Motion Planning for Dynamic Environments

Part I - Motion Planning: Living in C-Space

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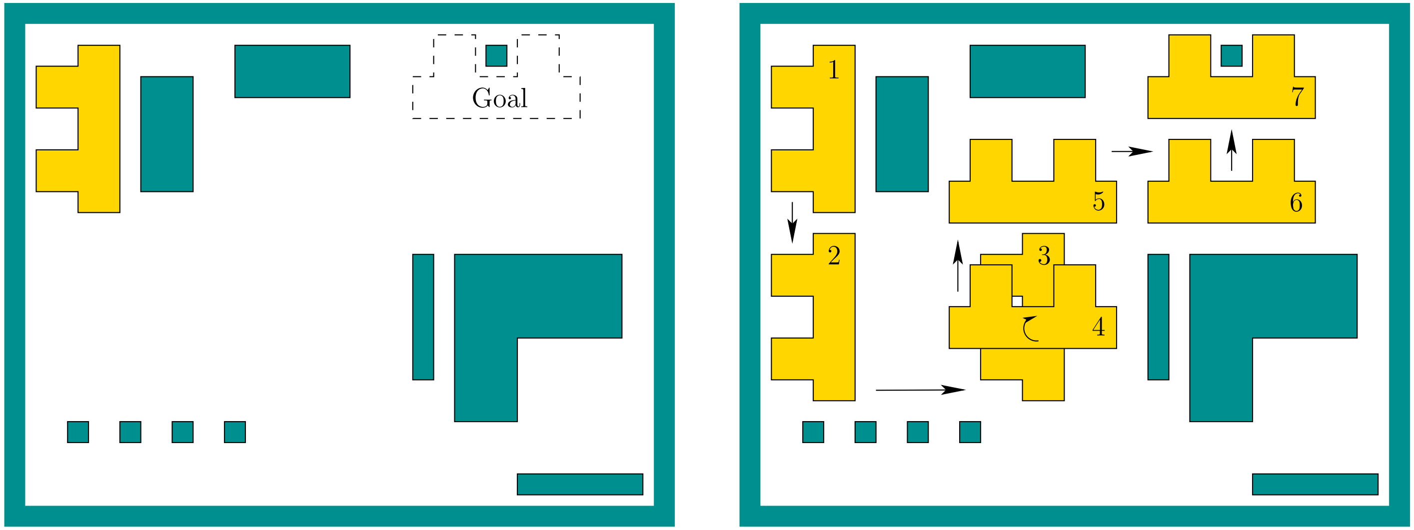
The Basic Path Planning Problem



Given obstacles, a robot, and its motion capabilities, compute collision-free robot motions from the start to goal.

Geometric Models

The robot and obstacles live in a *world* or *workspace* $\mathcal{W}$.
Usually, $\mathcal{W} = \mathbb{R}^2$ or $\mathcal{W} = \mathbb{R}^3$.
The *obstacle region* $\mathcal{O} \subset \mathcal{W}$ is a closed set.
The *robot* $\mathcal{A}(q) \subseteq \mathcal{W}$ is a closed set.
(placed at configuration q).

Representation issues:

- Can it be obtained automatically or with little processing?
- What is the complexity of the representation?
- Can collision queries be efficiently resolved?
- Can a solid or surface be easily inferred?

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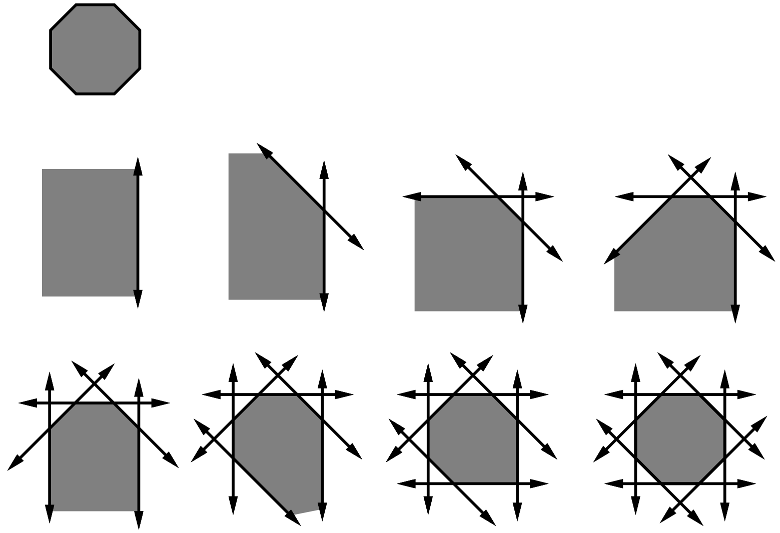
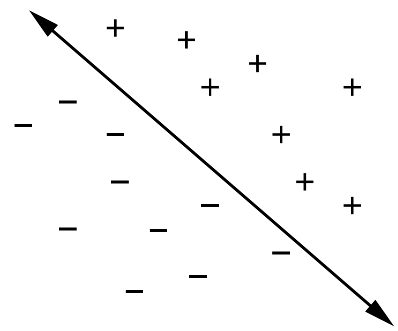
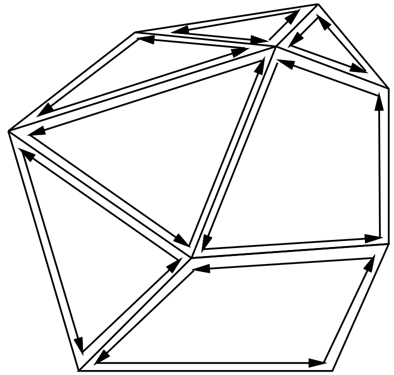
Geometric Models: Linear Primitives

$f(x, y) = ax + by + c$ Inside: $f(x, y) \le 0$



Intersections make convex polygons or polyhedra.





Notions of inside and outside are clear.

Geometric Models: Semi-Algebraic Sets

Consider primitives of the form:

$$H_i = \{(x, y, z) \in \mathcal{W} \mid f_i(x, y, z) \leq 0\},$$

which is a *half-space* is f_i is linear.

Now let f_i be any polynomial, such as $f(x, y) = x^2 + y^2 - 1$.

Obstacles can be formed from finite intersections:

$$\mathcal{O} = H_1 \cap H_2 \cap H_3 \cap H_4.$$

And from finite unions of those:

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_n.$$

$\mathcal{O}$ could then become any *semi-algebraic* set.

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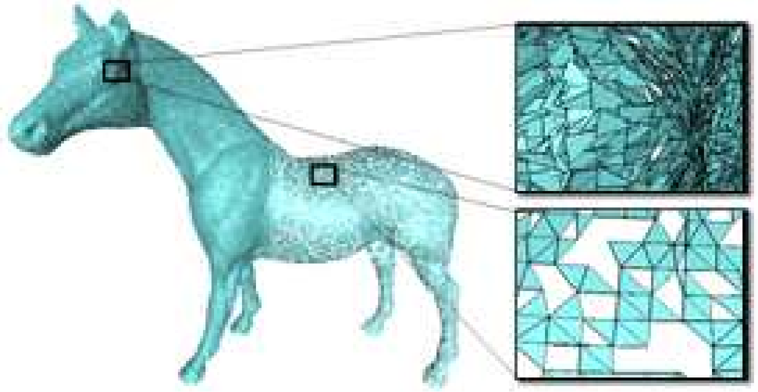
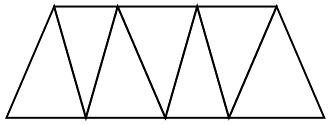
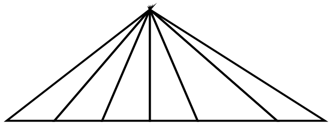
In CAD models inside-outside may not be clearly defined



Throw it all into a collision checker and hope for the best...

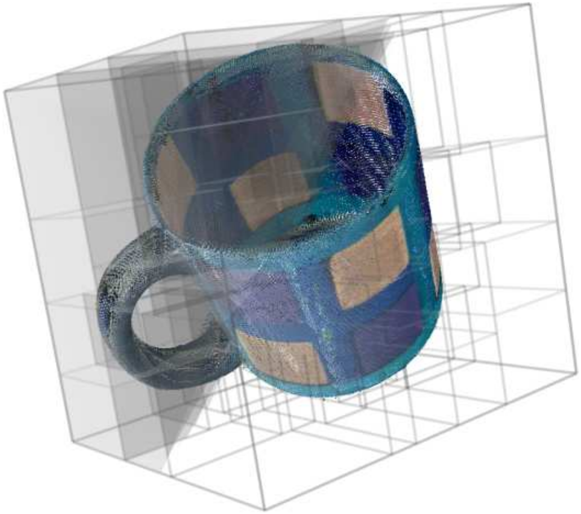
A typical representation: Triangle strips and fans





Geometric Models: Point Clouds

The most natural: Take data straight from range sensors



See the Point Cloud Library.

Problem: Hard to define and test for “collision”

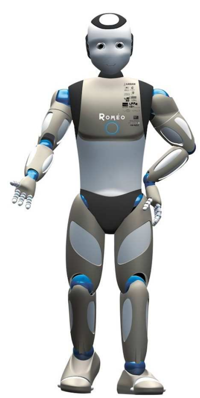
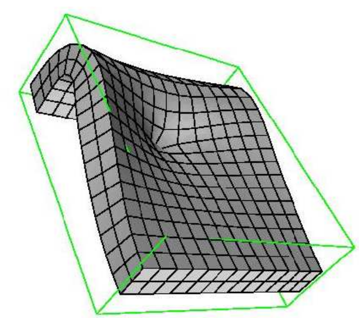
Transforming Robots

Transforming Robots

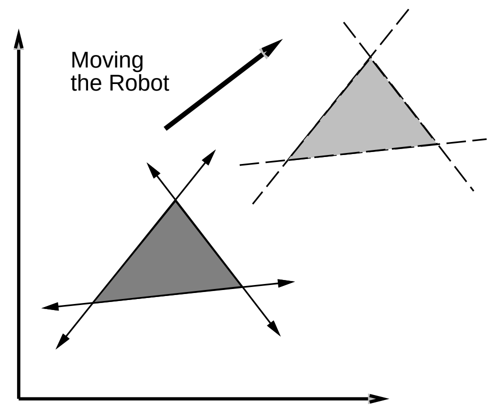
May be rigid, articulated, deformable, reconfigurable, ...

The *degrees of freedom* is important.

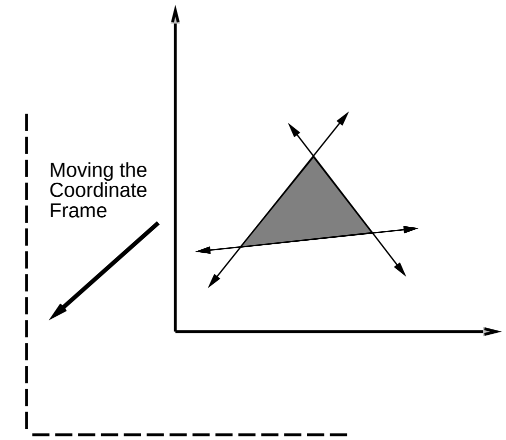
  

Transforming Robots: Planar Rigid Body

Consider $\mathcal{W} = \mathbb{R}^2$ and $\mathcal{A} \subset \mathbb{R}^2$.



Translation of the robot



Translation of the frame

Translation:

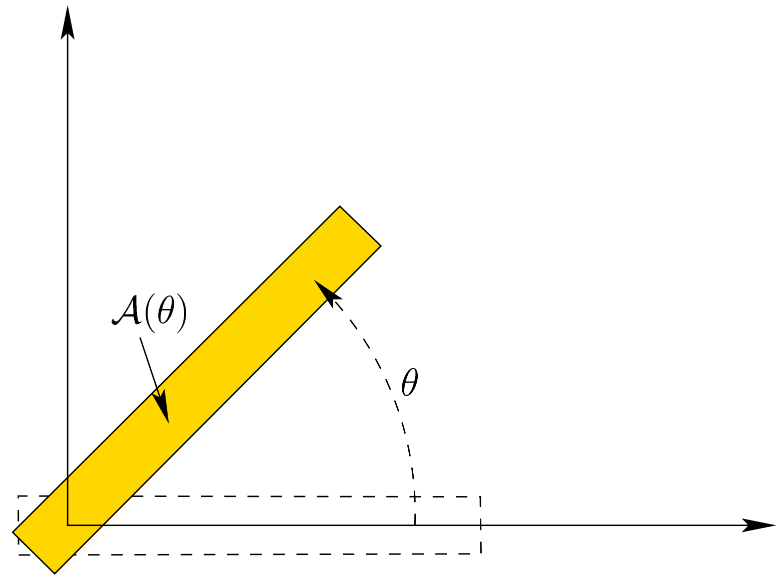
Translate $\mathcal{A}$ by $x_t \in \mathbb{R}$ and $y_y \in \mathbb{R}$.
This means for every $(x, y) \in \mathcal{A}$, we obtain

$$(x, y) \mapsto (x + x_t, y + y_t)$$

The result is denoted as $\mathcal{A}(x_t, y_t)$.

Transforming Robots: Planar Rigid Body

Rotation: Rotate $\mathcal{A}$ by $\theta \in [0, 2\pi)$



This means for every $(x, y) \in \mathcal{A}$, we obtain

$$(x, y) \mapsto (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta)$$

The result is $\mathcal{A}(\theta)$.

Combining Translation and Rotation

Important: Rotate first, then translate

$$(x, y) \mapsto \begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \end{pmatrix}$$

The operations can be performed by a matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \\ 1 \end{pmatrix}$$

Technically: A rigid body transformation is an orientation-preserving, isometric embedding.

Homogeneous Transformation Matrix

The 3 by 3 matrix

$$T(x_t, y_t, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix}$$

contains a rotation matrix in the upper left and a translation column vector on the right.

$$T(x_t, y_t, \theta) = \begin{pmatrix} R(\theta) & v \\ 0 & 1 \end{pmatrix}$$

in which

$$R(\theta) = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

and $v = (x_y, y_t)$.

Transforming Robots: 3D Body

Now, $\mathcal{W} = \mathbb{R}^3$ and $\mathcal{A} \subset \mathbb{R}^3$.

Translation:

Translate $\mathcal{A}$ by $x_t, y_t, z_t \in \mathbb{R}$.

This means for every $(x, y) \in \mathcal{A}$, we obtain

$$(x, y) \mapsto (x + x_t, y + y_t, z + z_t)$$

The result is denoted as $\mathcal{A}(x_t, y_t, z_t)$.

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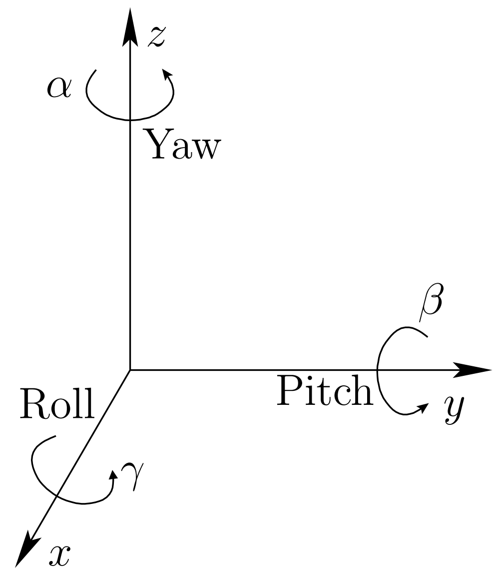
Rotation:



Yaw: Rotation of α about the z-axis:

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Transforming Robots: 3D Body

Pitch: Rotation of β about the y-axis:

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}.$$

Roll: Rotation of γ about the x-axis:

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}.$$

Transforming Robots: 3D Body

Combining them is sufficient to produce any rotation:

$$R(\alpha,\beta,\gamma) = R_z(\alpha)\,R_y(\beta)\,R_x(\gamma) =$$

$$\begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}.$$

Every rotation matrix must have:

- Unit column vectors
- Pairwise orthogonal columns
- Determine 1

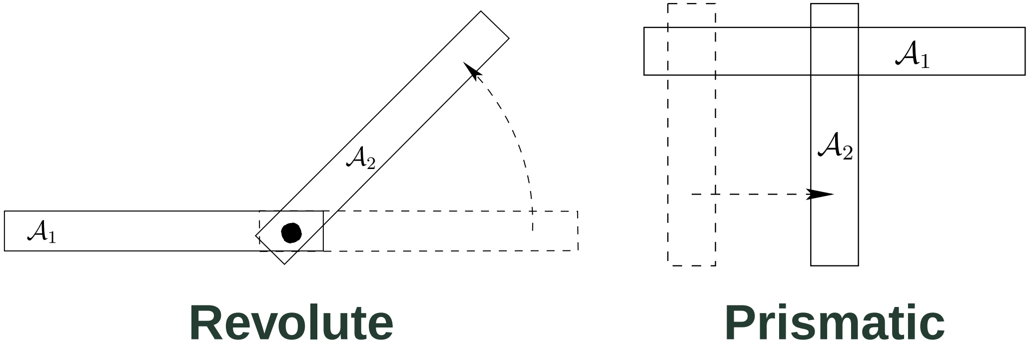
Transforming Robots: 3D Body

We now obtain a 4 by 4 homogeneous transformation matrix:

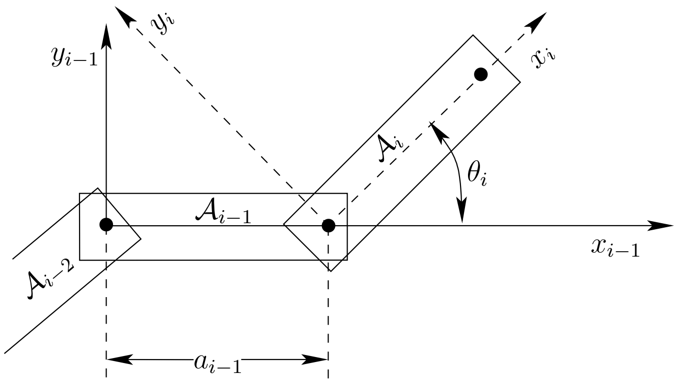
$$T(\alpha, \beta, \alpha, x_t, y_t, z_t) = \begin{pmatrix} R(\alpha, \beta, \gamma) & v \\ 0 & 1 \end{pmatrix}.$$

For n independent bodies, just use n separate homogeneous transformation matrices.

However, if they are non-rigidly attached:



then use specialized, chained transformations.



One matrix for each link:

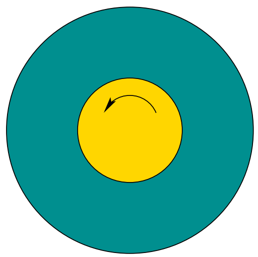
$$T_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & x_t \\ \sin\theta_1 & \cos\theta_1 & y_t \\ 0 & 0 & 1 \end{pmatrix}$$

A chain of matrices for the chain of links:

$$T_1 T_2 \cdots T_m \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

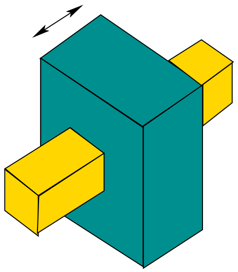
Transforming Robots: Multiple Bodies

In three dimensions, bodies may be non-rigidly attached in many ways:



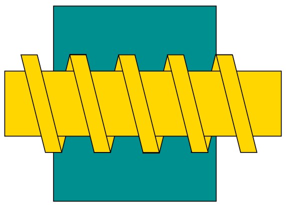
Revolute

1 Degree of Freedom



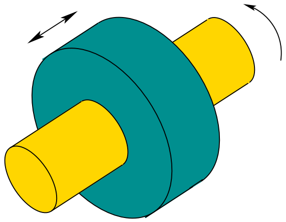
Prismatic

1 Degree of Freedom



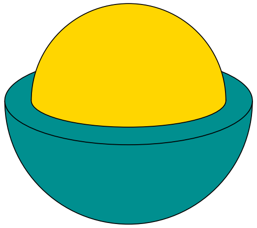
Screw

1 Degree of Freedom



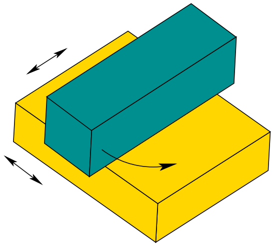
Cylindrical

2 Degrees of Freedom



Spherical

3 Degrees of Freedom



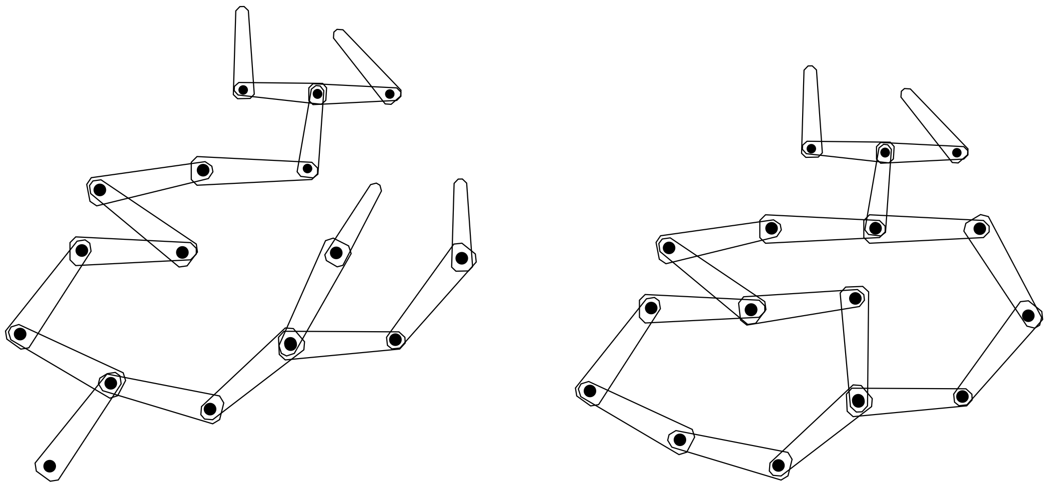
Planar

3 Degrees of Freedom

Nevertheless, systems of parametrizations are developed:

Denavit-Hartenburg, Khalil-Kleinfinger, ...

Transforming Robots: Trees and Loops



Tree of bodies   Closed kinematic chains

General idea: Need to find good parametrizations of the freedom of motion between attached links.

Warning: Extremely hard for closed chains.

Topology

The Space of All Transformations

- Path planning becomes a search on a space of transformations
- What does this space look like?
- How should it be represented?
- What alternative representations are allowed and how do they affect performance?

The C-Space

Three views of the configuration space:

1. As a topological manifold
2. As a metric space
3. As a differentiable manifold

Number 3 is too complicated! There is no calculus in basic path planning.

Topological Spaces

Start with any set X.

Declare some of the sets in $\mathrm{pow}(X)$ to be *open* sets.
If these hold:

1. The union of **any number** of open sets is an open set.
2. The intersection of a **finite number** of open sets is an open set.
3. Both X and $\emptyset$ are open sets.

then X is a *topological space*.

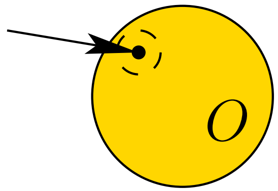
A set $C \subseteq X$ is *closed* if and only if $X \setminus C$ is open.

Many subsets of X could be neither open nor closed.

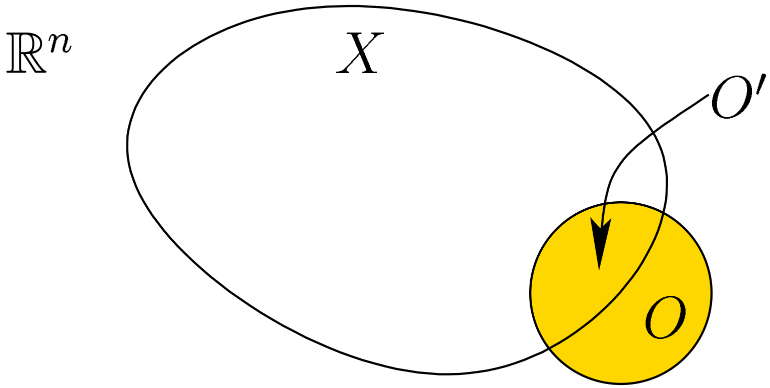
What Topology to Use?

Although elegant, the previous definition was much too general.

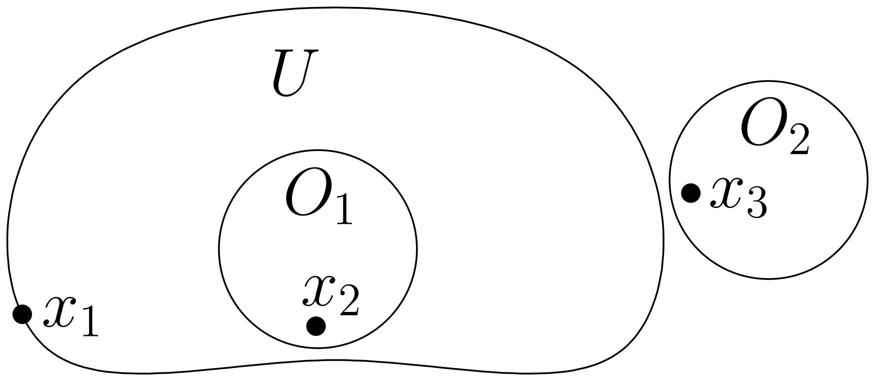
We will only consider spaces of the form $X \subseteq \mathbb{R}^n$.
$\mathbb{R}^n$ comes equipped with standard open sets:
A set O is open if every $x \in O$ is contained in a ball that is contained in O.



To get the open sets of X, take every open set $O \subseteq \mathbb{R}^n$ and form $O' = O \cap X$.



Interior, Exterior, Boundary

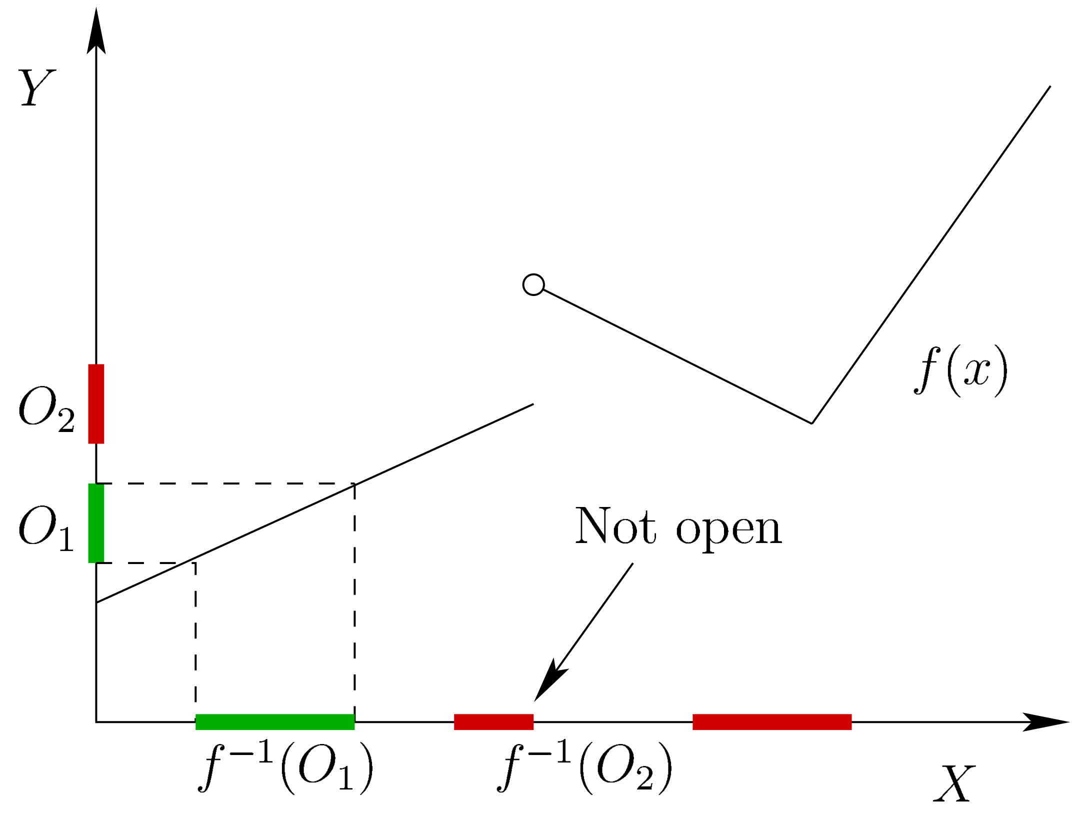


With respect to a subset $U \subseteq X$, a point $x \in X$ may be:

- a *boundary point*, as in x_1 above,
- an *interior point*, as in x_2,
- or an *exterior point*, as in x_3.

Let X and Y be any topological spaces.

A function $f : X \to Y$ is called *continuous* if for any open set $O \subseteq Y$, the preimage $f^{-1}(O) \subseteq X$ is an open set.

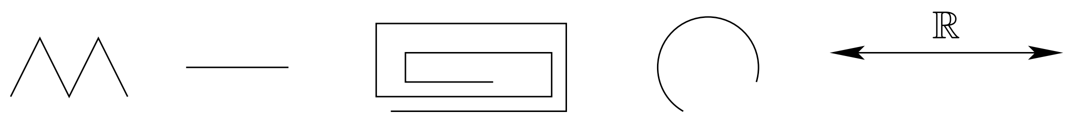


Homeomorphism

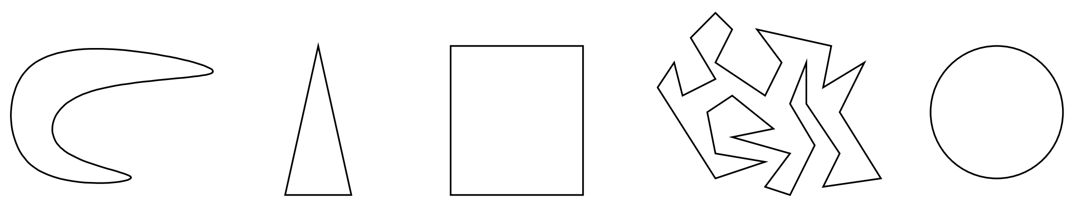
A bijection $f : X \to Y$ is called a *homeomorphism* if both f and f^{-1} are continuous.

If f exists, then X and Y are *homeomorphic*.

Example: For $X = (-1, 1)$ and $Y = \mathbb{R}$, let $x \mapsto 2\tan^{-1}(x)/\pi$ $(-1, 1)$.

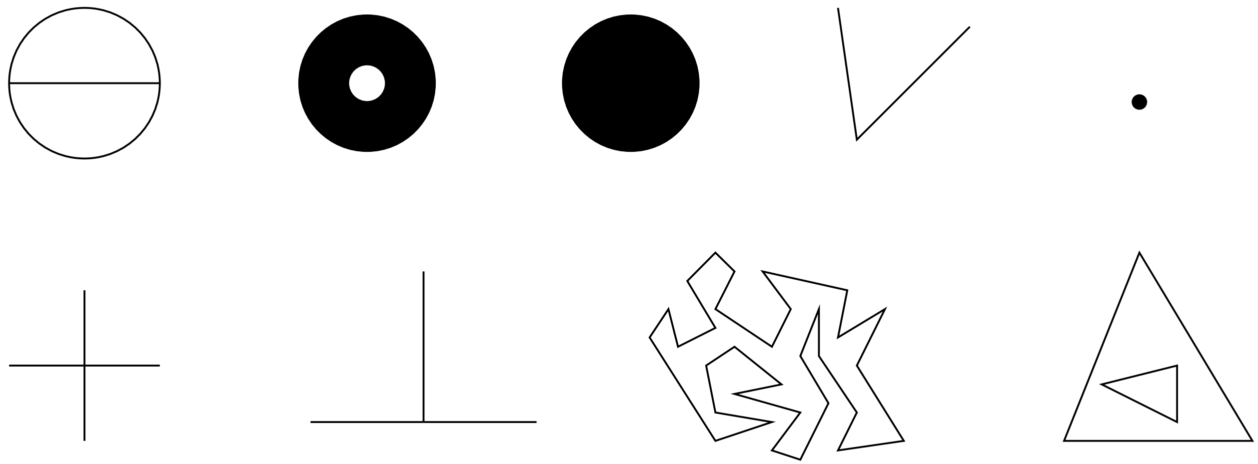


These are all homeomorphic subspaces of $\mathbb{R}^2$.



These are homeomorphic, but not with the ones above them.

Homeomorphism Examples

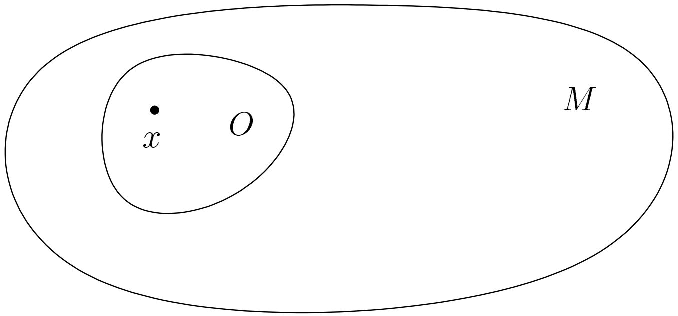


These are all mutually non-homeomorphic

Manifolds

Let $M \subseteq \mathbb{R}^m$ be any set that becomes a topological space using the subset topology.

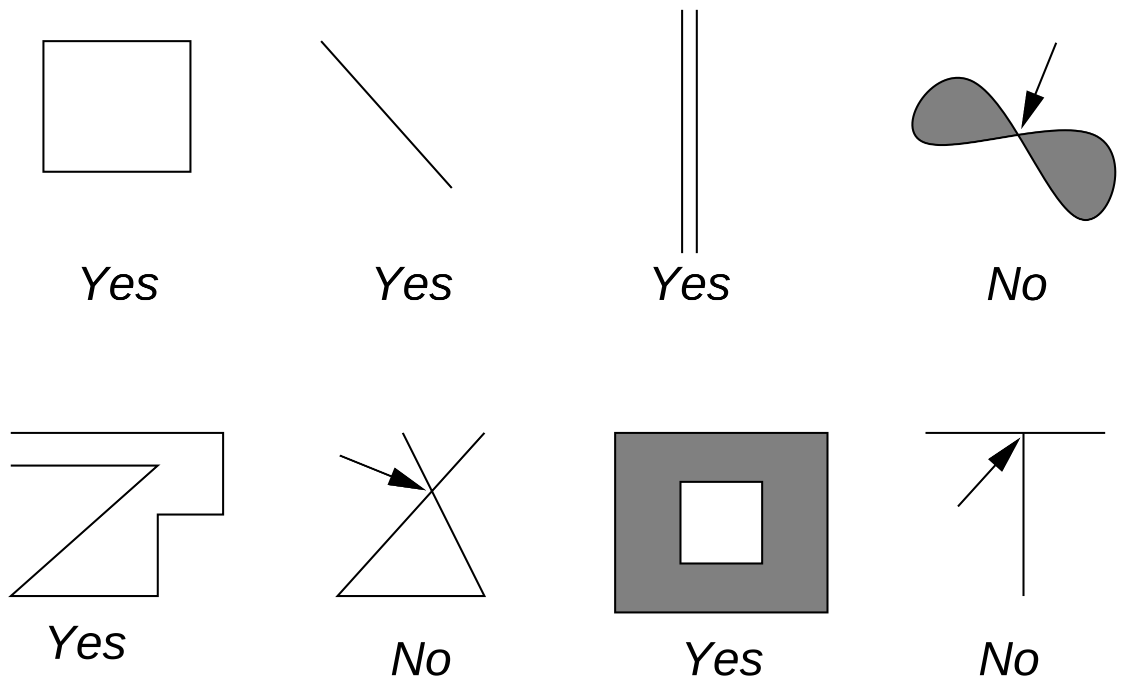
M is called a *manifold* if for every $x \in M$, an open set $O \subset M$ exists such that: 1) $x \in O$, 2) O is homeomorphic to $\mathbb{R}^n$, and 3) n is fixed for all $x \in M$.



It "feels like" $\mathbb{R}^n$ around every $x \in M$.

Manifold or Not?

Subspaces of $\mathbb{R}^2$:



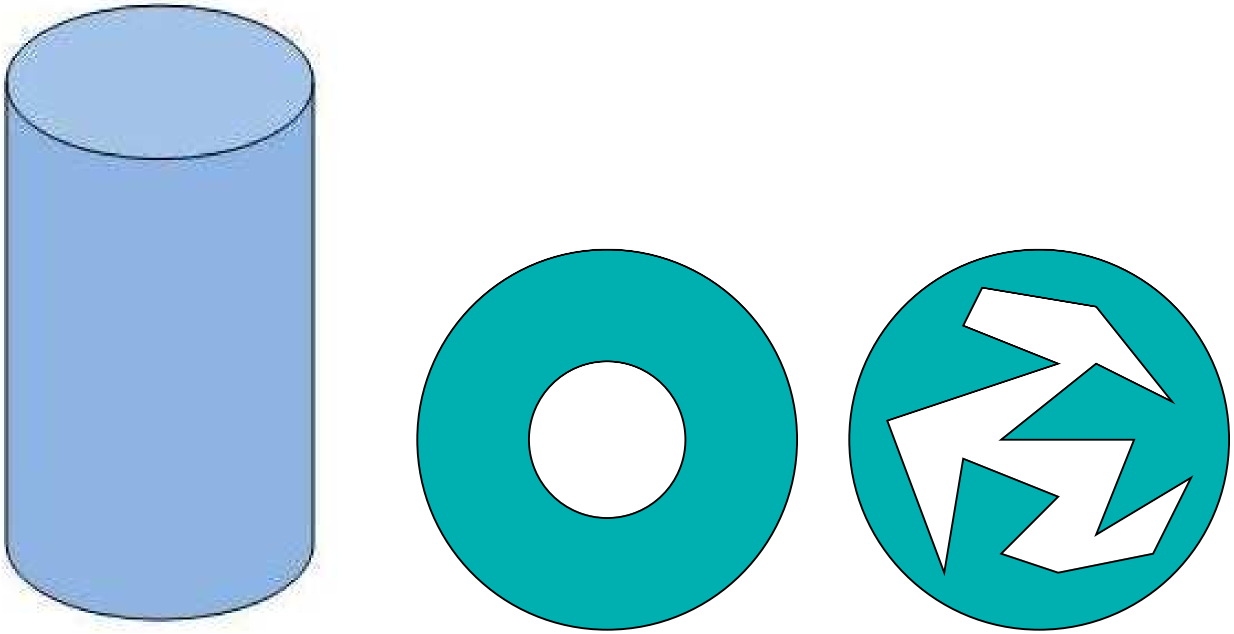
All it takes is one bad point to fail the manifold test.

Manifold Examples

$\mathbb{R}^n$ is a distinct manifold for each n

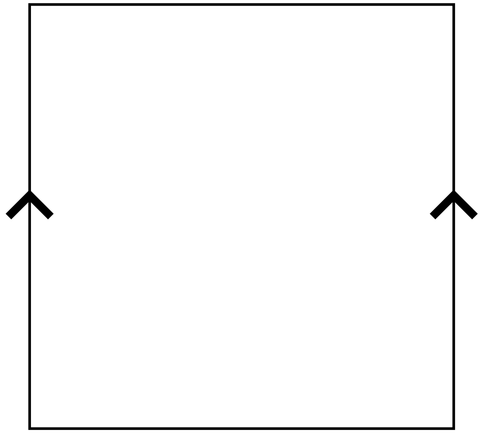
$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a circle manifold

Here are some 2D cylinders (all homeomorphic!):



Another one: $M = \mathbb{R}^2 \setminus \{(0,0)\}$ (the punctured plane)

Start with an open square $(0,1)^2 \subset \mathbb{R}^2$



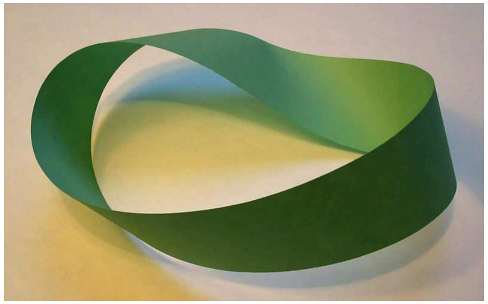
Let (x, y) denote a point on the manifold.

Include the $x = 0$ points and define equivalence relation $\sim$:

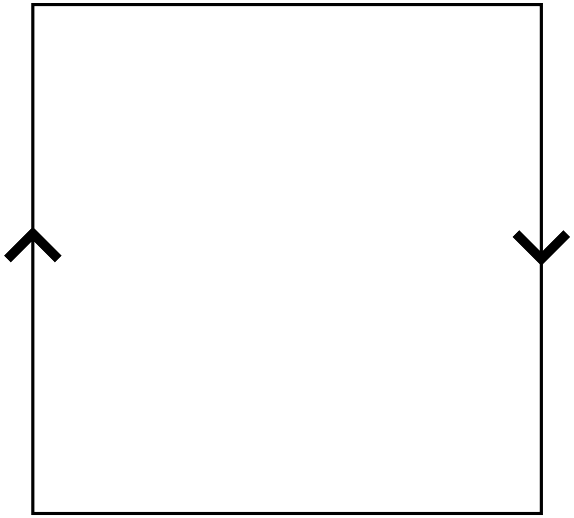
$$(0, y) \sim (1, y)$$

for all $y \in (0, 1)$.

Flat Möbius Band



Typical appearance



A flat representation

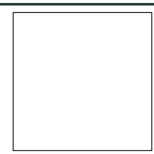
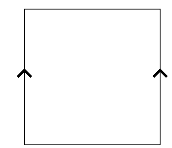
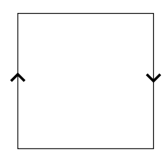
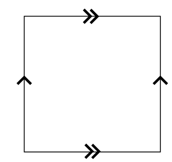
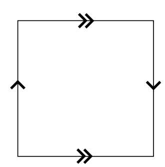
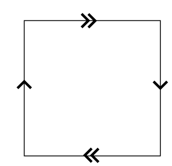
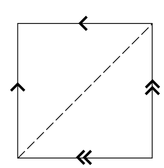
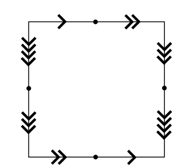
Change the equivalence relation to

$$(0, y) \sim (1, 1 - y)$$

for all $y \in (0, 1)$.

More Flat Manifolds

Many useful, distinct manifolds can be made by identifying edges of a polytope.

	Plane, $\mathbb{R}^2$		Cylinder, $\mathbb{R} \times S^1$
	Möbius band		Torus
	Klein bottle		Projective plane, $\mathbb{RP}^2$
	Two-sphere, S^2		Double torus

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C-Spaces

C-Spaces for Rigid Bodies

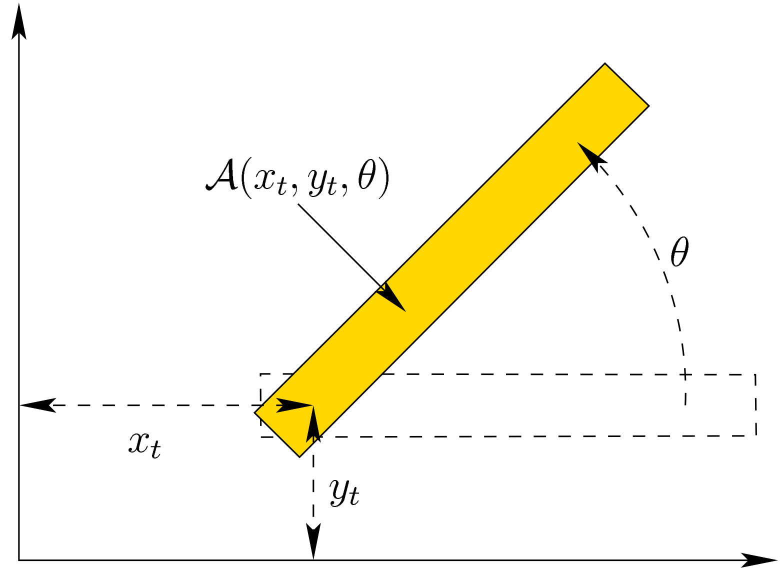
A simple way to describe the manifold of all transformations

$$T(q) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}$$

$SE(n)$ is the group of all $(n+1)$ by $(n+1)$ dimensional homogeneous transformation matrices.

Thus, $SE(2)$ is just a subset of $\mathbb{R}^9$ and $SE(3)$ is a subset of $\mathbb{R}^{16}$.
But which matrices are allowed? Is there a nice parametrization?

The *configuration space* $\mathcal{C}$ is the set of all allowable robot transformations.



Translation parameters: $x_t, y_t \in \mathbb{R}$
Rotation parameter: $\theta \in [0, 2\pi]$

Using the homeomorphism $\theta \mapsto (\cos\theta, \sin\theta)$, the space of all rotations is S^1.
The configuration space is $\mathcal{C} = \mathbb{R}^2 \times S^1$.
Note "=" here means "homeomorphic to"

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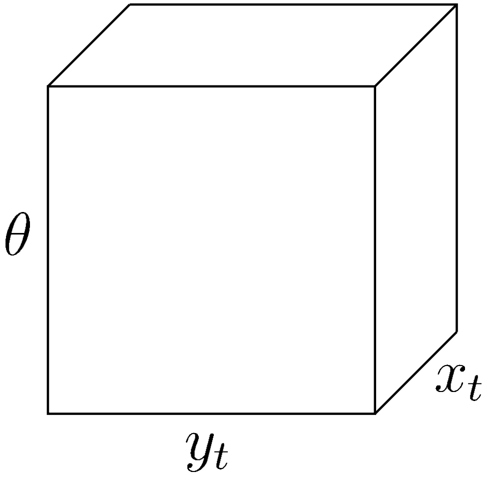
Recall that $\mathbb{R} \times S^1$ is a cylinder.
$\mathcal{C} = \mathbb{R}^2 \times S^1$ can be imagined as a “thick” cylinder.

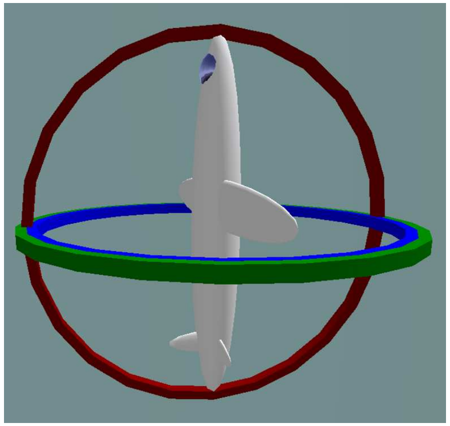
Or a square box with the top and bottom identified:



C-Space for 3D Rigid Body

Translation parameters: $x_t, y_t, z_t \in \mathbb{R}$ Rotation parameters: yaw, pitch, roll?

Gimbal lock problem: An infinite number of YPR parameters map to the same rotation.



When the pitch is 90°, yaw and roll become the same.

(First roll, then pitch, then yaw)

The Space of 3D Rotations

Consider the mapping:

$$(a, b, c, d) \mapsto \begin{pmatrix} 2(a^2 + b^2) - 1 & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & 2(a^2 + c^2) - 1 & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & 2(a^2 + d^2) - 1 \end{pmatrix}$$

in which $a, b, c, d \in \mathbb{R}$.

Enforce the constraint $a^2 + b^2 + c^2 + d^2 = 1$.

In this case, the mapping above is two-to-one everywhere onto $SO(3)$. (a, b, c, d) and $(-a, -b, -c, -d)$ map to the same rotation.

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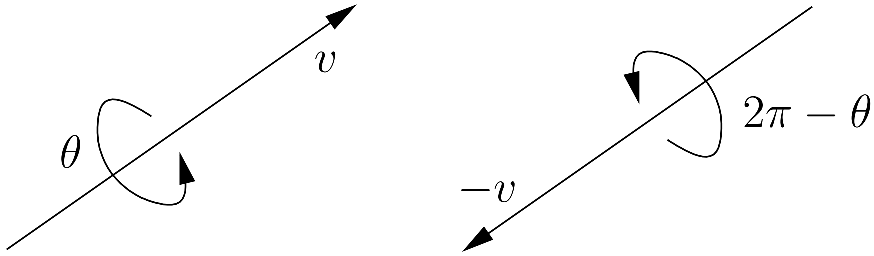
$$(a, b, c, d) = (\cos\frac{\theta}{2}, \left(v_1 \sin\frac{\theta}{2}\right), \left(v_2 \sin\frac{\theta}{2}\right), \left(v_3 \sin\frac{\theta}{2}\right))$$



These are the same rotation.

If you like algebra, consider (a, b, c, d) as a *quaternion*.

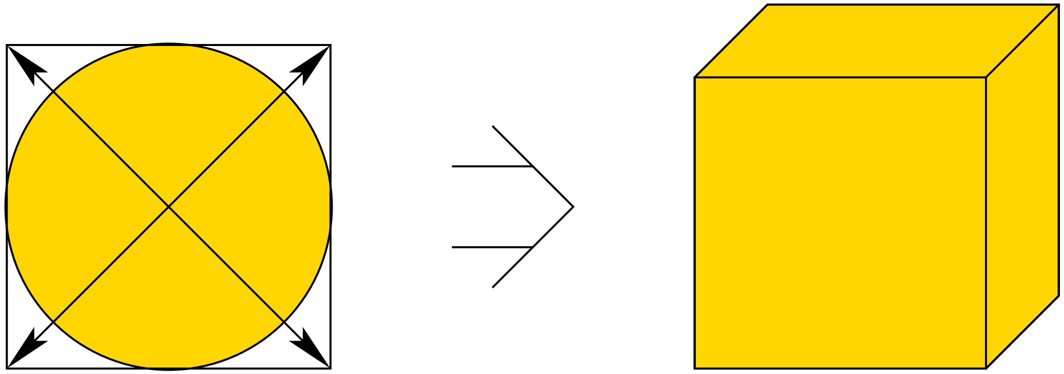
Use upper half of S^3: $d \geq 0$ and $a^2 + b^2 + c^2 + d^2 = 1$

Project down: $(a, b, c, d) \mapsto (a, b, c, 0)$.

The result is a 3D ball: $B_3 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 \leq 1\}$.

However, on the boundary of B_3 we have $(a, b, c) \sim (-a, -b, -c)$.

Stretching B_3 out to make a cubes.



Opposite faces are reverse identified; hence, $B_3 = \mathbb{R}P^3$.

Alternatively, could stretch S^3 out to the faces of the 4-cube.

The 4-cube as 8 faces, but only 4 $3D$ cubes are needed.

The C-Space for Rigid Bodies

For a rigid body that translates and rotates in $\mathbb{R}^3$:

$$\mathcal{C} = \mathbb{R}^3 \times \mathbb{R}P^3$$

The $\mathbb{R}^3$ components arise from translation.
The $\mathbb{R}P^3$ component arises from rotation.

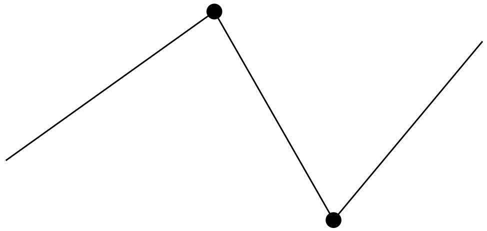
The C-Space for Multiple Bodies

For independent bodies, $\mathcal{A}_1$ and $\mathcal{A}_2$, take the Cartesian product:

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$$

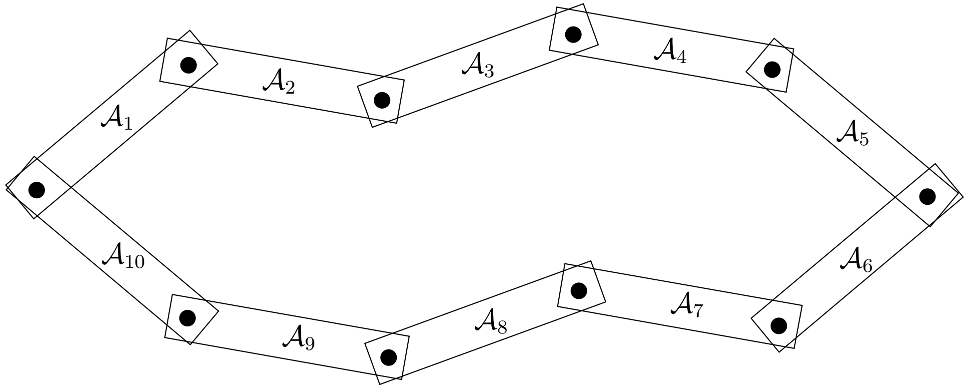
If they are attached to make a kinematic chain, then take the Cartesian product of their components:

$$\mathcal{C} = \mathbb{R}^2 \times S^1 \times S^1 \times S^1$$



The C-Space for Closed Kinematic Chains

The case of closed kinematic chains often arises in redundant robots, manipulation, protein folding, ...



A manifold may result, but it may be difficult to obtain an efficient parametrization.

Comparing Representations

- Convenient parametrizations preferred
- Geometric distortion should be minimized

How should be distortion be described? Metric space.

Metric Spaces

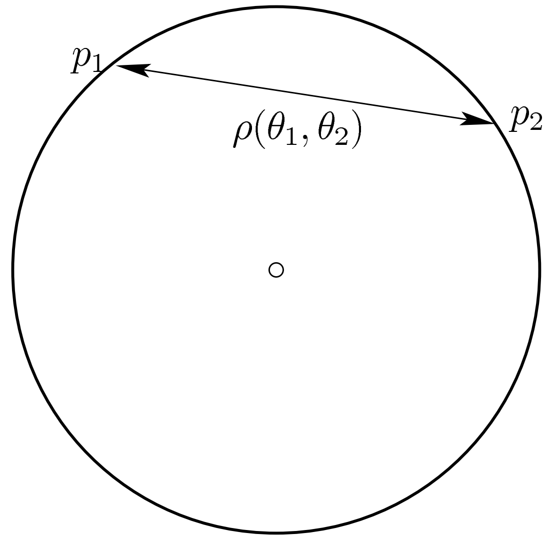
Metric Spaces

A *metric space* (X, ρ) is a topological space X equipped with a function $\rho : X \times X \to \mathbb{R}$ such that for any $a, b, c \in X$:

1. **Nonnegativity:** $\rho(a, b) \geq 0$.
2. **Reflexivity:** $\rho(a, b) = 0$ if and only if $a = b$.
3. **Symmetry:** $\rho(a, b) = \rho(b, a)$.
4. **Triangle inequality:** $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$.

Example: Euclidean distance in $\mathbb{R}^n$
More examples: L_p metrics in $\mathbb{R}^n$

Map onto a unit circle, and then use Euclidean distance:



$p_1 = (\cos\theta_1, \sin\theta_1)$

$p_2 = (\cos\theta_2, \sin\theta_2)$

Direct comparison of angles in $\mathbb{R}$:

$$\rho(\theta_1, \theta_2) = \min\left\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\right\}$$

or

$$\rho(a_1, b_1, a_2, b_2) = \cos^{-1}(a_1 a_2 + b_1 b_2),$$

in which $a_i = \cos\theta_i$ and $b_i = \sin\theta_i$.

Distances in $SO(3)$

Comparing rotations in $SO(3)$ works in a similar way, using the $h = (a, b, c, d)$ representation:

$$\rho_s(h_1, h_2) = \cos^{-1}(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2) \tag{1}$$

However, must consider identification of antipodal points:

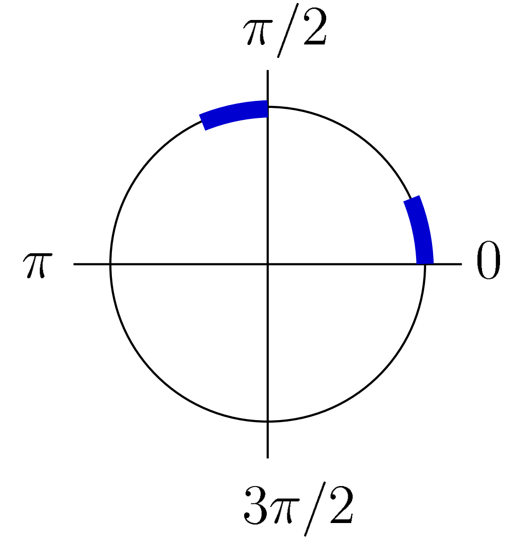
$$\rho(h_1, h_2) = \min\left\{\rho_s(h_1, h_2), \rho_s(h_1, -h_2)\right\}. \tag{2}$$

Other possibilities: Euclidean distance in yaw-pitch-roll space, Euclidean distance in $\mathbb{R}^9$ (the space of 3 by 3 matrices).

Some metrics are more “natural” than others. How to formalize?

Let G be a matrix group, such as $SO(n)$ or $SE(n)$.
Let μ be a *measure* on G. In could, for example, assign volumes by using the metric function.

If for any measurable subset $A \subseteq G$, and any element $g \in G$, $\mu(A) = \mu(gA) = \mu(Ag)$, then μ is called the *Haar measure*.
The Haar measure exists for any locally compact topological group and is unique up to scale.
Example for $SO(2)$ using the unit circle S^1:



For 3D rotations, recall the mapping

$$(a, b, c, d) \mapsto SO(3) \tag{3}$$

The Haar measure for $SO(3)$ is obtained as the standard area (or 3D volume) on the surface of S^3.

Uniform random points on S^3 yield uniform random rotations on $SO(3)$ that are comatible with the Haar measure (it is the right way to sample).

Comparing Rotations to Translations

Let (X, ρ_x) and (Y, ρ_y) be two metric spaces.
A metric space for the Cartesian product $Z = X \times Y$ is formed as

$$\rho_z(z, z') = \rho_z(x, y, x', y') = c_1 \rho_x(x, x') + c_2 \rho_y(y, y'), \qquad (4)$$

in which c_1, c_2 are positive constants.

If $X = \mathbb{R}^2$ from translation and $Y = S^1$ from rotation, what should c_1 and c_2 be?

Perhaps $c_2 = c_1 / r$, in which r is the point on $\mathcal{A}$ that is furthest from the origin.

What should the constants be for a long kinematic chain?

C-Space Obstacles

Obstacle Region

Given world $\mathcal{W}$, a closed obstacle region $\mathcal{O} \subset \mathcal{W}$, closed robot $\mathcal{A}$, and configuration space $\mathcal{C}$.
Let $\mathcal{A}(q) \subset \mathcal{W}$ denote the placement of the robot into configuration q.

The *obstacle region* $\mathcal{C}_{obs}$ in $\mathcal{C}$ is

$$\mathcal{C}_{obs} = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{O} \neq \emptyset\},$$

which is a closed set.

The *free space* $\mathcal{C}_{free}$ is an open subset of $\mathcal{C}$:

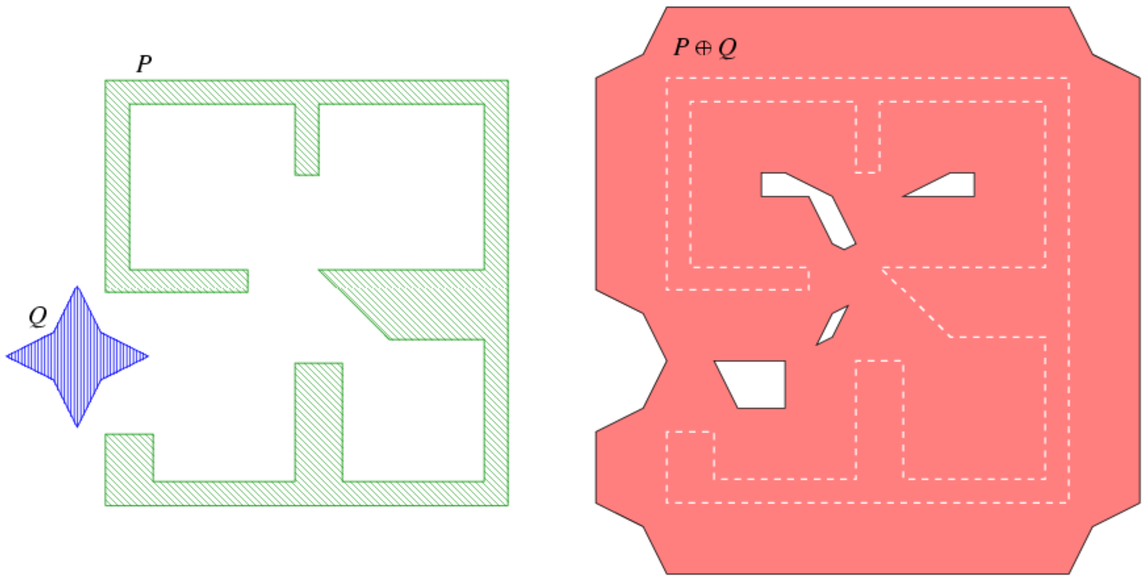
$$\mathcal{C}_{free} = \mathcal{C} \setminus \mathcal{C}_{obs}$$

We want to keep the configuration in $\mathcal{C}_{free}$ at all times!

Consider $\mathcal{C}_{obs}$ for the case of translation only.

The Minkowski sum of two sets is defined as

$$X \oplus Y = \{x + y \in \mathbb{R}^n \mid x \in X \text{ and } y \in Y\} \tag{5}$$

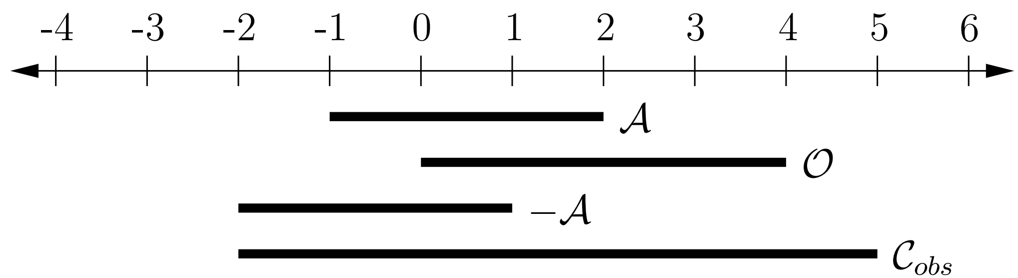


(from the CGAL manual)

The Minkowski difference of two sets is defined as

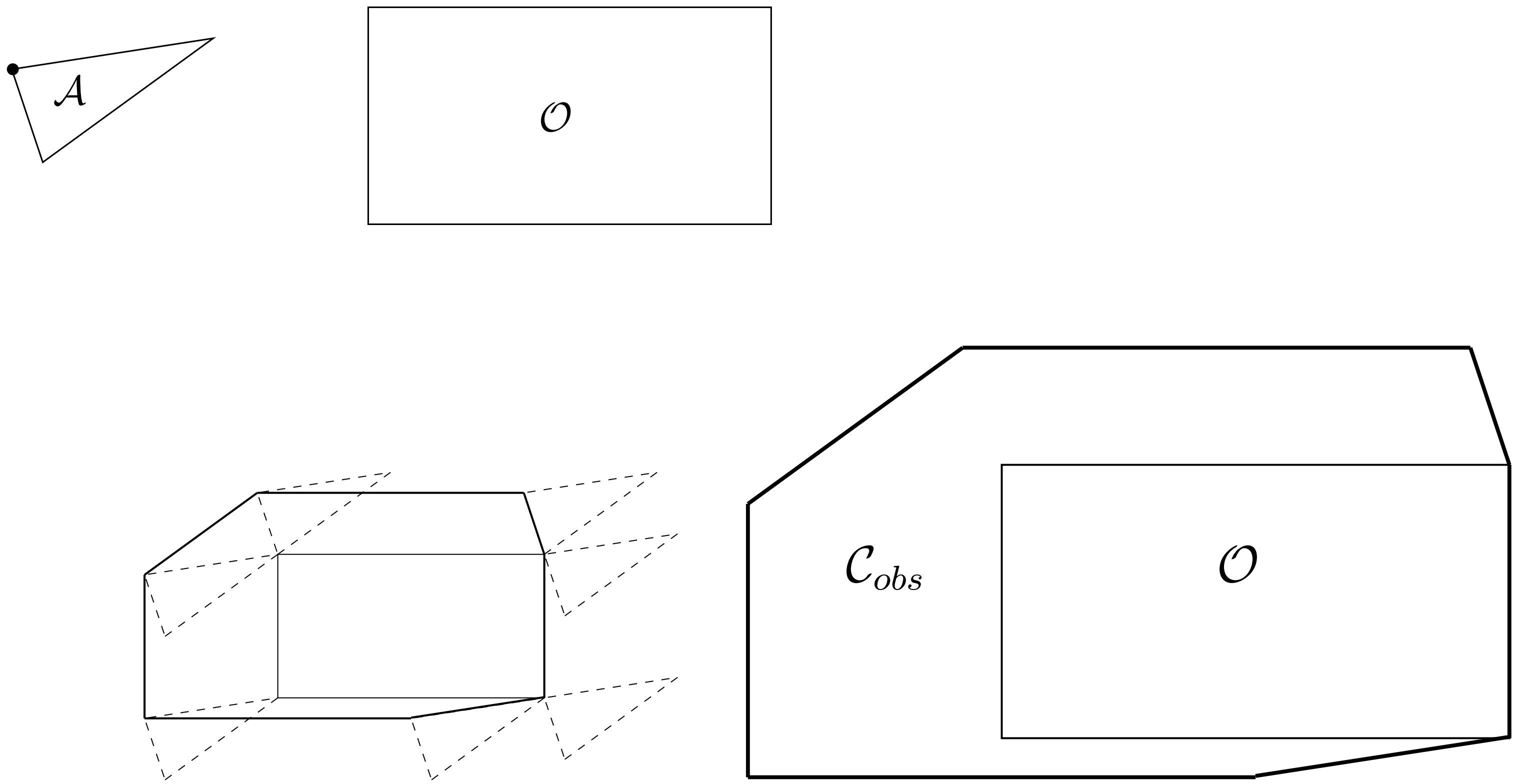
$$X \ominus Y = \{x - y \in \mathbb{R}^n \mid x \in X \text{ and } y \in Y\} \tag{6}$$

A one-dimensional example:



Sometimes called convolution.

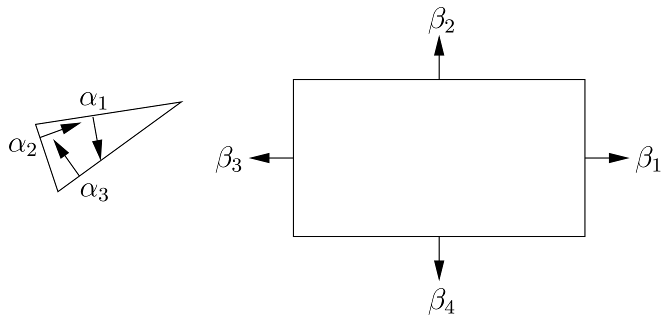
The C-Space Obstacle



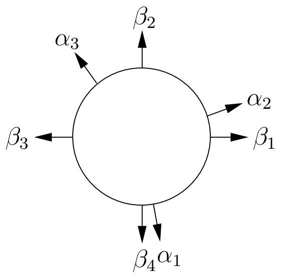
The C-Space Obstacle

Geometric Models
Transforming Robots
Topology
C-Spaces
Metric Spaces
C-Space Obstacles

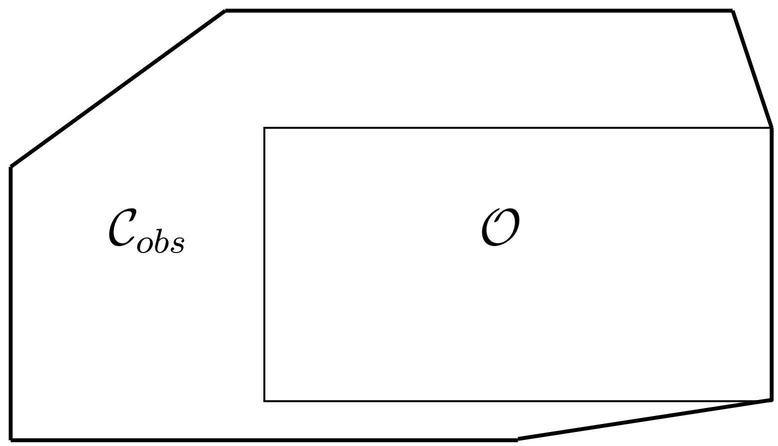
A simple algorithm to compute the obstacle.



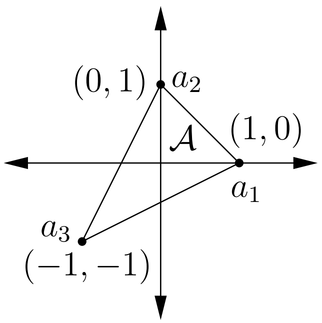
Inward and outward normals

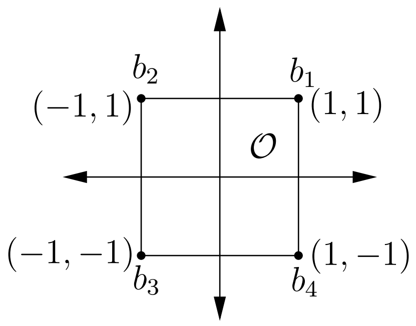


Sorted around S^1



The C-Space Obstacle



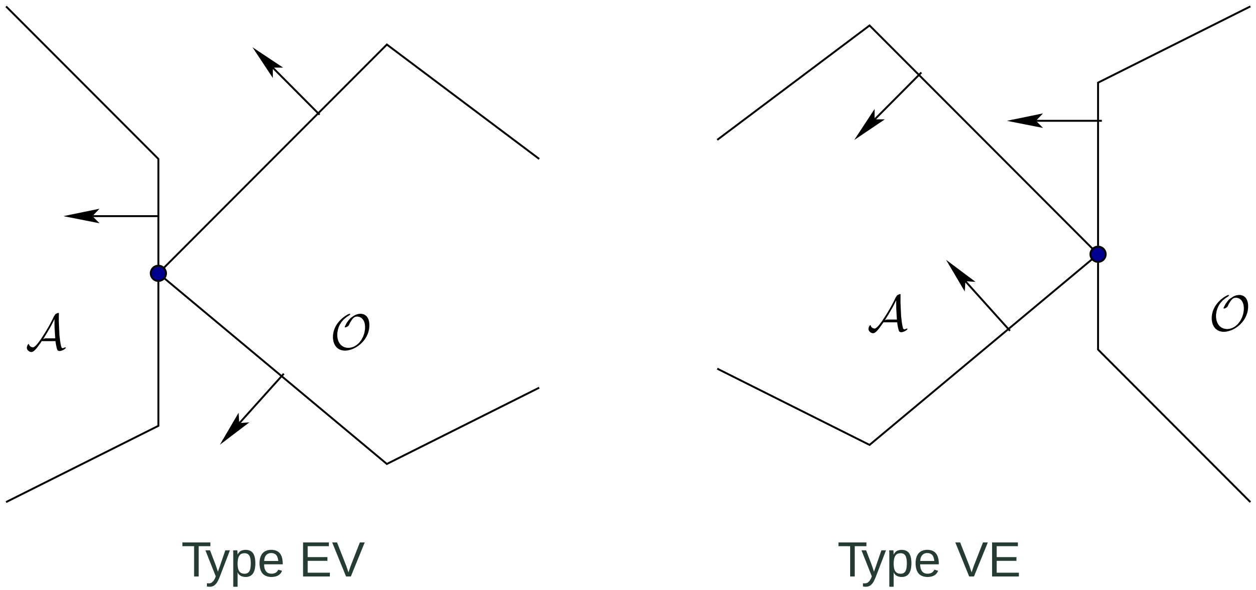


Type	Vtx.	Edge	n	v	Half-Plane
VE	a_3	b_4-b_1	$[1,0]$	$[x_t-2,y_t]$	$\{q\in\mathcal{C} \mid x_t-2\leq 0\}$
VE	a_3	b_1-b_2	$[0,1]$	$[x_t-2,y_t-2]$	$\{q\in\mathcal{C} \mid y_t-2\leq 0\}$
EV	b_2	a_3-a_1	[1,-2]	$[-x_t,2-y_t]$	$\{q\in\mathcal{C} \mid -x_t+2y_t-4\leq 0\}$
VE	a_1	b_2-b_3	$[-1,0]$	$[2+x_t,y_t-1]$	$\{q\in\mathcal{C} \mid -x_t-2\leq 0\}$
EV	b_3	a_1-a_2	$[1,1]$	$[-1-x_t,-y_t]$	$\{q\in\mathcal{C} \mid -x_t-y_t-1\leq 0\}$
VE	a_2	b_3-b_4	$[0,-1]$	$[x_t+1,y_t+2]$	$\{q\in\mathcal{C} \mid -y_t-2\leq 0\}$
EV	b_4	a_2-a_3	$[-2,1]$	$[2-x_t,-y_t]$	$\{q\in\mathcal{C} \mid 2x_t-y_t-4\leq 0\}$

The C-Space Obstacle

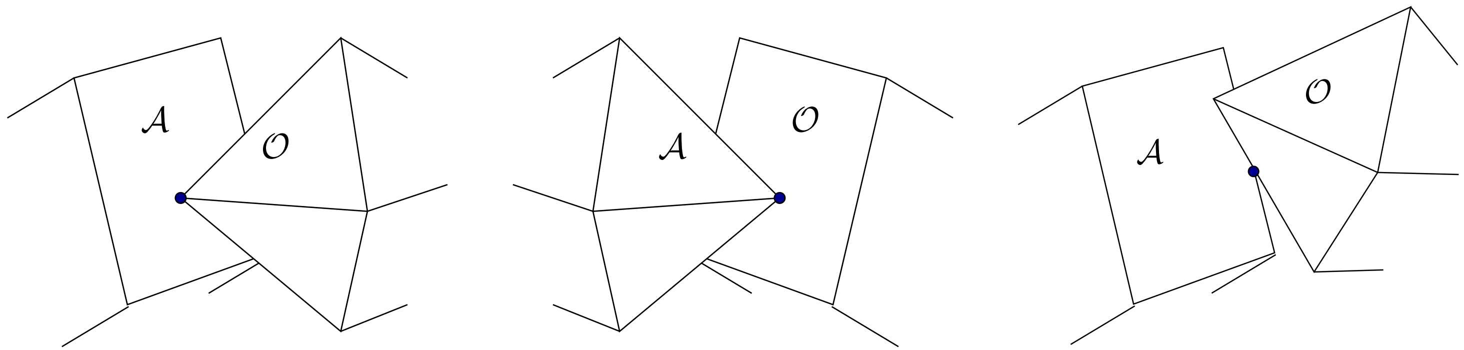
What about translation and rotation?
Obtain a 3D subset of $\mathbb{R}^2 \times S^1$.

Two contact types:



Equations polynomial in x_t, y_t, a, b arise.
($a = \cos\theta$ and $b = \sin\theta$)
Forms the boundary of a 3D semi-algebraic obstacle in $\mathcal{C} = \mathbb{R}^2 \times S^1$

In 3D, there are three contact types:



Type FV Type VF Type EE

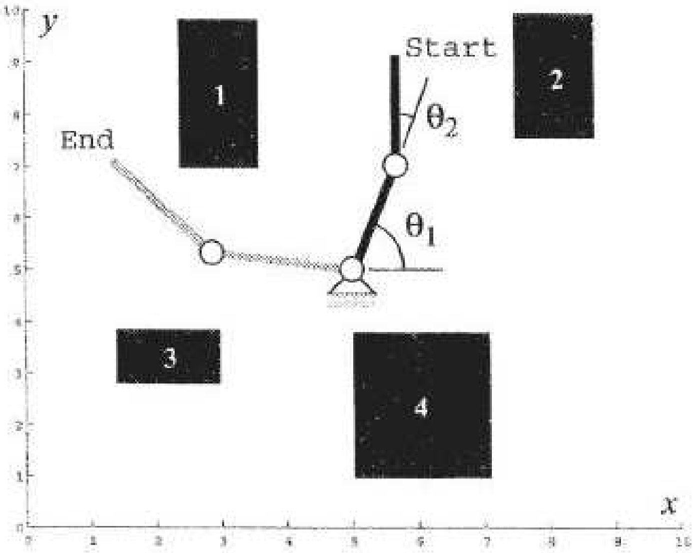
Forms the boundary of a 6D semi-algebraic obstacle in $\mathcal{C} = \mathbb{R}^3 \times \mathbb{R}P^3$

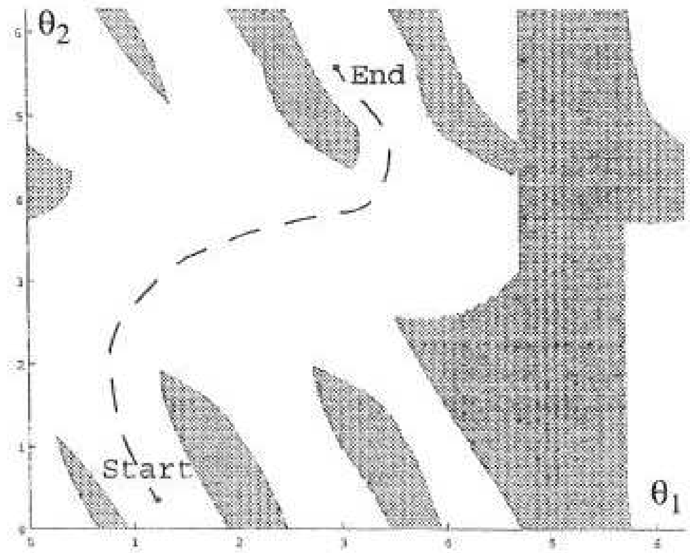
Three different kinds of contacts that each lead to half-spaces in $\mathcal{C}$:

1. **Type FV:** A face of $\mathcal{A}$ and a vertex of $\mathcal{O}$
2. **Type VF:** A vertex of $\mathcal{A}$ and a face of $\mathcal{O}$
3. **Type EE:** An edge of $\mathcal{A}$ and an edge of $\mathcal{O}$.

The Obstacles in C-Space Can Be Complicated

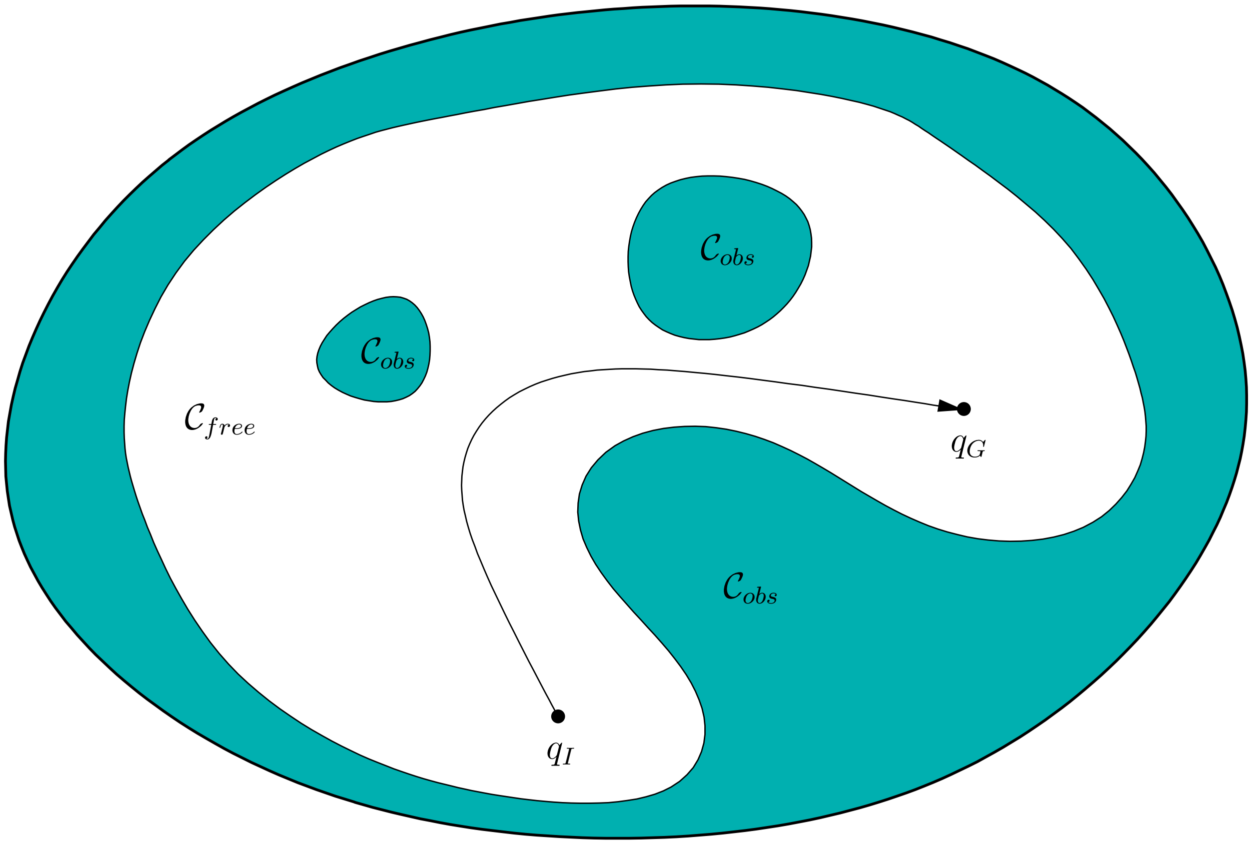
For the case of two-links, $\mathcal{C} = S^1 \times S^1$, but the obstacle region can quickly become strange and complicated:





Basic Motion Planning Problem

Given robot $\mathcal{A}$ and obstacle $\mathcal{O}$ models, C-space $\mathcal{C}$, and $q_I, q_G \in \mathcal{C}_{free}$.



Automatically compute a path $\tau : [0,1] \rightarrow \mathcal{C}_{free}$ so that $\tau(0) = q_I$ and $\tau(1) = q_G$.

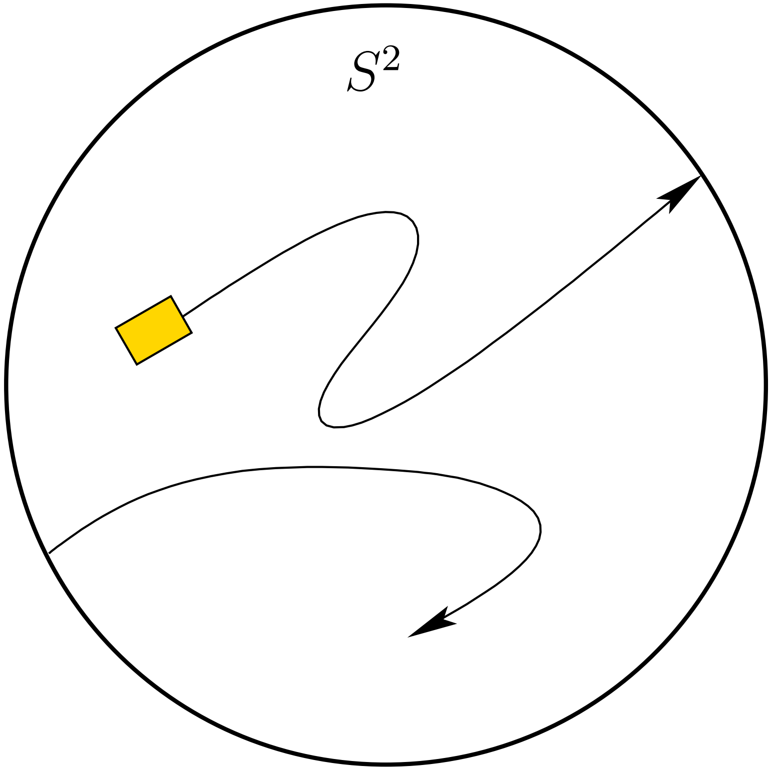
Summary of Part I

- Geometric representations are an important first step.
- Planning is a search on the space of transformations.
- Think like a topologist when it comes to C-space.

More details: *Planning Algorithms*, Chapters 3 and 4.

Homework 1: Solve During Coffee Break

A car driving on a gigantic sphere:



The sphere is large enough so that the car does not wobble.

The car can achieve any position and orientation on the sphere.

What is the C-space?