

# A MONOTONE MESHFREE FINITE DIFFERENCE METHOD FOR LINEAR ELLIPTIC PDES VIA NONLOCAL RELAXATION

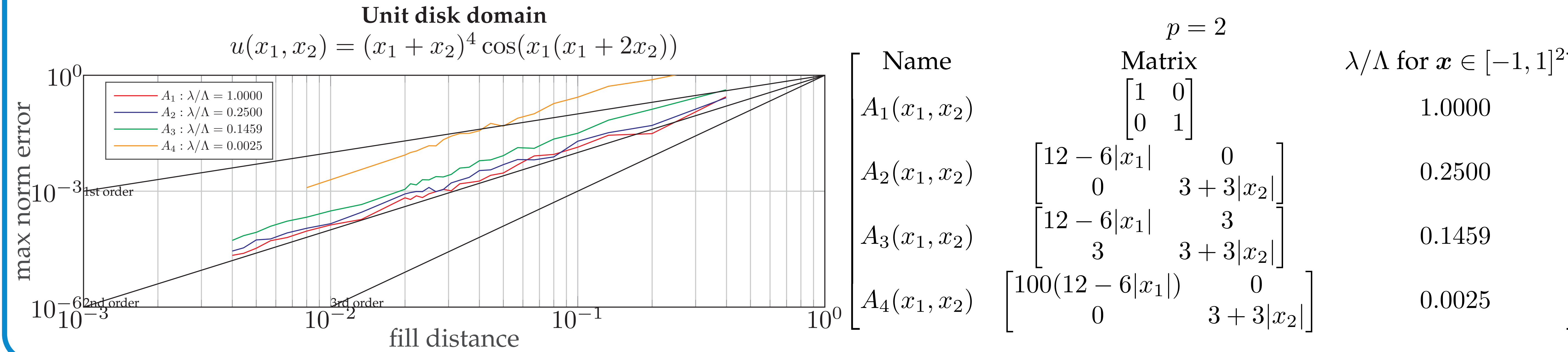
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## NUMERICAL RESULTS



## ANALYSIS(YE-TIAN, 2022)

**Lemma 1:** Assume  $\bar{S}_{\delta,h,p}$  is not empty and  $C > 0$  is a generic constant.

1. If  $p \geq 2$  and  $u \in C^2(\bar{\Omega})$ , then  $|\mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) - Lu(\mathbf{x}_i)| \rightarrow 0$  as  $\delta \rightarrow 0$  for all  $\mathbf{x}_i \in \Omega$ .
2. If  $p \geq 2$  and  $u \in C^{2,\alpha}(\bar{\Omega})$  for  $\alpha \in (0, 1]$ , then  $|\mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) - Lu(\mathbf{x}_i)| \leq C|u|_{C^{2,\alpha}(\bar{\Omega})}\delta^\alpha$  for all  $\mathbf{x}_i \in \Omega$ .
3. If  $p \geq 3$  and  $u \in C^{3,\alpha}(\bar{\Omega})$  for  $\alpha \in (0, 1]$ , then  $|\mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) - Lu(\mathbf{x}_i)| \leq C|u|_{C^{3,\alpha}(\bar{\Omega})}\delta^{1+\alpha}$  for all  $\mathbf{x}_i \in \Omega$ .

**Theorem 2:** In  $d = 2$ , there exists a constant  $C > 0$  such that if

$$h \leq C\delta\sqrt{\lambda/\Lambda}$$

then  $\bar{S}_{\delta,h,2}$  is not empty.

**Theorem 3:** In  $d = 2$ , assume  $\bar{S}_{\delta,h,p}$  is not empty, let  $u$  be the real solution and  $u_\delta^h$  be the solution solved by the discrete operator and  $C > 0$  is a generic constant.

1. If  $p \geq 2$  and  $u \in C^{2,\alpha}(\bar{\Omega})$  for  $\alpha \in (0, 1]$ , then  $\max_{\mathbf{x}_i \in \Omega} |u(\mathbf{x}_i) - u_\delta^h(\mathbf{x}_i)| \leq C|u|_{C^{2,\alpha}(\bar{\Omega})} \left(\sqrt{\lambda/\Lambda}\right)^{-\alpha} h^\alpha$
2. If  $p \geq 3$  and  $u \in C^{3,\alpha}(\bar{\Omega})$  for  $\alpha \in (0, 1]$ , then  $\max_{\mathbf{x}_i \in \Omega} |u(\mathbf{x}_i) - u_\delta^h(\mathbf{x}_i)| \leq C|u|_{C^{3,\alpha}(\bar{\Omega})} \left(\sqrt{\lambda/\Lambda}\right)^{-(1+\alpha)} h^{1+\alpha}$

## BASIC IDEAS

**MAIN GOAL:** Solve the second-order linear elliptic equations in non-divergence form

$$\begin{cases} -Lu(\mathbf{x}) := -\sum_{i,j=1}^d a^{ij}(\mathbf{x})\partial_{ij}u(\mathbf{x}) = f(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases},$$

for an open bounded domain  $\Omega \in \mathbb{R}^d$ . The matrix  $A(\mathbf{x}) = (a^{ij}(\mathbf{x}))_{i,j=1}^d$  is assumed to be symmetric and positive definite satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \xi^T A(\mathbf{x})\xi \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d$$

for positive constants  $\lambda, \Lambda$  with ratio  $\lambda/\Lambda \leq 1$ .

Denote  $M(\mathbf{x}) := (A(\mathbf{x}))^{1/2}$ .

**NONLOCAL RELAXATION METHOD:** The nonlocal elliptic operator[2, 3, 4] can be defined as

$$\begin{aligned} \mathcal{L}_\delta u(\mathbf{x}) &= \int_{B_\delta(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|\mathbf{z}|}{\delta}\right) (u(\mathbf{x} + M(\mathbf{x})\mathbf{z}) - u(\mathbf{x})) d\mathbf{z} \\ &= \int_{\mathcal{E}_\delta^\mathbf{x}(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|M(\mathbf{x})^{-1}\mathbf{y}|}{\delta}\right) \det(M(\mathbf{x}))^{-1} (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y} \\ &:= \int_{\mathcal{E}_\delta^\mathbf{x}(\mathbf{0})} \rho_\delta(\mathbf{x}, \mathbf{y}) (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y}. \end{aligned}$$

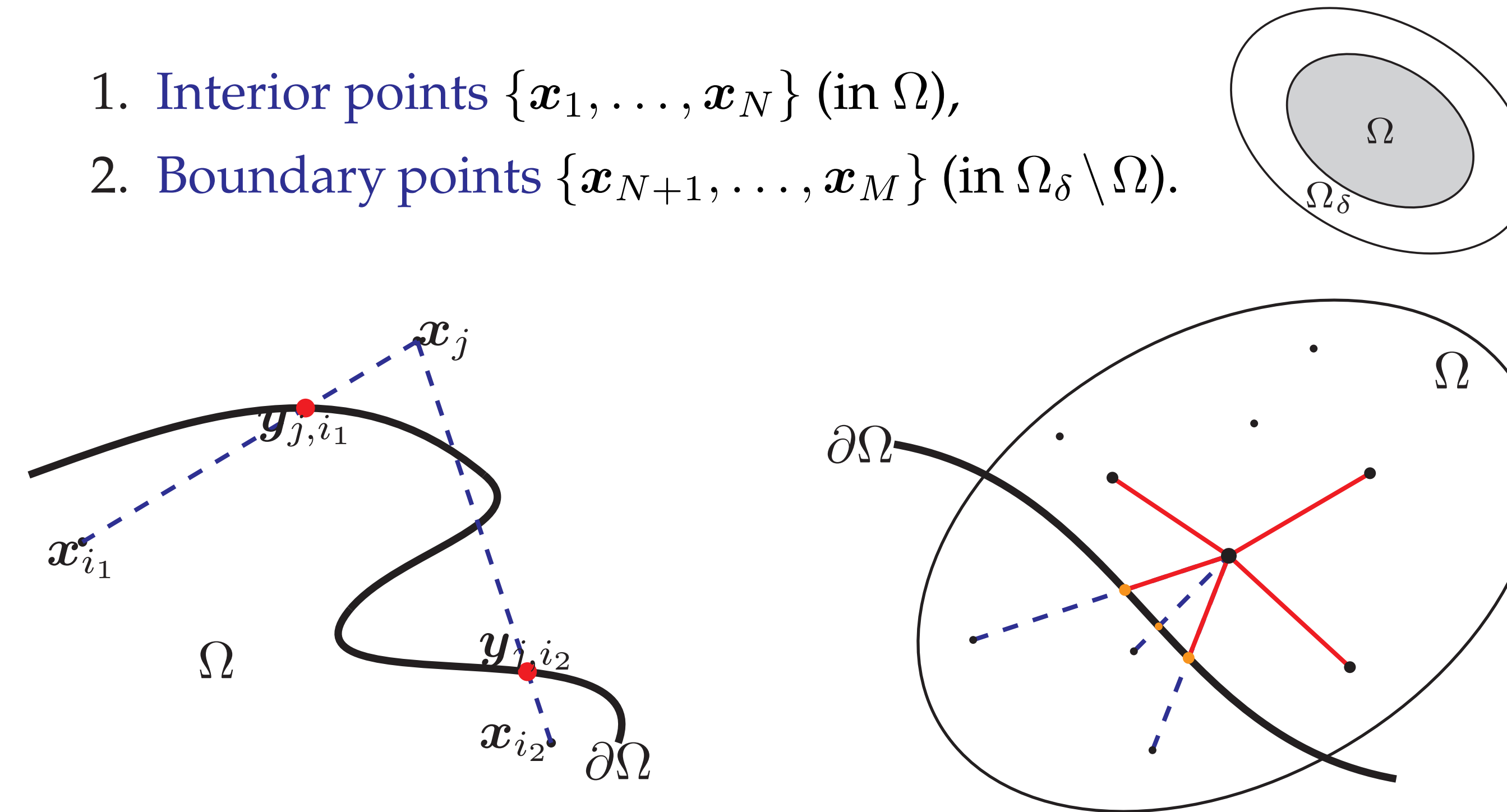
It can be shown that

$$\{\mathbf{y} \in \mathbb{R}^d : M(\mathbf{x})^{-1}\mathbf{y} \in B_\delta(\mathbf{0})\} =: \mathcal{E}_\delta^\mathbf{x}(\mathbf{0})$$

$$\mathcal{L}_\delta u(\mathbf{x}) \rightarrow Lu(\mathbf{x}) \quad \text{as } \delta \rightarrow 0.$$

**BOUNDARY TREATMENT:** Point cloud contains

1. **Interior points**  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  (in  $\Omega$ ),
2. **Boundary points**  $\{\mathbf{x}_{N+1}, \dots, \mathbf{x}_M\}$  (in  $\Omega_\delta \setminus \Omega$ ).



**OPTIMIZATION BASED MESHFREE METHOD:** We use the following minimization problem[1, 6] to select a stencil for interior point  $\mathbf{x}_i$  and  $p$  the order of the polynomial space:

$$\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \bar{S}_{\delta,h,p}} \sum_j \frac{\beta_{j,i}}{\rho_\delta(\mathbf{x}_i, \mathbf{y}_{j,i} - \mathbf{x}_i)},$$

where  $h$  is the fill distance[5] and

$$\mathbf{y}_{j,i} = \begin{cases} \mathbf{x}_j & , \mathbf{x}_j \in \bar{\Omega} \\ \text{projection from } \mathbf{x}_j \text{ to } \mathbf{x}_i \text{ at } \partial\Omega & , \mathbf{x}_j \in \Omega_\delta \setminus \bar{\Omega} \end{cases},$$

$$\bar{S}_{\delta,h,p} := \left\{ \{\beta_{j,i}\} : \beta_{j,i} \geq 0 \text{ and } \mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) = \mathcal{L}_\delta u(\mathbf{x}_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d) \right\},$$

$$\mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{E}_\delta^\mathbf{x}(\mathbf{x}_i)} \beta_{j,i} (u(\mathbf{y}_{j,i}) - u(\mathbf{x}_i)).$$

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