Notes of Math 719: Partial Differential Equation
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Contents

1 Laplace's equation

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Elliptic PDEs:

Example 1.1. Laplacian:

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial}{\partial x_n^2} \\ &= div\nabla \end{aligned}$$

Example 1.2. n = 3: Newtonian gravity

$$\begin{aligned} \text{magnitutde} &= \frac{\kappa m m_1}{\|a - a_1\|} \quad \text{(inverse square)} \\ m\ddot{a} &= \frac{\kappa m m_1}{\|a - a_1\|} \frac{a_1 - a}{\|a_1 - a\|} \\ &= -m\nabla u(a) \\ \text{where} \quad u(x) &= \frac{-k m_1}{\|x - a_1\|} \end{aligned}$$

N masses $m_1, \ldots, m_N \geqslant 0$, location $a_1, \ldots, a_N \in \mathbb{R}^3$

$$u(x) = -\kappa \sum_{k=1}^{N} \frac{m_k}{\|x - a_k\|}$$

Example 1.3. continuous distribution of mass

$$\varrho(x) \geqslant 0$$
$$u(x) = -\kappa \int_{\mathbb{R}^3} \frac{\varrho(y)}{\|x-y\|}\, dy$$

suppose supp $\varrho \subseteq \omega$ bdd open set

Remark. $supp\varrho := \overline{\{x \in \mathbb{R}^3 : S(x) \neq 0\}}$

Laplace

$$\Delta u(x) = 0\, for x \in \mathbb{R}^3 \bar{\omega}$$

$$\Delta_x u(x) = -\kappa \int_{\omega} \varrho(y) \Delta_x \frac{1}{\|x-y\|}\, dy$$

$$\Delta \frac{1}{\|x\|} = div \nabla \frac{1}{\|x\|}$$
$$= div(-\frac{x}{\|x\|^3})$$
$$= \big(-\frac{3}{\|x\|^3} + 3x \frac{x^2}{\|x\|^5})$$

Example 1.4.

$$\varrho = \text{const on } B_{\mathbb{R}}$$
$$= \frac{m}{4\pi\mathbb{R}^3}$$
$$\text{and} u = -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|x-y\|}\, dy$$

Claim 1.1.

$$u(x) = u(Ox)\, \text{where} O \in SO(3)$$
$$u(Ox) = -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|Ox - Oz\|}\, dz$$
$$= -\kappa m \int_{B_{\mathbb{R}}} \frac{1}{\|x - z\|}\, dz$$

$$u(x) = u(r)$$
$$u : \mathbb{R}^3 \to \mathbb{R}$$

$$u(x) = f(r)$$

[**YW:** TODO: Complete notes between Sep 4 and Sep 13]

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Last time: Interior estimate

Suppose $R > 0$ and $u \in C^2(B_{2R})$ is harmonic, then

$$\|\nabla^k u(x)\| \leqslant C_k \frac{1}{R^{k+3}} \int_{B_{2R} \setminus B_R} \|u\| \, dy \quad \forall x \in B_R \tag{1}$$

Question: Why estimate of this form?

Symmetries of $\Delta u = 0$.

1. homeoneity/scaler mult

$$u \mapsto \mu \quad (m \in \mathbb{R})$$

2. scaling symmetry

$$u_{\lambda(x)} := u(\lambda x)$$
$$u \mapsto u_\lambda \; (u(\lambda \cdot))$$

   Here $\Delta u_\lambda = \Delta[u(\lambda x)] = \lambda^2 \Delta u(\lambda u) = 0$

3. translation:

$$u \mapsto u(\cdot - x_0) \quad (x_0 \in \mathbb{R}^3)$$

4. rotation and reflection

$$u \mapsto u(\mathcal{O}^{-1}) \quad (\mathcal{O} \in O(n))$$

1. $k = 0, R = 1$, What if

$$\|mu\|(x) \leqslant C \int_{B_2 \setminus B_1} \|mu\|^2 \, dy \quad \text{in } B_1$$
$$\implies \|u(x)\| \leqslant Cm \int_{B_2 \setminus B_1} \|u\|^2 \, dy$$
$$\implies u = 0$$

Then this is nonsense. We need the power in the integral to be 1, i.e.,

$$\|mu\|(x) \leqslant C \int_{B_2 \backslash B_1} \|mu\| \, dy \quad \text{in } B_1$$

To prove (1), it suffices to prove it for

$$\int_{B_{2R} \backslash B_R} \|u\| = 1$$

Suppose (1) holds for all harmonic function u s.t.

$$\int_{B_{2R} \backslash B_R} \|u\| = 1$$

Given arbitrary harmonic function v, we define

$$u := \frac{v}{\int_{B_{2R} \backslash B_R} \|v\| \, dy}$$

Then u is still harmonic

$$\Longrightarrow \frac{\|v\|}{\int_{B_{2R} \backslash B_R} \|v\| \, dy} = \|u\| \leqslant C \frac{1}{R^3}$$

To prove (1), it suffices to prove it for $R = 1$

Given harmonic function v on B_{2R}

$$u(x) := v(Rx)$$

Here $u(x)$ is a harmonic function on B_2.

Then

$$\begin{aligned} \|\nabla^k u(x)\| &\leqslant C \int_{B_2 \backslash B_1} u(y) \, dy \\ &= \int_{B_2 \backslash B_1} \|v(Ry)\| \, dy \\ \text{(Change of Variable: } z = Ry) \\ &= \frac{C}{R^3} \int_{B_{2R} \backslash B_R} \int \|v(z)\| \, dz \end{aligned}$$

where $\|\nabla^k (v(Rx))\| := R^k \|(\nabla^k v)(Rx)\| = R^k \|(\nabla^k v)(q)\|$ where $q = Rx \in B_R$

To prove (1), it's enough to do it for $R = 1$ and $\int_{B_2 \backslash B_1} \|u\| \, dy = 1$

1.1

Given

$$\phi u = 2\Delta * (\Delta \phi u) + G * (\Delta \phi u) \tag{2}$$

where $u \in C^2$ and $\Delta u = 0$ and $\phi \in C_0^\infty$

One option $u \in L^1_{loc}$ ($u \in L^1(K)$) where K is compact , it's called locally integrable, is harmonic if (2) holds $\forall \phi \in C_0^\infty$

Another option $u \in L^1_{loc}$ is harmonic if

$$\nabla(u * \phi_\epsilon) = 0 \qquad \text{for all } \epsilon$$

Example 1.5.

$\frac{1}{|x|}$ is $L^1_{loc}(\mathbb{R}^3)$

$(1 + |x|)^{-3-\epsilon} \in L^1(\mathbb{R}^3)$

Definition 1.1. $\Omega \subseteq R^3$ open, $u \in L^1_{loc}(\Omega)$ is weakly harmonic if

$$\int u \nabla \phi \, dy = 0 \quad \forall \phi \in C_0^\infty(\Omega)$$

Remark.

$$\begin{aligned} \int \Delta u \cdot \phi &= \int (div \nabla u) \phi \\ &= -\int \nabla u \nabla \phi = \int u \nabla \phi \end{aligned}$$

This is integration by parts.

If you need a generalization, you need to make it easy to check and easy to work with.

Lemma 1.2 (Weyl's lemma)**.**

If u is weakly harmonic in Ω, then u is smooth and $\Delta u = 0$ in Ω

To prove this, we need the following claim:

Claim 1.3.

1. If u is C^2 and $\Delta u = 0$, then u is weakly harmonic
2. If u is C^2 and weakly harmonic, them $\Delta u = 0$

Proof of Claim 1.3. Suppose not.

$$\int u\Delta\phi = 0 \quad \forall\phi$$

But $\exists x_0$ s.t. $\Delta u(x_0) \neq 0$

$$\int \Delta u\phi = 0 \quad \phi$$

Choose ϕ s.t.

$$\int \Delta u\phi \neq 0$$

Countradiction. □

proof of Wely's lemma.

(3) $u \in L^1_{loc}$ is weakly harmonic, then $\phi_\epsilon * u$ is also weakly harmonic.

$$\implies \phi_\epsilon * u \text{ is strongly harmonic}$$

We need to check

$$\int (u * \phi_\epsilon)\Delta\psi \, dy = 0 \quad \forall\psi$$

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Remark. $\Delta(f * \phi_\epsilon) = \Delta f * \phi_\epsilon \implies$ Mollify harmonic function , get a harmonic function.

Enough to work with balls. Enough to work in B_3 and prove smoothness in B_1.
Because of translation and scalling symmetry.
$u \in L^1(B_3)$ weakly harmonic. We define

$$u_\epsilon(x) = u * \phi_\epsilon(x) \quad \text{for } x \in B_2 \text{ and } 0 < \epsilon \leqslant \frac{1}{2}$$

Want to check u_ϵ is weakly harmonic in B_2.

$$\begin{aligned}
\forall\psi \in C_0^\infty(B_2): \int_{B_2} u_\epsilon(x)\Delta\psi(x)\,dx &= \int_{B_2}\int_{\mathbb{R}^3} \phi_\epsilon(x-y)u(y)\Delta\psi(x)\,dy\,dx \\
&= \int u(y)(\phi_\epsilon(-\cdot) * \Delta\psi)(y)\,dy \\
&= \int_{\mathbb{R}^2} u(y)\Delta(\phi_\epsilon(-\cdot)\psi)(y)\,dy \\
&= 0 \text{ (by definition of weakly harmonic)}
\end{aligned}$$

$\square$

Use interior estimates on u_ϵ

$$|\nabla^k u_\epsilon| \leqslant C_k \int_{B_2 \setminus B_1} \int |u_\epsilon| \, dy \leqslant C_k \int_{B_3} |u| \, dy$$

Use Arzela-Ascoli: For all k, $\nabla^k u_\epsilon \longrightarrow \nabla^k u$ uniformly in B_1

Weak version of $\Delta u = f$?

Definition 1.2. $\Omega \subseteq \mathbb{R}^3$ open, $u, f \in L^1_{loc}(\Omega)$. Then we say that

$$\Delta u = f \quad \text{weakly harmonic in } \Omega$$

if

$$\int u \Delta \psi = \int f \psi \quad \forall \psi \in C_0^\infty(\Omega)$$

Previously: $f \in C^2$ and compactly supported

$$\Delta(G * f) = f$$

General f holds?

Example 1.6. $G * f$ makes sense for $f \in L^1$

$$-\frac{1}{4\pi} \int \frac{f(y)}{|x-y|} \, dy$$

Notice that $\frac{1}{|x|}$ is not integrable.

Now let $-\frac{1}{4\pi|x|} = \text{G}$. And $G_1 = G\mathbf{1}_{B_1} \in L^1 \bigcap L^{3-}$, $G_2 = G\mathbf{1}_{\mathbb{R}^3 \setminus B_1} \in L^\infty \bigcap L^{3+}$, where L^3 means that $L^{3-\epsilon} \quad \forall \epsilon > 0$.

$$\int_{B_1} \frac{1}{|x|^3} = c \int_{r=0}^{1} r^{-3} r^2 \mathrm{d}x = \infty$$
$$\int_{\mathbb{R}^3 \setminus B_1} = c \int_{r=1}^{\infty} r^{-3} r^2 \mathrm{d}x = \infty$$

And $G * f = \underbrace{G_1 * f}_{\in L^1 \bigcap L^{3-}} + \underbrace{G_2 * f}_{\in L^{3+} \bigcap L^\infty} \in L^1_{loc}$

Exercise 1.1. $f \in L^1 + L^p$ if $p < \frac{n}{2}$

Claim 1.4. $\Delta(G * f) = f$ is weakly harmonic in $\mathbb{R}^3$

Proof. <u>To check:</u>

$$\int (G * f)\Delta\varphi = \int f\varphi \forall \varphi \in C_0^\infty(\mathbb{R}^3)$$

$$\begin{aligned}\int (G * f)\Delta\varphi = \lim_{\epsilon\to 0^+} \int (K_\epsilon * f)\Delta\varphi = \lim_{\epsilon\to 0^+} \int (\Delta K_\epsilon) * f\varphi \\ = \int f\varphi\end{aligned}$$

□

Proposition 1.5. Suppose $u_1, u_2,\ f \in L^1_{loc}$ and $\Delta u_1 = \Delta u_2 = f$ weakly harmonic in $\mathbb{R}^3$ Then

1. $u_1 - u_2$ is smooth and harmonic
2. If u_1, u_2 are bounded, $u_1 - u_2$ is constant.
3. If $|u_1|, |u_2| \longrightarrow 0$ as $|x| \longrightarrow \infty$, (say $u_1, u_2 \in L^1 + L^p, p < \infty$) then $u_1 \equiv u_2$.