s)$$

where κ is the diffusivity

Change variable: $t = \kappa s$

$$\begin{aligned} \partial_t &= \frac{1}{\kappa}\partial_s \\ \Longrightarrow \partial_t v - \Delta v &= 0 \quad (\kappa = 1) \end{aligned}$$

Here we have $v(x,t) = u(x,s)$

Consider

$$\partial_t u - \Delta u = 0$$

Fundamental solution? We consider two equivalent formulations:

$$\begin{aligned} \partial_t \Gamma - \Delta \Gamma &= \delta_{(0,0)}(x,t) \\ &= \delta_0 \delta_0(t) \end{aligned}$$

The above is solved in $\mathbb{R}^n \times \mathbb{R}$

$$\begin{cases} \partial_t \Gamma - \Delta \Gamma = 0, \\ \Gamma|_{t=0} = \delta_0(x), \end{cases} \tag{19}$$

The above is solved in $\mathbb{R}^n \times (0, +\infty)$

Similarly, consider the ODE:

$$\begin{aligned} \dot{x} &= ax + \delta_0(t) \\ \text{or solve:} &\begin{cases} \dot{x} = ax, \\ x(0) = 1, \end{cases} \quad \Longrightarrow x(t) = e^{at} \end{aligned} \tag{20}$$

Symmetry of the heat equation:

1. space, time translation

2. $O(n)$ spatial rotation and reflection

3. homogeneity $u \mapsto mu$ where $m \in \mathbb{R}$

4. Scaling symmetry $u \mapsto u(\lambda x, \lambda^2 t)$, $\lambda > 0$ (Consider the former setup: $[\kappa] = \frac{L^2}{\Gamma}$)

What's the meaning of a function which is homogeneous?

$$\begin{aligned} f(\lambda x) = \lambda^\alpha f(x), \quad \lambda > 0 \\ \iff f \text{ is } \alpha - \text{homogeneous} \end{aligned}$$

Example 1.9. $\frac{1}{|x|}$ is (-1)-homogeneous.

$$\frac{1}{\lambda|x|} = \frac{1}{\lambda}\frac{1}{|x|} = \lambda^{-1}\frac{1}{|x|}$$

Here for $\delta_0(x)$:

$$\begin{aligned} \delta_0(x) &= \lim_{\epsilon \to \infty} \phi_\epsilon(x) \\ &= \lim_{\epsilon \to o^+} \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon}) \end{aligned}$$

Homogeneity of $\delta_0(x)$:

$$\begin{aligned} \delta_0(\lambda x) &= \lim_{\epsilon \to 0^+} \phi_\epsilon(\lambda x) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^n}\phi(\frac{\lambda x}{\epsilon}) \\ &= \lim_{\frac{\epsilon}{\lambda} \to 0^+} \frac{1}{\lambda^n}(\frac{\lambda}{\epsilon})^n\phi(\frac{\lambda x}{\epsilon}) \\ &= \lambda^{-n}\delta_0(x) \end{aligned}$$

This shows that δ_0 is $(-n)$-homogeneous.

Then with the scalling symmetry, we have:

$$\begin{aligned} u_{\lambda,m}(x,t) &= mu(\lambda x, \lambda^2 t) \\ u_\lambda(x,t) &:= \lambda^m u(\lambda x, \lambda^2 t) \\ (m = \lambda^n) \quad f_\lambda(x) &:= \lambda^n f(\lambda x) \\ f_\lambda = f &\iff (-n) - \text{homogeneous} \end{aligned}$$

Observation:

$$(\delta_0)_\lambda = \delta_0(x)$$

Expect: $\Gamma_\lambda = \Gamma$

In Evans:

$$\Gamma = \frac{1}{t^\alpha} F(\frac{x}{t^\beta})$$

It seems we "guess" this form for heat equation. But by the derivation we've done above for scaling symmetry, we know this must be the form.

By symmetry, we know that:

$$\begin{aligned}
\Gamma(x,t) &= \lambda^n \Gamma(\lambda x, \lambda^2 t), \quad \forall \lambda > 0 \\
(\text{Choose } \lambda &= \frac{1}{t^{\frac{1}{2}}}) \\
\Gamma(x,t) &= \frac{1}{t^{\frac{n}{2}}} \Gamma(\frac{x}{t^{\frac{1}{2}}}, 1) \\
&= \frac{1}{t^{\frac{n}{2}}} F(\frac{x}{t^{\frac{1}{2}}}) \\
F(y) &= \Gamma(y, 1)
\end{aligned}$$

Here we define: $y := \frac{x}{t^{\frac{1}{2}}}$

$$\begin{aligned}
\partial_t \Gamma - \Delta \Gamma = 0 \\
\partial_t (t^{-\frac{n}{2}} F(\frac{x}{t^{\frac{1}{2}}})) - \Delta (t^{-\frac{n}{2}} F(\frac{x}{t^{\frac{1}{2}}})) = 0
\end{aligned} \tag{21}$$

Differentiate it:

$$\begin{aligned}
&\partial_t (t^{-\frac{n}{2}} F(\frac{x}{t^{\frac{1}{2}}})) - \Delta (t^{-\frac{n}{2}} F(\frac{x}{t^{\frac{1}{2}}})) = -\frac{n}{2} t^{-\frac{n}{2}-1} F + t^{-\frac{n}{2}} (\nabla_y F)(-\frac{1}{2} x t^{-\frac{3}{2}}) - t^{-\frac{n}{2}} (\Delta_y F) t^{-1} = 0 \\
&\implies -\frac{n}{2} F - \frac{1}{2} y \nabla_y F - \Delta_y F = 0
\end{aligned}$$

Here we know F radial: $F = F(r)$, $r = |y|$, $y \nabla_y = r \partial_r$

Solve the ODE:

$$\begin{aligned}
\frac{n}{2} F + \frac{1}{2} r \partial_r F + \partial_r^2 F + \frac{n-1}{r} \partial_r F &= 0 \\
\frac{1}{2r^{n-1}} (r^n F)' + \frac{1}{r^{n-1}} (r^{n-1} F')' &= 0 \\
\frac{1}{2} r^n F + r^{n-1} F' &= A = 0 \\
F' &= -\frac{1}{2} r F \\
F &= B e^{\frac{-r^2}{4}}
\end{aligned}$$

What is B?

$$\partial_t \int \Gamma - \int \Delta \Gamma = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^n}\Gamma = 0$$
$$\Longrightarrow \int \Gamma\,dx = \text{ constant in time}$$

$$\int_{\mathbb{R}^n} F(|y|)\,dy = 1 \Longrightarrow B = \frac{1}{(4\pi)^{\frac{n}{2}}}$$

The Heat Kernel:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

$$\Gamma(x,\epsilon^2) = \phi_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon}) \text{ is a Mollifier.}$$

Theorem 1.13. u_0 is bounded and continuous on $\mathbb{R}^n$. Define

$$\begin{aligned} u(x,t) &:= (\Gamma(\cdot,t) * u_0)(x_0) \\ &= \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)\,dy \end{aligned}$$

, which solves heat equation and

$$u \longrightarrow u_0 \text{ as } t \longrightarrow 0^+ \quad \text{locally uniformly.}$$

[**Date:** Oct 2, 2024]

Heat Kernel:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{|x|^2}{4t})$$

Solve: IVP

$$\begin{cases} \partial_t u - \Delta u = 0, \\ u| = u_t, \\ t = 0 \end{cases} \tag{22}$$

$u(x,t) = \Gamma(,t) * u_0 = \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)dy$

$$\begin{cases} \partial_t u - \Delta u = f, \\ u| = 0, \\ t = 0 \end{cases} \tag{23}$$

$f : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ extend by zero.

We have

$$\Gamma *_{x,t} f = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \Gamma(x-y, t-s) f(y,s) dy \mathrm{d}s$$

We only care when $t \geqslant s$. And we know $s \geqslant 0$.

Thus we have

$$\Longrightarrow = \int_{s=0}^{t} \int_{\mathbb{R}^n} \Gamma(x-y, t-s) f(y,s) ds$$

Define

$$\Gamma_1 \begin{cases} \Gamma, & |x|^2 + t \geqslant 1 \\ smooth, & |x|^2 + t \leqslant 1 \end{cases} \tag{24}$$

Define $\Gamma_\epsilon(x,t) = \frac{1}{\epsilon^n}\Gamma(\frac{x}{\epsilon}, \frac{t}{\epsilon^2})$

Consider $(\partial_t - \Delta)\Gamma_\epsilon = \phi_\epsilon = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon}, \frac{t}{\epsilon^2})$

WTS: $\Delta(G * f) = f$

$$\Delta(G_\epsilon * f)$$

Proposition 1.14. Suppose $f \in C_1^2(\mathbb{R}^n \times [0,T))$ and f is compactly supported in $\mathbb{R}^n \times [0,T)$

Remark. Notice we let the function touches the initial time.

$$u = \Gamma *_{x,t} f$$
$$u \in C_1^2 \text{ solves}$$
$$\begin{cases} \partial_t u - \Delta u = f, \\ u| = 0, \\ t = 0 \end{cases} \tag{25}$$

Remark. Suppose we have $u \in L^1_{loc}(\mathbb{R}^n \times [0,T))$, then we can say:

$$\partial_t u - \Delta u = f \text{ weakly in } \mathbb{R}^n \times [0,T)$$

If $\int u(-\partial_t - \Delta)\varphi = \int f\varphi \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ (don't see initial time).

$$f \in L^1_{x,t}(\mathbb{R}^n \times (0,T))$$
$$\Longrightarrow u \text{ solves equation weakly.}$$

Now think about analogy of harmonic function: $\Delta u = 0$. We call them caloric function satisfying: $\partial_t u - \Delta u = 0$.

1. Interior estimates
2. Liouville theorem
3. maximum principle
4. Harneck
5. Weyl's lemma.

Now let's talk about interior estimates: if $u \in C_1^2$&compactly supported, then

$$\Gamma *_{x,t} (\partial_t - \Delta)u = u$$

Now we define $B_r := \{x \in \mathbb{R}^n : |x| < r\}$ and $Q_r := B_r \times (-r^2, 0)$.

If u is caloric function on Q_{2R}, then

$$\sup_{Q_R} |\partial_t \nabla_x^k u| \leqslant C_{j,k} \frac{1}{R^{2j+k}} \fint_{Q_{2R} \backslash Q_R} |u| dxdt \quad j, k \geqslant 0 \tag{26}$$

We think about $u = \phi u = \Gamma * (\partial_t - \Delta)(\phi u) = \Gamma * (\partial_t \phi u - \Delta \phi u - 2\nabla\phi\nabla u)$ where $\phi \equiv 1$ on Q_R and $\phi \equiv 0$ outside Q_{2R}.

Theorem 1.15 (Liouville)**.** If u is bounded and ancient sol to $\partial_t u - \Delta u = 0$ on $\mathbb{R}^n \times (-\infty, 0)$, then $u \equiv \text{const}$

Remark. We know

$$\leqslant C_{j,k} \frac{1}{R^{2j+k}} \fint_{Q_{2R} \backslash Q_R} |u| \leqslant C \frac{1}{R^2} \sup_{\mathbb{R}^n \times (-\infty, 0)} |u|$$

Weyl's lemma: weakly caloric function is smooth.

Theorem 1.16 (Maximum principle)**.** Ω is bounded domain $\subseteq \mathbb{R}^n$. Suppose $u \in C_1^2(\Omega \times (0,T)) \bigcap C(\overline{\Omega \times [0,T]})$. (This is kind of a cylinder).

$\partial_{par}(\Omega \times (0,T)) = \underbrace{(\partial\Omega \times [0,T] \times (\bar{\Omega} \times \{0\}))}_{\text{sides and bottom}}$

$$\text{False: } \max_{\Omega \times [0,T]} > \max_{\partial_{par}(\Omega \times [0,T])} u$$

If $u(x_0, t_0) = \max_{\Omega \times [0,T]} u$ and $(x_0, t_0) \notin \partial_{par}(\Omega \times [0,T])$, then

$$u \equiv \text{const}$$

[**Date:** Oct 4, 2024] Parabolic max principle: understanding

Suppose u has local max at x_0, t_0

1. In the domain Ω. $\partial_t u(x_0, t_0) = 0$ and $\nabla u(x_0, t_0) = 0$ We have

$$u(x,t) = u(x_0,t_0) + \frac{1}{2}(\nabla^2 u(x_0,t_0)(x-x_0,t-t_0))(x-x_0,t-t_0)$$

   And we have

$$\underbrace{\partial_t u}_{=0} - \Delta u(x_0,t_0) = 0$$

   We actually have

$$-\Delta u(x_0,t_0) < 0$$

   We will add $\epsilon(t + |x|^2)$.

2. $\partial_t u(x_0,t_0) \geqslant 0$ and we still have $\nabla_x u(x_0,t_0)$. Consider heat function

$$0 = \partial_t u - \Delta u(x_0,t_0) \geqslant -\Delta u(x_0,t_0)$$

   It's a parabola or saddle.

Theorem 1.17 (Uniqueness)**.** (We prove uniqueness by maximum principle) $u \in C_1^2(\mathbb{R}^n \times (0,T))$ and $u \in C(\overline{\mathbb{R}^n \times [0,T]})$, $u|_{t=0} = 0$ and $|u| \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Then $u \equiv 0$.

We prove uniquesness by another way

$$\dot{x} = Ax$$

where $x \in R^{10}, A \in R^{10\times 10}$

$$\begin{cases} u = x, & \in L^2 \\ \Delta = A, \end{cases} \tag{27}$$

By this example, we can try Parabolic PDF $\iff$ ODEs in ∞-dim

$$\begin{cases} \dot{x} = Ax, \\ x(0) = \vec{x}, \end{cases} \tag{28}$$

defined on $(0,T)$. We solve it and obtain $x(t) = e^{At}\vec{0} = \vec{0}$

We solve

$$\begin{cases} -\dot{y} = A^* y + f(t), & \text{on } (0,T) \\ y(T) = 0, \end{cases} \tag{29}$$

We have this "final time" problem $\longrightarrow$ solve backward.

Suppose $\forall f$ solve adj problem existence.

Existence for adj problem $\Longrightarrow$ uniqueness for forward problem.

Given x. Let f be arbitrary ($f \in C_0^\infty((0,T))$).

$$\begin{cases} \dot{x} = Ax, \\ \int_0^T \langle \frac{dx}{dt}, y \rangle = \int_0^T \langle Ax, y \rangle, \end{cases} \tag{30}$$

$$\begin{aligned} -\int_0^T \langle x, \dot{y} \rangle &= \int_0^T \langle x, A^* y \rangle \\ \Longrightarrow \int_0^T \langle x, \dot{y} + A^* y \rangle \, dt &= 0 \\ \int_0^T \langle x(t), -f(t) \rangle dt &= 0 \end{aligned}$$

This is true for $\forall f$. Then $\Longrightarrow$ $x \equiv 0$.

Theorem 1.18. $u \in C_1^2(\mathbb{R}^n \times (0,T)) \bigcap C(\overline{\mathbb{R}^n \times [0,T]})$. $u|_{t=0} = 0$, $|u| \leqslant Ce^{A|x|^2}$.

Then $u \equiv 0$.

We are solving

$$v(x,t) \cdots$$

Let $f \in C_0^\infty(\mathbb{R}^n \times (0,T))$

$$\begin{cases} -\partial_t v = \Delta v + f, \quad \text{on } \mathbb{R}^n \times (0,T) \\ v|_{t=T} = 0 \end{cases} \tag{31}$$

Let $w(x,t) = v(x, T-t)$ and $\tilde{f}(x,t) = f(x, T-t)$ where $w(x,t) = \Gamma *_{x,t} \tilde{f}$

$$\begin{aligned} w(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y, t-s) \tilde{f}(y,s) \, dyds \\ \leqslant \int_0^t \int_{B_R} e^{\frac{-|x-y|^2}{4(t-s)}} |\tilde{f}(y,s)| \end{aligned}$$

existence ✓.

$$|w| \leqslant \tilde{C} e^{\frac{-|x|^2}{Ct}}$$

Consider

$$\partial_t u - \Delta u = 0$$

$$\int_0^T \int \partial_t u \cdot v - \int_0^T \int \Delta u v = 0$$

$$\implies \underbrace{\int_{\mathbb{R}^n} u(x,T)v(x,T)}_{=0} - \int_{\mathbb{R}^n} u(0)v(0) - \int \int u(\partial_t v + \Delta v) = 0$$

$$\implies \int \int u(-f)\, dxdt = 0$$

This is true $\forall f \in C_0^\infty(\mathbb{R}^n) \times (0,T) \implies u \equiv 0$.