# A Monotone Meshfree Finite Difference Method for Linear Elliptic PDEs via Nonlocal Relaxation


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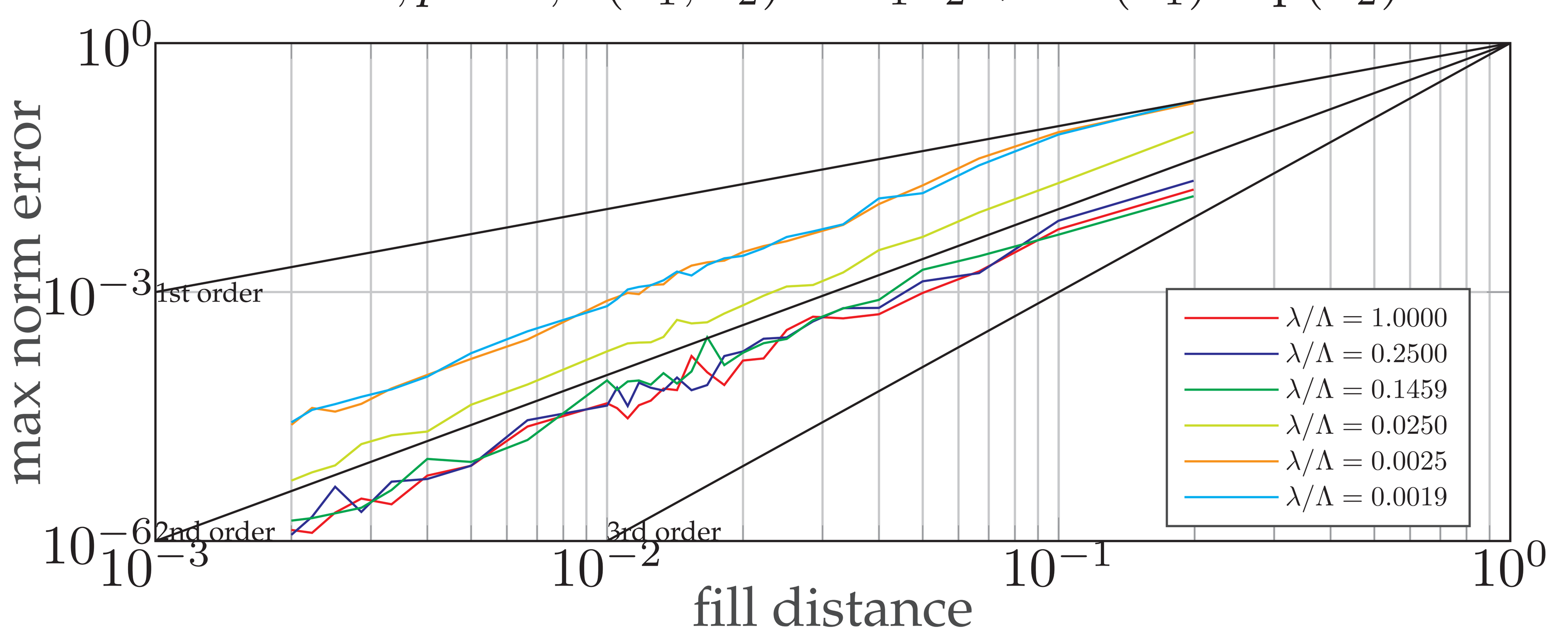
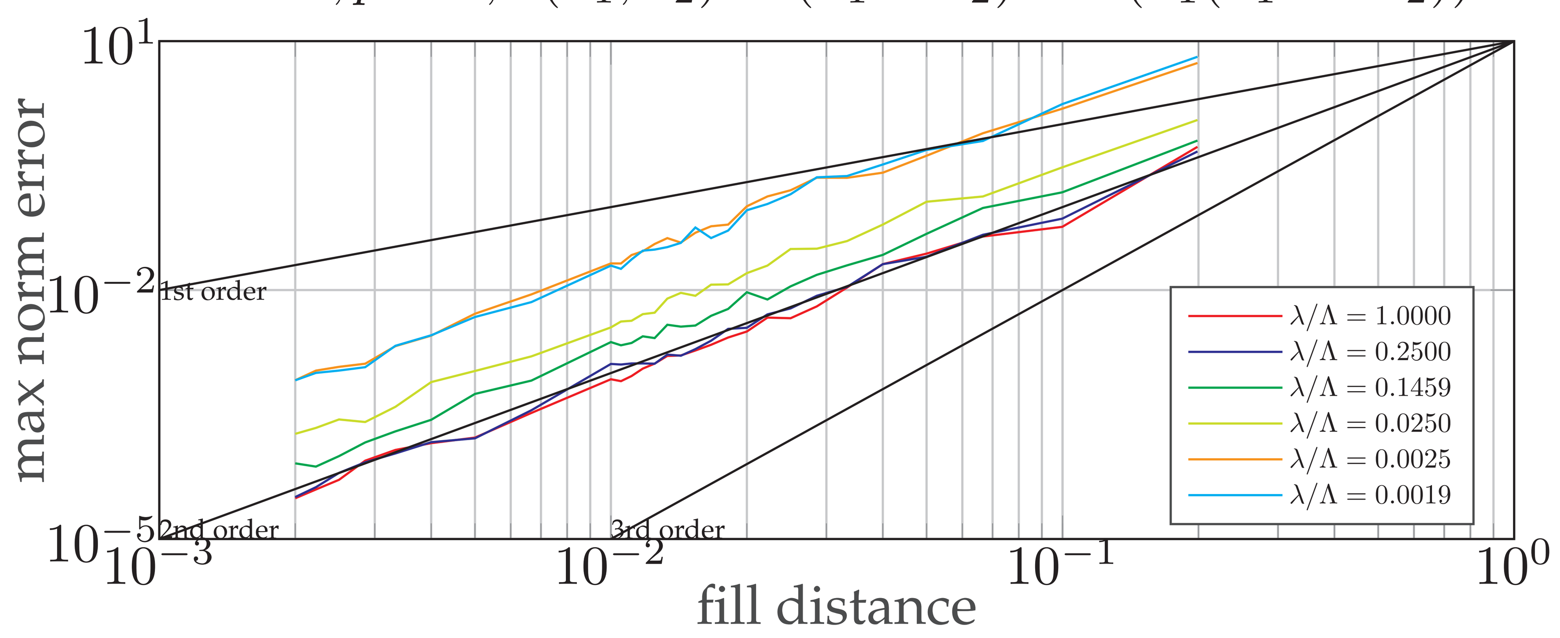
## Numercial Results




## Basic Ideas

**MAIN GOAL**: Solve the second-order linear elliptic equations in non-divergence form

$$\begin{cases} -Lu(\boldsymbol{x}) := -\displaystyle\sum_{i,j=1}^{d} a^{ij}(\boldsymbol{x})\partial_{ij}u(\boldsymbol{x}) = f(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \\ u(\boldsymbol{x}) = g(\boldsymbol{x}) & \boldsymbol{x}\in\partial\Omega \end{cases},$$

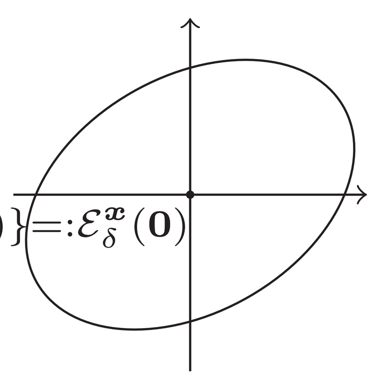
for an open bounded domain $\Omega\in\mathbb{R}^d$. The matrix $A(\boldsymbol{x}) = (a^{ij}(\boldsymbol{x}))_{i,j=1}^d$ is assumed to be symmetric and positive definite satisfying the uniform ellipticity condition

$$\lambda|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \le \Lambda|\boldsymbol{\xi}|^2 \qquad \forall\boldsymbol{\xi}\in\mathbb{R}^d$$

for positive constants $\lambda,\Lambda$ with ratio $\lambda/\Lambda\le 1$.
Denote $M(\boldsymbol{x}) := (A(\boldsymbol{x}))^{1/2}$.

**NONLOCAL RELAXATION METHOD**: The nonlocal elliptic operator[2, 3, 4] can be defined as

$$\mathcal{L}_\delta u(\boldsymbol{x}) = \int_{B_\delta(\mathbf{0})} \frac{1}{\delta^{d+2}}\gamma\left(\frac{|\boldsymbol{z}|}{\delta}\right)(u(\boldsymbol{x}+M(\boldsymbol{x})\boldsymbol{z})-u(\boldsymbol{x}))\,d\boldsymbol{z}$$
$$= \int_{\mathcal{E}_\delta^{\boldsymbol{x}}(\mathbf{0})} \frac{1}{\delta^{d+2}}\gamma\left(\frac{|M(\boldsymbol{x})^{-1}\boldsymbol{y}|}{\delta}\right)\det(M(\boldsymbol{x}))^{-1}(u(\boldsymbol{x}+\boldsymbol{y})-u(\boldsymbol{x}))\,d\boldsymbol{y}$$
$$:= \int_{\mathcal{E}_\delta^{\boldsymbol{x}}(\mathbf{0})} \rho_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x}+\boldsymbol{y})-u(\boldsymbol{x}))\,d\boldsymbol{y}.$$



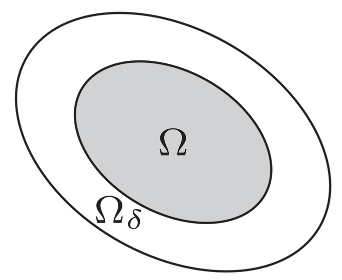
$\{\boldsymbol{y}\in\mathbb{R}^d : M(\boldsymbol{x})^{-1}\boldsymbol{y}\in B_\delta(\mathbf{0})\} =: \mathcal{E}_\delta^{\boldsymbol{x}}(\mathbf{0})$

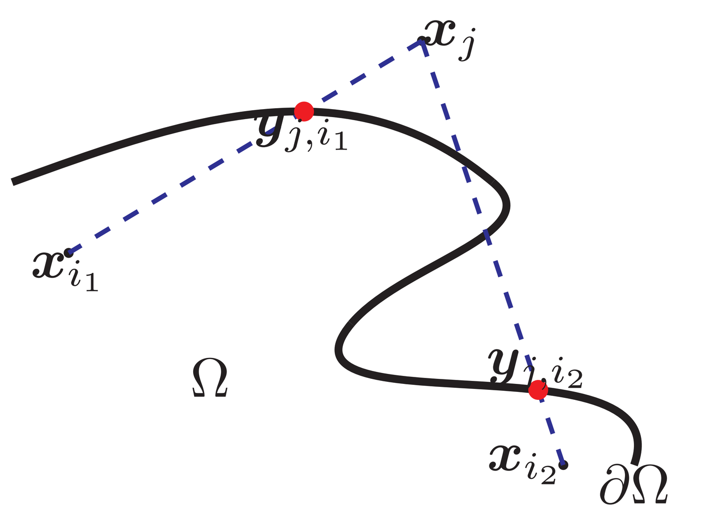
It can be shown that

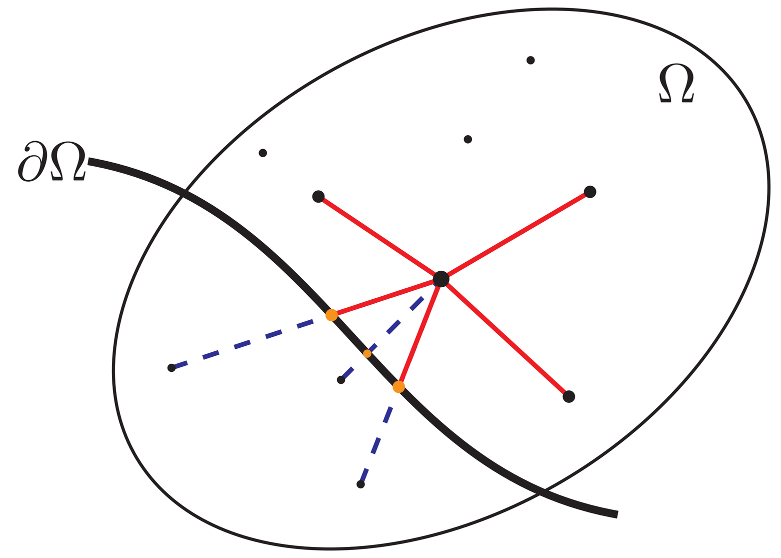
$$\mathcal{L}_\delta u(\boldsymbol{x}) \to Lu(\boldsymbol{x}) \quad \text{as} \quad \delta\to 0.$$

**BOUNDARY TREATMENT**: Point cloud contains

1. Interior points $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\}$ (in $\Omega$),
2. Boundary points $\{\boldsymbol{x}_{N+1},\dots,\boldsymbol{x}_M\}$ (in $\Omega_\delta\setminus\Omega$).




**OPTIMIZATION BASED MESHFREE METHOD**: We use the following minimization problem[1, 6] to select a stencil for interior point $\boldsymbol{x}_i$ and $p$ the order of the polynomial space:

$$\{\beta_{j,i}\} = \underset{\{\beta_{j,i}\}\in\overline{S}_{\delta,h,p}}{\arg\min} \sum_j \frac{\beta_{j,i}}{\rho_\delta(\boldsymbol{x}_i,\boldsymbol{y}_{j,i}-\boldsymbol{x}_i)},$$

where $h$ is the fill distance[5] and

$$\boldsymbol{y}_{j,i} = \begin{cases} \boldsymbol{x}_j & ,\boldsymbol{x}_j\in\overline{\Omega} \\ \text{projection from } \boldsymbol{x}_j \text{ to } \boldsymbol{x}_i \text{ at } \partial\Omega & ,\boldsymbol{x}_j\in\Omega_\delta\setminus\overline{\Omega} \end{cases},$$

$$\overline{S}_{\delta,h,p} := \left\{\{\beta_{j,i}\} : \beta_{j,i}\ge 0 \text{ and } \mathcal{L}^h_{\delta,\Omega}u(\boldsymbol{x}_i) = \mathcal{L}_\delta u(\boldsymbol{x}_i)\ \forall u\in\mathcal{P}_p(\mathbb{R}^d)\right\},$$

$$\mathcal{L}^h_{\delta,\Omega}u(\boldsymbol{x}_i) = \sum_{\boldsymbol{x}_j\in\mathcal{E}_\delta^{\boldsymbol{x}_i}(\boldsymbol{x}_i)} \beta_{j,i}(u(\boldsymbol{y}_{j,i})-u(\boldsymbol{x}_i)).$$

## Analysis(Ye-Tian, 2022)

**Lemma 1:** Assume $\overline{S}_{\delta,h,p}$ is not empty and $C>0$ is a generic constant.

1. If $p\ge 2$ and $u\in C^2(\overline{\Omega})$, then $|\mathcal{L}^h_{\delta,\Omega}u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)|\to 0$ as $\delta\to 0$ for all $\boldsymbol{x}_i\in\Omega$.
2. If $p\ge 2$ and $u\in C^{2,\alpha}(\overline{\Omega})$ for $\alpha\in(0,1]$, then $|\mathcal{L}^h_{\delta,\Omega}u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)| \le C|u|_{C^{2,\alpha}(\overline{\Omega})}\delta^\alpha$ for all $\boldsymbol{x}_i\in\Omega$.
3. If $p\ge 3$ and $u\in C^{3,\alpha}(\overline{\Omega})$ for $\alpha\in(0,1]$, then $|\mathcal{L}^h_{\delta,\Omega}u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)| \le C|u|_{C^{3,\alpha}(\overline{\Omega})}\delta^{1+\alpha}$ for all $\boldsymbol{x}_i\in\Omega$.

**Theorem 2:** In $d=2$, there exists a constant $C>0$ such that if

$$h\le C\delta\sqrt{\lambda/\Lambda}$$

then $\overline{S}_{\delta,h,2}$ is not empty.

**Theorem 3:** In $d=2$, assume $\overline{S}_{\delta,h,p}$ is not empty, let $u$ be the real solution and $u_\delta^h$ be the solution solved by the discrete operator and $C>0$ is a generic constant.

1. If $p\ge 2$ and $u\in C^{2,\alpha}(\overline{\Omega})$ for $\alpha\in(0,1]$, then
$$\max_{\boldsymbol{x}_i\in\Omega}|u(\boldsymbol{x}_i)-u_\delta^h(\boldsymbol{x}_i)| \le C|u|_{C^{2,\alpha}(\overline{\Omega})}\left(\sqrt{\lambda/\Lambda}\right)^{-\alpha}h^\alpha$$
2. If $p\ge 3$ and $u\in C^{3,\alpha}(\overline{\Omega})$ for $\alpha\in(0,1]$, then
$$\max_{\boldsymbol{x}_i\in\Omega}|u(\boldsymbol{x}_i)-u_\delta^h(\boldsymbol{x}_i)| \le C|u|_{C^{3,\alpha}(\overline{\Omega})}\left(\sqrt{\lambda/\Lambda}\right)^{-(1+\alpha)}h^{1+\alpha}$$

## References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