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Motion planning for control-affine systems satisfying low-order controllability conditions

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ABSTRACT
This paper is devoted to the motion planning problem for control-affine systems by using trigonometric polynomials as control functions. The class of systems under consideration satisfies the controllability rank condition with the Lie brackets up to the second order. The approach proposed here allows to reduce a point-to-point control problem to solving a system of algebraic equations. The local solvability of that system is proved, and formulas for the parameters of control functions are presented. Our local and global control design schemes are illustrated by several examples.

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Motion planning; Volterra series; control-affine system; Lie algebra rank condition; degree theory

1. Introduction

The motion planning problem for nonlinear systems has become an important research area over the last three decades due to its significant geometric features and applications in robotics. In spite of the number of studies, it still remains a challenging problem to construct control laws for general classes of systems, and the development of new approaches attracts considerable interest from both theoretical and applied points of view.

Let us briefly overview some related results in this area with a special emphasis on nonholonomic systems. Brockett (1981) proposed an optimal control law that steers first-order Lie bracket canonical systems. The construction of such optimal controls is also shown in the book by Bloch (2003). In Murray and Sastry (1990), an open-loop algorithm for steering first- and higher order chained-form systems using sinusoidal inputs has been proposed. A related method has been described in Sussmann and Liu (1991) for a more general class of driftless systems. In Liu (1997), a family of highly oscillatory high-amplitude inputs has been used for solving the problem of approximate tracking for a driftless control system. Highly oscillatory sinusoids are also applied in Gurvits and Li (1993) to compute time-periodic solutions for the nonholonomic motion planning problem with obstacle avoidance. A method for steering chained-form systems by piecewise-constant inputs is presented in Lafferriere and Sussmann (1991). Such type of controllers are used for the case of nilpotent systems as well as for the approximate steering problem of general nonholonomic systems. In Chumachenko and Zuyev (2009), the steering problem is solved for several examples of nonholonomic systems with piecewise-constant controls. Sinusoidal and polynomial inputs that steer a three-input system in two-chained form are constructed in Bushnell, Tilbury, and Sastry (1995). A globally convergent steering algorithm, based on nilpotent approximations, is proposed in Chitour, Jean, and Long (2013) and developed in the monograph by Jean (2014). The concept of interpolation entropy is introduced in Gauthier, Jakubczyk, and Zakalyukin (2010) to measure the asymptotics of the minimum length of admissible curves connecting the endpoints for the motion planning problem. In particular, it is shown that the entropy of a motion planning problem is equivalent to that of its nilpotent approximation. Estimates of the entropy and the metric complexity are obtained for generic motion planning problems by constructing their nilpotent approximations in Boizot and Gauthier (2013). A Lie algebraic method for motion planning exploiting the generalised Campbell–Baker–Hausdorff–Dynkin formula is described in Duleba, Khefifi, and Karcz-Duleba (2012).

To the best of our knowledge, only partial results are available for the control design of control-affine systems with drift. In Godhavn, Balluchi, Crawford, and Sastry (1999), motion planning algorithms with band-bang controls are presented for a class of Lagrangian systems with a cyclic coordinate. Another time-state controller for such type of systems is developed in Kiyota and Sampeio (1998). In Bloch and Reyhanoglu

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(1990), open-loop controls are obtained for a small-time locally controllable (STLC) system describing the motion of a knife edge on a flat surface. The paper by Matsuno and Saito (2000) is devoted to the study of a class of control-affine systems with three states and two inputs. To produce a control law, the authors use a special chained-form transformation. The steering problem is considered in Basto-Gonçalves (1999) for control-affine systems under second-order STLC conditions. A discontinuous control law is developed in ur Rehman (2005) to steer a class of control-affine systems with zero drift at the origin. In Michalska and Torres-Torriti (2003), an approach for solving the stabilisation problem by a time-varying feedback law is proposed with the use of sampling strategy and nilpotent approximations of control-affine systems. The time-varying feedback law is constructed there by a concatenation of piecewise constant controllers. The parameters of such piecewise constant controllers are obtained from solving the 'satisficing problem'. An important step in this control design scheme requires the knowledge of solutions to the control-affine system with these parameters. Sufficient Lie algebraic conditions for the stabilisability of control-affine systems have been proposed in Tsinias and Theodosis (2015) by using sampled-data feedback laws and infinite partitions of the time interval.

In this paper, we consider a class of control-affine systems whose vector fields together with their first- and second-order Lie brackets satisfy Hörmander's condition. To solve a point-to-point control problem, we use a Volterra series development for solutions of the system with time-varying trigonometric inputs. The main contribution of this work concerns the construction of steering controls in Sections 3 and 4. This construction allows to compute the parameters of control functions in terms of solutions to auxiliary algebraic equations (Theorems 3.1 and 4.1). To the best of our knowledge, no solvability results have been available for this class of problems. Local solvability results (Theorems 3.2 and 4.2) are proved by exploiting the degree theory, and solutions to the approximate path-following problem are presented in Theorems 2.1 and 4.3. In Section 6, the results obtained are applied to solving the motion planning problem for several mechanical examples. Some technical details are presented in the Appendices.

2. Problem statement and approximation theorem

Consider a control-affine system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \quad x \in D \subseteq \mathbb{R}^n, \; m < n, \tag{1}$$

where $x = (x_1, \dots, x_n)^*$ is the state vector, $u = (u_1, \dots, u_m)^* \in \mathbb{R}^m$ is the control, and '$*$' denotes the transpose. All vector fields $f_i : D \to \mathbb{R}^n$ are assumed to be of class C^3 in a domain D.

For $x^0 \in D$ and an admissible control $u : [0, \tau] \to \mathbb{R}^m$, we denote by $x(t; x^0, u) \in D$ the solution of system (1) with initial data $x|_{t=0} = x^0$ and control $u = u(t)$, $0 \le t \le \tau$. We also use the notation $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}$ for an ε-neighbourhood of a point $x \in \mathbb{R}^n$, $\rho(x, \gamma) = \inf_{y \in \gamma} \|x - y\|$, and $B_\epsilon(\gamma) = \cup_{y \in \gamma} B_\epsilon(y)$ for an ε-neighbourhood of a set $\gamma \subseteq \mathbb{R}^n$. Here, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. To study the local steering problem, we introduce the class $\mathcal{K}$ whose elements are continuous strictly increasing functions $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta(0) = 0$, $\mathbb{R}^+ = [0, +\infty)$.

Problem 2.1 (Local approximate steering problem)**:** For a given $x^\alpha \in D$, $\epsilon_0 > 0$, and $x^\omega \in B_{\epsilon_0}(x^\alpha) \subset D$, the goal is to construct a smooth control $u^{x^\alpha x^\omega}(t) \in \mathbb{R}^m$, defined on $0 \le t \le \tau = \tau(x^\alpha, x^\omega)$, such that the following conditions hold

$$\|x(\tau; x^\alpha, u^{x^\alpha x^\omega}) - x^\omega\| \le r\|x^\alpha - x^\omega\|, \tag{2}$$

$$\|x(t; x^\alpha, u^{x^\alpha x^\omega}) - x^\alpha\| \le \theta\left(\|x^\alpha - x^\omega\|\right) \quad \text{for all } t \in [0, \tau], \tag{3}$$

with a constant $r < 1$ and a function $\theta \in \mathcal{K}$.

It is clear that if system (1) is locally controllable at a point $x^\alpha \in D$, then, for small enough $\epsilon_0 > 0$ and any $x^\omega \in B_{\epsilon_0}(x^\alpha)$, there exists a control $u^{x^\alpha x^\omega} \in L^\infty[0, \tau]$ such that condition (2) holds with $r = 0$. However, in this paper, we treat the construction of controllers in Problem 2.1 as *an algorithm* that computes a smooth function $u^{x^\alpha x^\omega}(t)$ in terms of solutions to certain algebraic equations whose coefficients depend on the vector fields $f_0(x)$, $f_1(x)$,..., $f_m(x)$, and, possibly, their Lie brackets at a point $x = x^\alpha$. We will also extend such an algorithm in order to follow a given curve γ in the state space D.

Problem 2.2 (Approximate path-following problem)**:** For a given curve $\gamma \subset D$ with the endpoints x^0 and x^T, and a given $\varepsilon > 0$, the goal is to construct a piecewise-smooth control $u : [0, T] \to \mathbb{R}^m$ such that $\|x(T; x^0, u) - x^T\| < \varepsilon$ and $\rho(x(t; x^0, u), \gamma) < \varepsilon$ for all $t \in [0, T]$.

For solving this problem, we use a partition π of γ with a finite number of points $x^j \in \gamma$, $j = 0, 1, \dots, N$: $\pi : x^0 \prec x^1 \prec \dots \prec x^N = x^T$, where '$\prec$' denotes the natural order on γ. We assume for the moment that there are $\eta > 0$ and $\epsilon_0 > 0$ such that Problem 2.1 is solvable for each $x^\alpha \in B_\eta(\gamma)$ and $x^\omega \in B_{\epsilon_0}(x^\alpha)$ by a family of controls $\{u^{x^\alpha x^\omega}(\cdot)\}$, and that the mesh of π, defined as

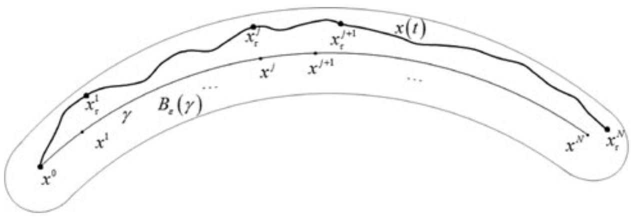

Figure 1. Approximate path-following trajectory $x(t) = x(t; x^0, u_\pi)$.

$\Delta(\pi) = \max_{1\le j\le N} \|x^j - x^{j-1}\|$, is small enough. Under these assumptions, we introduce the following definition.

Definition 2.1: *A π-approximating control* is the function $u_\pi : [0, T] \to \mathbb{R}^m$ defined as follows:

$$\begin{aligned} u_\pi(t) &= u^{x^0x^1}(t) \text{ for } t \in [t_0, t_1],\\ &\quad t_0 = 0,\ t_1 = \tau(x^0, x^1),\\ u_\pi(t) &= u^{x^j_\tau x^{j+1}}(t - t_j) \text{ for } t \in (t_j, t_{j+1}],\\ &\quad t_{j+1} = t_j + \tau(x^j_\tau, x^{j+1}),\ \ j = 1, 2, \dots, N-1, \end{aligned}$$

where the family of controls $u^{x^j_\tau x^{j+1}}(t)$ $(0 \le t \le \tau(x^j_\tau, x^{j+1}))$ solves Problem 2.1, $T = t_N$, $x^0_\tau = x^0$, and $x^{j+1}_\tau = x(\tau(x^j_\tau, x^{j+1}); x^j_\tau, u^{x^j_\tau x^{j+1}})$ for $j = 0, 1, \dots, N-1$ (see Figure 1).

As we will show in the proof of Theorem 2.1, the above construction is well defined if $\Delta(\pi)$ is small enough.

Theorem 2.1: *Let $\gamma \subset D$ be a curve with the endpoints x^0 and x^T, and let positive numbers η, ϵ_0 be such that Problem 2.1 is solvable for each $x^\alpha \in B_\eta(\gamma) \subset D$ and $x^\omega \in B_{\epsilon_0}(x^\alpha)$ by a family of controls $\{u^{x^\alpha x^\omega}(\cdot)\}$. Assume, moreover, that the constant $r \in (0, 1)$ and the function $\theta \in \mathcal{K}$ in formulas (2) and (3) may be chosen independently of $x^\alpha \in B_\eta(\gamma)$.*

Then, for any $\varepsilon > \varepsilon_1 > 0$, there exists a $\bar\Delta = \bar\Delta(\varepsilon, \varepsilon_1) > 0$ such that, for any partition $\pi : x^0 \prec x^1 \prec \dots \prec x^N = x^T$ of γ with $\Delta(\pi) < \bar\Delta$, the corresponding π-approximating control $u_\pi(t)$ is well defined on $t \in [0, T]$, and

$$\|x(t_j; x^0, u_\pi) - x^j\| < \varepsilon_1, \quad j = 0, 1, \dots, N, \tag{4}$$

$$\rho(x(t; x^0, u_\pi), \gamma) < \varepsilon, \quad t \in [0, T], \tag{5}$$

where t_j and T are introduced in Definition 2.1.

Proof: Without loss of generality, we assume that

$$\varepsilon_1 < \min\{\epsilon_0, \eta\},\ \theta(\varepsilon_1) < \varepsilon - \varepsilon_1, \tag{6}$$

otherwise, we take a smaller ε_1 such that condition (6) holds. As the continuous function $\theta \in \mathcal{K}$ is strictly increasing on $\mathbb{R}^+$ and $\theta(0) = 0$, then the inverse function $\theta^{-1}(s)$ is well defined on some semi-interval $s \in [0, \bar\varepsilon)$, $\bar\varepsilon \le +\infty$. We choose the following value for $\bar\Delta = \bar\Delta(\varepsilon, \varepsilon_1) > 0$:

$$\bar\Delta = \begin{cases} \min\left\{\left(\dfrac{1}{r} - 1\right)\varepsilon_1, \epsilon_0 - \varepsilon_1\right\}, & \text{if } \varepsilon - \varepsilon_1 \ge \bar\varepsilon,\\ \min\left\{\left(\dfrac{1}{r} - 1\right)\varepsilon_1, \epsilon_0 - \varepsilon_1,\right. \\ \qquad\qquad \left.\theta^{-1}(\varepsilon - \varepsilon_1) - \varepsilon_1\right\}, & \text{if } \varepsilon - \varepsilon_1 < \bar\varepsilon. \end{cases} \tag{7}$$

Let π: $x^0 \prec x^1 \prec \dots \prec x^N = x^T$ be a partition of γ such that $\Delta(\pi) < \bar\Delta$. We prove by induction that the π-approximating control $u_\pi(t)$, introduced in Definition 2.1, is well defined. It follows from formula (7) that $\|x^0 - x^1\| < \bar\Delta < \epsilon_0$, and thus the control $u_\pi(t) = u^{x^0x^1}(t)$ is well defined for $t \in [0, t_1]$, $t_1 = \tau(x^0, x^1)$. We denote $x^1_\tau = x(t_1; x^0, u^{x^0x^1})$ and observe that

$$\|x^1_\tau - x^1\| \le r\Delta(\pi) < r\bar\Delta \le (1 - r)\varepsilon_1 < \varepsilon_1 \tag{8}$$

because of inequality (2) and formula (7). Assume that the control $u_\pi(t)$ has been already defined for $0 \le t \le t_j$, and that $x^j_\tau = x(t_j; x^0, u_\pi) \in B_{\varepsilon_1}(x^j)$ for some $j \in \{1, 2, \dots, N-1\}$. Then, the control $\tilde u(t) = u^{x^j_\tau x^{j+1}}(t)$ is well defined for $0 \le t \le \tau(x^j_\tau, x^{j+1})$ as $\varepsilon_1 < \eta$ and

$$\|x^j_\tau - x^{j+1}\| \le \|x^j_\tau - x^j\| + \|x^j - x^{j+1}\| < \varepsilon_1 + \bar\Delta \le \epsilon_0. \tag{9}$$

Now we extend $u_\pi(t)$ to the segment $0 \le t \le t_{j+1} = t_j + \tau(x^j_\tau, x^{j+1})$ by assuming $u_\pi(t) = \tilde u(t - t_j)$ for $t \in (t_j, t_{j+1}]$. Then, we estimate the distance between $x^{j+1}_\tau = x(t_{j+1}; x^0, u_\pi)$ and x^{j+1} by using inequalities (2), (9) and formula (7):

$$\|x^{j+1}_\tau - x^{j+1}\| \le r\|x^j_\tau - x^{j+1}\| < r\left(\varepsilon_1 + \bar\Delta\right) \le \varepsilon_1. \tag{10}$$

Thus, by applying the above process for $j = 1, 2, \dots, N-1$, we construct the control $u_\pi(t)$ for all $t \in [0, T]$, $T = t_N$. Note that the corresponding solution $x(t) = x(t; x^0, u_\pi)$ of system (1) is well defined on $t \in [0, T]$ as $x(t) = x(t - t_j; x^j_\tau, u^{x^j_\tau x^{j+1}})$ for $t \in [t_j, t_{j+1}]$, and inequality (4) follows from estimates (8) and (10).

To complete the proof, we consider an arbitrary $t \in [t_j, t_{j+1}]$ $(0 \le j \le N-1)$, and use the triangle inequality together with inequalities (3), (4) and (9):

$$\begin{aligned} \rho(x(t), \gamma) &\le \|x(t) - x^j\| \le \|x(t) - x^j_\tau\| + \|x^j_\tau - x^j\|\\ &< \theta(\|x^j_\tau - x^{j+1}\|) + \varepsilon_1 < \theta(\varepsilon_1 + \bar\Delta) + \varepsilon_1. \end{aligned}$$

Thus, to prove that $\rho(x(t), \gamma) < \varepsilon$, it suffices to show that

$$\theta(\varepsilon_1 + \bar{\Delta}) \le \varepsilon - \varepsilon_1. \tag{11}$$

If $\bar{\varepsilon} = \sup_{s\in\mathbb{R}^+}\theta(s) \le \varepsilon - \varepsilon_1$, then inequality (11) is satisfied with any $\bar{\Delta} > 0$. Otherwise, as the function $\theta \in \mathcal{K}$ is strictly increasing on $\mathbb{R}^+$, inequality (11) is equivalent to $\bar{\Delta} \le \theta^{-1}(\varepsilon - \varepsilon_1) - \varepsilon_1$. The above inequality is satisfied, provided that condition (6) holds and $\bar{\Delta} > 0$ is given by formula (7). As $0 \le j \le N-1$ and $t \in [t_j, t_{j+1}]$ may be taken arbitrarily, we have proved that $\rho(x(t), \gamma) < \varepsilon$ for all $t \in [0, T]$. ■

Remark 2.1: The proof of Theorem 2.1 remains valid for general systems of the form $\dot{x} = f(x, u)$, $x \in D$, $u \in U$, as the main idea is just based on the group property: the translation of a trajectory is a trajectory for time-invariant control systems.

As we see, Theorem 2.1 justifies the possibility of reducing the approximate steering problem to successive concatenations of local controllers. Such local controllers will be constructed in this paper by exploiting the representation of solutions of system (1) by the Volterra series. Namely, if $u(t)$ $(0 \le t \le \tau)$ is an admissible control for system (1), then the corresponding solution $x(t; x^0, u)$ may be approximated by the Volterra series as follows (Lamnabhi-Lagarrigue, 1996; Nijmeijer & van der Schaft, 1990):

$$\begin{aligned} x(t; x^0, u) &= x^0 + \sum_{i=0}^m f_i(x^0)\int_0^t u_i(s)ds + \sum_{i,j=0}^m \frac{\partial f_i(x)}{\partial x} f_j(x)\Big|_{x=x^0} \\ &\quad \times \int_0^t\int_0^s u_i(s)u_j(p)\,dp\,ds \\ &\quad + \sum_{i,j,l=0}^m \frac{\partial}{\partial x}\Big(\frac{\partial f_i(x)}{\partial x} f_j(x)\Big) f_l(x)\Big|_{x=x^0} \\ &\quad \times \int_0^t\int_0^v\int_0^s u_i(v)u_j(s)u_l(p)\,dp\,ds\,dv \\ &\quad + R(t; x^0, u), \quad t \in [0, \tau], \end{aligned} \tag{12}$$

where we introduce an artificial control $u_0 \equiv 1$ for convenience of notation, $\frac{\partial f_i(x)}{\partial x}$ is the Jacobian matrix, and $R(t; x^0, u)$ is the remainder.

In Sections 3 and 4, we will present solutions to the local steering problem within the class of trigonometric polynomials as control inputs. Then, we will show how such controls can be used for solving the path-following problem in Section 4.

3. Controllability conditions with the first-order Lie brackets

For the local steering problem, our goal is to propose a control algorithm that steers system (1) from a given initial point $x^\alpha \in D$ to a small neighbourhood of a target point $x^\omega \in D$ at some time $\tau > 0$. In order to solve this problem explicitly, we assume that there are sets of indices $S_0 \subseteq \{1, 2, \dots, m\}$, $S_1 \subseteq \{1, 2, \dots, m\}$, and $S_2 \subseteq \{1, 2, \dots, m\}^2$ such that $|S_0| + |S_1| + |S_2| = n$. Without loss of generality, we assume that the elements of S_2 are ordered such that $i < j$ for each pair $(i, j) \in S_2$.

Definition 3.1: Control system (1) satisfies the *(S_0, S_1, S_2)-rank condition at a point $x \in D$* if

$$\begin{aligned} \operatorname{span}\big\{ f_i(x),\ [f_0, f_j](x),\ [f_k, f_l](x) \mid i \in S_0, \\ j \in S_1,\ (k, l) \in S_2 \big\} = \mathbb{R}^n. \end{aligned} \tag{13}$$

Here, and in the sequel, $[f_i, f_j](x) = \frac{\partial f_j(x)}{\partial x} f_i(x) - \frac{\partial f_i(x)}{\partial x} f_j(x)$ denotes the Lie bracket of vector fields $f_i(x)$ and $f_j(x)$.

Note that if system (1) satisfies the (S_0, S_1, S_2)-rank condition at a point $x^\alpha \in D$ and $f_0(x^\alpha) = 0$, then system (1) is STLC at x^α due to Proposition 7.4 of Sussmann (1987).

We consider the following family of controls:

$$\begin{aligned} u_k(t) &= \sum_{i\in S_0}\delta_{ki}v_i + \sum_{i\in S_1}\delta_{ki}a_i \sin\left(\frac{2\pi K_i t}{\tau}\right) \\ &\quad + \sum_{(i,j)\in S_2} a_{ij}\left\{\delta_{ki}\cos\left(\frac{2\pi K_{ij} t}{\tau}\right)\right. \\ &\quad \left. + \ \delta_{kj}\sin\left(\frac{2\pi K_{ij} t}{\tau}\right)\right\}, \end{aligned} \tag{14}$$

where $k = 1, 2, \dots, m$, v_i, a_i, a_{ij} are real coefficients, K_i and K_{ij} are nonzero integers, and δ_{ki} is the Kronecker delta. For given $x^\alpha, x^\omega \in D$ and $\tau > 0$, we will define the vector of coefficients

$$a = \big((v_i)_{i\in S_0}, (a_i)_{i\in S_1}, (a_{ij})_{(i,j)\in S_2}\big)^* \in \mathbb{R}^n$$

and parameters $K = \big((K_i)_{i\in S_1}, (K_{ij})_{(i,j)\in S_2}\big)^* \in (\mathbb{Z}\setminus\{0\})^{|S_1|+|S_2|}$ for formula (14) by using the following system of algebraic equations:

$$\tau\left(f_0(x^\alpha) + \sum_{i\in S_0} v_i f_i(x^\alpha)\right) + \frac{\tau^2}{2}V_{20} + \frac{\tau^2}{2\pi}V_{21} = x^\omega - x^\alpha \tag{15}$$

with

$$
\begin{aligned}
V_{20} &= \frac{\partial f_0(x^\alpha)}{\partial x} f_0(x^\alpha) \\
&\quad + \sum_{i\in S_0} v_i \left(\frac{\partial f_0(x^\alpha)}{\partial x} f_i(x^\alpha) + \frac{\partial f_i(x^\alpha)}{\partial x} f_0(x^\alpha) \right) \\
&\quad + \sum_{(i,j)\in {S_0}^2} v_i v_j \frac{\partial f_j(x^\alpha)}{\partial x} f_i(x^\alpha), \\
V_{21} &= \sum_{(i,j)\in S_1\times S_0} \frac{v_j a_i}{K_i} [f_i, f_j](x^\alpha) \\
&\quad - \sum_{i\in S_1} \frac{a_i}{K_i} [f_0, f_i](x^\alpha) - \sum_{(i,j)\in S_2} \frac{a_{ij}}{K_{ij}} [f_0, f_j](x^\alpha) \\
&\quad + \sum_{(i,j,k)\in S_2\times S_0} \frac{v_k a_{ij}}{K_{ij}} [f_j, f_k](x^\alpha) \\
&\quad + \frac{1}{2} \sum_{(i,j)\in S_2} \frac{a_{ij}^2}{K_{ij}} [f_i, f_j](x^\alpha),
\end{aligned} \tag{16}
$$

where $\frac{\partial f_i(x^\alpha)}{\partial x}$ stands for the Jacobian matrix $\frac{\partial f_i(x)}{\partial x}$ evaluated at $x = x^\alpha$.

To formulate the basic result concerning the local steering problem, we need a non-resonance assumption concerning integer parameters K_l and K_{ij}.

Assumption 3.1: *For each $l, q \in S_1$ and $(i_1, j_1) \in S_2$, $(i_2, j_2) \in S_2$ such that $l \neq q$ and $(i_1, j_1) \neq (i_2, j_2)$, the following inequalities hold: $|K_l| \neq |K_q| \neq |K_{i_1 j_1}| \neq |K_{i_2 j_2}|$.*

Theorem 3.1: *Assume that, for x^α, $x^\omega \in D$ and positive numbers τ, ε, ε_1, the vectors $a \in \mathbb{R}^n$ and $K \in (\mathbb{Z} \setminus \{0\})^{|S_1|+|S_2|}$ satisfy the system of algebraic equations (15) and Assumption 3.1, and that the following conditions hold:*

$$
\left\| \frac{\partial f_i(x)}{\partial x} \right\| \le M_1, \quad \left\| \frac{\partial^2 f_{ij}(x)}{\partial^2 x} \right\| \le M_2, \quad \text{for all } x \in \bar{B}_\varepsilon(x^\alpha) \subset D,\ i = \overline{0, m},\ j = \overline{1, n}, \tag{17}
$$

$$
\frac{M_0}{M_1} \left\{ e^{M_1 \bar{U}} - \frac{1}{2}\left((M_1 \bar{U} + 1)^2 + 1 \right) \right\} + \frac{M_2 M_0^2 \sqrt{n}}{4 M_1^3} \left\{ \left(e^{M_1 \bar{U}} - 2 \right)^2 + 2 M_1 \bar{U} - 1 \right\} \le \varepsilon_1, \tag{18}
$$

$$
\bar{U} \le \tfrac{1}{M_1} \ln\left(\tfrac{M_1 \varepsilon}{M_0} + 1 \right), \tag{19}
$$

where

$$
\bar{U} = \left(1 + \sum_{i\in S_0} |v_i| + \sum_{i\in S_1} |a_i| + \sqrt{2} \sum_{(i,j)\in S_2} |a_{ij}| \right) \tau, \quad M_0 = \max_{0\le i\le m} \| f_i(x^\alpha) \|. \tag{20}
$$

Then, $\|x(\tau; x^\alpha, u) - x^\omega\| \le \varepsilon_1$ and $\|x(t; x^\alpha, u) - x^\alpha\| \le \varepsilon$ for all $t \in [0, \tau]$, where the control $u(t)$ $(0 \le t \le \tau)$ is given by formula (14).

Here, $\bar{B}_\varepsilon(x^\alpha)$ stands for the closure of $B_\varepsilon(x^\alpha)$, and $\frac{\partial^2 f_{ij}(x)}{\partial^2 x}$ is the Hessian matrix of the j-th component of $f_i(x)$. The proof of Theorem 3.1 is given in Section 5.

Remark 3.1: By using the Taylor expansion, we conclude that condition (18) is equivalent to $\frac{M_0(M_1^2 + M_2 M_0 \sqrt{n})}{6} \bar{U}^3 + O(\bar{U}^4) < \varepsilon_1$, for small values of $\bar{U}$ given by formula (20).

To study the solvability of algebraic equations (15), we introduce new variables

$$
w_i = \tau v_i, \quad \tilde{a}_i = -\frac{\tau^2 a_i}{2\pi K_i}, \quad \tilde{a}_{ij} = \frac{\tau^2 a_{ij}^2}{4\pi K_{ij}}, \quad \varkappa_{ij} = \sqrt{|K_{ij}|}, \tag{21}
$$

and denote column vectors $w = (w_i)_{i\in S_0} \in \mathbb{R}^{n_0}$, $\tilde{a} = \binom{(\tilde{a}_k)_{k\in S_1}}{(\tilde{a}_{ij})_{(i,j)\in S_2}} \in \mathbb{R}^{n_1}$, $\xi = \binom{w}{\tilde{a}} \in \mathbb{R}^n$, $n_0 = |S_0|$, $n_1 = |S_1| + |S_2|$. As each $\varkappa_{ij}$ is the square root of a positive integer in (21), we will use the notation $\varkappa_{ij} \in \mathbb{N}_{1/2}$, where $\mathbb{N}_{1/2} = \bigcup_{k=1}^\infty \{\sqrt{k}\}$. We also introduce the $n \times n$-matrix

$$
A(x^\alpha) = \left((f_i)_{i\in S_0}, [f_0, f_j]_{j\in S_1}, [f_k, f_l]_{(k,l)\in S_2} \right), \tag{22}
$$

whose columns are formed by the vector fields from the rank condition (13) evaluated at $x = x^\alpha$. Then, we exploit formulas (16) and (21) to rewrite algebraic equations (15) in the following form:

$$
\begin{aligned}
&A(x^\alpha)\xi \\
&= x^\omega - x^\alpha - \tau f_0 - \frac{\tau^2}{2} \frac{\partial f_0}{\partial x} f_0 \\
&\quad - \frac{\tau}{2} \sum_{i\in S_0} w_i \left(\frac{\partial f_0}{\partial x} f_i + \frac{\partial f_i}{\partial x} f_0 \right) - \frac{1}{2} \sum_{(i,j)\in {S_0}^2} w_i w_j \frac{\partial f_j}{\partial x} f_i \\
&\quad + \frac{1}{\tau} \sum_{(i,j)\in S_1\times S_0} w_j \tilde{a}_i [f_i, f_j] + \frac{\tau}{\sqrt{\pi}} \sum_{(i,j)\in S_2} \frac{\sqrt{|\tilde{a}_{ij}|}}{\varkappa_{ij}} [f_0, f_j] \\
&\quad - \frac{1}{\sqrt{\pi}} \sum_{(i,j,k)\in S_2\times S_0} \frac{w_k \sqrt{|\tilde{a}_{ij}|}}{\varkappa_{ij}} [f_j, f_k],
\end{aligned} \tag{23}
$$

where all $f_i(x)$, $\frac{\partial f_i(x)}{\partial x}$, and $[f_i, f_j](x)$ are evaluated at $x = x^\alpha$. If $\xi = \binom{w}{\tilde{a}} \in \mathbb{R}^n$ is a solution of algebraic equation (23) with some $\varkappa_{ij} > 0$, then formula (21) implies that

$$v_i = \frac{w_i}{\tau},\ a_i = -\frac{2\pi K_i \tilde{a}_i}{\tau^2},$$
$$a_{ij} = \frac{2\varkappa_{ij}\sqrt{\pi|\tilde{a}_{ij}|}}{\tau}\operatorname{sign}\tilde{a}_{ij},\ K_{ij} = \varkappa_{ij}^2\operatorname{sign}\tilde{a}_{ij}$$

satisfy Equation (15).

We will prove the following local solvability result for system (23).

Theorem 3.2: *Let the (S_1, S_2, S_3)-rank condition (13) be satisfied at a point $x^\alpha \in D$, and let*

$$\|A^{-1}(x^\alpha)\|^2 \cdot \|f_0(x^\alpha)\| \cdot \left(\sum_{(i,j)\in S_1\times S_0} \|[f_i, f_j](x^\alpha)\|^2\right)^{1/2} < \frac{1}{2}. \tag{24}$$

Then, for any small enough $\epsilon_0 > 0$ and any $x^\omega \in B_{\epsilon_0}(x^\alpha)$, the system of algebraic equations (23) has a solution $\xi \in \mathbb{R}^n$ with some $\tau = O(\epsilon_0)$ and $\varkappa_{ij} \in \mathbb{N}_{1/2}$, $(i,j) \in S_2$ such that $\|\xi\| = O(\epsilon_0)$ and

$$\varkappa_{ij} \neq \varkappa_{i'j'} \quad \text{for each } (i,j) \neq (i',j') \in S_2.$$

Proof: If the (S_1, S_2, S_3)-rank condition (13) is satisfied, then the matrix $A(x^\alpha)$ given by (22) is nonsingular, and

$$\|A(x^\alpha)\xi\| \geq \frac{\|\xi\|}{\|A^{-1}(x^\alpha)\|} \geq c(\|w\| + \|\tilde{a}\|),$$
$$c = \frac{\sqrt{2}}{2\|A^{-1}(x^\alpha)\|},\quad \text{for all } \xi = \begin{pmatrix} w\tilde{a}\end{pmatrix} \in \mathbb{R}^n. \tag{25}$$

In order to prove the solvability of equations (23), we show that there exist positive numbers $\tau, \epsilon_0, \bar{\varkappa}, \epsilon_w, \epsilon_a$ such that

$$c(\|w\| + \|\tilde{a}\|) > \epsilon_0 + \tau k_0 + \tau^2 k_1 + \tau k_2\|w\| + k_3\|w\|^2 + \frac{k_5}{\tau}\|\tilde{a}\|\cdot\|w\| + (\tau k_4 + k_6\|w\|)\frac{\sqrt{\|\tilde{a}\|}}{\bar{\varkappa}} \text{ for all } \xi \in \partial W, \tag{26}$$

where $W = \{\xi \in \mathbb{R}^n \mid \|w\| < \epsilon_w,\ \|\tilde{a}\| < \epsilon_a\}$,

$$k_0 = \|f_0(x^\alpha)\|,\ k_1 = \frac{1}{2}\left\|\frac{\partial f_0(x^\alpha)}{\partial x} f_0(x^\alpha)\right\|,$$

$$k_2 = \frac{1}{2}\left(\sum_{i\in S_0}\left\|\frac{\partial f_0(x^\alpha)}{\partial x} f_i(x^\alpha) + \frac{\partial f_i(x^\alpha)}{\partial x} f_0(x^\alpha)\right\|^2\right)^{1/2},$$

$$k_3 = \frac{1}{2}\left(\sum_{(i,j)\in {S_0}^2}\left\|\frac{\partial f_j(x^\alpha)}{\partial x} f_i(x^\alpha)\right\|^2\right)^{1/2},$$

$$k_4 = \frac{|S_2|^{1/4}}{\sqrt{\pi}}\left(\sum_{(i,j)\in S_2}\|[f_0, f_j](x^\alpha)\|^2\right)^{1/2},$$

$$k_5 = \left(\sum_{(i,j)\in S_1\times S_0}\|[f_i, f_j](x^\alpha)\|^2\right)^{1/2},$$

$$k_6 = \frac{|S_2|^{1/4}}{\sqrt{\pi}}\left(\sum_{(i,j,k)\in S_2\times S_0}\|[f_j, f_k](x^\alpha)\|^2\right)^{1/2}.$$

Let us first consider the limiting case $\bar{\varkappa} \to \infty$. Then, inequality (26) takes the form

$$g_\tau(\|\tilde{a}\|, \|w\|) > \epsilon_0 + \tau k_0 + \tau^2 k_1, \tag{27}$$

with $g_\tau(p,q) = (c - \tau k_2)q + cp - k_3 q^2 - \frac{k_5}{\tau}pq$. To show that inequality (27) holds for all $\xi = \binom{w}{\tilde{a}} \in \partial W$, it suffices to find positive numbers $\tau, \epsilon_0, \epsilon_w, \epsilon_a$ such that

$$\inf_{\xi\in\partial W} g_\tau(\|\tilde{a}\|, \|w\|) = \inf_{(p,q)\in l_p\cup l_q} g_\tau(p,q) > \epsilon_0 + \tau k_0 + \tau^2 k_1, \tag{28}$$

where $l_p : p \in [0, \epsilon_a], q = \epsilon_w,\ l_q : p = \epsilon_a, q \in [0, \epsilon_w]$.

We see that $g_\tau(p,q)$ is increasing along l_p and l_q if

$$\frac{\partial g_\tau(p, \epsilon_w)}{\partial p} = c - \frac{k_5\epsilon_w}{\tau} \geq 0 \ \text{ and}$$
$$\frac{\partial g_\tau(\epsilon_a, q)}{\partial q} = c - \tau k_2 - \frac{k_5}{\tau}\epsilon_a - 2k_3 q \geq 0 \quad \text{for } q \in [0, \epsilon_w]. \tag{29}$$

If these conditions are satisfied, then formula (28) is reduced to

$$\inf_{(p,q)\in l_p\cup l_q} g_\tau(p,q) = \min\left\{g_\tau(0,\epsilon_w), g_\tau(\epsilon_a, 0)\right\}$$
$$= \min\{(c - \tau k_2)\epsilon_w - k_3\epsilon_w^2, c\epsilon_a\} > \epsilon_0 + \tau k_0 + \tau^2 k_1. \tag{30}$$

In particular, condition (29) holds for

$$\epsilon_w = \frac{c}{k_5}\tau,\ \epsilon_a = \frac{\tau}{k_5}\left(c - \frac{k_2 k_5 + 2k_3 c}{k_5}\tau\right). \tag{31}$$

With this choice of ϵ_a and ϵ_w, the inequalities $(c - \tau k_2)\epsilon_w - k_3\epsilon_w^2 > \epsilon_0 + \tau k_0 + \tau^2 k_1$ and $c\epsilon_a > \epsilon_0 +$

$\tau k_0 + \tau^2 k_1$ from formula (30) will be satisfied if

$$d := \frac{c^2}{k_5} - k_0 > 0 \tag{32}$$

and

$$\tau - \gamma_1 \tau^2 > \frac{\epsilon_0}{d}, \ \tau - \gamma_2 \tau^2 > \frac{\epsilon_0}{d}, \tag{33}$$

where $\gamma_1 = \frac{k_1 k_5^2 + c(k_2 k_5 + k_3 c)}{k_5^2 d}$, $\gamma_2 = \gamma_1 + \frac{k_3 c^2}{k_5^2 d}$. Note that condition (24) implies that $d > 0$ in formula (32). To satisfy condition (33), we observe that $\tau - \gamma\tau^2 > \frac{\tau}{2}$ for $\tau \in \left(0, \frac{1}{2\gamma}\right)$, $\gamma > 0$. This inequality implies that both conditions in (33) are satisfied for positive $\gamma_1 \le \gamma_2$ if

$$\epsilon_0 < \frac{1}{2\gamma_2} \text{ and } \frac{2\epsilon_0}{d} \le \tau \le \frac{1}{2\gamma_2}. \tag{34}$$

We note also that ϵ_a and ϵ_w are positive in formulas (31) if and only if

$$0 < \tau < \frac{ck_5}{k_2 k_5 + 2k_3 c}. \tag{35}$$

Thus, by putting together the inequalities in (34) and (35), we conclude that, for any positive ϵ_0 such that $\epsilon_0 < \min\left\{\frac{1}{2\gamma_2}, \frac{d}{4\gamma_2}, \frac{dck_5}{2(k_2k_5+2k_3c)}\right\}$, inequality (28) holds with $\tau = \frac{2\epsilon_0}{d}$ and positive numbers ϵ_a, ϵ_w given by formula (31). It means also that property (26) is satisfied provided that

$$\begin{aligned} \bar{\varkappa} &> \frac{\sqrt{\epsilon_a}(k_4\tau + k_6\epsilon_w)}{\delta}, \\ \delta &= \min\{(c - \tau k_2)\epsilon_w - k_3\epsilon_w^2, c\epsilon_a\} \\ &\quad - \epsilon_0 - \tau k_0 - \tau^2 k_1 > 0. \end{aligned} \tag{36}$$

Then, we choose the parameters $\varkappa_{ij} \ge \bar{\varkappa}$ such that $\varkappa_{ij} \in \mathbb{N}_{1/2}$ for each $(i,j) \in S_2$ and $\varkappa_{ij} \ne \varkappa_{i'j'}$ whenever $(i, j) \ne (i', j')$.

Under our choice of parameters ϵ_0, τ, ϵ_a, ϵ_w, $\varkappa_{ij}$, property (26) implies that $\|A(x^\alpha)\xi\| > \Phi_{x^\omega}(\xi)$, for each $\xi \in \partial W$, for each $x^\omega \in B_{\epsilon_0}(x^\alpha)$, where $\Phi_{x^\omega}(\xi)$ denotes the right-hand side of Equation (23). Thus, the vector fields $A(x^\alpha)\xi$ and $\Psi(\xi) = A(x^\alpha)\xi - \Phi_{x^\omega}(\xi)$ are homotopic on ∂W, so the rotation of $\Psi(\xi)$ on ∂W is equal to sign $|A(x^\alpha)| \ne 0$ (Krasnosel'skij & Zabrejko, 1984). Then, the principle of nonzero rotation implies that there exists a $\xi \in W$ such that $\Psi(\xi) = 0$ (Zabrejko, 1997, Theorem 1), which completes the proof. ■

4. Second-order rank condition: nonholonomic systems

In this section, we consider a driftless control system

$$\dot{x} = \sum_{k=1}^{m} u_k f_k(x), \quad x \in D \subseteq \mathbb{R}^n, \ m < n. \tag{37}$$

Although the solvability of motion planning problems for nonholonomic systems has been already established under rather general controllability assumptions (Jean, 2014; Liu, 1997), our aim is to propose an explicit control design scheme and perform all necessary computations analytically. For this purpose, we restrict our analysis to a class of bracket-generating systems of step 3.

Let $S_2 \subseteq \{1, 2, \dots, m\}^2$ and $S_3 \subseteq \{1, 2, \dots, m\}^3$ be subsets of indices such that $|S_2| + |S_3| = n - m$. Without loss of generality, we assume that the elements of sets S_2 and S_3 are ordered as $j_1 < j_2$ for all $(j_1, j_2) \in S_2$, and $l_1 < l_2$ whenever $(l_1, l_2, l_3) \in S_3$.

Definition 4.1: Control system (37) satisfies the *(S_2, S_3)-rank condition at a point $x \in D$* if

$$\begin{aligned} &\operatorname{span}\bigl\{f_i(x), [f_{j_1}, f_{j_2}](x), \left[[f_{l_1}, f_{l_2}], f_{l_3}\right](x) \mid \\ &i = 1, 2, \dots, m, (j_1, j_2) \in S_2, (l_1, l_2, l_3) \in S_3\bigr\} = \mathbb{R}^n. \end{aligned} \tag{38}$$

In order to solve Problem 2.1, we apply the following family of control functions:

$$\begin{aligned} u_k(t) = a_k + &\sum_{(i,j)\in S_2} a_{ij}\left(\delta_{ki}\cos\frac{2\pi K_{ij}t}{\tau} + \delta_{kj}\sin\frac{2\pi K_{ij}t}{\tau}\right) \\ &+ \sum_{(i,j,l)\in S_3} a_{ijl}\left(\delta_{ki}\cos\frac{2\pi K_{1ijl}t}{\tau} + \delta_{kj}\sin\frac{2\pi K_{2ijl}t}{\tau}\right. \\ &\left. + \delta_{kl}\cos\frac{2\pi K_{1ijl}t}{\tau}\sin\frac{2\pi K_{2ijl}t}{\tau}\right), \\ &k = 1, 2, \dots, m, \ t \in [0, \tau], \end{aligned} \tag{39}$$

where a_k, a_{ij}, a_{ijl} are real coefficients, K_{ij}, K_{1ijl}, K_{2ijl} are nonzero integer parameters. To define the vector of coefficients $a = (a_k|_{k\in\{1,\dots,m\}}, a_{ij}|_{(i,j)\in S_2}, a_{ijl}|_{(i,j,l)\in S_3})^* \in \mathbb{R}^n$, and parameters $K = (K_{ij}|_{(i,j)\in S_2}, K_{1ijl}, K_{2ijl}|_{(i,j,l)\in S_3})^* \in (\mathbb{Z}\setminus\{0\})^{|S_2|+2|S_3|}$ for formula (39), we introduce the following system of algebraic equations:

$$\tau \sum_{k=1}^{m} f_k(x^\alpha) a_k + \frac{\tau^2}{4\pi} \sum_{(i,j)\in S_2} [f_i, f_j](x^\alpha) \frac{a_{ij}^2}{K_{ij}} + \frac{\tau^3}{16\pi^2} \sum_{(i,j,l)\in S_3} \big[[f_i, f_j], f_l\big](x^\alpha) \frac{a_{ijl}^3}{K_{2ijl}^2 - K_{1ijl}^2} + \frac{\tau^2}{2}\Omega(a, x^\alpha, \tau) = x^\omega - x^\alpha, \tag{40}$$

where the expression for Ω is given in Appendix 2. We also need an extra non-resonance assumption on the frequencies of the sine and cosine functions, so that there are no low-order resonances among the frequency multipliers K_{ij}, K_{1ijl}, K_{2ijl}, and $K_{1ijl} \pm K_{2ijl}$.

Assumption 4.1: *If c_{ij}, $c_{1ijl}, \dots, c_{4ijl}$ are any integers such that $\sum_{(i,j)\in S_2} |c_{ij}| + \sum_{(i,j,l)\in S_3} (|c_{1ijl}| + |c_{2ijl}| + |c_{3ijl}| + |c_{4ijl}|) > 0$ and*

$$\sum_{(i,j)\in S_2} c_{ij} K_{ij} + \sum_{(i,j,l)\in S_3} \big((c_{1ijl} + c_{3ijl} + c_{4ijl}) K_{1ijl} + (c_{2ijl} + c_{3ijl} - c_{4ijl}) K_{2ijl}\big) = 0,$$

then,

$$\Bigg(\sum_{(i,j)\in S_2} |c_{ij}| + \sum_{\substack{(i,\,j,\,l)\,\in\, S_3,\\ 1\le \nu\le 4}} |c_{\nu ijl}| > 3\Bigg) \text{ or}$$

$$\Bigg(\sum_{(i,j)\in S_2} |c_{ij}| + \sum_{\substack{(i,\,j,\,l)\,\in\, S_3,\\ 1\le \nu\le 4}} |c_{\nu ijl}| = 3 \text{ and } \sum_{(i,j)\in S_2} |c_{ij}| > 0\Bigg).$$

Our basic result concerning solutions of the local steering problem for nonholonomic case is as follows.

Theorem 4.1: *Assume that, for x^α, $x^\omega \in D$ and positive numbers τ, ε, ε_1, the vectors $a \in \mathbb{R}^n$ and $K \in (\mathbb{Z} \setminus \{0\})^{|S_2|+2|S_3|}$ satisfy the system of algebraic equations (40) and Assumption 4.1, and that the following conditions hold:*

$$\left\| \frac{\partial f_i}{\partial x}(x) \right\| \le M_1, \quad \left\| \frac{\partial^2 f_{ik}}{\partial x^2}(x) \right\| \le M_2,$$
$$\frac{1}{6} \sum_{|\alpha|=3} \left| \frac{\partial^3 f_{ik}(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| \le M_3,$$
$$\text{for all } x \in \bar{B}_\varepsilon(x^\alpha) \subset D,\ 1 \le i \le m,\ 1 \le k \le n, \tag{41}$$

$$\phi(\bar{U}) = \sqrt{n} M_0 \bar{U}^3 (e^{M_1\bar{U}} - 1) \left\{ \frac{M_0^2 M_3 (e^{M_1\bar{U}} - 1)^2}{\bar{U}^2} + \frac{M_0 M_1 M_2 \left(e^{M_1\bar{U}} - 1\right)(3n^{3/2} + 2M_1\bar{U})}{12\bar{U}} + \frac{M_1(M_1^2 + 2M_0M_2)}{6} \right\} \le \varepsilon_1, \tag{42}$$

$$\bar{U} \le \frac{1}{M_1} \ln\left(\frac{M_1\varepsilon}{M_0} + 1\right), \tag{43}$$

where

$$\bar{U} = \left(\sum_{i=1}^{m} |a_i| + \sqrt{2} \sum_{(i,j)\in S_2} |a_{ij}| + 3 \sum_{(i,j,k)\in S_3} |a_{ijk}| \right)\tau,$$
$$M_0 = \max_{1\le i\le m} \| f_i(x^\alpha) \|. \tag{44}$$

Then, $\|x(\tau; x^\alpha, u) - x^\omega\| \le \varepsilon_1$ and $\|x(t; x^\alpha, u) - x^\alpha\| \le \varepsilon$ for all $t \in [0, \tau]$, where the control $u(t)$ $(0 \le t \le \tau)$ is given by formula (39).

The proof of this result is given in Section 5.

Remark 4.1: For small values of $\bar{U}$, condition (42) is reduced to the following one:

$$\phi(\bar{U}) = \frac{\sqrt{n} M_0 M_1^2 (2M_1^2 + 12M_0^2 M_1 M_3 + 4M_0M_2 + 3n^{3/2} M_0 M_1 M_2)}{12} \bar{U}^4 + O(\bar{U}^5) < \varepsilon_1. \tag{45}$$

A crucial assumption of Theorem 4.1 is that the coefficients of control (39) satisfy the system of algebraic equations (40). To prove the solvability of system (40), we introduce new variables

$$\tilde{a} = \left(\tilde{a}_k\big|_{k\in\{1,\dots,m\}},\ \tilde{a}_{ij}\big|_{(i,j)\in S_2},\ \tilde{a}_{ijl}\big|_{(i,j,l)\in S_3} \right)^* \in \mathbb{R}^n$$

and parameters $K^+ = (K_{ij}^+|_{(i,j)\in S_2}, K_{1ijl}^+, K_{2ijl}^+|_{(i,j,l)\in S_3})^* \in \mathbb{N}^{|S_2|+2|S_3|}$, where

$$\begin{aligned}
\tilde{a}_k &= \tau a_k \text{ for } k = 1, 2, \dots, m,\\
\tilde{a}_{ij} &= \frac{\tau^2 a_{ij}^2}{4\pi K_{ij}} \text{ for } (i, j) \in S_2,\\
\tilde{a}_{ijl} &= \frac{\tau^3 a_{ijl}^3}{16\pi^2 \big(K_{2ijl}^2 - K_{1ijl}^2\big)} \text{ for } (i, j, l) \in S_3,\\
K_{ij}^+ &= |K_{ij}| \text{ for } (i, j) \in S_2,\\
K_{\nu ijl}^+ &= |K_{\nu ijl}| \text{ for } (i, j, l) \in S_3,\ \nu = 1, 2.
\end{aligned}$$

In new variables, we write system (40) in the following form:

$$\sum_{k=1}^{m}\tilde{a}_k f_k(x^\alpha) + \sum_{(i,j)\in S_2}\tilde{a}_{ij}[f_i, f_j](x^\alpha) + \sum_{(i,j,l)\in S_3}\tilde{a}_{ijl}\big[[f_i, f_j], f_l\big](x^\alpha) + \tilde{\Omega}(\tilde{a}, x^\alpha) = x^\omega - x^\alpha, \quad (46)$$

where $\tilde{\Omega}(\tilde{a}, x^\alpha)$ does not contain terms of order less than 4/3 with respect to $\tilde{a}$ (see Appendix 2). We assume that the (S_2, S_3)-rank condition is satisfied, therefore, the matrix

$$F(x^\alpha) = \Big(f_1(x^\alpha), \dots, f_m(x^\alpha), [f_i, f_j](x^\alpha)\big|_{(i,j)\in S_2}, \big[[f_i, f_j], f_l\big](x^\alpha)\big|_{(i,j,l)\in S_3}\Big) \quad (47)$$

is nonsingular. Then, we define the integers K_{ij}^+ and K_{1ijl}^+, K_{2ijl}^+ according to Assumption 4.1. Thus, if $\tilde{a}$ is a solution of system (46) for given $x^\alpha, x^\omega \in D$, then the components of a solution of system (40) are

$$\begin{aligned} a_k &= \tau^{-1}\tilde{a}_i \text{ for } k = 1, 2, \dots, m,\\ a_{ij} &= 2\tau^{-1}\mathrm{sign}(\tilde{a}_{ij})\sqrt{\pi K_{ij}^+|\tilde{a}_{ij}|} \text{ for } (i, j) \in S_2,\\ a_{ijl} &= 2\sqrt[3]{2\pi^2}\tau^{-1}\sqrt[3]{({K_{2ijl}^+}^2 - {K_{1ijl}^+}^2)\tilde{a}_{ijl}}\\ &\quad \text{for } (i, j, l) \in S_3, \end{aligned} \quad (48)$$

$$\begin{aligned} K_{ij} &= K_{ij}^+\mathrm{sign}(\tilde{a}_{ij}) \text{ for } (i, j) \in S_2,\\ K_{\nu ijl} &= K_{\nu ijl}^+ \text{ for } (i, j, l) \in S_3,\ \nu = 1, 2, \end{aligned} \quad (49)$$

where $\mathrm{sign}(\tilde{a}_{ij}) = 1$ if $\tilde{a}_{ij} \ge 0$ and $\mathrm{sign}(\tilde{a}_{ij}) = -1$ otherwise. So, the solvability problem for system (40) is reduced to the study of system (46). The formula for $\tilde{\Omega}(\tilde{a}, x)$ in Appendix 2 implies that there exists a function $C(x) > 0$, which is continuous in D, such that

$$\|\tilde{\Omega}(\tilde{a}, x)\| \le C(x)\|\tilde{a}\|^{4/3} \quad \text{for all } x \in D,\ \tilde{a} \in \bar{B}_1(0) \subset \mathbb{R}^n. \quad (50)$$

We derive the following corollary of Theorem 4.1 for solving Problem 2.1.

Theorem 4.2: *Assume that the rank condition (38) holds at $x = x^\alpha \in D$ and that inequality (41) is satisfied in $\bar{B}_\varepsilon(x^\alpha)$ for some $\varepsilon > 0$. Then, for any $r \in (0, 1)$ and $\tau > 0$, there exist $\epsilon_0 > 0$ and $\theta \in \mathcal{K}$ such that:*

(1) *for any $x^\omega \in B_{\epsilon_0}(x^\alpha)$, there exists a solution $a \in \mathbb{R}^n$ of algebraic system (40) with some $K \in (\mathbb{Z} \setminus \{0\})^{|S_2|+2|S_3|}$ that satisfy Assumption 4.1;*

(2) *if $u(t)$ is the control given by formula (39) with the above $a \in \mathbb{R}^n$ and $K \in (\mathbb{Z} \setminus \{0\})^{|S_2|+2|S_3|}$, then,*

$$\|x(\tau; x^\alpha, u) - x^\omega\| \le r\|x^\alpha - x^\omega\|, \quad (51)$$

$$\|x(t; x^\alpha, u) - x^\alpha\| \le \theta(\|x^\alpha - x^\omega\|) \quad \text{for all } t \in [0, \tau]. \quad (52)$$

Proof: Let $x^\alpha \in D$, $\varepsilon > 0$, $r \in (0, 1)$, and $\tau > 0$ be given. To prove assertion (1), we note that solutions of algebraic systems (40) and (46) are related by transformation (48). We choose a vector $K^+ \in \mathbb{N}^{|S_2|+2|S_3|}$ in such a way that Assumption 4.1 is satisfied. Then, we rewrite system (46) as $\Phi(\tilde{a}) = 0$, where

$$\Phi(\tilde{a}) = \tilde{a} + F^{-1}(x^\alpha)(\tilde{\Omega}(\tilde{a}, x^\alpha) + x^\alpha - x^\omega).$$

In the trivial case $x^\omega = x^\alpha$, it is easy to see that $\tilde{a} = 0 \in \mathbb{R}^n$ is a root of algebraic equation (46). If $\|x^\alpha - x^\omega\| > 0$ is small enough, we will use the principle of nonzero rotation to prove that the equation $\Phi(\tilde{a}) = 0$ has a root $\tilde{a} \in B_d(0)$ for some $d > 0$. For this purpose, we show that the maps $\Phi(\tilde{a})$ and $\Psi(\tilde{a}) = \tilde{a}$ are homotopic on the sphere $S_d = \partial B_d(0)$. A sufficient condition for the homotopy equivalence reads as follows (cf. Krasnosel'skij & Zabrejko, 1984):

$$\|\Phi(\tilde{a}) - \tilde{a}\| < \|\tilde{a}\| \quad \text{for all } \tilde{a} \in S_d. \quad (53)$$

We estimate the left-hand side of inequality (53) by using estimate (50) and assuming that $d \le 1$:

$$\begin{aligned} &\|\Phi(\tilde{a}) - \tilde{a}\|\\ &\quad \le \|F^{-1}(x^\alpha)\|\big(\|\tilde{\Omega}(\tilde{a}, x^\alpha)\| + \|x^\alpha - x^\omega\|\big)\\ &\quad \le \|F^{-1}(x^\alpha)\|\Big(C(x^\alpha)d^{4/3} + \|x^\alpha - x^\omega\|\Big). \end{aligned}$$

Thus, inequality (53) follows from the conditions

$$\|x^\alpha - x^\omega\| < \mu_{x^\alpha}(d),\ d \le 1, \quad (54)$$

where

$$\mu_{x^\alpha}(d) = \frac{d}{\|F^{-1}(x^\alpha)\|} - C(x^\alpha)d^{4/3}. \quad (55)$$

We see that the function $\mu_{x^\alpha}(d)$ is positive and increasing on $d \in (0, d_{max}]$, where

$$\begin{aligned} d_{\max} &= \min\left\{d_{\max}^+, 1\right\},\\ d_{\max}^+ &= \left(\frac{3}{4\|F^{-1}(x^\alpha)\|C(x^\alpha)}\right)^3,\ \mu'_{x^\alpha}(d_{\max}^+) = 0. \end{aligned} \quad (56)$$

As $\mu_{x^\alpha}(d)$ is strictly concave on $\mathbb{R}^+$ and $\mu_{x^\alpha}(0)=0$, condition (54) is satisfied with $\|x^\alpha - x^\omega\| = \frac{\mu_{x^\alpha}(d_{\max})d}{d_{\max}}$, $0 < d < d_{\max}$, or, equivalently, if

$$d = \frac{d_{\max}}{\mu_{x^\alpha}(d_{\max})}\|x^\alpha - x^\omega\| < d_{\max}. \tag{57}$$

Thus, we conclude that if

$$\|x^\alpha - x^\omega\| < \mu_{x^\alpha}(d_{\max}), \tag{58}$$

then, condition (53) holds on the sphere S_d of radius d given by formula (57). Thus, the maps $\Phi(\tilde a)$ and $\Psi(\tilde a) = \tilde a$ are homotopic on the sphere S_d, and the rotation of $\Phi(\tilde a)$ is equal to 1. Applying the principle of nonzero rotation, we conclude that there exists an $\tilde a \in B_d(0)$ such that $\Phi(\tilde a)=0$ (see, e.g. Krasnosel'skij & Zabrejko, 1984). Then, we define the vectors $a\in\mathbb{R}^n$ and $K \in (\mathbb{Z}\setminus\{0\})^{|S_2|+2|S_3|}$ by formulas (48) and (49) and observe that the system of algebraic equations (40) and Assumption 4.1 are satisfied. This completes the proof of assertion (1).

It remains to show that assertion (2) follows from Theorem 4.1. Indeed, for given $r \in (0,1)$ and $\varepsilon > 0$, our goal is to find an $\epsilon_0 > 0$ such that the conditions of Theorem 4.1 hold with $\varepsilon_1 = r\|x^\alpha - x^\omega\|$ if $\|x^\alpha - x^\omega\| < \epsilon_0$. Condition (43) follows from inequality (59) if

$$\frac{C_1 d_{\max}^{1/3}\|x^\alpha - x^\omega\|^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})} \le \frac{1}{M_1}\ln\left(\frac{M_1\varepsilon}{M_0}+1\right). \tag{61}$$

It is easy to see that the function $\phi(\bar U)$, given by formula (42), is increasing on $\mathbb{R}^+$ (as all its Taylor coefficients at $\bar U = 0$ are non-negative). Hence, by exploiting inequalities (58) and (59), we conclude that condition (42) holds with $\varepsilon_1 = r\|x^\alpha - x^\omega\|$ if

$$\phi\left(\frac{C_1 d_{\max}^{1/3}\|x^\alpha - x^\omega\|^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})}\right) \le r\|x^\alpha - x^\omega\| < r\mu_{x^\alpha}(d_{\max}). \tag{62}$$

Let $\bar\epsilon_0$ be the positive root of the equation $\frac{1}{\bar\epsilon_0}\phi\left(\frac{C_1 d_{\max}^{1/3}\bar\epsilon_0^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})}\right) = r$. It follows from the Taylor expansion (45) that

$$\bar\epsilon_0 \approx \frac{1728\mu_{x^\alpha}^4(d_{\max})}{n^{3/2}C_1^{12}d_{\max}^4 M_0^3 M_1^6\left(2M_1^2 + 12M_0^2M_1M_3 + 4M_0M_2 + 3n^{3/2}M_0M_1M_2\right)^3} r^3 \quad \text{as } r\to 0.$$

Under our choice of the coefficients $a\in\mathbb{R}^n$ of control (39), the expression $\bar U$ in formula (44) is estimated as follows:

$$\begin{aligned}\bar U &\le \sum_{k=1}^m |\tilde a_k| + 2\sqrt{2\pi}\sum_{(i,j)\in S_2}\sqrt{K_{ij}^+|\tilde a_{ij}|} \\ &\quad + 6\sqrt[3]{2\pi^2}\sum_{(i,j,l)\in S_3}\sqrt[3]{({K_{2ijl}^+}^2 - {K_{1ijl}^+}^2)|\tilde a_{ijl}|} \\ &\le \sum_{k=1}^m |\tilde a_k|^{1/3} + 2\sqrt{2\pi}\sum_{(i,j)\in S_2}\sqrt{K_{ij}^+}|\tilde a_{ij}|^{1/3} \\ &\quad + 6\sqrt[3]{2\pi^2}\sum_{(i,j,l)\in S_3}\sqrt[3]{({K_{2ijl}^+}^2 - {K_{1ijl}^+}^2)|\tilde a_{ijl}|},\end{aligned}$$

where we have used formula (48) and the inequality $\|\tilde a\| < d \le 1$. Furthermore, by applying Hölder's inequality with exponents $\left(6, \frac{6}{5}\right)$ and exploiting condition (57), we get

$$\bar U \le C_1\|\tilde a\|^{1/3} \le C_1 d^{1/3} = \frac{C_1 d_{\max}^{1/3}\|x^\alpha - x^\omega\|^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})}, \tag{59}$$

$$C_1 = \Bigg(m + 2^{9/5}\pi^{3/5}\sum_{(i,j)\in S_2}(K_{ij}^+)^{3/5} + 2^{8/5}3^{6/5}\pi^{4/5}\sum_{(i,j,l)\in S_3}\left({K_{2ijl}^+}^2 - {K_{1ijl}^+}^2\right)^{2/5}\Bigg)^{5/6}, \tag{60}$$

provided that condition (58) holds.

Now, we choose

$$\epsilon_0 = \min\left\{\bar\epsilon_0,\ \mu_{x^\alpha}(d_{\max}),\ \frac{\mu_{x^\alpha}(d_{\max})}{M_1^3C_1^3 d_{\max}}\ln^3\left(\frac{M_1\varepsilon}{M_0}+1\right)\right\} > 0. \tag{63}$$

Let $\|x^\alpha - x^\omega\| < \epsilon_0$, and let $x(t;x^\alpha,u)$ be the solution of system (37) corresponding to the control $u=u(t)$ given by formula (39) with the coefficients $a\in\mathbb{R}^n$ and parameters $K\in(\mathbb{Z}\setminus\{0\})^{|S_2|+2|S_3|}$ from assertion (1). The assumptions of Theorem 4.1 are satisfied because of inequalities (61) and (62), which proves condition (51). It is easy to see that estimate (52) is satisfied with the following function $\theta = \theta_{x^\alpha}(s)$ of class $\mathcal{K}$:

$$\theta_{x^\alpha}(s) = \frac{M_0}{M_1}\left(\exp\left\{\frac{M_1C_1d_{\max}^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})}s^{1/3}\right\} - 1\right). \tag{64}$$

Indeed, let us denote $s = \|x^\alpha - x^\omega\| < \epsilon_0$, then $\bar U \le \frac{C_1 d_{\max}^{1/3}s^{1/3}}{\mu_{x^\alpha}^{1/3}(d_{\max})}$ because of inequality (59), and condition (43) of Theorem 4.1 holds with $\bar\varepsilon = \theta_{x^\alpha}(s)$. Thus, Theorem 4.1 implies that

$$\|x(t;x^\alpha,u) - x^\alpha\| \le \bar\varepsilon = \theta_{x^\alpha}(\|x^\alpha - x^\omega\|) \quad \text{for all } t\in[0,\tau],$$

which completes the proof. ■

We will show below that the construction of local controllers in Theorem 4.2 can be used to satisfy the conditions of Theorem 2.1 for solving the approximate path-following problem.

Theorem 4.3: *Let $\gamma \subset D$ be a curve with the endpoints x^0 and x^T, and let the rank condition (38) be satisfied at each $x \in \gamma$. Then, for any $\tau > 0$ and $\varepsilon > \varepsilon_1 > 0$, there exists a $\bar{\Delta} > 0$ such that, for any partition $\pi : x^0 \prec x^1 \prec \ldots \prec x^N = x^T$ of γ with $\Delta(\pi) < \bar{\Delta}$, the corresponding π-approximating control $u_\pi(t)$ is well defined on $t \in [0, T]$, $T = N\tau$, and*

$$\|x(j\tau; x^0, u_\pi) - x^j\| < \varepsilon_1, \quad j = 1, 2, \ldots, N, \quad (65)$$

$$\rho(x(t; x^0, u_\pi), \gamma) < \varepsilon, \quad t \in [0, T]. \quad (66)$$

Here, the control $u_\pi(t)$ is constructed as in Definition 2.1 by using the concatenation of local controllers $u(t) = u^{x^\alpha x^\omega}(t)$ of form (39) whose coefficients are defined by the system of algebraic equations (40).

Proof: As the rank condition (38) holds on $\gamma \subset D$ and all the vector fields $f_i(x)$ are of class $C^3(D)$, there exists an $\eta > 0$ such that $\Gamma = \bar{B}_\eta(\gamma) \subset D$ and condition (38) also holds at each $x \in \Gamma$. For a compact subset Γ of domain D, we choose a positive $\bar{\varepsilon}$ such that $D_0 = \bar{B}_{\bar{\varepsilon}}(\Gamma) \subset D$. Then, the numbers

$$M_1 = \max_i \left(\sup_{x \in D_0} \left\| \frac{\partial f_i}{\partial x}(x) \right\| \right),$$

$$M_2 = \max_{i,k} \left(\sup_{x \in D_0} \left\| \frac{\partial^2 f_{ik}}{\partial x^2}(x) \right\| \right),$$

$$M_3 = \frac{1}{6} \max_{i,k} \left(\sup_{x \in D_0} \sum_{|\alpha|=3} \left| \frac{\partial^3 f_{ik}(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| \right),$$
$$1 \le i \le m, \ 1 \le k \le n, \quad (67)$$

are finite by the Weierstrass theorem. We see that the conditions of Theorem 4.2 are satisfied for each $x^\alpha \in \Gamma$ with the above choice of M_1, M_2, and M_3. Let us now fix arbitrary $r \in (0, 1)$, $\tau > 0$, and show that the number $\epsilon_0 > 0$ and function $\theta \in \mathcal{K}$ in Theorem 4.2 may be chosen independently of $x^\alpha \in \Gamma$.

Since all the vector fields appearing in the rank condition (38) are continuous on the compact $\Gamma \subset D$, there exists a vector $K^+ \in \mathbb{N}^{|S_2|+2|S_3|}$ satisfying Assumption 4.1 such that the matrix $F(x^\alpha)$ is nonsingular for each $x^\alpha \in \Gamma$. As in the proof of Theorem 4.2, we fix such $K^+ \in \mathbb{N}^{|S_2|+2|S_3|}$ and introduce the function $\mu(d) = \frac{d}{c_1} - c_2 d^{4/3}$, where $c_1 = \sup_{x \in \Gamma} \|F^{-1}(x)\| > 0$, $c_2 = \sup_{x \in \Gamma} C(x) > 0$. It follows from the construction of $\mu(d)$ that

$$\mu(d) \le \mu_{x^\alpha}(d) \quad \text{for all} \ \ x^\alpha \in \Gamma, \ d \ge 0, \quad (68)$$

and $\mu(d) > 0$ is strictly increasing on $d \in (0, \bar{d}_{\max}]$, $\bar{d}_{\max} = \min\left\{1, \left(\frac{3}{4c_1c_2}\right)^3\right\}$. Following the proof of Theorem 4.2 with the use of inequality (68), we conclude that its assertions (1) and (2) remain true for each $x^\alpha \in \Gamma$ and $x^\omega \in B_{\epsilon_0}(x^\alpha)$ if, instead of formula (63), we define

$$\epsilon_0 = \min\left\{\hat{\epsilon}_0, \mu(\bar{d}_{\max}), \frac{\mu(\bar{d}_{\max})}{M_1^3 C_1^3 \bar{d}_{\max}} \ln^3\left(\frac{M_1 \bar{\varepsilon}}{M_0} + 1\right)\right\} > 0, \quad (69)$$

where $\hat{\epsilon}_0$ is the positive root of the equation $\frac{1}{\hat{\epsilon}_0}\phi\left(\frac{C_1 \bar{d}_{\max}^{1/3} \hat{\epsilon}_0^{1/3}}{\mu^{1/3}(\bar{d}_{\max})}\right) = r$, the constants C_1 and M_1 are given by formulas (60) and (67), respectively, and

$$M_0 = \max_{1 \le i \le m} \sup_{x \in \Gamma} \|f_i(x)\| > 0. \quad (70)$$

Thus, expression (69) defines the constant $\epsilon_0 > 0$ for Theorem 4.2 independently of $x^\alpha \in \Gamma$. It remains to verify that there exists a $\theta \in \mathcal{K}$ such that the estimate

$$\theta_{x^\alpha}(s) \le \theta(s), \quad s \in \mathbb{R}^+, \quad (71)$$

holds for each $x^\alpha \in \Gamma$ and $\theta_{x^\alpha}(s)$ given by formula (64). Indeed, straightforward computations with the use of inequality (68) show that the function

$$\theta(s) = \frac{M_0}{M_1}\left(\exp\left\{\frac{M_1 C_1 \hat{d}}{\mu^{1/3}(\bar{d}_{\max})} s^{1/3}\right\} - 1\right) \quad (72)$$

satisfies property (71), where M_0 is defined in (70), $\hat{d} = \min\left\{1, \frac{3}{4\inf_{x\in\Gamma}(\|F^{-1}(x)\| C(x))}\right\} > 0$. Thus, we have shown that formulas (69) and (72) define the constant $\epsilon_0 > 0$ and function $\theta \in \mathcal{K}$ for Theorem 4.2 independently of $x^\alpha \in \Gamma$.

Now, the assertion of Theorem 4.3 follows from Theorem 2.1. ■

In Section 6, we demonstrate the approach of Theorem 4.3 with several examples, where the system of algebraic equations (40) will be solved numerically.

5. Auxiliary results and proofs

To prove Theorem 3.1, we rewrite the Volterra series (12) by using the first-order Lie brackets as follows:

$$\begin{aligned}
&x(t;x^0,u)\\
&= x^0 + \sum_{k=0}^{m} f_k(x^0)\int_0^t u_k(s)ds\\
&+ \frac{1}{2}\sum_{i,j=0}^{m}\frac{\partial f_j(x^0)}{\partial x}f_i(x^0)\int_0^t u_i(s)ds\int_0^t u_j(s)ds\\
&+ \frac{1}{2}\sum_{i<j}[f_i,f_j](x^0)\int_0^t\int_0^s\big\{u_j(s)u_i(v)\\
&- u_i(s)u_j(v)\big\}\,dv\,ds + R_2(t),\quad t\in[0,\tau],
\end{aligned}\tag{73}$$

where $R_2(t)$ is the sum of the last two terms of formula (12).

We need two auxiliary lemmas, whose proofs can be found in Zuyev (2016).

Lemma 5.1: *Let $\tilde D\subset\mathbb{R}^n$ be a closed convex domain, and let $x(t)\in\tilde D$, $0\le t\le\tau$, be the solution of system (1) corresponding to initial value $x(0)=x^0\in\tilde D$ and control $u\in C[0,\tau]$. If the vector fields $f_0(x), f_1(x),\dots, f_m(x)$ satisfy assumptions*

$$\left\|\frac{\partial f_i(x)}{\partial x}\right\|\le M_1,\quad \left\|\frac{\partial^2 f_{ij}(x)}{\partial^2 x}\right\|\le M_2,\quad i=\overline{0,m},\ j=\overline{1,n},\tag{74}$$

in $\tilde D$ with some positive constants M_1 and M_2, then the remainder $R_2(\tau)$ of the Volterra expansion (73) satisfies the following estimate:

$$\begin{aligned}
\|R_2(\tau)\| &\le \frac{M_0}{M_1}\left\{e^{M_1U\tau}-\frac{1}{2}\left((M_1U\tau+1)^2+1\right)\right\}\\
&\quad+\frac{M_2M_0^2\sqrt{n}}{4M_1^3}\left\{\left(e^{M_1U\tau}-2\right)^2+2M_1U\tau-1\right\}\\
&= \frac{M_0(M_1^2+M_2M_0\sqrt{n})}{6}U^3\tau^3+O(U^4\tau^4).
\end{aligned}\tag{75}$$

Here, $M_0=\max_{0\le i\le m}\|f_i(x^0)\|,\ U=1+\max_{0\le t\le\tau}\sum_{i=1}^m|u_i(t)|$.

Lemma 5.2: *Let $x(t)\in\tilde D\subset\mathbb{R}^n$, $0\le t\le\tau$, be a solution of system (37) with a control $u\in C[0,\tau]$, and let $\|f_i(x')-f_i(x'')\|\le M_1\|x'-x''\|$, $M_1>0$, for all $x',x''\in\tilde D$, $i=1,2,\dots,m$. Then,*

$$\|x(t)-x(0)\|\le\frac{M_0}{M_1}(e^{M_1\tilde U t}-1),\quad t\in[0,\tau],\tag{76}$$

where $M_0=\max_{1\le i\le m}\|f_i(x(0))\|,\ \tilde U=\max_{0\le t\le\tau}\sum_{i=1}^m|u_i(t)|$.

Proof of Theorem 3.1: By substituting controls (14) into formula (73) with $x^0=x^\alpha\in D$ and computing the integrals, we obtain

$$\begin{aligned}
x(\tau;x^\alpha,u) &= x^\alpha+\tau\left(f_0(x^\alpha)+\sum_{i\in S_0}v_if_i(x^\alpha)\right)+\frac{\tau^2}{2}V_{20}\\
&\quad+\frac{\tau^2}{2\pi}V_{21}+R_2(\tau),
\end{aligned}\tag{77}$$

where the terms V_{20} and V_{21} are given by formulas (16) provided that Assumption 3.1 holds. For given $x^\alpha, x^\omega\in D$ and $\tau>0$, we assume that the vector $a=\big((v_i)_{i\in S_0},(a_i)_{i\in S_1},(a_{ij})_{(i,j)\in S_2}\big)^*\in\mathbb{R}^n$ satisfies the system of algebraic equations (15) and $K\in(\mathbb{Z}\setminus\{0\})^{|S_1|+|S_2|}$ satisfies Assumption 3.1. Then, formulas (15) and (77) imply that $x(\tau;x^\alpha,u)=x^\omega+R_2(\tau)$, where $x(t;x^\alpha,u)$ is the solution of system (1) with the control $u=u(t)$ of form (14). Thus, it suffices to prove that

$$\|R_2(\tau)\|\le\varepsilon_1\tag{78}$$

and

$$\|x(t;x^\alpha,u)-x^\alpha\|\le\varepsilon,\quad t\in[0,\tau].\tag{79}$$

We estimate the sum of components $|u_i(t)|$ in formula (14) as follows:

$$\begin{aligned}
&\sum_{i=1}^m|u_i(t)|\\
&\le\sum_{i\in S_0}|v_i|+\sum_{i\in S_1}|a_i|\left|\sin\left(\frac{2\pi K_it}{\tau}\right)\right|\\
&\quad+\sum_{(i,j)\in S_2}|a_{ij}|\left(\left|\cos\left(\frac{2\pi K_{ij}t}{\tau}\right)\right|+\left|\sin\left(\frac{2\pi K_{ij}t}{\tau}\right)\right|\right)\\
&\le\sum_{i\in S_0}|v_i|+\sum_{i\in S_1}|a_i|+\sqrt{2}\sum_{(i,j)\in S_2}|a_{ij}|.
\end{aligned}$$

Hence, $U\tau=\big(1+\max_{0\le t\le\tau}\sum_{i=1}^m|u_i(t)|\big)\tau\le\big(1+\sum_{i\in S_0}|v_i|+\sum_{i\in S_1}|a_i|+\sqrt{2}\sum_{(i,j)\in S_2}|a_{ij}|\big)\tau=\bar U$, where $\bar U$ is given in (20). As the right-hand side of inequality (75) is strictly increasing with respect to $U\in\mathbb{R}^+$ and $U\tau\le\bar U$, inequality (78) follows from condition (18) because of Lemma 5.1 with $\tilde D=\bar B_\varepsilon(x^\alpha)$. To show that inequality (79) holds, we apply a modification of estimate (76) for system (1). Indeed, the assertion of

Lemma 5.2 for system (1) can be formulated as follows:

$$\|x(t; x^\alpha, u) - x^\alpha\| \le \frac{M_0}{M_1}(e^{M_1\bar{U}t/\tau} - 1), \quad t \in [0, \tau], \tag{80}$$

where $\bar{U}$, M_0, and M_1 are defined in (17) and (20). Now, inequality (79) follows from conditions (19) and (80). ∎

In order to prove Theorem 4.1, we rewrite formula (12) by using the Lie brackets as follows:

$$\begin{aligned} x(t; x^0, u) &= x^0 + \sum_{k=1}^{m} f_k(x^0)\int_0^t u_k(s)ds \\ &+ \frac{1}{2}\sum_{i<j}[f_i, f_j](x^0)\int_0^t\int_0^\tau (u_j(\tau)u_i(s) \\ &- u_i(\tau)u_j(s))ds\,d\tau + \frac{1}{3}\sum_{i<j}\sum_{l=1}^{m}\left[[f_i, f_j], f_l\right](x^0) \\ &\times \int_0^t\int_0^\tau\int_0^s \Big(u_l(\tau)\big(u_j(s)u_i(p) \\ &- u_i(s)u_j(p)\big)\Big)dp\,ds\,d\tau + G(t) + R(t). \end{aligned} \tag{81}$$

The proof of this fact is presented in Appendix 1 together with the expression for $G(t)$, and the remainder $R(t)$ is estimated by the following lemma.

Lemma 5.3: *Let $\tilde{D} \subset \mathbb{R}^n$ be a closed convex domain, and let $x(t) \in \tilde{D}$, $0 \le t \le \tau$, be a solution of system (37) corresponding to the initial value $x(0) = x^0 \in \tilde{D}$ and control $u \in C[0, \tau]$. Assume that the vector fields $f_1(x), \ldots, f_m(x)$ satisfy conditions*

$$\begin{aligned} &\left\|\frac{\partial f_i}{\partial x}(x)\right\| \le M_1, \quad \left\|\frac{\partial^2 f_{ik}}{\partial x^2}(x)\right\| \le M_2, \\ &\frac{1}{6}\sum_{|\alpha|=3}\left|\frac{\partial^3 f_{ik}(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right| \le M_3, \ 1 \le i \le m, \ 1 \le k \le n, \end{aligned} \tag{82}$$

with some positive constants M_1, M_2, M_3, for all $x \in \tilde{D}$. Then, the remainder of the Volterra expansion (81) satisfies the estimate

$$\begin{aligned} \|R(t)\| \le \frac{\sqrt{n}M_0(e^{M_1\bar{U}} - 1)}{\bar{U}}&\left\{\frac{M_0^2M_3(e^{M_1\bar{U}} - 1)^2}{\bar{U}^2}\right. \\ &+ \frac{M_0M_1M_2\left(e^{M_1\bar{U}} - 1\right)(3n^{3/2} + 2M_1\bar{U})}{12\bar{U}} + \\ &\left.+ \frac{M_1(M_1^2 + 2M_0M_2)}{6}\right\}U^4t^4 \quad \text{if } 0 \le Ut \le \bar{U}, \end{aligned} \tag{83}$$

where $M_0 = \max_{1\le i\le m}\|f_i(x^0)\|$, $U = \max_{t\in[0,\tau]}(|u_1(t)| + \cdots + |u_m(t)|)$.

Proof: Let us denote by $R_i^{(N+1)}(x)$ the remainder term for the N-th order Taylor expansion of $f_i(x)$ at a point $x^0 \in \tilde{D}$. If $f_i(x)$ is of class C^{N+1} in a convex domain $\tilde{D}$, then $R^{(N+1)}(x)$ may be represented in the Lagrange form of the remainder as follows:

$$\begin{aligned} &R_i^{(N+1)}(x) = \frac{1}{(N+1)!}\sum_{|\alpha|=N+1}\frac{\partial^{|\alpha|}f_i(\theta)}{\partial\theta_1^{\alpha_1}\cdots\partial\theta_n^{\alpha_n}}\Delta x_1^{\alpha_1}\cdots\Delta x_n^{\alpha_n}, \\ &\quad \Delta x_j = x_j - x_j^0, \ \theta \in \bar{B}_{\|\Delta x\|}(x^0), \\ &\quad \alpha = (\alpha_1, \ldots, \alpha_n), \ |\alpha| = \alpha_1 + \cdots + \alpha_n. \end{aligned} \tag{84}$$

To prove the assertion of Lemma 5.3, we use the integral representation of system (37) with initial conditions $x(0) = x^0$ and the Taylor expansion for $f_{ik}(x)$:

$$\begin{aligned} x_k(t) &= x_k^0 + \sum_{i=1}^{m}\int_0^t u_i(v)f_{ik}(x(v))dv = x_k^0 \\ &+ \sum_{i=1}^{m}\int_0^t u_i(v)\Bigg\{f_{ik}(x^0) + \frac{\partial f_{ik}(x)}{\partial x}\bigg|_{x=x^0} \\ &\times \Bigg(\sum_{j=1}^{m}\int_0^v u_j(s)\bigg(f_j(x^0) + \frac{\partial f_j(x)}{\partial x}\bigg|_{x=x^0} \\ &\times \Bigg(\sum_{l=1}^{m}\int_0^s u_l(p)\big(f_l(x^0) + R_l^{(1)}(x(p))\big)dp\Bigg) \\ &+ R_j^{(2)}(x(s))\bigg)ds\Bigg) + \frac{1}{2}\Bigg(\sum_{j=1}^{m}\int_0^v u_j(s)\Big(f_j(x^0) \\ &+ R_j^{(1)}(x(s))\Big)ds\Bigg)^* \frac{\partial^2 f_{ik}(x)}{\partial x^2}\bigg|_{x=x^0} \\ &\times \Bigg(\sum_{j=1}^{m}\int_0^v u_j(s)\Big(f_j(x^0) + R_j^{(1)}(x(s))\Big)ds\Bigg) \\ &+ R_{ik}^{(3)}(x(v))\Bigg\}dv, \end{aligned} \tag{85}$$

where the gradient $\frac{\partial f_{ik}(x)}{\partial x}$ is treated as a row vector. After several transformation, expression (85) takes the form (12) with

$$\begin{aligned} R_k(t) &= \sum_{i=1}^{m}\int_0^t R_{ik}^{(3)}(x(v))u_i(v)dv + \sum_{i,j=1}^{m}\frac{\partial f_{ik}(x)}{\partial x}\bigg|_{x=x^0} \\ &\times \int_0^t\int_0^v R_j^{(2)}(x(s))u_i(v)u_j(s)ds\,dv \\ &+ \sum_{i,j,l=1}^{m}\frac{\partial f_{ik}(x)}{\partial x}\frac{\partial f_j(x)}{\partial x}\bigg|_{x=x^0}\int_0^t\int_0^v\int_0^s R_l^{(1)}(x(p)) \end{aligned}$$

$$\times u_i(v)u_j(s)u_l(p)dpdsdv$$
$$+\sum_{i,j=1}^{m} f_j^*(x^0)\left.\frac{\partial^2 f_{ik}(x)}{\partial x^2}\right|_{x=x^0}\int_0^t\left(\int_0^v u_j(s)ds\right)$$
$$\times\left(\sum_{l=1}^{m}\int_0^v u_l(s)R_l^{(1)}(x(s))ds\right)u_i(v)dv$$
$$+\frac{1}{2}\sum_{i=1}^{m}\int_0^t\left(\sum_{j=1}^{m}\int_0^v u_j(s)R_j^{(1)*}(s)ds\right)$$
$$\times\left.\frac{\partial^2 f_{ik}(x)}{\partial x^2}\right|_{x=x^0}\left(\sum_{j=1}^{m}\int_0^v u_j(s)R_j^{(1)}(s)ds\right)$$
$$\times u_i(v)dv. \tag{86}$$

By estimating the absolute value of $R_k(t)$ term-by-term in (86) with the use of (84), we get

$$|R_k(t)| \le M_3\|\Delta x(t)\|^3 Ut + \frac{M_1\bar{M}_2}{2}\|\Delta x(t)\|^2U^2t^2$$
$$+\frac{M_1^3}{6}\|\Delta x(t)\|U^3t^3 + \frac{M_0M_1M_2}{3}\|\Delta x(t)\|U^3t^3$$
$$+\frac{M_1^2M_2}{6}\|\Delta x(t)\|^2U^3t^3, \tag{87}$$

where $\bar{M}_2 = \frac{1}{2}\sup_{x\in\tilde{D}}\sum_{|\alpha|=2}\left|\frac{\partial^2 f_{ik}(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right|$. The Cauchy–Schwarz inequality implies that

$$\bar{M}_2 \le \frac{n\sqrt{n}}{2}M_2. \tag{88}$$

The norm of $\Delta x(t) = x(t) - x^0$ is estimated by Lemma 5.2 as follows:

$$\|\Delta x(t)\| \le \frac{M_0}{M_1}\left(e^{M_1Ut}-1\right), \quad t\ge 0. \tag{89}$$

As the function $\psi(\beta) = e^\beta - 1$ is convex, it follows from (89) that

$$\|\Delta x(t)\| \le \frac{M_0(e^{M_1\bar{U}}-1)}{M_1\bar{U}}Ut, \quad 0\le Ut\le \bar{U}. \tag{90}$$

Component-wise estimates (87) together with inequalities (88) and (90), and $U^2t^2 \le \bar{U}Ut$, $0\le Ut\le\bar{U}$ imply estimate (83) for the Euclidean norm of $R(t)$. ■

Proof of Theorem 4.1: By substituting the control $u = u(t)$ of form (39) into the Volterra series (81) with $x^0 = x^\alpha \in D$, we get

$$x(\tau; x^\alpha, u) = x^\alpha + \tau\sum_{k=1}^{m} f_k(x^\alpha)a_k + \frac{\tau^2}{4\pi}$$
$$\times\sum_{(i,j)\in S_2}[f_i, f_j](x^\alpha)\frac{a_{ij}^2}{K_{ij}} + \frac{\tau^3}{16\pi^2}$$
$$\times\sum_{(i,j,l)\in S_3}\left[[f_i, f_j], f_l\right](x^\alpha)\frac{a_{ijl}^3}{K_{2ijl}^2 - K_{1ijl}^2}$$
$$+\frac{\tau^2}{2}\Omega(a, x^\alpha, \tau) + R(\tau), \tag{91}$$

provided that Assumption 4.1 is satisfied (the explicit formula for Ω is in Appendix 2). It is easy to see that the system of algebraic equations (40) is equivalent to the following condition in terms of representation (91):

$$x(\tau; x^\alpha, u) = x^\omega + R(\tau).$$

Therefore, if the vectors $a\in\mathbb{R}^n$ and $K\in(\mathbb{Z}\setminus\{0\})^{|S_2|+2|S_3|}$ satisfy the system of algebraic equations (40) and Assumption 4.1, then it remains to show that

$$\|R(\tau)\|\le\varepsilon_1 \quad \text{and} \quad \frac{M_0}{M_1}(e^{M_1Ut}-1)\le\varepsilon, \quad t\in[0,\tau], \tag{92}$$

because of Lemma 5.2 with $\tilde{D} = \bar{B}_\varepsilon(x^\alpha)$, where $U = \max_{1\le i\le m}\sum_{i=1}^{m}|u_i(t)| \le \bar{U}/\tau$, the constants M_i are given in formulas (41) and (44). To complete the proof, we conclude that conditions (92) follow from Lemma 5.3 and inequalities (42) and (43). ■

6. Examples

6.1 Ball on the plane

Consider a unit ball rolling on the plane. As it was shown in Li and Canny (1990), the kinematic equations take the following form:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \tag{93}$$

where $x = (x_1, x_2, x_3, x_4, x_5)^*$, $f_1(x) = \big(0, \sec x_1, -\sin x_5, -\cos x_5, \operatorname{tg} x_1\big)^*$, $f_2(x) = (-1, 0, -\cos x_5, \sin x_5, 0)^*$. Here, $(x_1, x_2)\in\mathbb{R}^2$ and $(x_3, x_4)\in\mathbb{R}^2$ define the Gaussian frames, and $x_5\in(-\frac{\pi}{2}, \frac{\pi}{2})$ is the angle of contact. The controls u_1 and u_2 are related to components of the angular velocity. By computing the first- and the second-order Lie brackets, we observe that

$$\operatorname{span}\{f_1(x), f_2(x), [f_1, f_2](x), [[f_1, f_2], f_1](x),$$
$$[[f_1, f_2], f_2](x)\} = \mathbb{R}^5,$$

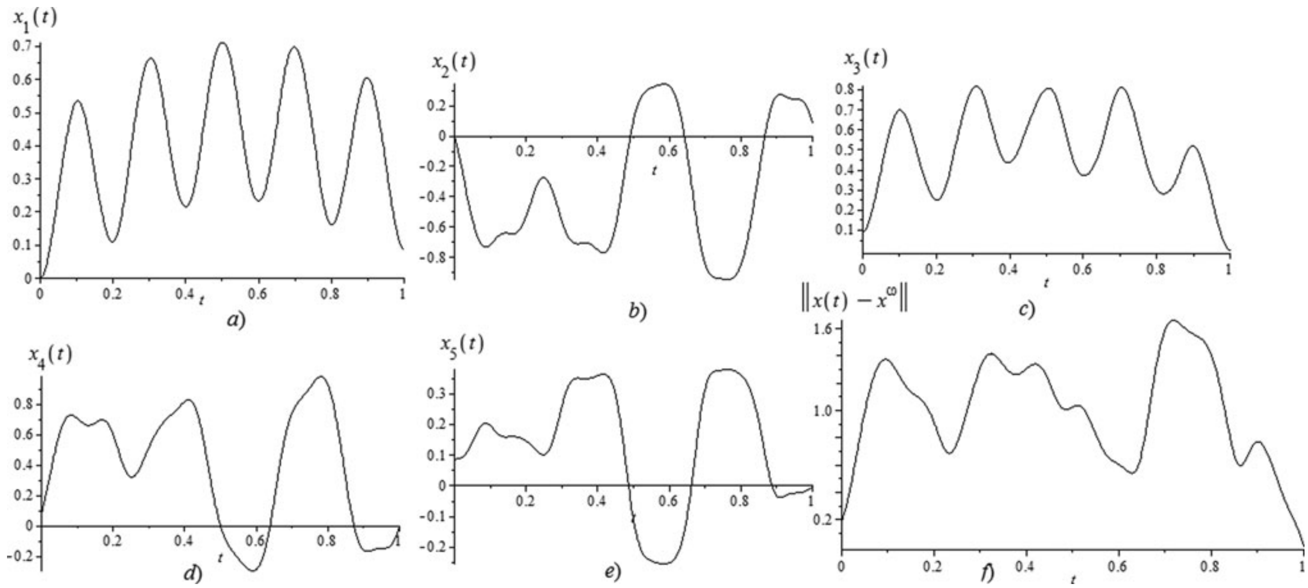

Figure 2. (a)–(e) Components $x_i(t)$ of the solution of system (93) with the initial condition $x(0) = x^\alpha$ and controls (94); (f) time-plot of $\|x(t) - x^\omega\|$.

for all $x \in \mathbb{R}^5$ such that $x_1 \neq \frac{\pi}{2} (\mathrm{mod}\, \pi)$. Thus, the (S_2, S_3)-rank condition (Definition 4.1) is satisfied with $S_2 = \{(1, 2)\}$ and $S_3 = \{(1, 2, 1), (1, 2, 2)\}$ for all $x \in D$, $D = \{x \in \mathbb{R}^5 \mid |x_1| < \pi/2\}$. Following the approach of Section 4 for steering system (93) from $x^\alpha \in D$ to $x^\omega \in D$, we use controls of the form (39):

$$\begin{aligned} u_1(t) &= a_1 + a_{12} \cos \frac{2\pi K_{12} t}{\tau} + a_{121} \left(1 + \sin \frac{2\pi K_{2121} t}{\tau}\right) \\ &\quad \times \cos \frac{2\pi K_{1121} t}{\tau} + a_{122} \cos \frac{2\pi K_{1122} t}{\tau}, \\ u_2(t) &= a_2 + a_{12} \sin \frac{2\pi K_{12} t}{\tau} + a_{121} \sin \frac{2\pi K_{2121} t}{\tau} \\ &\quad + a_{122} \left(1 + \cos \frac{2\pi K_{1122} t}{\tau}\right) \sin \frac{2\pi K_{2122} t}{\tau}, \end{aligned} \tag{94}$$

with the coefficients $a = (a_1, a_2, a_{12}, a_{121}, a_{122})^* \in \mathbb{R}^5$ and parameters $K = (K_{12}, K_{1121}, K_{2121}, K_{1122}, K_{2122})^* \in (\mathbb{Z} \setminus \{0\})^5$. For any $x^\alpha \in D$ and $x^\omega \in D$ such that $\|x^\alpha - x^\omega\|$ is small enough, there exists a solution $a \in \mathbb{R}^5$ of the system of algebraic equation (40) with some $K \in (\mathbb{Z} \setminus \{0\})^5$ satisfying Assumption 4.1 by Theorem 4.2.

As an example, let us fix $x^\alpha = \left(0, 0, \frac{\pi}{36}, \frac{\pi}{36}, \frac{\pi}{36}\right)^*$, $x^\omega = \left(\frac{\pi}{36}, \frac{\pi}{36}, 0, 0, 0\right)^*$, and $\tau = 1$. It is easy to check that Assumption 4.1 is satisfied with

$$K_{12} = 1,\ K_{1121} = 3,\ K_{2121} = 5,\ K_{1122} = 12,\ K_{2122} = 19, \tag{95}$$

and a numerical solution of the system of algebraic equation (40) with these parameters is

$$a_1 \approx 0.07,\ a_2 \approx -0.08,\ a_{12} \approx -0.56,\ a_{121} \approx -7.7,\ a_{122} \approx -0.37. \tag{96}$$

To illustrate that the above controls solve the local approximate steering problem (Problem 2.1), we solve the Cauchy problem for system (93) numerically with the initial condition $x(0) = x^\alpha$ and the controls represented by (94)–(96) (see Figure 2).

The value of $\|x(\tau) - x^\omega\|$ from Figure 2(f) can be used to evaluate the relative accuracy of our local steering algorithm: $\tilde{r} = \|x(\tau) - x^\omega\| / \|x^\alpha - x^\omega\| \approx 0.027 < 1$. Note that a theoretical upper bound for $\tilde{r}$ is given by the r constant in (2) (Problem 2.1 formulation). This constant can be estimated from Theorem 4.1 as $r = \phi(\bar{U}) / \|x^\alpha - x^\omega\|$, where the computation of $\phi(\bar{U})$ by formula (42) is based on the coefficients a of the control (94) and the upper bounds of the derivatives of $f_i(x)$. Similarly, the maximal overshoot is estimated by inequality (3): $\|x(t) - x^\alpha\| \leq \theta(\|x^\alpha - x^\omega\|)$ for all $t \in [0, \tau]$, where the right-hand side can be estimated as $\theta(s) = \theta_{x^\alpha}(s)$ by formula (64) from the proof of Theorem 4.2.

6.2 Rigid body with oscillators

Consider a control system

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_2^2 u_1 - x_1^2 u_2,\quad x \in \mathbb{R}^3,\ u \in \mathbb{R}^2. \tag{97}$$

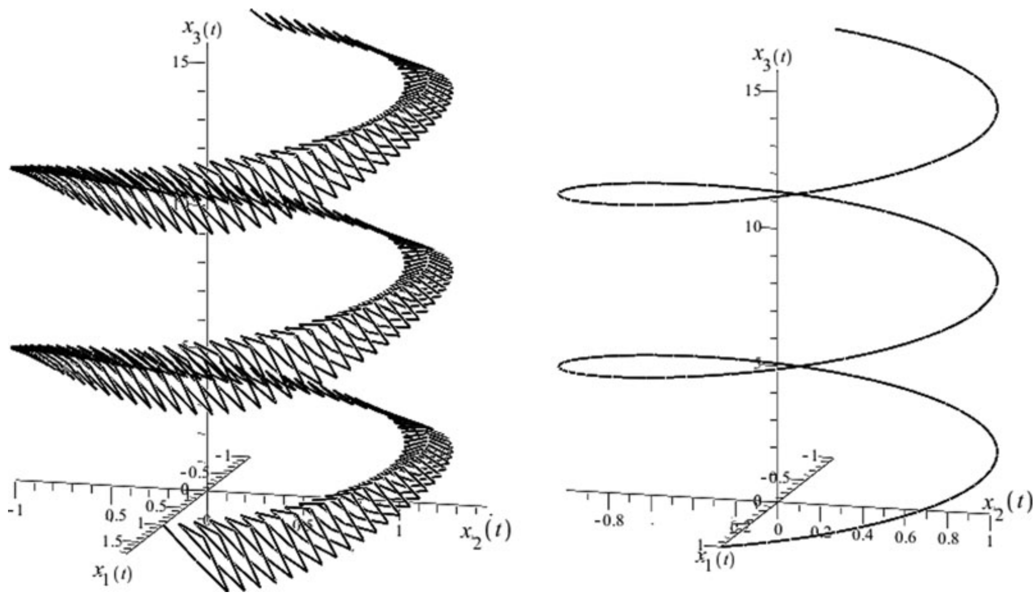

Figure 3. The trajectory of system (97) with controls (98) (left figure) and the helix γ (right figure).

These equations describe the motion of a planar rigid body with two oscillators (Carinena, Clemente-Gallardo, & Ramos, 2003; Yang, Krishnaprasad, & Dayawansa, 1996). The vector fields of system (97) are: $f_1(x) = (1, 0, x_2^2)^*$, $f_2(x) = (0, 1, -x_1^2)^*$, $[f_1, f_2](x) = (0, 0, -2(x_1 + x_2))^*$, $[[f_1, f_2], f_1](x) = (0, 0, 2)^*$. As one can see, the first-order Lie bracket does not generate the remaining direction if $x_1 = -x_2$. However, control system (97) satisfies the (S_2, S_3)-rank condition (Definiton 4.1) with $S_2 = \emptyset$ and $S_3 = \{(1, 2, 1)\}$:

$$\operatorname{span}\{f_1(x), f_2(x), [[f_i, f_j], f_l](x) \mid (i, j, l) \in S_3\} = \mathbb{R}^3 \quad \text{for all } x \in D = \mathbb{R}^3.$$

In this section, we apply controls (39) to solve the approximate path-following problem for system (97) from the point $x^0 = (1, 0, 0)^*$ to $x^T = (1, 0, 5\pi)^*$ along the helix $\gamma = \{(\cos s, \sin s, s)^* \mid s \in [0, 5\pi]\}$. The conditions of Theorem 4.3 are satisfied, and we illustrate its assertion for $\tau = 1$ and a uniform partition of the curve γ with $N = 200$, such that $x^j = (\cos(0.025\pi j), \sin(0.025\pi j), 0.025\pi j)^*$, $j = \overline{0, 200}$. For this purpose, we construct the π-approximating control for $t \in [0, 200]$ in the sense of Definition 2.1 and Theorem 2.1:

$$\begin{aligned} u_\pi(t) &= u^{x^0 x^1}(t) \quad \text{for } t \in [0, 1], \\ u_\pi(t) &= u^{x_\tau^{j-1} x^j}(t - j + 1) \quad \text{for } t \in (j-1, j], \\ & j = \overline{2, 200}, \end{aligned} \tag{98}$$

where $u^{x_\tau^{j-1} x^j}(t)$ are defined by formula (39) with $K_{1121} = 2$, $K_{2121} = 3$, for all $j \in \overline{1, 200}$:

$$\begin{aligned} u_1^{x_\tau^{j-1} x^j}(t) &= a_1^j + a_{121}^j \cos 4\pi t \big(1 + \sin 6\pi t\big), \\ u_2^{x_\tau^{j-1} x^j}(t) &= a_2^j + a_{121}^j \sin 6\pi t, \quad t \in [0, 1]. \end{aligned}$$

Here, a_1^j, a_2^j, a_{121}^j satisfy algebraic equation (46), that is,$a_1^j = x_1^j - x_1^{j-1}$, $a_2^j = x_2^j - x_2^{j-1}$, and a_{112}^j is a real solution of the following cubic equation:

$$\begin{aligned} x_3^j - x_3^{j-1} &= a_1^j {x_2^{j-1}}^2 - a_2^j {x_1^{j-1}}^2 + a_1^j a_2^j \big(x_2^{j-1} - x_1^{j-1}\big) \\ &\quad + \frac{a_{112}^j}{\pi}\left(\frac{a_1^j}{3} - \frac{3a_2^j}{5}\right)\big(x_1^{j-1} + x_2^{j-1}\big) \\ &\quad + \frac{1}{3} a_1^j a_2^j (a_2^j - a_1^j) + \frac{a_{112}^j}{\pi}(a_1^j + a_2^j) \\ &\quad \times \left(\frac{a_1^j}{6} + \frac{a_2^j(5 - 12\pi)}{40\pi}\right) \\ &\quad + \frac{{a_{121}^j}^2}{\pi^2}\left(\frac{11a_1^j}{192} - \frac{381a_2^j}{1600}\right) + \frac{{a_{121}^j}^3}{40\pi^2}. \end{aligned}$$

Figure 3 illustrates the nature of assertions of Theorems 2.1 and 4.3: the trajectory of system (97) with controls (98) remains in some small ε-neighbourhood of

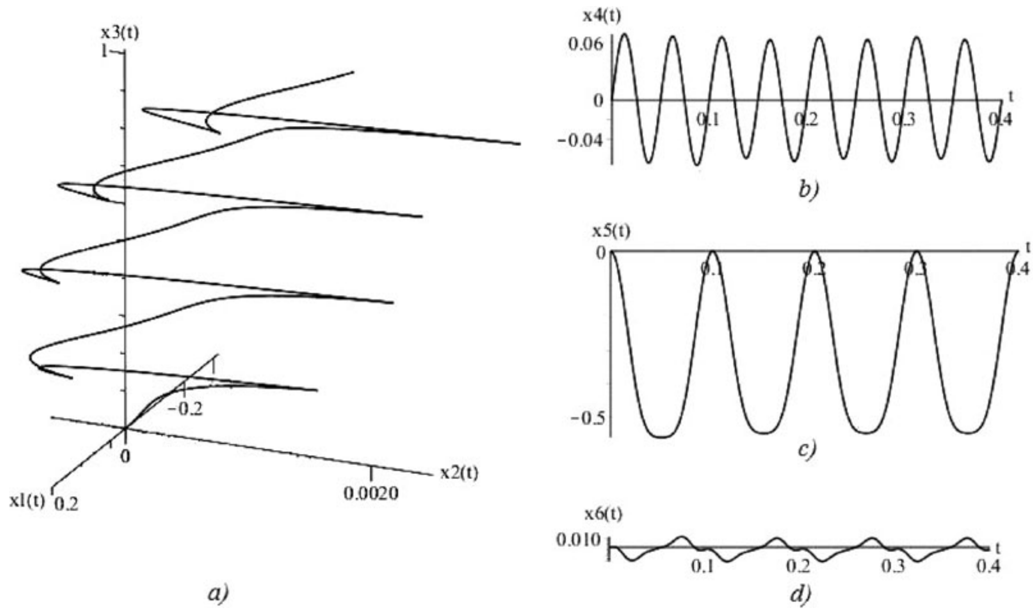

Figure 4. Components of the solution of system (99) with the control $u = u_\pi(t)$: (a) $(x_1(t), x_2(t), x_3(t))$, (b) $(t, x_4(t))$, (c) $(t, x_5(t))$, (d) $(t, x_6(t))$.

the helix γ for all $t \in [0, T]$, and closely approaches the target x^T at $T = 250$.

6.3 Underwater vehicle

In this subsection, we illustrate the possibility of using local controllers of Section 3 for the control design scheme described in Theorem 2.1. For this purpose, we consider the equations of motion for an autonomous 3D underwater vehicle:

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2 + f_3(x)u_3,$$
$$x = (x_1, \dots, x_6)^* \in \mathbb{R}^6, \; u = (u_1, u_2, u_3)^* \in \mathbb{R}^3, \quad (99)$$

where x_1, x_2, x_3 are the coordinates of the centre of mass, and x_4, x_5, x_6 specify the Euler angles,

$$f_0(x) = (0, 0, 0, u_0 \cos x_4 \mathrm{tg}\, x_5, -u_0 \sin x_4, u_0 \cos x_4 \sec x_5)^*,$$
$$f_2(x) = (0, 0, 0, 1, 0, 0)^*,$$
$$f_1(x) = (\cos x_5 \cos x_6, \cos x_5 \sin x_6, -\sin x_5, 0, 0, 0)^*,$$
$$f_3(x) = (0, 0, 0, \sin x_4 \mathrm{tg}\, x_5, \cos x_4, \sin x_4 \sec x_5)^*.$$

Note that system (99) is a modification of the equations considered in Nalamura and Savant (1991) for the case when the angular velocity component along the x_3 axis is not controlled ($u_0 = \mathrm{const}$). Therefore, our controls are the translational velocity $u_1 = v$ along the Ox_1 axis and two angular velocity components: $u_2 = \omega_1$ and $u_3 = \omega_2$.

It is easy to see that

$$\mathrm{span}\{f_1(x), f_2(x), f_3(x), [f_0, f_1](x), [f_1, f_3](x), [f_2, f_3](x)\} = \mathbb{R}^6,$$

for all $x \in \mathbb{R}^6$ such that $x_5 \neq \frac{\pi}{2} (\mathrm{mod}\, \pi)$, so that the (S_0, S_1, S_2)-rank condition (Definition 3.1) holds with $S_0 = \{1, 2, 3\}$, $S_1 = \{1\}$, $S_2 = \{(1, 3), (2, 3)\}$ for all $x \in D = \{x \in \mathbb{R}^6 \,|\, |x_5| < \frac{\pi}{2}\}$. We illustrate the possibility of solving the path-following problem (Problem 2.2) for system (99) by using controls of the type (14):

$$u_1(t) = v_1 + a_1 \sin\left(\frac{2\pi K_1 t}{\tau}\right) + a_{13} \cos\left(\frac{2\pi K_{13} t}{\tau}\right),$$
$$u_2(t) = v_2 + a_{23} \cos\left(\frac{2\pi K_{23} t}{\tau}\right),$$
$$u_3(t) = v_3 + a_{13} \sin\left(\frac{2\pi K_{13} t}{\tau}\right) + a_{23} \sin\left(\frac{2\pi K_{23} t}{\tau}\right), \quad (100)$$

with the vector of coefficients $a = (v_1, v_2, v_3, a_1, a_{13}, a_{23})^* \in \mathbb{R}^6$ and parameters $K = (K_1, K_{13}, K_{23})^* \in (\mathbb{Z} \setminus \{0\})^3$.

In particular, to steer system (99) with $u_0 = 0.25$ from the origin to the target point $x^T = (0, 0, 1, 0, 0, 0)^*$ along the segment $\gamma = \{(0, 0, x_3, 0, 0, 0)^* \,|\, x_3 \in [0, 1]\}$, we construct the control $u_\pi(t)$ as in Definition 2.1 and Theorem 2.1 for the partition of γ with $x^j = (0, 0, j/4, 0,$

$0, 0)^*$, $j = \overline{0, 4}$. At each step $j = 1, 2, 3, 4$, we apply controls of the form (100) for $(j-1)\tau < t \le j\tau$, $\tau = 0.1$, with the following parameters:

$$\text{Step } j = 1: \ v_1 = 0, v_2 \approx 0.087, v_3 \approx 0.001,$$

$$a_1 = 0, a_{13} \approx -17.724, a_{23} \approx 8.395;$$

$$\text{Step } j = 2\ v_1 \approx 0.766, v_2 \approx 0.072, v_3 \approx -0.0003,$$

$$a_1 \approx 2.781, a_{13} \approx -17.336, a_{23} \approx 7.879;$$

$$\text{Step } j = 3\ v_1 \approx 0.772, v_2 \approx 0.077, v_3 \approx 0,$$

$$a_1 \approx 4.923, a_{13} \approx -17.312, a_{23} \approx 7.924;$$

$$\text{Step } j = 4\ v_1 \approx 0.771, v_2 \approx 0.076, v_3 \approx 0,$$

$$a_1 \approx 8.713, a_{13} \approx -17.313, a_{23} \approx 7.923.$$

The above parameters are obtained by solving the system of algebraic equation (15) with $x^\alpha = x((j-1)\tau)$, $x^\omega = x^j$, and the integer parameters being chosen as $K_1 = 3$, $K_{13} = 1$, $K_{23} = -2$ (these parameters clearly satisfy Assumption 3.1). We see in Figure 4 that the controller proposed is able to solve the approximate path-following problem for system (99) with the accuracy $\|x(T) - x^T\| < \varepsilon_1 \approx 0.002$ at the final time $T = 0.4$.

7. Conclusion

In this paper, we have proposed an explicit reduction of the motion planning problem to systems of algebraic equations for classes of bracket-generating systems of steps 2 and 3. To the best of our knowledge, no general results concerning the solvability of such algebraic systems of an arbitrary dimension have been published so far. On the one hand, it has been already proved in Liu (1997) that any trajectory of the Lie bracket extension can be approximated by trajectories of the original system with highly oscillatory inputs. On the other hand, we do not use any sequence of trigonometric polynomials with unbounded amplitudes and frequencies here. It should be also emphasised that our construction provides explicit formulas for controls and does not use any specific changes of coordinates (e.g. canonical coordinates corresponding to the P. Hall basis). Thus, our solvability result provides a novel contribution towards the justification of the use of trigonometric controls for local and global steering problems. Note that the proofs of Theorems 3.2 and 4.2 are based on the degree theory, as the standard implicit function theorem is not applicable (the nonlinear part of the corresponding vector function is not differentiable at $\tilde{a} = 0$).

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Appendices

A.1 Representation of the Volterra series using the Lie brackets

Lemma A1: *Formula (12) for the solution of system (37) with initial condition $x|_{t=0} = x^0$ can be rewritten in the form (81) with*

$$\begin{aligned}
G(t) &= \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x)\bigg|_{x=x^0} \int_0^t u_i(s)ds \int_0^t u_j(s)ds \\
&+ \frac{1}{6}\sum_{i,j,l=1}^{m} \frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right) f_l(x)\bigg|_{x=x^0} \int_0^t u_i(s)ds \\
&\times \int_0^t u_j(s)ds \int_0^t u_l(s)ds + \frac{1}{6}\sum_{i<j}\sum_{l=1}^{m} \\
&\times \left(\frac{\partial f_l(x)}{\partial x}[f_i, f_j](x) + 2\frac{\partial}{\partial x}\left([f_i, f_j](x)\right) f_l(x)\right)\bigg|_{x=x^0} \\
&\times \int_0^t u_l(s)ds \int_0^t \int_0^\tau (u_j(s)u_i(p) \\
&- u_i(s)u_j(p))dpds.
\end{aligned} \tag{A1}$$

Proof: Indeed, straightforward computations show that

$$\begin{aligned}
&\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x) \int_0^t \int_0^\tau u_i(\tau)u_j(p)dpd\tau \\
&= \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x) \int_0^t u_i(s)ds \int_0^t u_j(s)ds \\
&+ \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x) \int_0^t \int_0^\tau \big(u_i(\tau)u_j(p) - u_j(\tau)u_i(p)\big)dpd\tau \\
&= \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x) \int_0^t u_i(s)ds \int_0^t u_j(s)ds \\
&+ \frac{1}{2}\sum_{i<j}[f_i, f_j](x) \int_0^t \int_0^\tau \big(u_j(\tau)u_i(s) - u_j(\tau)u_i(p)\big)dpd\tau.
\end{aligned}$$

Analogously, using the formula

$$\begin{aligned}
&6\int_0^t \int_0^\tau \int_0^s u_i(\tau)u_j(s)u_l(p)dpdsd\tau \\
&= \int_0^t u_i(s)ds \int_0^t u_j(s)ds \int_0^t u_l(s)ds + \int_0^t u_j(s)ds \\
&\times \int_0^t \int_0^s \big(u_i(s)u_l(p) - u_l(s)u_i(p)\big)dsds
\end{aligned}$$

$$+\int_0^t u_l(s)ds\int_0^t\int_0^s \big(u_i(s)u_j(p)-u_j(s)u_i(p)\big)dsds$$
$$+3\int_0^t\int_0^\tau\int_0^s u_i(\tau)\big(u_j(s)u_l(p)-u_l(s)u_j(p)\big)dpdsd\tau$$
$$+\int_0^t\int_0^\tau\int_0^s u_j(\tau)\big(u_l(s)u_i(p)-u_i(s)u_l(p)\big)dpdsd\tau$$
$$+\int_0^t\int_0^\tau\int_0^s u_l(\tau)\big(u_j(s)u_i(p)-u_i(s)u_j(p)\big)dpdsd\tau,$$

we transform the remaining part of (12) and obtain expression (81) with $G(t)$ defined by (11). ■

A2. Formulas for $\Omega(a, x, \tau)$ and $\tilde\Omega(\tilde a, x)$

$$\Omega(a,x,\tau)=\sum_{i,j=1}^m \frac{\partial f_i(x)}{\partial x}f_j(x)\Bigg\{a_ia_j+\sum_{(q,r)\in S_2}\frac{a_{qr}}{\pi K_{qr}}(a_i\delta_{jr}$$
$$-a_j\delta_{ir})+\sum_{(q,r,s)\in S_3}\frac{a_{qrs}}{\pi}\left(a_i\left(\frac{\delta_{jr}}{K_{2qrs}}+\frac{\delta_{js}K_{2qrs}}{K_{3qrs}}\right)\right.$$
$$\left.-\ a_j\left(\frac{\delta_{ir}}{K_{2qrs}}+\frac{\delta_{is}K_{2qrs}}{K_{3qrs}}\right)\right)\Bigg\}+\frac{\tau}{2}\sum_{i,j,l=1}^m$$
$$\times\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x}f_j(x)\right)f_l(x)\Bigg\{\frac{2}{3}a_ia_ja_l$$
$$+\sum_{(q,r)\in S_2}\left(\frac{a_{qr}}{\pi^2K_{qr}^2}\sigma_1^{(2)}(a)+\frac{a_{qr}}{\pi K_{qr}}\sigma_2^{(2)}(a)\right.$$
$$+\frac{a_{qr}^2}{4\pi^2K_{qr}^2}\sigma_1^{(1)}(a)+\frac{a_{qr}^2}{2\pi K_{qr}}\sigma_2^{(1)}(a)$$
$$+\sum_{\substack{(k,p)\in S_2\\(k,p)\neq(q,r)}}\left(\frac{a_{qr}}{\pi K_{qr}}\frac{a_{kp}}{\pi K_{kp}}\sigma_3^{(1)}(a)\right.$$
$$\left.\left.+\frac{a_{qr}a_{kp}}{\pi^2}\sigma_4^{(1)}(a)\right)\right)+\frac{1}{\pi^2}\sum_{(q,r,s)\in S_3}$$
$$\times\left(a_{qrs}\sigma_3^{(2)}(a)+a_{qrs}^2\sigma_5^{(1)}(a)\right.$$
$$\left.+\sum_{\substack{(k,p,z)\in S_3\\(q,r,s)\neq(k,p,z)}}a_{qrs}a_{kpz}\sigma_6^{(1)}(a)\right)+\frac{1}{\pi^2}$$
$$\times\sum_{\substack{(q,r)\in S_2\\(k,p,z)\in S_3}}a_{kpz}\left(\frac{a_{qr}}{K_{qr}}\sigma_7^{(1)}(a)+a_{qr}K_{qr}\sigma_8^{(1)}(a)\right.$$
$$\left.\left.+a_{qr}\sigma_9^{(1)}(a)\right)\right\},$$

where $K_{3qrs}=K_{2qrs}^2-K_{1qrs}^2$, $\sigma_k^{(2)}$ are quadratic forms with respect to a_i, and $\sigma_s^{(1)}$ are linear forms with respect to a_i, $i=\overline{1,m}$:

$$\sigma_1^{(2)}(a)=a_ia_j\delta_{lq}+a_ja_l\delta_{iq}-2a_ia_l\delta_{jq},$$
$$\sigma_2^{(2)}(a)=a_ia_j\delta_{lr}-a_ja_l\delta_{ir},$$
$$\sigma_3^{(2)}(a)=a_ia_j\left(\frac{\delta_{lq}}{K_{1qrs}^2}+\frac{\pi\delta_{lr}}{K_{2qrs}}+\frac{\pi\delta_{ls}K_{2qrs}}{K_{3qrs}}\right)$$
$$+a_ja_l\left(\frac{\delta_{iq}}{K_{1qrs}^2}-\frac{\pi\delta_{ir}}{K_{2qrs}}-\frac{\pi\delta_{is}K_{2qrs}}{K_{3qrs}}\right)-\frac{2a_ia_l\delta_{jq}}{K_{1qrs}^2},$$
$$\sigma_1^{(1)}(a)=a_i(\delta_{jq}\delta_{lq}+3\delta_{jr}\delta_{lr})+a_l(\delta_{iq}\delta_{jq}+3\delta_{ir}\delta_{jr})$$
$$-2a_j(\delta_{iq}\delta_{lq}+3\delta_{ir}\delta_{lr}),$$
$$\sigma_2^{(1)}(a)=a_i(\delta_{jr}\delta_{lq}-\delta_{jq}\delta_{lr})+a_l(\delta_{ir}\delta_{jq}-\delta_{iq}\delta_{jr}),$$
$$\sigma_3^{(1)}(a)=\frac{K_{kp}^2\delta_{jr}}{K_{kp}^2-K_{qr}^2}(a_i\delta_{lp}-a_l\delta_{ip})-a_j\delta_{ir}\delta_{lp},$$
$$\sigma_4^{(1)}(a)=\frac{a_i\delta_{jq}\delta_{lk}-a_l\delta_{iq}\delta_{jk}}{K_{kp}^2-K_{qr}^2},$$
$$\sigma_5^{(1)}(a)=\frac{1}{4K_{1qrs}^2}(a_i\delta_{jq}\delta_{lq}+a_l\delta_{iq}\delta_{jq}-2a_j\delta_{iq}\delta_{lq})$$
$$+\frac{3}{4K_{2qrs}^2}(a_i\delta_{jr}\delta_{lr}+a_l\delta_{ir}\delta_{jr}-2a_j\delta_{ir}\delta_{lr})$$
$$+\frac{\delta_{jr}}{4K_{2qrs}^2-K_{1qrs}^2}(a_i\delta_{ls}+a_l\delta_{is})$$
$$-\frac{K_{1qrs}^2+5K_{2qrs}^2}{8K_{3qrs}}(a_i\delta_{js}\delta_{ls}+a_l\delta_{is}\delta_{js}-2a_j\delta_{is}\delta_{ls})$$
$$-\frac{3\delta_{js}K_{2qrs}^2}{K_{3qrs}(K_{1qrs^2}-4K_{2qrs}^2)}(a_i\delta_{lr}+a_l\delta_{ir})$$
$$-\frac{a_j}{K_{3qrs}}(\delta_{ir}\delta_{ls}+\delta_{is}\delta_{lr}),$$

$$
\begin{aligned}
\sigma_6^{(1)}(a) = {} & \frac{\delta_{jq}}{K_{1kpz}^2 - K_{1qrs}^2}(a_i\delta_{lk} + a_l\delta_{ik}) + \frac{\delta_{jr}K_{2kpz}}{K_{2qrs}(K_{2kpz}^2 - K_{2qrs}^2)}(a_i\delta_{lp} + a_l\delta_{ip}) \\
& + \frac{K_{2kpz}}{\big((K_{1kpz}+K_{2kpz})^2 - K_{2qrs}^2\big)\big((K_{1kpz}-K_{2kpz})^2 - K_{2qrs}^2\big)}\left(\frac{\delta_{jr}(K_{3kpz} - K_{2qrs}^2)}{K_{2qrs}}(a_i\delta_{lp}\right. \\
& \left. + a_l\delta_{ip}) - \frac{\delta_{jz}K_{2qrs}(3K_{1kpz}^2 + K_{2kpz}^2 - K_{2qrs}^2)}{K_{3kpz}}(a_i\delta_{lr} + a_l\delta_{ir})\right) + \frac{\delta_{js}K_{2kpz}}{2}(a_i\delta_{lz} + a_l\delta_{iz}) \\
& \times \left(\frac{(K_{1qrs}-K_{2qrs})^2 - K_{3kpz}}{(K_{1qrs}-K_{2qrs})\big((K_{1kpz}+K_{2kpz})^2 - (K_{1qrs}-K_{2qrs})^2\big)\big((K_{1kpz}-K_{2kpz})^2 - (K_{1qrs}-K_{2qrs})^2\big)}\right. \\
& \left. + \frac{(K_{1qrs}+K_{2qrs})^2 - K_{3kpz}}{(K_{1qrs}+K_{2qrs})\big((K_{1kpz}+K_{2kpz})^2 - (K_{1qrs}+K_{2qrs})^2\big)\big((K_{1kpz}-K_{2kpz})^2 - (K_{1qrs}+K_{2qrs})^2\big)}\right) \\
& - a_j\left(\frac{\delta_{ir}\delta_{lp}}{K_{2qrs}K_{2kpz}} + \frac{\delta_{is}\delta_{lz}K_{2qrs}K_{2kpz}}{K_{3qrs}K_{3kpz}} + \frac{K_{2kpz}(\delta_{ir}\delta_{lz} + \delta_{iz}\delta_{lr})}{K_{2qrs}K_{3kpz}}\right), \\
\sigma_7^{(1)}(a) = {} & \frac{\delta_{jr}K_{2kpz}}{K_{2kpz}^2 - K_{qr}^2}(a_i\delta_{lp} + a_l\delta_{ip}) - \frac{\delta_{jr}K_{2kpz}(K_{qr}^2 - K_{3kpz})}{\big((K_{1kpz}+K_{2kpz})^2 - K_{qr}^2\big)\big((K_{1kpz}-K_{2kpz})^2 - K_{qr}^2\big)}(a_i\delta_{lz} \\
& + a_l\delta_{iz}) - a_j\left(\frac{K_{2kpz}}{K_{3kpz}}(\delta_{ir}\delta_{lz} + \delta_{lr}\delta_{iz}) + \frac{\delta_{ir}\delta_{lp} + \delta_{lr}\delta_{ip}}{K_{qr}K_{2kpz}}\right), \\
\sigma_8^{(1)}(a) = {} & \frac{\delta_{jp}}{K_{2kpz}(K_{qr}^2 - K_{2kpz}^2)}(a_i\delta_{lr} + a_l\delta_{ir}) \\
& - \frac{\delta_{jz}K_{qr}K_{kpz}(3K_{1kpz}^2 + K_{2kpz}^2 - K_{qr}^2)}{K_{3kpz}\big((K_{1kpz}+K_{2kpz})^2 - K_{qr}^2\big)\big((K_{1kpz}-K_{2kpz})^2 - K_{qr}^2\big)}(a_i\delta_{lr} + a_l\delta_{ir}), \\
\sigma_9^{(1)}(a) = {} & \frac{a_i(\delta_{jq}\delta_{lk} - \delta_{lq}\delta_{jk}) + a_l(\delta_{jq}\delta_{ik} - \delta_{iq}\delta_{jk})}{K_{1kpz}^2 - K_{qr}^2}.
\end{aligned}
$$

The expression for $\widetilde{\Omega}(\tilde{a}, x)$ is as follows:

$$
\begin{aligned}
\widetilde{\Omega}(\tilde{a}, x) = {} & \sum_{i,j=1}^{m} \frac{\partial f_i(x)}{\partial x} f_j(x)\left\{\frac{\tilde{a}_i\tilde{a}_j}{2} + \sum_{(q,r)\in S_2} \frac{|\tilde{a}_{qr}|^{1/2}}{\sqrt{\pi K_{qr}}}(\tilde{a}_i\delta_{jr} - \tilde{a}_j\delta_{ir})\right. \\
& \left. + \sum_{(q,r,s)\in S_3} \tilde{a}_{qrs}^{1/3}\sqrt[3]{\frac{\pi^2}{2}K_{3qrs}^+}\left(\tilde{a}_i\left(\frac{\delta_{jr}}{K_{2qrs}^+} + \frac{\delta_{js}K_{2qrs}^+}{K_{3qrs}^+}\right) - \tilde{a}_j\left(\frac{\delta_{ir}}{K_{2qrs}^+} + \frac{\delta_{is}K_{2qrs}^+}{K_{3qrs}^+}\right)\right)\right\} \\
& + \sum_{i,j,l=1}^{m} \frac{\partial}{\partial x}\Big(\frac{\partial f_i(x)}{\partial x} f_j(x)\Big) f_l(x)\left\{\frac{1}{6}\tilde{a}_i\tilde{a}_j\tilde{a}_l + \sum_{(q,r)\in S_2}\left(\frac{\operatorname{sign}(\tilde{a}_{qr})|\tilde{a}_{qr}|^{1/2}}{2(\pi K_{qr}^+)^{3/2}}\sigma_1^{(2)}(\tilde{a})\right.\right. \\
& + \frac{|\tilde{a}_{qr}|^{1/2}}{2\sqrt{\pi K_{qr}^+}}\sigma_2^{(2)}(\tilde{a}) + \frac{|\tilde{a}_{qr}|}{4\pi K_{qr}^+}\sigma_1^{(1)}(\tilde{a}) + \frac{\tilde{a}_{qr}}{2}\sigma_2^{(1)}(\tilde{a}) + \sum_{\substack{(k,p)\in S_2 \\ (k,p)\neq(q,r)}}\left(\frac{|\tilde{a}_{qr}\tilde{a}_{kp}|^{1/2}}{\pi\sqrt{K_{qr}^+K_{kp}^+}}\sigma_3^{(1)}(\tilde{a})\right. \\
& \left.\left. + \frac{\operatorname{sign}(\tilde{a}_{qr}\tilde{a}_{kp})|\tilde{a}_{qr}\tilde{a}_{kp}|^{1/2}}{\pi\sqrt{K_{qr}^+K_{kp}^+}}\sigma_4^{(1)}(\tilde{a})\right)\right) + \sum_{(q,r,s)\in S_3}\left(\tilde{a}_{qrs}^{1/3}\sqrt[3]{\frac{K_{3qrs}^+}{4\pi^4}}\sigma_3^{(2)}(\tilde{a}) + \tilde{a}_{qrs}^{2/3}\sqrt[3]{\frac{4K_{3qrs}^{+\ 2}}{\pi^2}}\sigma_5^{(1)}(\tilde{a})\right. \\
& \left. + \sum_{\substack{(k,p,z)\in S_3 \\ (q,r,s)\neq(k,p,z)}} \tilde{a}_{qrs}^{1/3}\tilde{a}_{kpz}^{1/3}\sqrt[3]{\frac{4K_{3qrs}^+K_{3kpz}^+}{\pi^2}}\sigma_6^{(1)}(\tilde{a})\right) + \sum_{\substack{(q,r)\in S_2 \\ (k,p,z)\in S_3}} \tilde{a}_{kpz}^{1/3}\sqrt[6]{\frac{4K_{3kpz}^{+\ 2}}{\pi^5}}\left(\frac{|\tilde{a}_{qr}|^{1/2}}{\sqrt{K_{qr}}}\sigma_7^{(1)}(\tilde{a})\right. \\
& \left.\left. + |\tilde{a}_{qr}|^{1/2}K_{qr}^{+\,3/2}\sigma_8^{(1)}(\tilde{a}) + \operatorname{sign}(\tilde{a}_{qr})|\tilde{a}_{qr}|^{1/2}\sqrt{K_{qr}^+}\sigma_9^{(1)}(\tilde{a})\right)\right\}, \quad \text{where } K_{3qrs}^+ = (K_{2qrs}^+)^2 - (K_{1qrs}^+)^2.
\end{aligned}
$$