Sand Piles

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1 Introduction

When a granular material, say, sand is poured vertically onto a horizontal table, it forms a conical pile with slant angle θ. The behaviour of sand and other granular materials cannot be modelled the same as solid objects or fluids. In this paper we will investigate the volume of stable shapes which piles of granular material form on different surfaces and shapes. We will also look at the volume of sand in a bucket with various holes in them. This will be done with a mixture of geometry and integration.

2 Volume of sand piles

All granular piles can be generalised as cones, this is because as new particles fall on top of the pile, whether they stay there or roll down is determined by the slope of the pile. Therefore if all the particles are relatively identical, the slope of the pile will be constant, forming a cone.

Figure 1: Experimental evidence supports the cone model, albeit there is a little rounding at the top

2.1 Angle of repose

As a sand pile is formed by continuously pouring sand on a flat surface, there will be a certain critical angle between the slope of the pile and the ground which will not be above a certain constant.

Definition 1. the *angle of repose* θ is the maximum angle between the ground and the slope of a granular pile

Definition 2. the *friction coefficient* μ is the proportion of normal force that the sand grains direct into frictional force

Suppose that a new grain of sand falls onto a slope, the forces acting on that particle will be such :

$$\begin{aligned} F_{net} &= mg\sin\theta - F_{friction} \\ F_{friction} &= \mu mg\cos\theta \end{aligned} \tag{1}$$

where μ is the friction coefficient of the sand such that $0 < \mu < 1$

$$\text{therefore } F_{net} = mg\sin\theta - \mu mg\cos\theta$$

Since the sand on a stable sand pile is stationary, the net force must be 0

$$\begin{aligned} \text{therefore } mg\sin\theta &= \mu mg\cos\theta \\ \tan\theta &= \mu \text{ (Important, remember for later)} \\ \theta &= \arctan\mu \end{aligned} \tag{2}$$

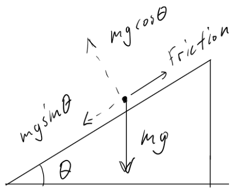


Figure 2: a decomposition of forces acting on a single grain of sand

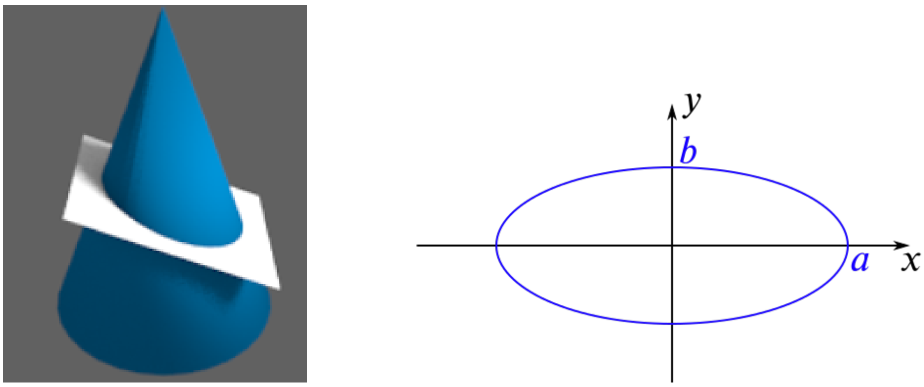
We found experimentally that the shape of a material didn't impact the overall conical structure. Therefore, we can account for different types of material - coffee grounds, rice, sand, salt - by measuring their friction coefficients or measuring their angles of repose.

2.2 If the spatial volume occupied by the pile is V, what is the height of the apex of the pile?

If the volume of a sand pile is given by V, is it possible to find the height of that pile? It is known that the formula of a cone is given by $V = \frac{\pi}{3} r^2 h$ where h is the height of the cone and r is the radius of its base. The angle of repose gives the relationship between the base and the height $h = r \tan\theta$.

$$\begin{aligned} \text{therefore } r &= \frac{h}{\tan\theta} \\ \implies V &= \frac{\pi h^3}{3\tan^2\theta} \\ \text{therefore } h &= \sqrt[3]{\frac{3V\tan^2\theta}{\pi}} \end{aligned} \tag{3}$$

2.3 Volume on a slanted surface?



(a) oblique view of the pile (b) base view of the pile

Figure 3: The shape of a conical sand pile on a slanted surface (not to scale)

On a slanted surface, the top of the sand pile is still a cone, but the base is slanted in the shape of an ellipse like in Figure 3b (we know this as the slanted cross section of a cone is one of the definitions for an ellipse) .

Definition 3. the *slope angle* α is the angle between the slanted surface and the flat ground(flat ground is the plane that the gravitational force vector is normal to)

The volume of this pile can be determined via integration of ellipses.

$$A_{ellipse} = \pi ab$$

In this case, b remains constant and a increases as the slant angle increases. To find the volume, we will make use of an integration trick.

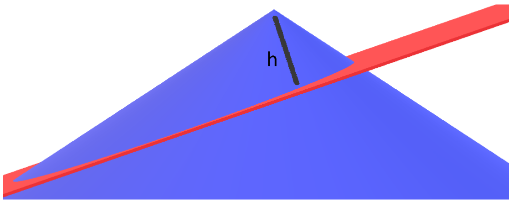


Figure 4: let h be the minimum distance between the tip of the cone to the plane which cuts through it

As we see in Figure 4 above, the area $a \propto h^2$therefore $\frac{a}{a_{max}} = \left(\frac{h}{h_{max}}\right)^2$, this allows us to set up the integral:

$$\begin{aligned}
V_{conic} &= \int_0^h A_{ellipse} dh \\
&= \int_0^h \pi ab\, dh \\
&= \pi b \int_0^h a\, dh \\
\frac{a}{a_{max}} &= \left(\frac{h}{h_{max}}\right)^2 \\
\text{therefore } V_{conic} &= \pi b \int_0^h \frac{a_{max} h^2}{h_{max}^2}\, dh \\
&= \pi b \frac{a_{max}}{h_{max}^2} \int_0^h h^2\, dh \\
&= \frac{\pi}{3} a_{max} bh
\end{aligned} \tag{4}$$

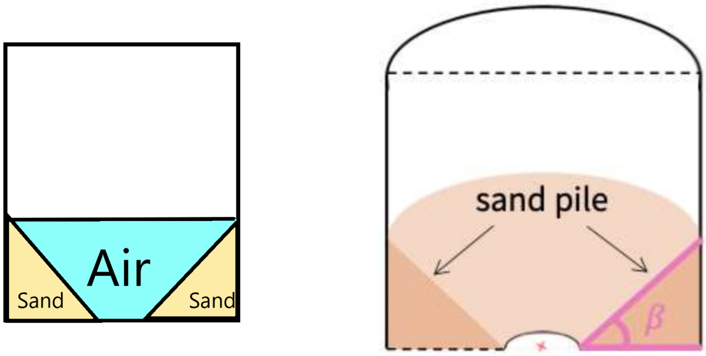
To put this in terms of h and θ, we note that $b = \frac{h}{\tan\theta}$ and that $a = \frac{1}{2}(\frac{h}{\tan\theta - \alpha} + \frac{h}{\tan\theta + \alpha})$. After substituting those values into our volume equation and rearranging it to make h the subject, we find that $h = \sqrt[3]{\frac{6V\tan\theta}{\pi(\cot(\theta-\alpha)+\cot(\theta+\alpha))}}$. This equation also verifies equation(3) when we set the incline angle α to 0.

2.4 Constraints

Experimentation (figure 1) showed that the top of the sand piles were not a perfect conical point, but rather a rounded top. The impact of this is insignificant for larger piles but nevertheless introduces an error in our cone model. Further investigation could be conducted to find out if there is a shape which fits what was observed.

3 Volume of Sand in a Bucket?

In a cylindrical bucket with a circular hole in the center, sand will drain out through that hole until the slope of the remaining sand in the bucket is equal to the angle of repose everywhere. How can we determine the volume of that sand left in the bucket? The space of the air in the bucket will form the frustum an upside down cone. Therefore, by finding the volume of that upside-down cone and subtracting it from the volume of the total cylinder shape, we can find the volume of the sand. Alternatively, we can rotate a straight line representing the slope of the sand around the axis that goes through the centre of the hole, thus forming a solid of revolution with the volume of the sand. Another alternative would be to find the total volume with a double integral, which is the most general solution.



(a) Cross sectional view of a bucket with a hole in its centre

(b) Oblique view of the bucket, where $\beta = \theta$

Figure 5: A bucket of sand has the same angle of repose as a sand pile

3.1 Volume of revolution method

If we draw a Cartesian plane on Figure 5 above, with the x-axis being the bottom of the bucket and the y-axis going through the centre of the hole, then we can draw a straight line from the bottom to the side of the bucket wall, and rotate it around the y axis to find the volume.
The x-axis intercept of the line will be the radius(r) of the hole itself, and the slope(m) is derived from its friction coefficient (μ).

$$\begin{aligned}
m &= \frac{rise}{run} = \tan\theta = \mu \\
y &= mx + c \\
\implies 0 &= (\mu)\, r + c \\
\implies c &= -r\mu \\
\text{therefore } y &= \mu x - r\mu \\
x &= \frac{y + r\mu}{\mu}
\end{aligned}$$

Now we integrate to find the volume of this solid of revolution, which will be the volume of the frustum of the air cone inside the bucket, therefore we need to subtract that from the total volume of the bucket to find the volume of sand. Our terminals will be from the height of the hole (0) to the maximum height of the sand in the bucket($\mu R - \mu r$).

$$\begin{aligned}
V &= \pi R^2(\mu R - \mu r) - \int_0^{\mu R - \mu r} \pi x^2 \, dy \\
&= \pi R^2(\mu R - \mu r) - \int_0^{\mu R - \mu r} \pi \left(\frac{y}{\mu} + r\right)^2 dy \\
&= \pi R^2(\mu R - \mu r) - \pi \left[\frac{\mu \left(\frac{y}{\mu} + r\right)^3}{3}\right]_0^{\mu R - \mu r} \\
&= \pi R^2(\mu R - \mu r) - \frac{1}{3}\pi\mu\left(R^3 - r^3\right) \\
&= \frac{\pi\mu\left(r^3 - 3R^2 r + 2R^3\right)}{3} \qquad (5)
\end{aligned}$$

This equation is neat, but only applicable if we have a centre of rotation within the bucket. That is, **only centred, circular holes can have their sand volumes calculated with this method**. We need a more general method for both off-centre holes and multiple holes.

3.2 Air cone volume method

As we mentioned at the start, the volume of the sand is the difference between the total volume of the bucket and the volume of the air. The air volume is in the shape of a cone, and can be calculated even if the hole is off-centre.

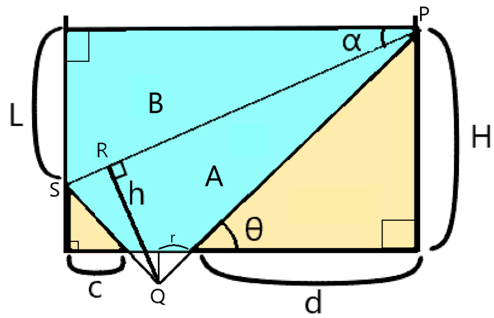


Figure 6: Cross sectional view of a bucket with a hole offset

Volume B is simply half the volume of a whole cylinder. Volume A is a slanted cone with a missing top, these shapes are already familiar to us from section 2.3.

$$\begin{aligned} V_B &= \frac{1}{2}\pi R^2 L \\ L &= H - c\tan\theta, \text{where } H = d\tan\theta \\ &= d\tan\theta - c\tan\theta, \text{where } \mu = \tan\theta \\ &= \mu\,(d-c) \\ \text{therefore } V_B &= \frac{1}{2}\pi R^2 \mu\,(d-c) \end{aligned}$$

From figure 6 we can see that V_A is composed of a slanted cone without the top. The top itself is another cone with the hole as its base, and therefore has the volume $\frac{\pi}{3}r^3\tan\theta$

$$\begin{aligned} V_A &= \frac{\pi}{3}abh - \frac{\pi}{3}r^3\tan\theta \\ b &= R \\ a &= \frac{1}{2}\sqrt{L^2 + (2R)^2} \\ &= \frac{1}{2}\sqrt{(\mu\,(d-c))^2 + (2R)^2} \end{aligned}$$

Note that h is the opposite side to $\angle RPQ$, therefore $h = |PQ|\sin(\angle RPQ)$. From the Pythagorean theorem, we can see that $|PQ| = \sqrt{(d+r)^2 + (H + r\tan\theta)^2}$. We can also see that $\angle RPQ = \theta - \alpha$

$$\begin{aligned}
h &= \sqrt{(d+r)^2 + (H + r\tan\theta)^2}\,(\sin(\theta-\alpha)) \\
&= \sqrt{(d+r)^2 + (d\tan\theta + r\tan\theta)^2}\,(\sin(\theta-\alpha)) \\
&= \sqrt{(1+\mu^2)(d+r)^2}\,(\sin(\theta-\alpha)) \\
&= \sqrt{(1+\mu^2)}\,(d+r)\sin(\theta-\alpha)
\end{aligned}$$

$$\begin{aligned}
\text{therefore } V_A &= \frac{\pi}{3}\left(\frac{1}{2}\sqrt{(\mu(d-c))^2 + (2R)^2}R\sqrt{(1+\mu^2)}\,(d+r)\sin(\theta-\alpha)\right) - \frac{\pi}{3}r^3\mu \\
&= \frac{\pi}{6}R\sqrt{(\mu(d-c))^2 + (2R)^2}\sqrt{(1+\mu^2)}\,(d+r)\sin(\theta-\alpha) - \frac{\pi}{3}r^3\mu
\end{aligned}$$

Therefore

$$\begin{aligned}
V_{sand} &= \pi R^2 H - (V_A + V_B) \\
&= \pi R^2\mu d - \frac{\pi}{6}R\sqrt{(\mu(d-c))^2 + (2R)^2}\sqrt{(1+\mu^2)}\,(d+r)\sin(\theta-\alpha) \\
&+ \frac{\pi}{3}r^3\mu - \frac{1}{2}\pi R^2\mu(d-c) \\
&= \pi\left(R^2\mu\left(\frac{d+c}{2}\right) - \frac{R\sqrt{(\mu(d-c))^2 + (2R)^2}\sqrt{(1+\mu^2)}\,(d+r)\sin(\theta-\alpha)}{6} + \frac{r^3\mu}{3}\right)
\end{aligned}$$

This equation allows us to calculate the volume of sand for off-centre, circular holes. However, this method drastically increases in difficulty when dealing with multiple holes, as it needs us to calculate the volume overlap of 2 or more conic sections. This is technically possible but not at all worth the effort considering there is a more general solution.

3.3 Double integration method

Let us consider a bird's eye view of the bucket, which is just a circle. In the case of 1 single, centred hole, the position of it is at the origin (0,0). The further away a point is from the hole , the taller the sand is at that particular position. Therefore we get the expression $H \propto \sqrt{x^2+y^2}$, where H is the height of the sand at any given position. Therefore, if we integrate H with respect to x then with respect to y, we will have the total volume of the sand.

$$V = \int\int H\,dx\,dy \tag{6}$$

This however does not account for the size of the hole. If the hole does have a size, then the integral overestimates the volume.

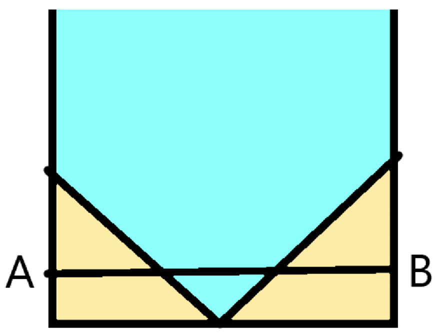


Figure 7: Equation 6 overestimates the volume of sand by the amount of sand under line AB

As we can see in figure 7 above, we can correct this by subtracting the volume of the cylinder under line AB, then add back the volume of the air cone under line AB. We know that the height of the cylinder and the cone is related to the radius of the hole and the angle of repose, therefore we can find the volume of the cylinder to be $V = \pi R^2 r\mu$ (recall that R is the radius of the bucket, r is the radius of the hole and $\mu = \tan\theta$. We can also calculate the volume of the air cone to be $V = \frac{\pi}{3}(r^2)(r\mu)$. Therefore, the equation for the volume of sand in a cylindrical bucket with a circular hole would be:

$$\begin{aligned} V &= \int\int H\,dx\,dy - \pi R^2 r\mu + \frac{\pi r^3 \mu}{3} \\ h &= \tan\theta\sqrt{x^2+y^2} = \mu\sqrt{x^2+y^2} \\ \text{therefore } V &= \int\int \mu\sqrt{x^2+y^2}\,dx\,dy + \pi r\mu\left(\frac{r^2}{3} - R^2\right) \end{aligned}$$

The beauty of this equation is that it can calculate the volume of sand for almost any shaped container just by changing the terminals of the integral. For example, a centred circular hole on a square bucket would be calculated as such:

$$V = \int_{-y}^{y}\int_{-x}^{x} \mu\sqrt{x^2+y^2}\,dx\,dy$$

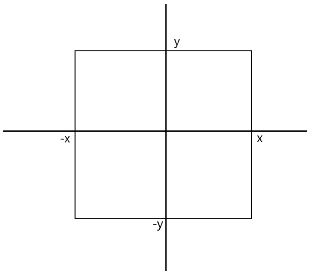


Figure 8: Bird's eye view of a square bucket with the origin at a hole in the centre

This is just an example of what changing the terminals to fit the shape looks like, we will now change the integral to fit a circular bucket. The integral above is left as an exercise for the reader.
For a cylindrical bucket, the top down view would be a circle with the equation $x^2+y^2 = R^2$. Therefore the terminals for the dx integral would be $x = \pm\sqrt{R^2-y^2}$.

$$\begin{aligned} V &= \int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \mu h\,dx\,dy - \pi R^2 r\mu + \frac{\pi r^3\mu}{3} \\ &= \int_{-R}^{R} \mu\left[\frac{y^2\sinh^{-1}\left(\frac{x}{|y|}\right) + x\sqrt{x^2+y^2}}{2}\right]_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dy + \pi r\mu\left(\frac{r^2}{3} - R^2\right) \\ &= \int_{-R}^{R} \mu\left(y^2\sinh^{-1}\left(\frac{\sqrt{R^2-y^2}}{y}\right) + R\sqrt{R^2-y^2}\right) dy + \pi r\mu\left(\frac{r^2}{3} - R^2\right) \\ &= \frac{2\mu\pi R^3}{3} + \pi r\mu\left(\frac{r^2}{3} - R^2\right) \end{aligned}$$

To solve for an off-centre hole, we can modify the equation for H and translate it such that $\sqrt{x^2+y^2} \to \sqrt{(x-x_1)^2+(y-y_1)^2}$, where (x_1, y_1) are the coordinates of the hole from a bird's eye view. Alternatively, terminals of the integral can be modified to account for a circular hole in any location. $[-R, R] \to [-R+y_1, R+y_1]$ and $\left[-\sqrt{R^2-y^2}, \sqrt{R^2-y^2}\right] \to$

$\left[-\sqrt{R^2-(y-y_1)^2}+x_1, \sqrt{R^2-(y-y_1)^2}+x_1\right]$ where (x_1, y_1) are the Cartesian coordinates of the hole.

$$V = \int_{-R+y_1}^{R+y_1} \int_{-\sqrt{R^2-(y-y_1)^2}+x_1}^{\sqrt{R^2-(y-y_1)^2}+x_1} \mu\sqrt{(x)^2+(y)^2}\,dx\,dy + \pi r\mu\left(\frac{r^2}{3}-R^2\right)$$

However, this integral is quite long when evaluated. Instead we can exploit the rotational symmetry of a cylindrical bucket by rotating the bucket until the Cartesian coordinates have no x-value.

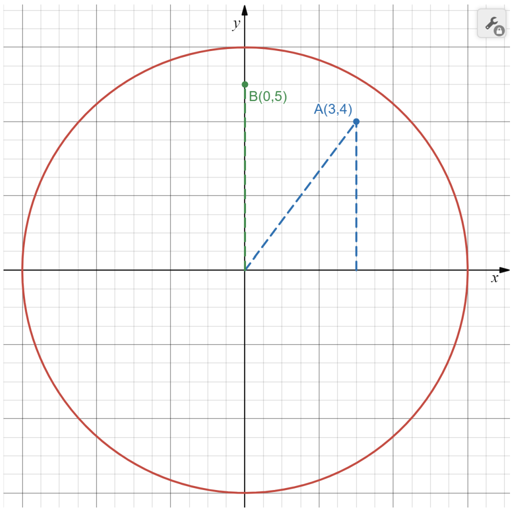


Figure 9: Bird's eye view of cylindrical bucket. The hole at point A gives the same volume as a hole at point B

Applying this technique, we can get rid of the translation in the x-axis, simplifying the integral.

$$\begin{aligned} V &= \int_{-R}^{R} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \mu\sqrt{x^2+(y-y_1)^2}\,dx\,dy + \pi r\mu\left(\frac{r^2}{3}-R^2\right) \\ &= \int_{-R}^{R} \left[\frac{(y-y_1)^2\sinh^{-1}\left(\frac{x}{|y-y_1|}\right)}{2} + \frac{x\sqrt{x^2+(y-y_1)^2}}{2}\right]_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dy + \pi r\mu\left(\frac{r^2}{3}-R^2\right) \\ &= \int_{-R}^{R} \left(y^2-2y_1y+y_1^2\right)\sinh^{-1}\left(\frac{\sqrt{R-y^2}}{y-y_1}\right) + \\ &\sqrt{2y_1y^3+\left(-y_1^2-R\right)y^2-2Ry_1y+Ry_1^2+R^2}\,dy + \pi r\mu\left(\frac{r^2}{3}-R^2\right) \end{aligned}$$

Unfortunately, the integral above has no analytic solution, and so we can only calculate approximate areas with that method. This method is only really worth doing for multiple holes.

3.4 Double integral method for multiple holes of the same size

Let there be 2 fixed points, A and B, anywhere on the Cartesian plane. And let there be another point C, how do we determine which point C is closest to? We simply draw a line segment AB, and then draw its perpendicular bisector. The bisector will divide the space in 2, the any point on the A side of the space will be closer to point A and any point on the B side of the space will be closer to point B.

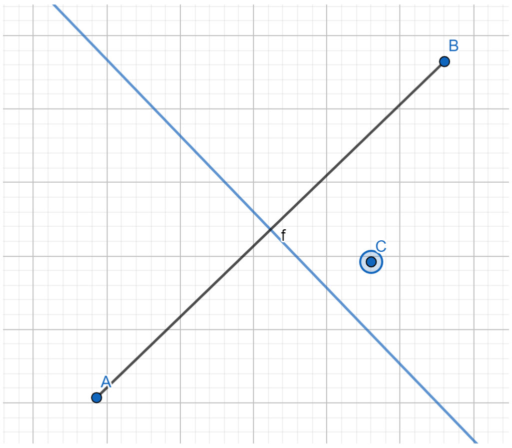


Figure 10: 2 points A and B with a perpendicular bisector between them, in this case C is shown to be closer to B

This may seem obvious, but this also works when we extend this example to multiple points. For example, in the case of 4 holes, we shall connect every single point together and draw the perpendicular bisector for every connection. Then go through each individual divided area systematically to determine which point is closest. For n points, the expression for the number of connecting lines drawn would be $\sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1)$. How does this relate to holes in buckets? Well, if each of those points were a hole, then would be able to solve for the volume of sand by first determining which areas are closest to which holes, then using the double integral for each hole in their specific areas, add all the volumes up to find the total volume of sand in a bucket with multiple holes.

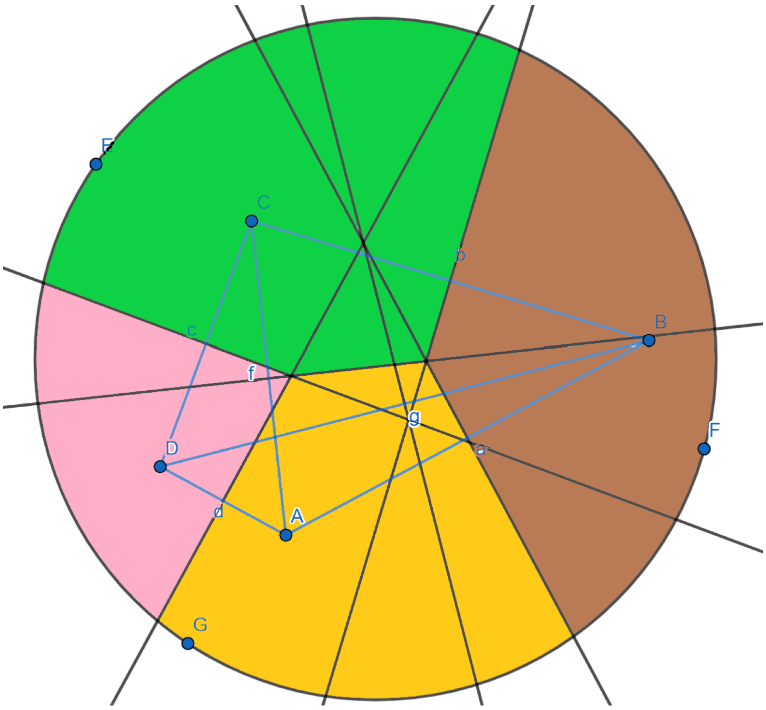


Figure 11: 4 points A, B, C, D. And a colour-coded breakdown of which areas are closest to each point in a circle. I.e. the green area is closest to C, the pink is closest to D, etc. The lines in blue connect every point to each other and the lines in black are their perpendicular bisectors

For each area around each hole, we will further divide it into easily solvable sections.

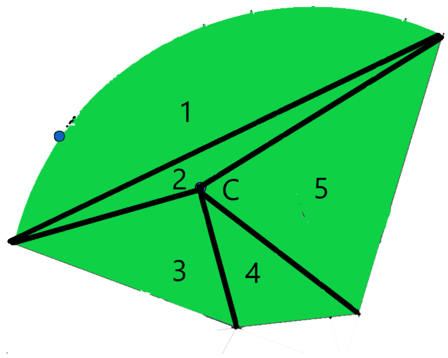


Figure 12: The area around point C can be split into 5 sections, 4 triangles and 1 minor segment

Focusing on section 3 (see figure 12), let us draw a Cartesian plane such that the origin is at the hole (point B) and the y axis is perpendicular to the side opposite the origin. The terminals of the x-axis integral would be equation the 2 straight lines AB and BC. These equations can be found as long as we have the distance between the holes and the position of each of the holes in the bucket, for the purpose of conciseness, we will leave the equations of lines AB and BC as $y_1 = m_1x_1$ and $y_2 = m_2x_2$ respectively. Therefore, the integral terminals would be $\left[\frac{y_2}{m_2}, \frac{y_1}{m_1}\right]$. As for the y-axis integral terminal, it will be from 0 to the y-value of the intersection between AC and the y-axis, let's call that value L. Therefore the y-axis terminals would be $[0, L]$. Therefore the total volume of sand enclosed by section 3 would be given by the integral:

$$\begin{aligned} V &= \int_0^L \int_{\frac{y_2}{m_2}}^{\frac{y_1}{m_1}} \mu h \, dx \, dy \\ &= \int_0^L \int_{\frac{y_2}{m_2}}^{\frac{y_1}{m_1}} \mu \sqrt{x^2 + y^2} \, dx \, dy \end{aligned}$$

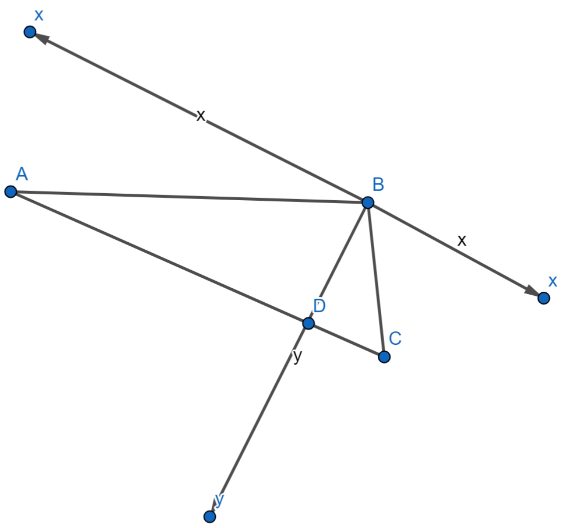


Figure 13: A close-up of section 3

As for the minor segment (Section 1), we can apply the same method, except our origin will be at the hole in section 1 and our x-axis will be parallel to the line AC. This leaves H to be $\sqrt{(x-x_1)^2+(y-y_1)^2}$, where x_1 is the distance from the midpoint of AC to point E and y_1 is the negative y-value of the centre of the bucket with regard to the hole at point B. And the y-axis terminals to be $[L_1, L_2]$, where L_1 is the distance $BE + y_1$ and L_2 is the distance $DE + y_1$.

$$\begin{aligned} V &= \int_{L_1+y_1}^{L_2+y_1} \int_{-\sqrt{R^2-y^2}+x_1}^{\sqrt{R^2-y^2}+x_1} \mu h \, dx \, dy \\ &= \int_{L_1+y_1}^{L_2+y_1} \int_{-\sqrt{R^2-y^2}+x_1}^{\sqrt{R^2-y^2}+x_1} \mu \sqrt{x^2+y^2} \, dx \, dy \end{aligned}$$

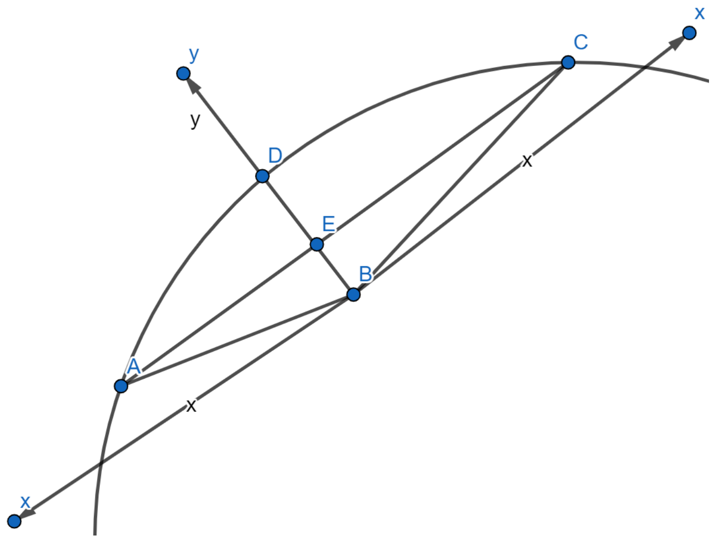


Figure 14: A close-up of section 1 (enclosed by ADCE)

With these 2 equations we can calculate the total volume of sand in the green section of Figure 11 (A sort of "zone of influence" of hole C). To account for the volume of the holes, we simply use the same method see in section 3.3. Subtract the overall volume of the overestimated cylinder $V = \pi R^2 r \mu$ then add back all the air cones produced by the holes. This only works with holes of identical size, which raises the problem of dealing with multiple, different sized holes.

3.5 Multiple holes of different sizes

To account for a different hole size, let us look back at figure 11. A bigger hole means a larger "zone of influence" since the perpendicular bisectors highlight regions of maximum sand height. The bisectors are simply translated in the direction of the smaller hole by half the difference in radii. From then on, we can apply the same steps in section 3.4 to find the volume of each sector independently.

However, recall that the double integral alone does not factor in the size of the hole. Recall from section 3.3 that to compensate for the size of the hole, we need to subtract the volume of the overall section, then add back the volume of the air cone. This cone still has the volume $V = \frac{\pi}{3} r^3 \mu$, but the overall volume subtracted is no longer a cylinder shape. Instead it is found by the area of the section times the height of the cone. I.E. to compensate for hole size, find the sum of the areas 1-5 in figure 12, then multiply by the height of the cone, which is given by $r\mu$. Do this for each individual area marked out by a hole.

3.6 Holes of different shapes

As we have seen in the previous sections, the double integral method can account for different bucket shapes through changing the integral terminals. But what about different shaped holes? Differently shaped holes would make the equation for the height of sand more complicated, as it is no longer solely dependant on the distance from the hole.
For holes in the shape of a regular polygon, we need to find the expression for H. Let us redefine r such that it is the distance **from the centre of a regular polygon to one of its vertices**, and let n be the number of sides of the polygon. Noting that H is proportional to the distance from the edge of the hole to (x_1, y_1), we begin by plotting the n-sided regular polygon; this will allow us to use the distance from the origin to (x_1, y_1), subtract the distance r from origin to the edge of the hole (n-sided regular polygon), and work some trigonometric magic to find H.

To plot the polygon, we will utilize polar coordinates, let $p(\theta)$ be the equation for a polygon: To find $p(\theta)$, consider an n-sided regular polygon with radius (distance from centre to vertex) r. The shape can be split up into n parts, each with internal angle $\frac{2\pi}{n}$. Splitting up each part into 2 equal halves, we get a right angled triangle with internal angle $\frac{\pi}{n}$, the adjacent length being the apothem a (the shortest distance from the centre to one of the sides), the hypotenuse being r, and the opposite length being half the side length of the hole.

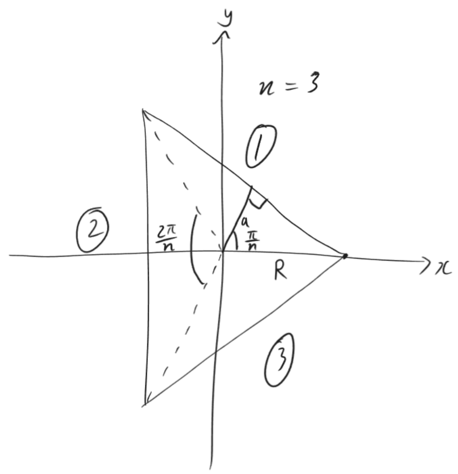


Figure 15: Triangle: n=3, so it has 3 parts

Therefore:

$$\cos\left(\frac{\pi}{n}\right) = \frac{a}{r} \text{ and } \cos\left(\frac{\pi}{n} - \theta\right) = \frac{a}{p\left(\theta\right)},\ 0 \leq \theta < \frac{2\pi}{n}$$

since $p\left(\theta\right)$ is the distance from the centre to the edge of the hole. Dividing these two equations, we get:

$$p\left(\theta\right) = r\frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n} - \theta\right)},\ 0 \leq \theta < \frac{2\pi}{n}$$

Note that this only plots 1 part of the polygon, where θ is in the 1st part (i.e $0 \leq \theta < \frac{2\pi}{n}$). Denote this $\theta = \theta_1$. Since the polygon repeats every $\frac{2\pi}{n}$, we must find a way to map all other values of θ to θ_1 (i.e $0 \leq \theta < \frac{2\pi}{n}$):

$$\theta_1 = \theta - k \cdot \frac{2\pi}{n},\ k \in \mathbb{Z}$$

Notice that θ_1 is in the first 'part' of the polygon, so $k = 0$. If θ is in the 2nd part, $k = 1$, and if θ is in the ith part, $k = i - 1$. Therefore we can say that if θ is in the ith part,

$$i - 1 \leq \frac{\theta}{\left(\frac{2\pi}{n}\right)} < i$$

$$k \leq \frac{\theta}{\left(\frac{2\pi}{n}\right)} < k+1$$

$$k = \left\lfloor \frac{n\theta}{2\pi} \right\rfloor$$

Thus:

$$\theta_1 = \theta - \frac{2\pi \left\lfloor \frac{n\theta}{2\pi} \right\rfloor}{n}$$

Plugging this into our original equation for $p\left(\theta\right)$, we get:

$$p\left(\theta\right) = r \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n} - \theta + \frac{2\pi\left\lfloor \frac{n\theta}{2\pi} \right\rfloor}{n}\right)}$$

for the equation of an n-sided regular polygon. Converting the polar coordinates to cartesian, we get:

$$p\left(x, y\right) = r \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n} - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor \frac{n\arctan\left(\frac{y}{x}\right)}{2\pi} \right\rfloor}{n}\right)}$$

However, H is proportional to the the distance from (x, y) to the edge of the hole, not to the origin. Therefore we must find the angle made by the line from $(0, 0)$ to (x, y) and the outer edge of the polygon to find the component of $\sqrt{x^2 + y^2} - p\left(x, y\right)$ that is perpendicular to the edge of the hole. Denote this angle α.

From the parts of the polygon demonstrated earlier being isosceles, we find that half its vertex angle is:

$$\frac{\pi - \frac{2\pi}{n}}{2} = \frac{\pi\left(n-2\right)}{2n}$$

Since vertically opposite angles are equal, we find:

$$\alpha = \pi - \theta - \frac{\pi\left(n-2\right)}{2n}$$

However, due to possible asymmetries in $p(\theta)$, we must once again re-calibrate θ to the first part of the polygon. This can be achieved via the same method as above:

$$\alpha = \pi - \theta + \frac{2\pi\left\lfloor \frac{n\theta}{2\pi} \right\rfloor}{n} - \frac{\pi\left(n-2\right)}{2n}$$

$$\alpha = \pi - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor \frac{n\arctan\left(\frac{y}{x}\right)}{2\pi} \right\rfloor}{n} - \frac{\pi\left(n-2\right)}{2n}$$

Therefore, the perpendicular distance from the edge of the hole to (x,y) is:

$$\sin\left(\alpha\right)\left(\sqrt{x^2 + y^2} - p\left(x, y\right)\right)$$

and thus

$$H\left(x, y\right) = \mu \sin\left(\alpha\right)\left(\sqrt{x^2 + y^2} - p\left(x, y\right)\right)$$

$$H(x,y) = \mu \sin\left(\pi - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor\frac{n\arctan\left(\frac{y}{x}\right)}{2\pi}\right\rfloor}{n} - \frac{\pi(n-2)}{2n}\right) * \left(\sqrt{x^2+y^2} - r\frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n} - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor\frac{n\arctan\left(\frac{y}{x}\right)}{2\pi}\right\rfloor}{n}\right)}\right)$$

Now that we have a solution for H, we can use it in the good old double integral.

$$V = \int\int H\,dx\,dy$$

If we recall from the previous section, we need to add some terms to this integral to account for the hole size. Since the function for H already accounts for the dimensions of the hole, we only need to correct the height summed by the integral when (x_1, y_1) is within the hole itself. In such cases, negative height is added by the integral, resulting in a negative volume. Since the magnitude of the height is relative to its distance from its respective edge, the negative volume is an upside-down pyramid with the hole as its base (though it is reflected along the y-axis in relation to the hole due to the term $\sqrt{x^2+y^2} - r\left(\arctan\left(\frac{y}{x}\right)\right)$ in $H(x,y)$). To rectify this, we must add the volume a pyramid with an n-sided regular polygon base, radius r, apothem a, side length s, and height $h = \mu a$. This is given by:

$$V_{correction} = \frac{n}{12}\cdot\mu a\cdot s^2\cdot\cot\left(\frac{\pi}{n}\right)$$

$$V_{correction} = \frac{n}{12}\cdot\mu r\cos\left(\frac{\pi}{n}\right)\cdot\left(2\cdot r\sin\left(\frac{\pi}{n}\right)\right)^2\cdot\cot\left(\frac{\pi}{n}\right)$$

$$V_{correction} = \frac{n}{3}\mu r_1^3\cos^2\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)$$

or:

$$V_{correction} = \frac{1}{3}Aa\mu$$

$$V_{correction} = \frac{A\mu\cos\left(\frac{\pi}{n}\right)}{3}\sqrt{\frac{A}{n\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}}$$

Where A is the area of the hole.

Therefore the total volume of sand in a bucket with a regular polygon hole of n sides in the middle of the circular bucket is:

$$V = \int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} H\,dx\,dy + V_{correction}$$

3.7 Maximum and minimum volume of sand?

If the total area of holes is kept fixed, is it possible to determine an arrangement of holes that leads to maximum and minimum volumes of stationary sand stored? We will split this question into three.

1. Does more holes result in less volume? (Assuming constant total area)
2. At which positions do holes yield the maximum and minimum volumes of sand.
3. Which shapes give the maximum and minimum volume?

Lets first just compare 2 holes together in this special case:
Total hole area: $2\pi\, m^2$, Number of holes: 2, Hole radius: $1\, m$, Bucket radius: $5\, m$, Material friction coefficient: 1, Hole coordinates: co-linear with centre of bucket, $2\, m$ either side of the centre.

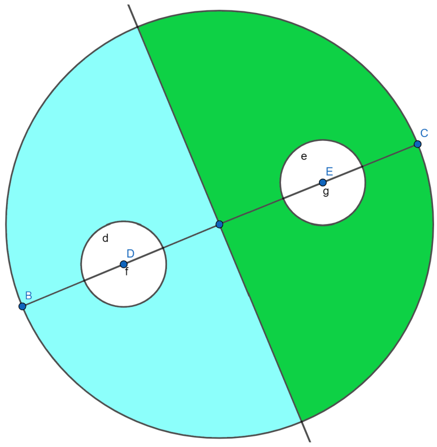


Figure 16: A bird's eye view of the bucket with holes described above

Since both sides are identical then we can just solve for 1 side and double the result.

$$
\begin{aligned}
V &= 2\int_0^5 \int_{-\sqrt{5^2-y^2}}^{\sqrt{5^2-y^2}} \mu H \, dx\, dy - \pi R^2 (r\mu) + \frac{2\pi}{3} r^2 (r\mu) \\
&= 2\int_0^5 \int_{-\sqrt{5^2-y^2}}^{\sqrt{5^2-y^2}} \sqrt{x^2 + (y-2)^2} \, dx\, dy - \pi R^2 r\mu + \frac{2\pi}{3} r^3 \mu \\
&= 2\int_0^5 \left(y^2 - 4y + 4\right) \operatorname{arsinh}\left(\frac{\sqrt{25-y^2}}{y-2}\right) + \sqrt{4y^3 - 29y^2 - 100y + 725}\, dy \\
&- \pi R^2 r\mu + \frac{2\pi}{3} r^3 \mu \\
&\approx 15.244\, m^3
\end{aligned}
$$

Now lets try the same area but for only 1 hole in the middle. All properties of the bucket and sand remain the same, except the hole in the middle has a radius calculated by $\pi r^2 = 2\pi$ therefore $r = \sqrt{2}$. We can plug this in and use any of the 3 ways to find the volume, we will use equation 5:

$$
\begin{aligned}
V &= \frac{\pi\mu\left(r^3 - 3R^2 r + 2R^3\right)}{3} \\
&= \frac{\pi\left(\sqrt{2}^3 - 3(5^2)\sqrt{2} + 2(5^3)\right)}{3} \\
&\approx 153.7\, m^3
\end{aligned}
$$

Now we shall answer the second question by offsetting the single hole 1 meter radially from the centre, and apply the inverted cone equation from section 3.4. Recall that

$R = 5, r = \sqrt{2}, \mu = 1,$

$$V = \pi\left(R^2\mu\left(\frac{d+c}{2}\right) - \frac{R\sqrt{(\mu(d-c))^2 + (2R)^2}\sqrt{(1+\mu^2)}(d+r)\sin(\theta-\alpha)}{6} + \frac{r^3\mu}{3}\right)$$

$$d = 6 - \sqrt{2}$$

$$c = 4 - \sqrt{2}$$

$$\theta = \tan^{-1}\mu = \frac{\pi}{4}$$

$$\alpha = \tan^{-1}\left(\frac{\mu(d-c)}{2R}\right) \approx 0.197$$

$$V \approx 158.9\, m^3$$

In fact, we can use the equation above to find out a more general solution. Since $d + c + 2r = 2R$ by definition (Going back to figure 6), we can substitute $d = 2R - 2r - c$ to get a function $V(c)$, since all other values are constants. Then to find the extrema (either maximum or minimum) of the volume by varying c, we evaluate

$$\frac{dV}{dc} = \frac{2\pi Rm(c + r - R)}{3} = 0$$

Therefore

$$c = R - r$$

Since $c = R - r$ only when the hole is at the centre, it means that a centred hole is an extrema. And from our previous comparison, it is a volume minima. This means that the closer to the edge the hole is, the more sand it will hold.

Now let's test the impact of different shapes on the holes. Retaining all the dimensions of the bucket and sand from before, a $2\pi\, m^2$ n-sided polygon hole would have a volume:

$$V(n) = \int_{-5}^{5}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} H\, dx\, dy + \frac{2\pi(1)\cos\left(\frac{\pi}{n}\right)}{3}\sqrt{\frac{2\pi}{n\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}}$$

$$H =$$

$$\mu\sin\left(\pi - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor\frac{n\arctan\left(\frac{y}{x}\right)}{2\pi}\right\rfloor}{n} - \frac{\pi(n-2)}{2n}\right) *$$

$$\left(\sqrt{x^2+y^2} - r\frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n} - \arctan\left(\frac{y}{x}\right) + \frac{2\pi\left\lfloor\frac{n\arctan\left(\frac{y}{x}\right)}{2\pi}\right\rfloor}{n}\right)}\right)$$

Plotting $V(n)$, we see that as $n \to \infty$, $V(n) \to 153.7\, m^3$.

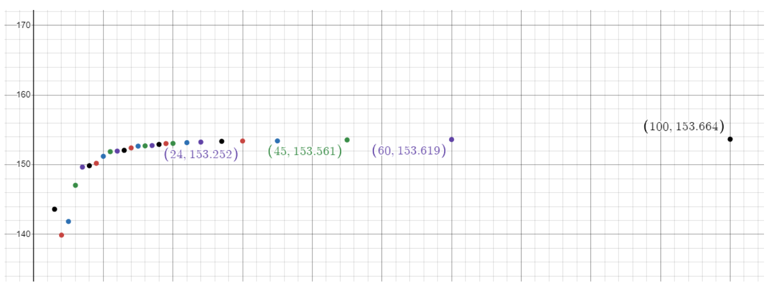


Figure 17: The graph $V(n)$ has a horizontal asymptote at $153.7\, m^3$ as $n \to \infty$

This is to be expected; at $n = \infty$, the hole becomes a circle with corresponding volume earlier shown to be $153.7\,m^3$. Thus for all values $n < \infty$, $V(n)$ is less than the volume of a circular hole with equal area.

Furthermore, the **minimum** volume is shown to occur at $n = 4$ where $V(n) = 139.9\,m^3$.

Figure 18: Volume vs n, the scatter plot $V(n)$ cannot have a graphical form as n only takes integer values greater than 2

This is also the only case in which $V(n-1) > V(n)$. At first it seems perplexing that a triangular hole doesn't follow the general trend of $V(n) < V(n+1)$, however, an intuitive understanding can be gained by considering the physical properties that allow the greatest volume of sand in the bucket. These properties are :

1. a; the smaller the apothem, the larger the distance from the edge of the bucket to its centre, thus the more space for sand to pile higher

2. n; the more sides of a polygon, the closer the ratio $r : a$ is to 1 (the holes edge is smoother) thus there are less jagged points of minimum height potential where the sharp edges of the hole are closest to the bucket's edge.

Plotting $\frac{a}{n}$ vs n and setting both its first and second derivative to 0, we see that there is a turning point at $2 < n < 3$ and a point of inflection at $3 < n < 4$. This explains why we see a minimum at $n = 4$; $n = 3$ is the only value of n where the $a : n$ ratio is increasingly decreasing, whereas from $n = 4$ and beyond, the ratio is decreasingly decreasing.

From these desk checks and derivatives we can conclude that

1. Less holes probably means more volume

2. More centred hole definitely means less volume (as shown via derivatives)

3. More sides on a polygon means more volume for a centred hole, with the exception of a triangular hole

3.8 Constraints

There is no single formula to find the volume of sand for multiple holes, and the process of finding the volume becomes extremely tedious for multiple holes. Further investigation could be conducted to find a quicker, more elegant method for the volume for multiple holes. Another area for further investigation could be a more definitive proof that multiple holes resulted in less volume.

4 Obstructions

What if there is a small obstruction on the table? We will look at rectangular prisms and see how they impact the shape of the sand pile. Let us classify 3 types of obstructions:

1. Fully submerged obstruction - An obstruction that is completely covered by sand, and is not visible from the outside.
2. Semi-submerged obstruction - A rectangular-prism shaped obstruction with its upward facing face partially covered by sand.
3. unsubmerged obstruction - A rectangular-prism shaped obstruction with its upward facing face uncovered by sand.

4.1 Fully submerged obstruction

The most straightforward one is the fully submerged obstruction, the cone shape of the sand pile remains unchanged, and so the volume of the sand pile becomes:

$$\begin{aligned}
V_{total} &= V_{sand} + V_{Obstruction} \\
\frac{\pi}{3} r^2 h &= V_{sand} + V_{Obstruction} \\
r &= \frac{h}{\tan\theta} \\
h &= \sqrt[3]{\frac{3\tan^2\theta\left(V_{sand} + V_{Obstruction}\right)}{\pi}}
\end{aligned}$$

This is exactly the same as equation 3, just with the volume of the obstruction added, this also works with any shape.

4.2 Unsubmerged obstruction

An unsubmerged obstruction can be thought of as an infinitely tall pillar marking out an area where there is no sand. If we imagine a bird's eye view, the sand cone will be a circle with its centre at the origin and the obstruction will be a rectangular area where there is no sand.

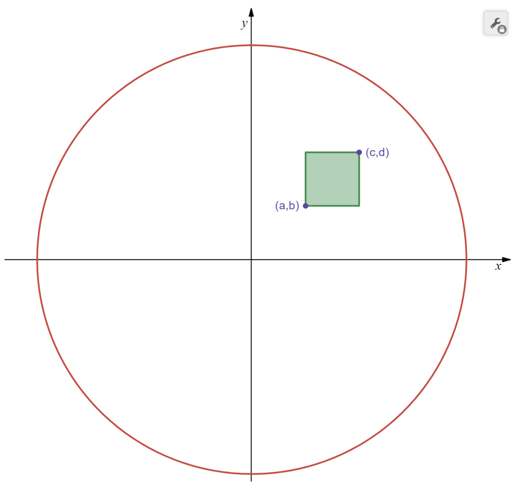


Figure 19: A bird's eye view of a sand pile cone with an obstruction marked out by the green area

This means we can reuse the double integral method from 3.3. All we need to find is an expression for H, instead of increasing with distance as in the instance of the bucket, it now decreases linearly with distance from the origin, and maximum height of the sand is equal to the height of the pile at the origin h . Therefore the expression is $H = h - \mu\sqrt{x^2 + y^2}$. The volume taken up by the obstruction is calculated by this H in double integral with the terminals $[a, c]$ for the x-axis and $[b, d]$ for the y-axis (refer to figure 17).

$$\begin{aligned} V_{sand} &= Vcone - V_{obstruction} \\ &= \frac{\pi}{3}r^2h - \int_b^d \int_a^c H\,dx\,dy \\ &= \frac{\pi h^3}{3\mu^2} - \int_b^d \int_a^c h - \mu\sqrt{x^2+y^2}\,dx\,dy \end{aligned} \tag{7}$$

4.3 Semi-submerged obstruction

When sand is poured around such an obstruction, it will eventually fill around it and leave a corner sticking out the top. We can utilise equation 7,and adding back the volume of sand directly on top of the obstruction. Since, in the case of a rectangular prism, the top face of the obstruction will be parallel with the ground, the shape traced out by the sand on the obstruction will be the segment of a circle (As all horizontal cross sections of a cone reveal a circle), let the radius of this smaller circle be p. Therefore the volume of sand on top of the obstruction can be calculated with the double integral formula with the x-axis terminals $\left[a, \sqrt{p^2 - y^2}\right]$ and the y-axis terminals $[b, d]$. The height of the sand H also needs to be decreased by the height of the obstruction, which we will call L to avoid confusion.

$$\begin{aligned} V_{sand} &= V_{cone} - V_{obstruction} + V_{top} \\ &= \frac{\pi}{3}r^2h - \int_b^d \int_a^c H\,dx\,dy + \int_b^d \int_a^{\sqrt{p^2-y^2}} H - L\,dx\,dy \\ &= \frac{\pi h^3}{3\mu^2} - \int_b^d \int_a^c h - \mu\sqrt{x^2+y^2}\,dx\,dy + \int_b^d \int_a^{\sqrt{p^2-y^2}} h - \mu\sqrt{x^2+y^2} - L\,dx\,dy \end{aligned} \tag{8}$$

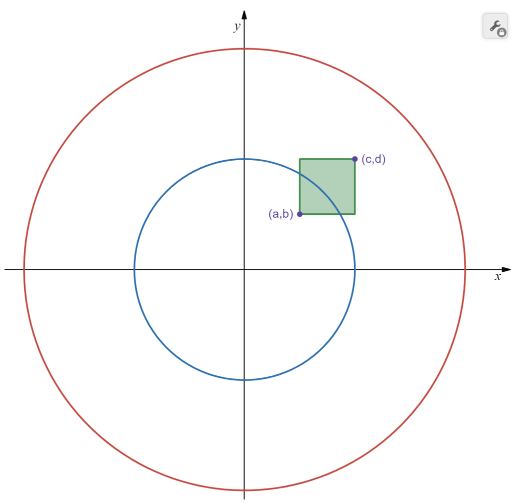


Figure 20: The blue circle represents the circle around the cone at the maximum height of the prism with radius p

4.4 Constraints

Unfortunately, the 2 equations listed for semi-submerged and unsubmerged obstructions are not accurate for cases where the obstructions are near the edge of the sand pile. As experimentation has shown, the grains do not wrap entirely around the obstacle, instead forming a depression behind it. Further investigation could be conducted to better account for the volume of the pile in such cases, such as finding a shape which matches the shape of the depression, perhaps 2 half-cones.

Figure 21: An obstruction leaves a depression in its wake