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Citation: Physics of Fluids A: Fluid Dynamics **5**, 3026 (1993); doi: 10.1063/1.858712
View online: http://dx.doi.org/10.1063/1.858712
View Table of Contents: http://aip.scitation.org/toc/pfa/5/12
Published by the American Institute of Physics

Chaotic motion of a solid through ideal fluid

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(Received 11 June 1993; accepted 9 August 1993)

Numerical evidence is presented that the motion of a solid body through incompressible, inviscid fluid, moving irrotationally and otherwise at rest, is chaotic.

The motion of a solid body through incompressible, inviscid, irrotational fluid otherwise at rest is a classical problem of great significance. It was shown by Kirchhoff[1,2] that the problem reduces to a set of *ordinary* differential equations which generalize Euler's equations[3] describing the motion of a solid body in a vacuum. The Kirchhoff equations present a most remarkable simplification of a problem that, in principle, involves an infinite number of degrees of freedom. Surprisingly, the literature exploring these equations from the point of view of dynamical systems theory is rather sparse.

In a significant paper Kozlov and Oniscenko[4] achieved a characterization of the integrable cases of Kirchhoff's equations. They suggest that when the conditions of their main theorem are not met, the motion of the body will be chaotic. Neither the exact nature of this chaotic motion, nor evidence for its existence, appear to have been given. It is the purpose of this Letter to provide numerical evidence that the Kirchhoff equations do, indeed, have chaotic solutions and to take a first step in providing a physical description of such motions.

Since the solutions of the potential flow problem about a body represent the "outer" solution of the "full" flow problem (i.e., the problem in the presence of viscosity), we believe that the distinction between regular and chaotic motion is of importance for such fundamental aspects of the flow about bodies as separation and for the nature of the pressure distribution over the bounding surface of the body. On the basis of this argument we expect to see vestiges of the chaotic motion found for the idealized problem of inviscid potential flow in the full viscous flow problem.

An important step is to cast Kirchhoff's equations in the familiar language of Hamiltonian dynamics. This is, in essence, achieved in Sec. 122 of Lamb's classic treatise.[5] We review some of the details here.

The equations of motion as written by Kirchhoff[1,2] were in terms of the vectors of velocity, **U**, and angular velocity, **Ω**, of the body referred instantaneously to a frame of coordinates moving with the body:

$$\frac{d}{dt}\left(\frac{\partial T_{\text{tot}}}{\partial \mathbf{U}}\right)+\boldsymbol{\Omega}\times\frac{\partial T_{\text{tot}}}{\partial \mathbf{U}}=\mathbf{0}, \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial T_{\text{tot}}}{\partial \boldsymbol{\Omega}}\right)+\boldsymbol{\Omega}\times\frac{\partial T_{\text{tot}}}{\partial \boldsymbol{\Omega}}+\mathbf{U}\times\frac{\partial T_{\text{tot}}}{\partial \mathbf{U}}=\mathbf{0}. \tag{2}$$

Here, T_{tot} is the kinetic energy of the solid body *and* the surrounding fluid,

$$T_{\text{tot}}=\tfrac{1}{2}\mathbf{U}\cdot(\mathbf{T}+m\mathbf{1})\mathbf{U}+\tfrac{1}{2}\boldsymbol{\Omega}\cdot(\mathbf{J}+\mathbf{I})\boldsymbol{\Omega}+\mathbf{U}\cdot\mathbf{S}\boldsymbol{\Omega}, \tag{3}$$

where m is the mass and **I** is the inertia tensor of the solid body, **1** is the unit 3×3 identity tensor, and **T**, **S**, and **J** make up the 6×6 *added mass tensor*, $T_{\alpha\beta}$, as follows:[5,6]

$$\{T_{\alpha\beta}\}=\begin{Bmatrix}\mathbf{T} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{J}\end{Bmatrix}, \tag{4}$$

with $\mathbf{S}^T$ the transpose of **S**. The elements of **T**, **S**, and **J** depend only on the body shape.[5,6] For certain simple body shapes, such as ellipsoids, these constants are known analytically. In writing (3) a common assumption of rigid body dynamics, that the origin of the moving coordinate system coincides with the center of mass of the body, has been made.

We may write these equations of motion more explicitly as follows. From (3) we introduce a generalized momentum (the *linear impulse*[5])

$$\mathbf{P}=\left(\frac{\partial T_{\text{tot}}}{\partial \mathbf{U}}\right)_{\boldsymbol{\Omega}}=(\mathbf{T}+m\mathbf{1})\mathbf{U}+\mathbf{S}\boldsymbol{\Omega}, \tag{5}$$

and a generalized angular momentum (the *angular impulse*[5])

$$\mathbf{L}=\left(\frac{\partial T_{\text{tot}}}{\partial \boldsymbol{\Omega}}\right)_{\mathbf{U}}=(\mathbf{J}+\mathbf{I})\boldsymbol{\Omega}+\mathbf{S}^T\mathbf{U}. \tag{6}$$

Equations (1) and (2) then take the form

$$\dot{\mathbf{P}}+\boldsymbol{\Omega}\times\mathbf{P}=\mathbf{0}, \tag{7}$$

and

$$\dot{\mathbf{L}}+\boldsymbol{\Omega}\times\mathbf{L}+\mathbf{U}\times\mathbf{P}=\mathbf{0}. \tag{8}$$

It is immediately seen from (7) and (8) that the quantities $\mathbf{P}\cdot\mathbf{P}$ and $\mathbf{L}\cdot\mathbf{P}$ are integrals of the motion. We note for later reference that

$$2T_{\text{tot}}=\mathbf{U}\cdot\mathbf{P}+\boldsymbol{\Omega}\cdot\mathbf{L}. \tag{9}$$

We adopt the point of view that the "generalized coordinates" of this system are the three components of **R** and the nine elements, A_{ij}, of the orthogonal matrix **A** that determines the orientation of the body-fixed coordinate frame relative to a fixed, inertial, "laboratory" frame. The orthogonality relations for **A** are then viewed as integrals of the motion of this system. Equations (7) and (8) constitute a system of six coupled ODE's for the components of **U** and **Ω**. In order to obtain the actual motion of the solid body relative to the "laboratory" frame, one must supplement these equations by additional ODE's for the position of the origin of the body-fixed system, $\mathbf{R}$, and for its orientation. These equations are

$$\dot{\mathbf{R}}=\mathbf{A}\mathbf{U}, \tag{10}$$

$$\dot{A}_{ij}=-\epsilon_{jkl}A_{il}\Omega_k. \tag{11}$$

where $\dot{\mathbf{R}}$ and $\dot{A}_{ij}$ are the time derivatives of $\mathbf{R}$ and A_{ij}, respectively, and ϵ_{jkl} is the completely antisymmetric tensor of rank three.

Kirchhoff's equations (1) and (2) are the Lagrangian equations of motion. The Lagrangian is T_{tot}. We may transform to Hamilton's canonical equations by considering the Legendre transform

$$\begin{aligned} H&=T_{\text{tot}}-\dot{\mathbf{R}}\cdot\frac{\partial T_{\text{tot}}}{\partial\dot{\mathbf{R}}}-\dot{A}_{ij}\frac{\partial T_{\text{tot}}}{\partial\dot{A}_{ij}}\\ &=T_{\text{tot}}-\mathbf{U}\cdot\frac{\partial T_{\text{tot}}}{\partial\mathbf{U}}-\mathbf{\Omega}\cdot\frac{\partial T_{\text{tot}}}{\partial\mathbf{\Omega}}. \end{aligned} \tag{12}$$

Hence, by (9)

$$H=T_{\text{tot}}-\mathbf{U}\cdot\mathbf{P}-\mathbf{\Omega}\cdot\mathbf{L}=-T_{\text{tot}}.$$

Since

$$dT_{\text{tot}}=\mathbf{P}\cdot d\mathbf{U}+\mathbf{L}\cdot d\mathbf{\Omega}, \tag{13}$$

we now have as complements to the definitions (5) and (6):

$$\begin{aligned} \mathbf{U}&=-\left(\frac{\partial H}{\partial\mathbf{P}}\right)_L=\left(\frac{\partial T_{\text{tot}}}{\partial\mathbf{P}}\right)_L;\\ \mathbf{\Omega}&=-\left(\frac{\partial H}{\partial\mathbf{L}}\right)_P=\left(\frac{\partial T_{\text{tot}}}{\partial\mathbf{L}}\right)_P. \end{aligned} \tag{14}$$

Combining these results with (7) and (8), we easily see that H itself is an integral of the motion.

The generalized coordinates being the components of $\mathbf{R}$ and the A_{ij}, we introduce generalized momenta conjugate to these by

$$\Pi_i=\frac{\partial T_{\text{tot}}}{\partial\dot{R}_i}=A_{ik}P_k;\quad B_{ij}=\frac{\partial T_{\text{tot}}}{\partial\dot{A}_{ij}}=\frac{1}{2}A_{im}\epsilon_{jmk}L_k. \tag{15}$$

The appropriate Poisson–Lie bracket for quantities f,g that do not depend on $\mathbf{R}$ is

$$\{f,g\}=\frac{\partial f}{\partial A_{ij}}\frac{\partial g}{\partial B_{ij}}-\frac{\partial f}{\partial B_{ij}}\frac{\partial g}{\partial A_{ij}}. \tag{16}$$

A somewhat tedious derivation using (15) then gives

$$\{f,g\}=\mathbf{L}\cdot\frac{\partial g}{\partial\mathbf{L}}\times\frac{\partial f}{\partial\mathbf{L}}+\mathbf{P}\cdot\left(\frac{\partial g}{\partial\mathbf{L}}\times\frac{\partial f}{\partial\mathbf{P}}-\frac{\partial f}{\partial\mathbf{L}}\times\frac{\partial g}{\partial\mathbf{P}}\right). \tag{17}$$

In particular, we see from (14) that

$$\{H,f\}=-\frac{\partial f}{\partial\mathbf{L}}\cdot(\mathbf{\Omega}\times\mathbf{L}+\mathbf{U}\times\mathbf{P})-\frac{\partial f}{\partial\mathbf{P}}\cdot\mathbf{\Omega}\times\mathbf{P}. \tag{18}$$

Thus we finally obtain the equations of motion (7) and (8) in the desired form

$$\dot{\mathbf{P}}=\{H,\mathbf{P}\};\quad\dot{\mathbf{L}}=\{H,\mathbf{L}\}. \tag{19}$$

This is the Hamiltonian formulation. Lamb[5] uses the notation T for our H, but the treatment is less transparent and the connection to Hamiltonian mechanics and a Poisson–Lie bracket is not made explicitly.

We note the "elementary" brackets

$$\begin{aligned} &\{P_i,P_j\}=0;\quad\{L_i,L_j\}=-\epsilon_{ijk}L_k;\\ &\{P_i,L_j\}=-\epsilon_{ijk}P_k. \end{aligned} \tag{20}$$

It follows from these that any function of $\mathbf{P}$ and $\mathbf{L}$ commutes with both $\mathbf{P}^2$ and $\mathbf{L}\cdot\mathbf{P}$. In particular, the two integrals $\mathbf{P}^2$ and $\mathbf{L}\cdot\mathbf{P}$ are in involution. The dynamics of Kirchhoff's equations can therefore be reduced to the subspace of the six-dimensional $(\mathbf{P},\mathbf{L})$ space given by $|\mathbf{P}|=\text{const}$, and $\mathbf{L}\cdot\mathbf{P}=\text{const}$. Along with the constancy of the total kinetic energy, this reduces the problem to motion on a three-dimensional manifold. This formulation is used as a basis for numerical exploration of the system and identification of integrable and chaotic solutions.

We have integrated Eqs. (7), (8), (10), and (11) numerically for various body shapes and a number of initial conditions. We chose an ellipsoid with semiaxes $(a,b,c)=(1,0.8,0.6)$, which does not satisfy the integrability condition of Kozlov and Oniscenko.[4] For all motions considered we chose $\mathbf{L}\cdot\mathbf{P}=0$. A Poincaré section is obtained by recording points (L_x,L_z) for which $P_z=0$. Three examples of such sections are shown in Fig. 1. The three sections correspond to three different values of the density ratio of the solid body (assumed homogeneous) to the fluid, $\delta\equiv\rho_{\text{fluid}}/\rho_{\text{solid}}$. This parameter determines the relative importance of fluid to solid inertia. If $\delta\ll 1$, we expect the problem to approach the integrable Euler equations for a body in a vacuum. On the other hand, for $\delta\gg 1$ the problem approaches that of a very light but rigid "bubble" moving through the fluid. In Fig. 1 we have (a) $\delta=0.5$; (b) 1.0; (c) 2.0. The increasing amount of apparent chaos in these sections corresponds with physical intuition.

Now, consider Fig. 1(b) in more detail. Section points for four different initial conditions are plotted. The two innermost elliptical curves correspond to two initial conditions that yielded regular motion. There follows a period-8 "island chain." Finally, the last initial condition used produced the chaotic splatter of points at the outer bound of the plot. Figure 2 provides a different view of these four motions. In all four panels of Fig. 2 we show the curve traced out by a "pointer," rigidly attached to the ellipsoid, on a sphere centered at the centroid of the ellipsoid and following along in its motion but *without* rotating relative to the laboratory frame. Hence, each panel in Fig. 2 gives a representation of the *change in orientation* of the solid ellipsoid as it moves through the fluid. Panels (a) and (b) of Fig. 2 correspond to the two regular motions seen in the section of Fig. 1(b). Panel (d) corresponds to the chaotic trajectory in Fig. 1(b). The period-8 island chain in Fig. 1(b) leads to panel (c) of Fig. 2, and one can almost pick out the period-8 motion in Fig. 2(c) as well. Figures 1 and 2 taken together provide numerical evidence for the existence of chaotic motions of the solid body, and show that

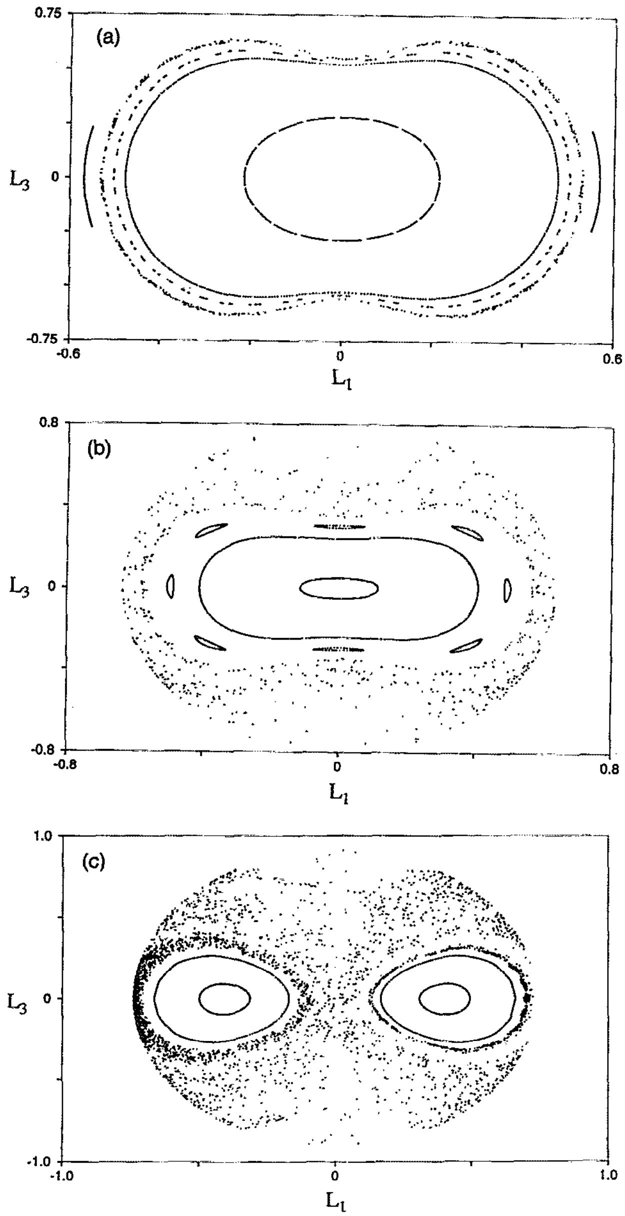

FIG. 1. Poincaré sections (L_x, L_z) for $P_z=0$ with $\mathbf{L}\cdot\mathbf{P}=0$. Several initial conditions are shown in each panel and (a) $\delta=0.5$; (b) 1.0; (c) 2.0.

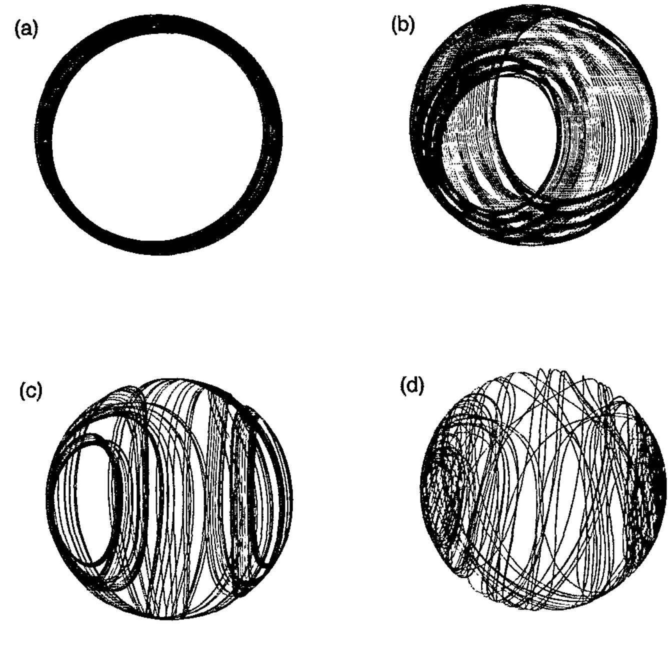

FIG. 2. Illustrations of the variation of body orientation for the four initial conditions in Fig. 1(b). The trajectories shown would be traced out by the tip of a "pointer," rigidly attached to the body, on a sphere that followed the motion of the body centroid but did not rotate relative to the laboratory frame.

one can observe this chaotic motion through the changes in orientation of the body as a function of time.

Due to the close coupling of orientation dynamics and the actual trajectory in space of the body [see Eqs. (10) and (11), in particular] one would expect to also be able to detect the progression from regular to chaotic motion seen in Fig. 1(b) in a suitable plot of the trajectory of the body centroid. This is indeed the case, as we have shown. Space limitations do not allow us to present these results here.

In summary, we have found compelling and consistent numerical evidence that Kirchhoff's equations are, in general, nonintegrable, and that chaotic motion of a solid body moving through a fluid can be detected either by monitoring changes in orientation of the body (regardless of displacement) or by monitoring the trajectory in space of a fixed point in the body (regardless of orientation). Several additional questions present themselves: Can chaotic body motion be detected by monitoring the pressure field on the body? When will the trajectory of a passively advected particle in the flow induced by the motion of the body be chaotic? Will the time series produced by the Eulerian velocity field at a point display chaos when the "Lagrangian" motion of the body is chaotic? And so on. The correspondence between chaotic motion of the body and chaos in other diagnostics of the flow is not *a priori* obvious, and is currently being explored.

The results presented here were first reported by one of us (SWJ) at the Midwestern Universities Fluid Mechanics retreat, 1–3 April 1993.

ACKNOWLEDGMENTS

We are indebted to D. D. Holm for discussions, for alerting us to Ref. 4, and for communicating to us prior to publication results obtained jointly with R. Camassa on certain integrable cases of Kirchhoff's equations. This research was supported by University of Illinois.

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