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Robot Optimal Trajectory Planning Based on Geodesics

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***Abstract*—Geometric characteristics of geodesics in the Riemannian surface are used to make robotic optimal trajectory planning in this paper. Distance length and kinetic energy are regarded as Riemannian metrics respectively, and the Riemannian surfaces are determined by the corresponding metrics, and they represent the robotics kinematics and dynamics respectively. The geodesics on the Riemannian surface are calculated and are regarded as the optimal trajectory. Geodesic is the necessary condition of the shortest length between two points on the Riemannian surface and the covariant derivative of the geodesic's tangent vector is zero. When to implement optimal trajectory planning with arc length as the Riemannian metric, geodesic makes the shortest length between two points. The end-effector's velocity is invariant along the geodesic and the acceleration is zero. So the motion is very smooth. When system's kinetic energy as the Riemannian metric, the geodesic between two points on the kinetic surface makes the kinetic energy remain invariant. Computer calculation and simulation verify that the method based on geodesic is good at trajectory planning especially when the trajectory is linear or certain index should be minimized.**

Keywords- Robot; Optimal trajectory planning; Geodesics

I. INTRODUCTION

The existing trajectory motion planning method is to use interpolating function, and each joint rotates or translates to the final position independently at the same time[1-7]. The coupling between joints is rarely taken into account. When to implement trajectory planning, we hope that the trajectory is unrelated to the world coordinates and it can optimize certain index. Milos Zefran et al[8] investigated the rigid body motion planning in three-dimensional space based on Lie group and covariant derivative. Rodnay and Rimon[9] visualized the dynamics properties of 2-dof robot in three-dimensional Euclidean space. They introduced geodesics on dynamic surface to represent the trajectory when robot system makes free motion. Rodnay and Rimon's paper did not provide a method of how to identify the geodesics under given initial conditions and how to plan robot trajectories based on these geodesics. Park[10] used the concept of Riemannian metric in the optimal design of open loop mechanism.

Riemannian metric, and geodesic are coordinate-free concepts, and geodesic is the necessary condition of the shortest length between two points on the Riemannian surface[11,12]. In the working process of the robot, we usually hope that the trajectory is the shortest or the energy is the minimum. In this article we regard arc length and kinetic energy as Riemannian metrics respectively according to task requirement. The given metric determines an exclusive Riemannian surface, then we obtain the corresponding Riemannian surface. We derive geodesic formula using covariant derivative. Moving along geodesics, the corresponding indices can be minimized. When planning a trajectory along geodesic, the relations between joints are no longer linearly independent, because the joints are coupled. It will improve the robot manipulability to investigate how to use the coupling effects between joints to make trajectory planning.

II. GEOMETRIC BACKGROUND

A. *Riemannian Surface and Riemannian Metric*

Riemannian surface is an abstract geometric surface. The tangent vectors on the surface make up of the tangent space. The rule of inner product is defined in this tangent space, and this rule is called Riemannian metric. It is a symmetric, positive definite quadratic form[11,12].

$$ds^2 = (d\theta_1 \quad d\theta_2)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix} \tag{1}$$

Equation (1) defines an inner product of two vectors in the tangent space of surface, s, ds, θ_1, $d\theta_1$, θ_2, $d\theta_2$ denote the arc length, derivative of arc length, rotatory angle of link 1, derivative of rotatory angle of link 1, rotatory angle of link 2, derivative of rotatory angle of link 2 respectively. And the vectors have the form

$$d\theta_1 \frac{\partial}{\partial\theta_1} + d\theta_2 \frac{\partial}{\partial\theta_2} \tag{2}$$

∂ refers to the partial differentiation, and E, F, G in (1) denote $(\frac{\partial}{\partial\theta_1})^2$, $(\frac{\partial}{\partial\theta_1})(\frac{\partial}{\partial\theta_2})$, $(\frac{\partial}{\partial\theta_2})^2$ respectively.

B. *Covariant Derivative and Geodesic*

We differentiate a vector X on the Riemannian surface and project it to the tangent space, and then the result vector is called the covariant derivative of the vector X. If the covariant derivative of a curve's tangent vector is zero, this curve is called geodesic. Consider a Riemannian surface with Riemannian metric

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$$ds^2 = \sum_{i,j} g_{ij} d\theta_i d\theta_j$$

where g_{ij} denotes Riemannian metric coefficient, equal to E, F, G in (1), for i, j =1, 2, …, n. Assume a curve

$$\theta_i = \theta_i(t) \tag{3}$$

on the Riemannian surface, and its tangent vector is

$$\sum_{i=1}^{n} d\theta_i \frac{\partial}{\partial \theta_i} \tag{4}$$

The curve's arc length is invariant, and it is convenient to regard arc length as the parameter of the curve. We calculate the covariant derivative of (4) and let the result be zero, and then the formulation of geodesic is obtained[12],

$$\frac{d^2\theta_i}{ds^2} + \Gamma^i_{kj} \frac{d\theta_k}{ds}\frac{d\theta_j}{ds} = 0 \tag{5}$$

where s is the arc length of the curve, θ_i is the curve coordinate, for i, j, k=1,2, … , n. The Christoffel symbols Γ^i_{kj} are given in terms of Riemannian metric by

$$\Gamma^i_{kj} = \frac{1}{2} g^{im} \left(\frac{\partial g_{km}}{\partial \theta_j} + \frac{\partial g_{jm}}{\partial \theta_k} - \frac{\partial g_{kj}}{\partial \theta_m}\right) \tag{6}$$

where g^{im} is the element of inverse matrix of the Riemannian metric coefficient matrix g_{ij}.

Geodesic is the shortest curve between two points on the Riemannian surface and the velocity along geodesic curve remains invariant. It is noticed that the meanings of velocities are different according to different Riemannian surfaces determined by different Riemannian metrics. When arc length is regarded as the metric, the velocity along geodesic refers to the normal velocity that we all know well. So the acceleration along geodesic is zero and the motion is smooth. When kinetic energy is regarded as Riemannian metric, the 'velocity' of curve on the surface determined by this metric shows the invariant of kinetic energy. For each point in the Riemannian surface, there is only one geodesic according to the given direction. That is, geodesic is determined by the initial conditions.

III. Optimal Trajectory Planning of 2R Robot

A. Arc Length as Riemannian Metric

A planar 2R robot is shown in Fig. 1. The forward kinematics is

$$\boldsymbol{r} = (l_1 c_1 + l_2 c_{12}, l_1 s_1 + l_2 s_{12}) \tag{7}$$

where $\boldsymbol{r}$ is the position vector of the end-effector of the robot, l_1 is the length of link 1, l_2 is the length of link 2, c_1, c_{12}, s_1, s_{12} denote $\cos(\theta_1)$, $\cos(\theta_1+\theta_2)$, $\sin(\theta_1)$, $\sin(\theta_1+\theta_2)$ respectively.

The Riemannian metric of arc length is given by

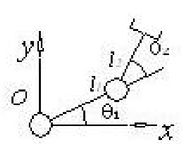

Figure 1 planar 2R robot

$$ds^2 = dr^2 = (d\theta_1 \; d\theta_2) G \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix} \tag{8}$$

where dr is the derivative of vector $\boldsymbol{r}$, G is the metric coefficient matrix,

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} l_1^2 + l_2^2 + 2 l_1 l_2 c_2 & l_1 l_2 c_2 + l_2^2 \\ l_1 l_2 c_2 + l_2^2 & l_2^2 \end{pmatrix} \tag{9}$$

We assume that the mass concentrates on the end of each linkage. Linkage length and mass are all 1. When the metric is given by (8), the corresponding geodesic on the Riemannian surface is determined by (5). The geodesic is solely identified by the given initial conditions.

For example, the initial conditions are given by

$$\theta_1 = \frac{1}{6}\pi, \theta_2 = \frac{2}{3}\pi, \dot{\theta}_1 = \dot{\theta}_2 = 1$$

For a 2R robot, equation (5) is a 2-order differential equation group which determines a geodesic of the robot under the initial conditions above. The numerical solution of this geodesic is shown in Fig. 2. The joints' angles and joints' rates are determined by our geodesic method. The robot moves according to this planning results will perform an optimal trajectory, that is, the shortest path of the end-effector between two given points. The simulation of the end-effector's trajectory by this geodesic method is shown in Fig. 3a.

Fig. 3a shows that the trajectory of the end-effector of the manipulator is a line, that is, the shortest curve on the plane. The velocity of end-effector is invariant along the geodesic and its acceleration is zero, so the end-effector's motion is smooth. If we use the interpolation method to make the trajectory planning, the corresponding result is shown in Fig. 3b. The calculation is simple, but the trajectory is not the shortest and the velocity of the end-effector is not invariant. If we try to make a line trajectory, we must choose more points on the line to interpolate.

B. Kinetic Energy as Riemannian Metric

In order to minimize the kinetic energy of robot system, we consider the kinetic energy as Riemannian metric. It is given by

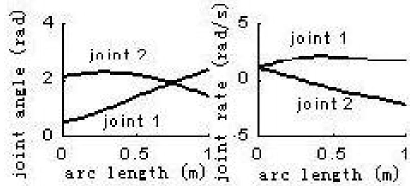

Figure 2 The result of optimal trajectory planning when arc length as Riemannian metric

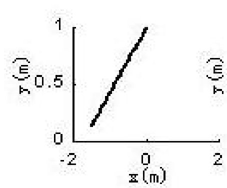

Figure 3a

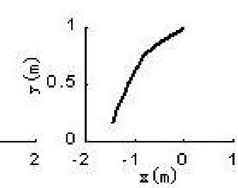

Figure 3b

$$ds^2 = (d\theta_1\ d\theta_2) M \begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix} \tag{10}$$

where M is the inertia matrix of the system, being as the metric coefficient matrix,

$$M = \begin{pmatrix} (m_1 + m_2)l_1^2 + 2m_2 l_1 l_2 c_2 + m_2 l_2^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{pmatrix} \tag{11}$$

m_1 and m_2 denote the mass of link 1 and link 2 respectively and they are set to 1 in this example as well as l_1 and l_2. The Riemannian surface determined by the metric (10) indicates the dynamic properties of the system. The geodesic on this surface is the optimal trajectory that minimizes the kinetic energy.

When the initial conditions are given by

$$\theta_1 = \frac{\pi}{6}, \theta_2 = \frac{2\pi}{3}, \dot{\theta}_1 = \dot{\theta}_2 = 1$$

the sole geodesic under above conditions is determined. This geodesic can be obtained by solve the differential equations (5) according to the initial conditions given above. The numerical solution of geodesic is shown in Fig. 4.

The Riemannian surface with kinetic energy as the metric can be visualized in Euclidean space. It is shown in Fig. 5.

The grids on the surface in Fig. 5 represent the curve coordinates θ_1 and θ_2, and the bold curve on the surface in Fig.5 is the geodesic with the given initial conditions. The kinetic energy of the system remains invariant when the robot moving along this geodesic. The simulation of the end-effector's optimal trajectory is shown in Fig. 6.

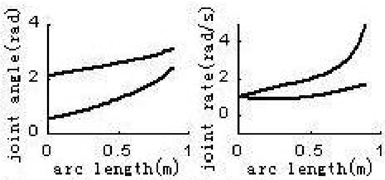

Figure 4 The result of optimal trajectory planning when kinetic energy as Riemannian metric

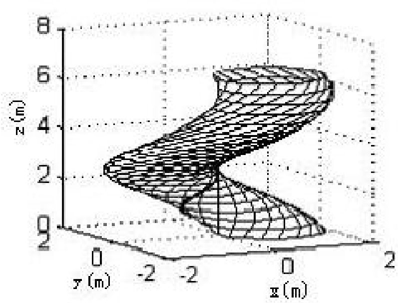

Figure 5 Visualization of Riemannian surface when kinetic energy as the metric

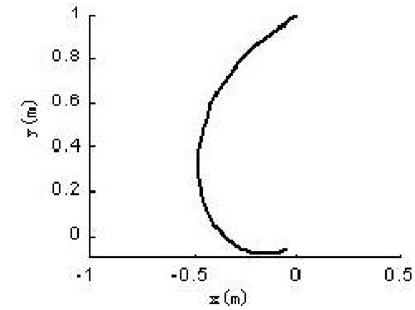

Figure 6 Simulation of the optimal trajectory when kinetic energy as Riemannian metric

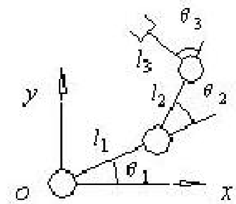

Figure 7 planar 3R robot

IV. Optimal Trajectory Planning of 3R Robot

The planar 3R robot is shown in Fig. 7. l_1, l_2, l_3, θ_1, θ_2, θ_3 denote linkage length and rotatory angle respectively.

The planar 3R robot possesses one redundant freedom and we can use this additional freedom to improve its dynamic manipulability. So we define a compound Riemannian metric made up of arc length and kinetic energy in order to optimize distance length and kinetic energy at the same time. The new Riemannian metric is given by

$$ds^2 = c(ds_1)^2 + d(ds_2)^2 \tag{12}$$

where $(ds_1)^2$ represents the arc length metric and $(ds_2)^2$ represents the kinetic energy metric. c and d are scale coefficients, and they determine the proportion of arc length and kinetic energy in the new Riemannian metric in (12). c and d are selected by the requirement of the task. We set $c = d = 1$, and linkage length and mass are all 1. The mass of each linkage is assumed to concentrated on the tip of each lingkage. The Riemannian metric of a planar 3R robot is given by

$$ds^2 = (d\theta_1\ d\theta_2\ d\theta_3) M (d\theta_1\ d\theta_2\ d\theta_3)^T \tag{13}$$

We can use the same method as the example of 2R robot to calculate the distance metric and kinetic metric of 3R robot respectively and then add them together to construct the combined Riemannian metric of (13). The metric coefficient matrix is

$$M = \begin{pmatrix} 9+6c_2+4c_3+4c_{23} & 5+3c_2+4c_3+2c_{23} & 2+2c_{23}+2c_3 \\ 5+3c_2+4c_3+2c_{23} & 5+4c_3 & 2+2c_3 \\ 2+2c_{23}+2c_3 & 2+2c_3 & 2 \end{pmatrix} \tag{14}$$

When the initial conditions are given by

$$\theta_1 = 0, \dot{\theta}_1 = 1, \theta_2 = 0, \dot{\theta}_2 = 1, \theta_3 = 0, \dot{\theta}_3 = 1$$

$\dot{q}_1, \dot{q}_2, \dot{q}_3$ denote the rates of joint 1, joint 2 and joint 3 respectively. The geodesic on the Riemannian surface is determined by the initial conditions above. In this example, the Riemannian metric is different from arc length metric and kinetic metric. It is a metric considering both arc length metric and kinetic metric, so the geodesic on this Riemannian surface is an optimal trajectory for a combining index of distance and energy. The numerical solution of the geodesic equation (5) for the given initial conditions is shown in Fig. 8. The simulation of the trajectory of the end-effector in Euclidean space is shown in Fig. 9a. For the redundant robot, the inverse kinematics has multiple solutions. The traditional redundant robot trajectory planning is Lagrange multiplier method. But this method needs to solve pseudo-inverse of inertia matrix and Jacobian matrix. The result using Lagrange method is shown in Fig. 9b. Our method is based on Riemannian metric and geodesics and does not need this pseudo-inverse calculation.

V. Conclusion

A new robot optimal trajectory planning method based on geodesic is introduced in this paper. Arc length of the trajectory and kinetic energy are regarded as Riemannian metrics respectively. We also define a general Riemannian metric in order to adapt different optimal task requirements. Examples of application of the new method to planar 2R and planar 3R robots are given.

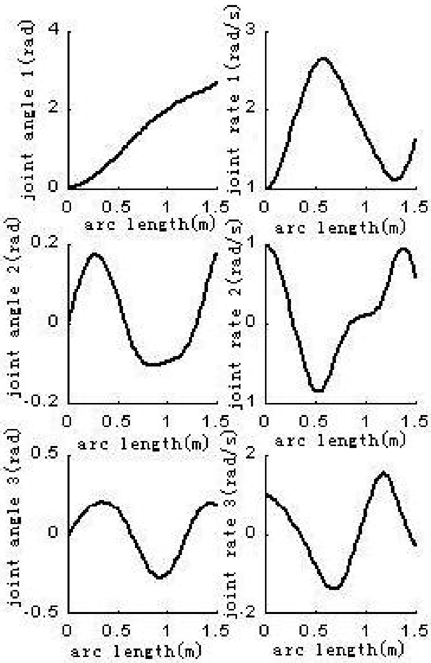

Figure 8 The result of optimal trajectory planning with a combined Riemannian metric

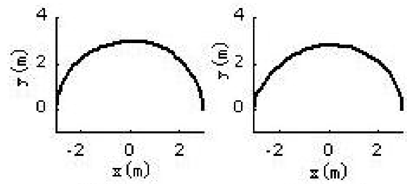

Figure 9a **Figure 9b**

Figure 9a: simulation result based on geodesic
Figure 9b: simulation result based on Lagrange method

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