Chapter 7
Data Envelopment Analysis

7.1 Introduction

The outcomes from the previous chapters provide useful information from the literature of operations research and economics on measuring the performance of a set of homogenous firms with multiple input factors and multiple output factors as well as ranking and benchmarking firms. If firms are not homogenous, the situation is the same as when each factor has a different unit of measurement from one firm to another, and therefore, no meaningful discrimination can be expressed, unless the simple conditions of discrimination, which are represented in Sect. 2.3, are satisfied. In this chapter, the concepts introduced in the previous chapters are adapted with the literature and the philosophical background is discussed.

7.2 Reestablishing the Introduced Phrases

In the previous chapters, several phrases are repeatedly used, such as, 'doing the job well', 'doing the well job' and 'doing the useful job'. Since different purposes of discrimination in real-life applications may introduce different meanings for these concepts, each concept should transparently be defined according to the purpose of discrimination to avoid causing any doubts of confusion in research, findings, statements, and so on. In other words, any confusion or misinterpretation about these concepts and phrases which are misleading, cause awkward outcomes, and unfair decision making.

In this book, the purpose of discrimination for a set of homogenous firms (factories, organization, divisions and so on), in which each firm has multiple input factors and multiple output factors, is to find the firms which have lesser values of the input factors and greater values of the output factors. For such an aim,

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the ratio of a linear combination of the output factors to a linear combination of the input factors is introduced as the measure for the purpose of discrimination. Therefore, the weight/price/worth of each factor is required to introduce the linear combination of the factors and allow measuring the performances of the firms, according to Types 1–6. Since the firms are homogenous, the weight/price/worth of each factor should not be varied from one firm to another, unless all differences known, and multiplied to the values of the factors before the evaluation, as illustrated in Sects. 2.3 and 4.3.

The concepts of ‘doing the job right’, ‘doing the job well’, ‘doing the useful job’ and so on, were similarly introduced in the literature of economics and operations research, but differently interpreted with several words and phrases in recent decades, such as, *efficiency*, *technical efficiency*, *price efficiency*, *productive efficiency*, *relative efficiency*, *economic efficiency*, *allocative efficiency*, *overall efficiency*, *productivity*, and so on.

Philosophically, we should avoid using several phrases for a concept, and should cautiously clarify whether the relationship between that concept and what we express is meaningful. For instance, the word ‘efficiency’ is commonly used instead of the phrases ‘technical efficiency’ and ‘relative efficiency’. If these phrases illustrate the same concept, the word ‘efficiency’ should be enough to mention that concept and any further terminology is redundant and misleading.

The same criticism can be illustrated for the phrases ‘price efficiency’, ‘overall efficiency’, ‘productive efficiency’ and so on.

According to the Cambridge English dictionary, the word ‘efficiency’ means “the condition or fact of producing the results you want without waste, or a particular way in which this is done”, the phrase ‘technical efficiency’ means “a situation in which a company or a particular machine produces the largest possible number of goods with the time, materials, labor, etc. that are available”, and the word ‘relative’ means “as judged or measured in comparison with something else”. Even from the literal definition, we can see the terms are clearly different in meaning. Therefore, it is vital to review the meaning of these phrases in the literature of economics, engineering and operations research, and reintroduce them with industry-wide accuracy and understanding.

7.2.1 The Technical Efficiency Measurement

Suppose that there are several homogenous firms which each firm uses a set of input factors to produce a set of output factors. In the literature of *the production theory*, a *Production Possibility Set* (PPS) is a set of all possible situations which a set of output factors can be produced from a set of input factors. A *production function* (*production frontier*) is also a function that gives the maximum possible values of the output factors from the values of the input factors. The points on the production function are called the *technically efficient* points, and this definition is matched to the literal definition as well. None of the coordinates of the technically efficient points can be improved without worsening another coordinate, that is, none of the values of the input (output) factors can be decreased (increased) without increasing the value of another input factor or decreasing the value of another output factor. A point which is in the PPS, and does not lie on the production function is called *technically inefficient*. When a point is technically inefficient, at least one of its input or output factors can be improved to reach the production function in order to be technically efficient. In other words, a technical inefficient point is dominated by one technical efficient point at least.

Figure 7.1 depicts the production function for a set of six firms, labeled A-F, which each firm has two input factors to produce a single constant output factor, as well as the related PPS and the technically efficient and inefficient firms.

The horizontal axis in Fig. 7.1 represents the values of the first input factor per unit of the output factor and the vertical axis represents the values of the second input factor per unit of the output factor. The curve SS' in Fig. 7.1 is called the production function and the above area of the curve is the related PPS.

The firms A-D which lie on the production function are technically efficient and the firms E and F, which are inside the PPS, are technically inefficient. Firms A-C wholly dominate E; for instance, E and C used the same value of the first input factor, but E used the greater value of the second input factor, and for this reason E is technically inefficient. In other words, the points which are inside the PPS can be

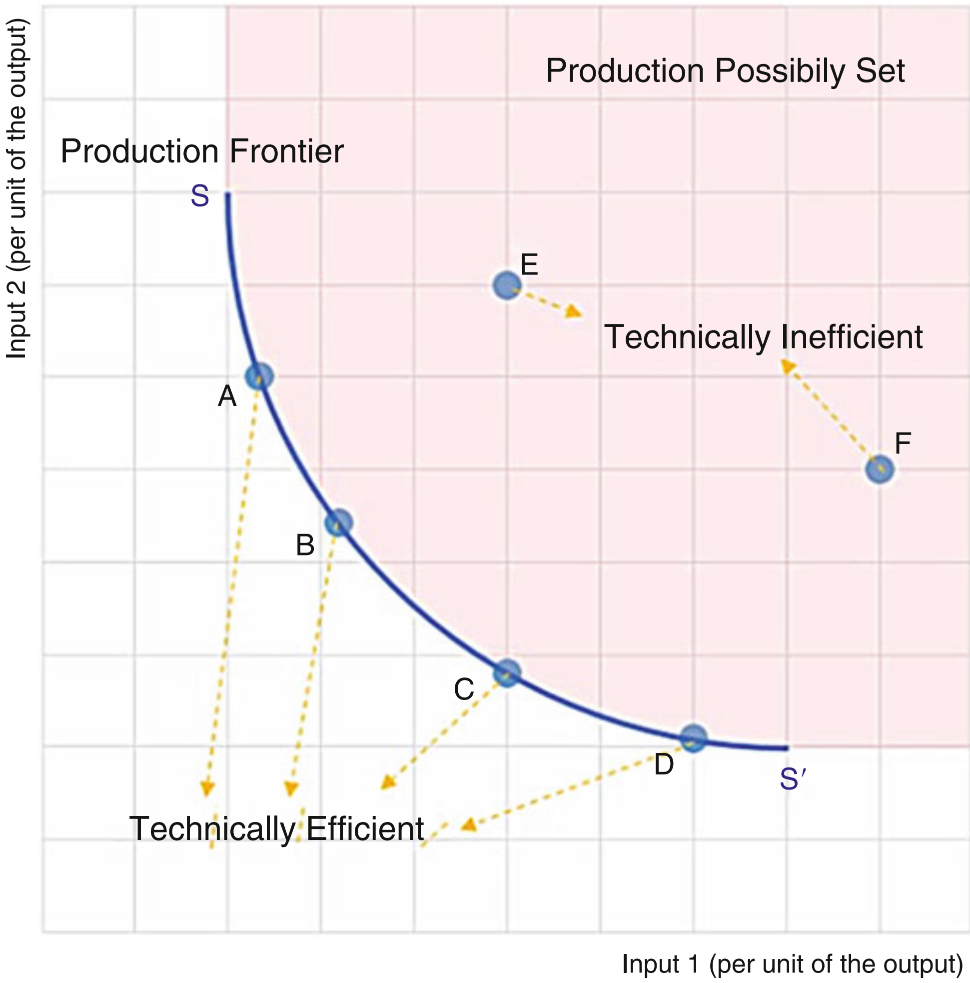

Fig. 7.1 A PPS of a set of six firms

compared with at least one point which is on the frontier of the PPS in Fig. 7.1. Is this phenomenon enough to discriminate between the firms A-F? Is the performance of D always better than the performance of E? Is there any point inside of the PPS which has a better performance than B?

The above illustrations are similar to the discussions about Figs. 3.5 and 3.6 in Chap. 3. Indeed, the PPS is the practical region; the production function is the frontier of the feasible area, and the technically efficient points are the points which have done the job right. In other words, the concept of doing the job right is the same as the concept of technical efficiency, and the word 'technical' refers to the used technology (approach) to introduce the practical points.

As illustrated in Chaps. 1–6, when a firm has done the job right, it means that the firm has produced the maximum possible values of the output factors from a set of the input factors. In addition, the concept of doing the job right depends upon the introduced approach to generate the practical points, and this is the same as the concept of technical efficiency which depends upon the use of technology to define the production function.

It is repeatedly illustrated in Chaps. 1–6 that the concept of doing the job right is only a necessary condition to discriminate between firms, and is not enough to introduce the firms which have done the job well. For instance, both firms A and D have done the job right in Fig. 7.1; however, if it is supposed that they have done the job well at the same time, a paradox is generated according to illustration in Sects. 4.2.5 and 4.3. If a firm lies on the production function, it does not logically say that the firm has done the job well. It is possible that a firm which does not lie on the production function has done the job better than the firm which lies on the production function, as illustrated in Chaps. 1–6. In other words, the discrimination between firms based on the production function only, (even if the production function is exactly available), is not valid. The important pros of technical efficiency are to estimate the production function and find the firms which can be candidates for the concept of doing the job well, (without introducing the firm which has done the job well). A firm which has done the job well in comparison with all other firms is technically efficient and lies on the production function, but the points on the production function, which are technically efficient, have not necessarily done the job well.

7.2.2 *Efficiency Measurement*

Let's suppose that the line $TT^{'}$ has the same slope as the ratio of the prices/weights/worth of the two input factors, as depicted in Fig. 7.2. From the figure, D has done the job well in comparison with all other firms.

Thus, the firms can be arranged from the highest rank to the lowest rank, given by D, C, B, A, F and E, respectively. As illustrated in Sect. 4.4, point P is not practical (according to the PPS), but has the same worth as D's performance, and allows discrimination between E and D. According to Type 5, E can increase the

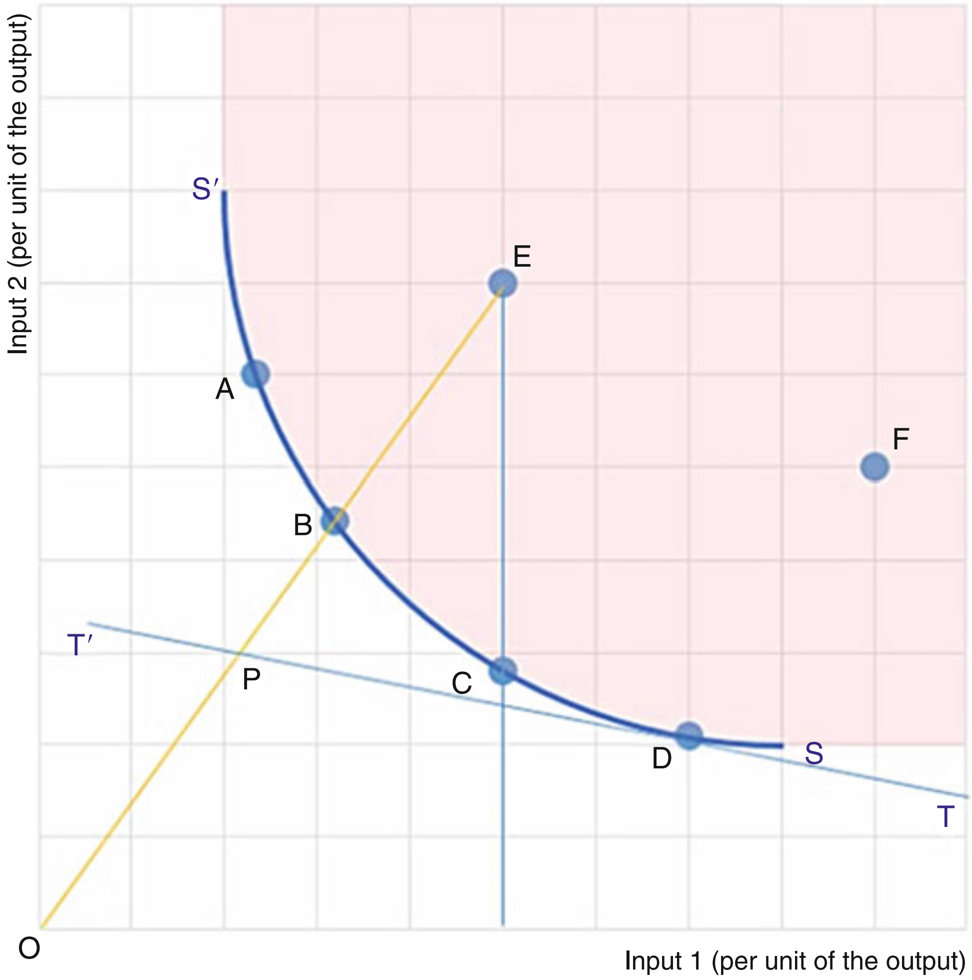

Fig. 7.2 The price/overall/relative/allocative/economic efficiency of E

value of the first input factor and decrease the value of the second output factor to perform as well as D within the PPS.

From the literature of economics, the ratio of *OB*/*OE* is called *the technical efficiency of E*, the ratio of *OP*/*OB* is called *the price efficiency of B* or *the allocative efficiency* of E, and the ratio of *OP*/*OE* is called the ratio of *the* o*verall efficiency of E* or *the economic efficiency* of E. All of these ratios are less than equal to 1.

As can be seen, the overall efficiency of E has exactly the same meaning as *the price efficiency of E*, which can also be measured by multiplying the technical efficiency of E and the allocative efficiency of E, that is, $(OB/OE) \times (OP/OB) = OP/OE$. Therefore, at least two of phrases, ‘price efficiency’, ‘overall efficiency’ or ‘economic efficiency’, are redundant.

The concept of allocative efficiency also does not provide a ranking tool similar to the concept of technical efficiency and just describes *the non-technical efficiency*; hence, there is no reason to use a new phrase. Indeed, *the price inefficiency of E* can be decomposed by *the technical inefficiency of E* and *the non-technical inefficiency of E*, thus we avoid using extra and superfluous phrases in this book.

From these definitions, the price (overall/economic) efficiency of D is 1 and the price efficiency of the other firms is always less than equal to 1, thus, the meaning of the price (overall/economic) efficiency can also be interpreted as ‘the relative efficiency’. As illustrated in Chap. 2, the equation $w_1x_1 + w_2x_2$ has the same value

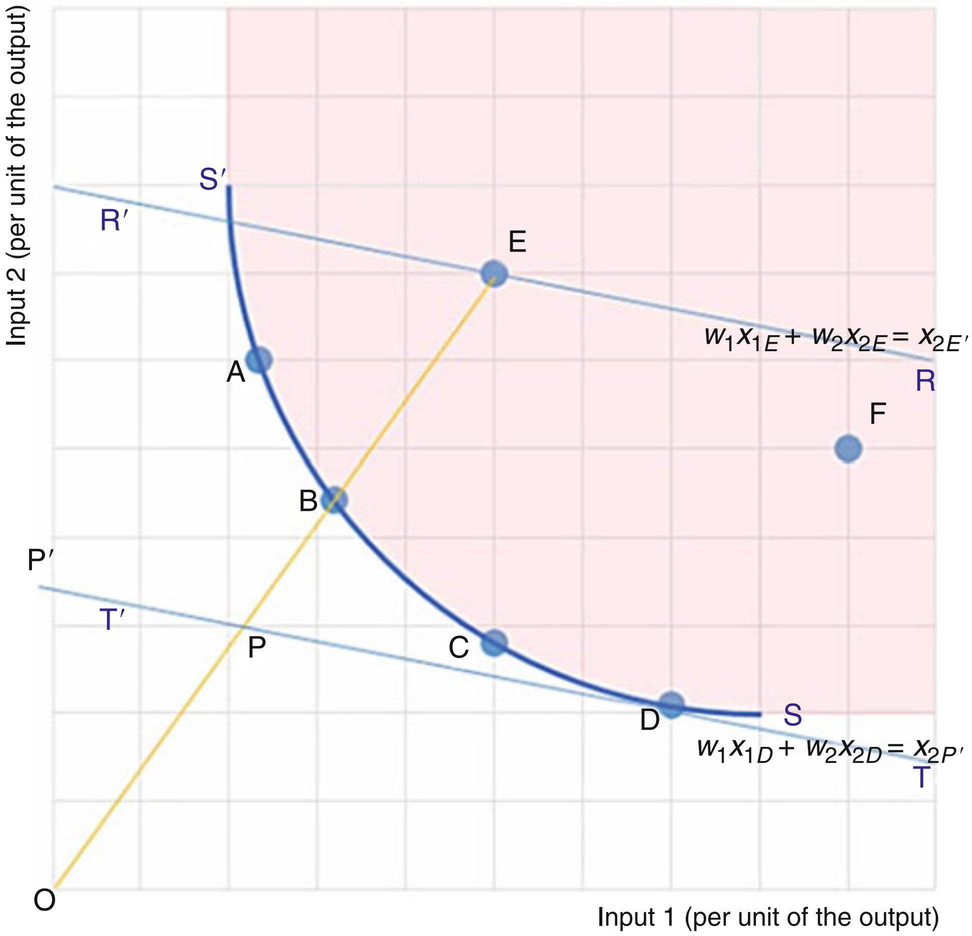

Fig. 7.3 The concept of doing the job well

for every point (x_1, x_2) on the line TT', and is equal to $w_1x_{1D} + w_2x_{2D}$, where w_1 and w_2 are the corresponded prices/weights/worth of the first input factor and the second input factor, respectively.

As Fig. 7.3 illustrates, the price (overall/economic) efficiency of E, OP/OE, is equal with the ratio of $(w_1x_{1P} + w_2x_{2P})/(w_1x_{1E} + w_2x_{2E})$, which is equal to $(w_1x_{1D} + w_2x_{2D})/(w_1x_{1E} + w_2x_{2E})$. Indeed, the lines TT' and RR' are parallel and the ratio of OP/OE is equal to the ratio of OP'/OE' in the triangle OEE'. Hence, the linear combination of the input factors of every point of the PPS is compared with the linear combination of D's input factors. Thus, the provided score is relatively meaningful, and the price (overall/economic) efficiency of E can be introduced as the relative efficiency of E as well. This outcome precisely defines the concept of doing the job well and can be expressed in one word 'efficiency', which is the condition or fact of producing the results that we would want without waste.

Note that, in the literature of operations research, 'relative efficiency' is usually considered as 'technical efficiency', which is incorrect, as illustrated in Chaps. 1–6. In definition of technical efficiency, no suitable discrimination between the points on the production function is introduced. The provided ratio for the technical efficiency, such as, OB/OE, is a fake relative score, and does not yield a valid comparison between E and other points in the PPS. In fact, the pros and cons of the technical efficiency are the same as that of doing the job right, and the provided score for the concept of doing the job right is not relatively meaningful.

Table 7.1 Reestablishing the concepts/phrases

The concept/approach in Chaps. 1, 2, and 3	Reestablishing
Doing the job right	Technical efficiency
Doing the job well	Efficiency
Doing the right job	Technical effectiveness
Doing the well job	Effectiveness
Doing the useful job	Productivity
The wholly dominant approach	Free disposal hull
The wholly dominant and convexity approaches	Variable returns to scale
The wholly dominant, convexity and radiate approaches	Constant returns to scale
The wholly dominant, convexity and inner radiate approaches	Decreasing returns to scale
The wholly dominant, convexity and outer radiate approaches	Increasing returns to scale

Since, the concept of doing the job well depends on the weights/prices/worth of the factors, and requires the concepts of the wholly dominant, the convexity and the radiate approaches to discriminate the firms linearly; the efficiency also depends on the weights/prices/worth of the factors, and at least requires the concepts of the wholly dominant, the convexity and the radiate approaches to discriminate the firms linearly.

In the literature of economics and operations research, the wholly dominant approach is called *Free Disposal Hull* (FDH) technology, the combination of the wholly dominant and the convexity approaches is called *Variable Returns to Scale* (VRS) technology, and the combination of the wholly dominant, the convexity and the radiate approaches is called *Constant Returns to Scale* (CRS) technology, as Table 7.1 illustrates. The technical efficiency (doing the job right) depends on the FDH, VRS or CRS technologies and does not provide discrimination between firms, but the efficiency (doing the job well) depends on the relationships between the factors and at least requires *the CRS technical efficiency* to discriminate firms linearly.

When a firm is not efficient, it is *inefficient*. If one desires to decompose the inefficiency of a firm, inefficiency can be decomposed by the CRS technical inefficiency and the non-CRS-technical inefficiency. The CRS technical inefficiency can also be decomposed by VRS technical inefficiency and non-VRS technical inefficiency, and so on. This topic is discussed in the upcoming sections.

In short, the meaning of CRS technical efficiency should not be misinterpreted as efficiency, similar to the concept of doing the job right which should not be misinterpreted with the concept of doing the job well. *As illustrated in Chap. 1, if using \$200 at most yields \$200, and \$220 yields \$700, the point (220, 700) is more efficient than the point (200, 200), and this is our suitable choice, regardless of whether we are applying VRS, FDH or any other approaches to define the production function.* In other words, when our purpose is to rank a set of homogenous firms, at least CRS technical efficiency should be measured. Of course, after finding the best firm and measure the concept of partially dominant, the exact returns to scale is required to estimate the production function.

Please note, from here forward in this book, instead of the phrase 'CRS technical efficiency' the phrase 'technical efficiency' will be exclusively used.

7.2.3 The Productivity Measurement

The concept of doing the job well is also introduced as '*productivity*' in the literature of economics. There is no problem if one desires to call 'doing the job well' as productivity and one can use the pair 'technical efficiency and productivity' instead of the pair 'technical efficiency and efficiency'; nonetheless, as illustrated in Sect. 2.3, after measuring the concept of doing the job well, there is still a need to measure whether the outcomes satisfy the goals of firms. Indeed, the concept of 'doing the well job' is different from the concept of 'doing the job well', and requires another meaningful name.

According to Cambridge English dictionary, "the ability to be successful and produce the intended results" is called '*effectiveness*'. Therefore, 'effectiveness' can be used for the concept of 'doing the well job' and 'productivity' can be used for the concept of 'doing the useful job', which is a combination of both efficiency and effectiveness. The word 'productivity' means "the rate at which a person, company or country does useful work", according to Cambridge English dictionary. Therefore, it is suggested that the commonly utilized phrases and concepts be reestablished according to the following table.

To clearly explain the above, *let's suppose that a set of 9 homogenous banks, labeled A-I, are selected. Assume that the aim of discrimination is (1) to find the banks which have used a smaller number of tellers to service a greater number of customers, and (2) to find the banks which have at least serviced* y_1 *number of customers in the period of evaluation. Suppose that the production function is available and the location of each bank in the Cartesian coordinate plane is depicted in* Fig. 7.4.

The blue curve represents the production function, the horizontal axis illustrates the number of tellers and the vertical axis displays the number of customers. The banks A-G lie on the production function, and are VRS-technically efficient, and the banks H and I are VRS-technically inefficient.

Similar to Chap. 1, by considering the ratio of the number of customers to the number of tellers, D is the most efficient bank followed by E, H and F, respectively. H is not technically efficient, but, for instance, H is more efficient than A which is VRS-technically efficient.

The banks C, D, E, F, G and H have at least serviced y_1 number of customers, and are *effective*. F is the most effective bank followed by G, E and H. Therefore, the banks C, D, E, F and H are the most productive banks which are most efficient and most effective at the same time in comparison with other banks.

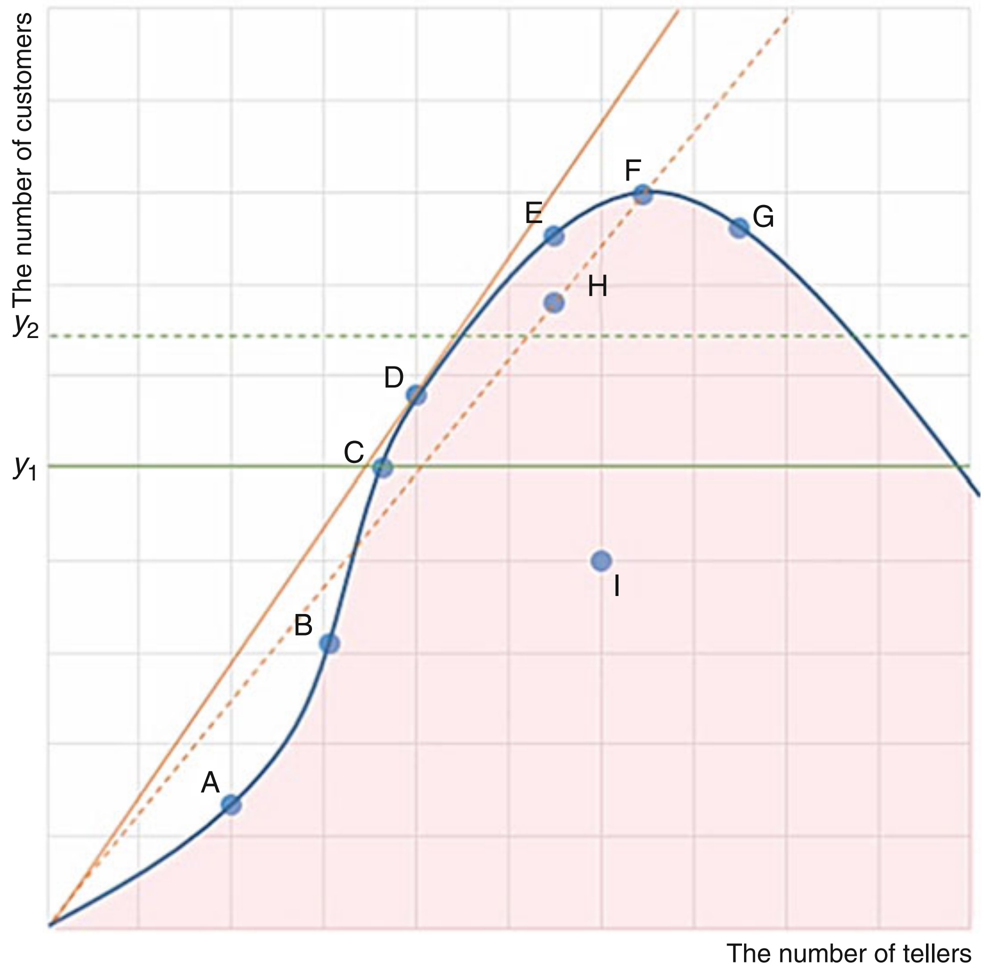

Fig. 7.4 The production function

The above results can also be seen in Fig. 7.5, where the horizontal axis displays the relative efficiency of each bank and the vertical axis represents the relative effectiveness of each bank. As can be seen, D has the relative score equal to 1 and F has the relative effectiveness equal to 1. The red area illustrates the non-productive banks, which are the banks A, B and I, and have relative effectiveness scores less than that of C and relative efficiency scores less than 0.8.

G is the most effective bank after F, but has the relative efficiency score less than 0.8. The (dark and light) green area represents the productive banks which have relative efficiency scores more than 0.8 and relative effectiveness scores more than that of C.

If the goal of evaluation, which is at least servicing y_1 number of customers in period of evaluation, is changed to at least servicing y_2 number of customers, as Figs. 7.4 and 7.5 illustrate, even the most efficient banks D and C are not called productive due to the lack of their effectiveness. In this case, the banks E, F and H are the most productive banks among the banks A-I, as the dark green area displays in Fig. 7.5.

The concept of effectiveness is always required in real-life applications, for instance, no bank desires to decrease *consumer satisfaction* and no firm works

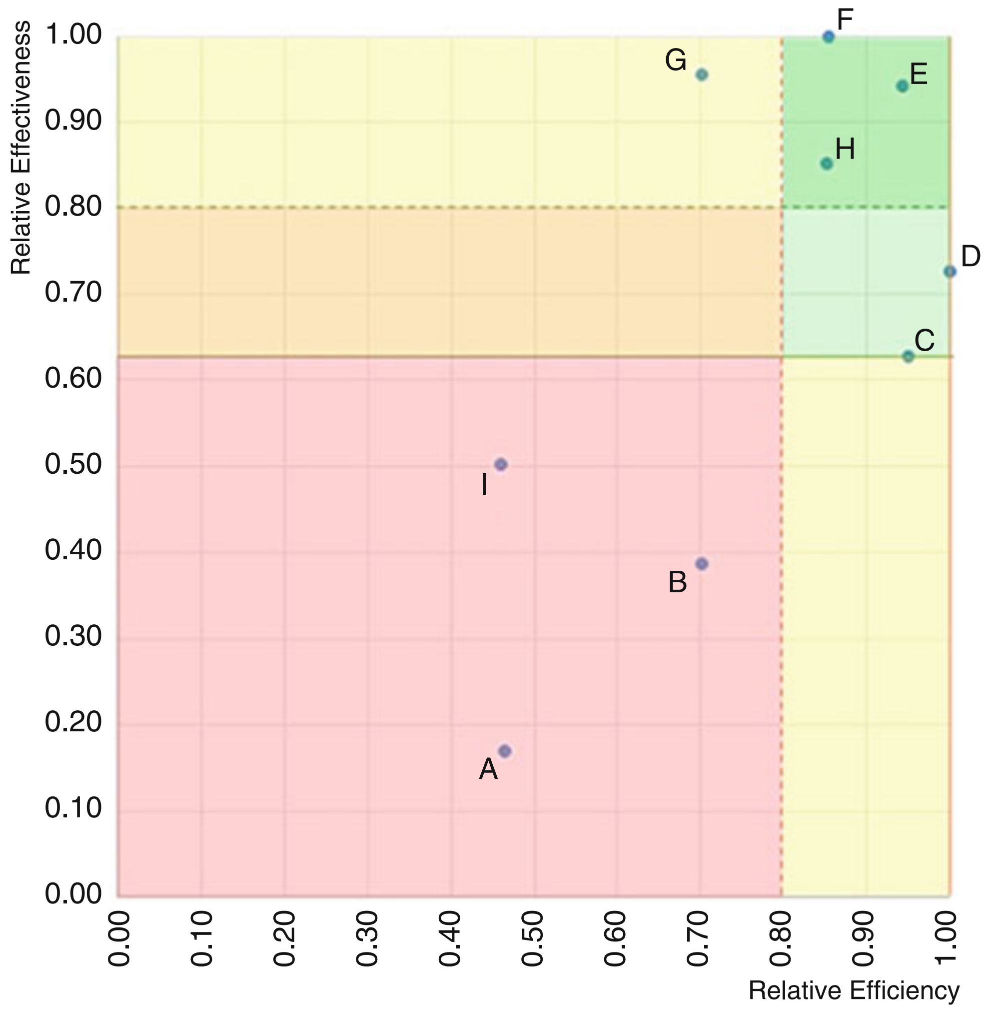

Fig. 7.5 The productivity measurement

without a plan or some requirements. It is also possible that each bank has different goals for effectiveness index; for instance, a bank may only prefer the production function values according to its set of input factors and in this case, if a different goal for each bank does not affect the homogeneity of the banks, the technical effectiveness index can also be calculated.

In short, there are several concepts which provide the most important indexes to discriminate the performance of homogenous firms. The concept of doing the job right, which is called technical efficiency in this book, is not enough to discriminate between firms. The technical efficiency is usually interpreted as efficiency in the literature, and if one would like to use such a term for the concept of doing the job right, one should be aware that such efficiency is neither enough to discriminate between firms, nor should the corresponded index be used to rank and benchmark the firms. In order to rank and benchmark a set of homogenous firms, the concept of doing the job well is required, and this concept is called efficiency in this book. The efficiency is usually interpreted as productivity in the literature of economics, and similarly if one would like to use the word productivity for the concept of doing the job well, one should know that there is still a need to measure the effectiveness of firms. Therefore, Table 7.1 is provided to reintroduce the concepts of efficiency, effectiveness and productivity with industry-wide accuracy and understanding.

7.3 Data Envelopment Analysis

A non-parametric technique to estimate the technical efficiency and the efficiency of a set of homogenous firms was proposed by Farrell (1957). His method estimates the production function non-parametrically, which was similarly suggested by Debreu (1951) and Koopmans (1951). Førsund and Hjalmarsson (1974) illustrated the notion of efficiency in the macro, the industry and the micro levels, and clearly displayed and demonstrated the differences between the production frontier and efficiency. Soon after, Charnes et al. (1978) proposed a mathematical construction and a linear programming model to introduce technically efficient firms with multiple input factors and multiple output factors. They called the mathematical construction '*Data Envelopment Analysis'* (DEA) and the model '*Charnes, Cooper and Rhodes*' (CCR). CCR generates a PPS based upon a set of available homogenous firms, and non-parametrically and linearly estimates the production function; thus, the firms which lie on the frontier of that PPS are called technically efficient. CCR is the same as Forms 1 and 2 (Eqs. 5.25 and 5.26) in Chap. 5 which only introduce technically efficient firms, and the provided scores by CCR (Forms 1 and 2) are neither relatively meaningful, nor can be used to rank and benchmark the firms, as explained in Chap. 5.

In addition, Forms 1 and 2 (CCR) even fail to measure the technical inefficiency completely, and only calculate the output view of technical efficiency (that is, increasing the values of output factors without measuring the excess of input factors) or calculate only the input view of technical efficiency (that is, decreasing the values of input factors without measuring the shortage of output factors). For this reason, Färe and Lovell (1978) noted on Farrell's measurement of technical efficiency and CCR, and proposed a *Russell measure*, to simultaneously deal with both input and output views of technical efficiency. Their proposed model is a non-linear programming, and difficult to solve; thus, Pastor et al. (1999) proposed *Enhanced Russell Measure* (ERM) (see Exercise 7.3) to measure technical efficiency of firms, and avoid computational and interpretative difficulties with the Russell measure.

On the other hand, Charnes et al. (1985) proposed an *Additive model* (ADD), which is the same as Form 3 (Eq. 5.31) in Chap. 5, to remove the shortcomings of CCR (Forms 1 and 2) to measure the technical inefficiency of firms. Nonetheless, ADD (Form 3) is also not a perfect model to measure the technical inefficiency, as illustrated in Chap. 5. Therefore, Tone (2001) proposed a *Slack-Based Measure* (SBM) model (Eq. 6.26) to measure technical inefficiency of firms. He proved that (1) ERM and SBM are equivalent in that the lambda's values that are optimal for one are also optimal for the other, (2) the SBM measurement corresponds to the mean proportional rate of input factors' reduction and the mean proportional rate of output factors' expansion, (3) the SBM measurement is monotone, decreasing in each input and output slack, and (4) it is invariant with respect to the unit of measurement of each input and output item.

As explained in Chap. 5, all proposed models, such as CCR (Forms 1 and 2), ADD (Form 3) and SBM (Eq. 6.26) are provided to measure the technical inefficiency which neither provides a ranking and benchmarking tool, nor introduces the relative efficiency scores for firms. Note that there are a lot of proposed models based on CCR (Forms 1 and 2) in the literature of operations research since 1978 which have the same (or more) mentioned shortcomings to focus on technical efficiency (doing the job right) instead of efficiency (doing the job well), in order to discriminate a set of homogenous firms. While these studies not be further upon here, but readers can examine several of these studies, and review their discrimination and ranking tools as simple exercises, see for example the topic in Exercise 7.4.

Sexton et al. (1986) wisely noted the shortcomings of CCR (Forms 1 and 2) and stated that DEA cannot be used to analyze or comment on a firm's (price) efficiency and a firm can be technically efficient, but (price) inefficient. Thus, they proposed a *cross efficiency* model, which was supposed to measure the score that a particular firm receives when it is rated by another firm. Nonetheless, the cross efficiency score for a firm is also not a relative score for that firm; it is an average value of the relative scores of that firm, according to some specified sets of weights, which neither should be used to rank firms, nor is relatively meaningful, similar to discussions in Sects. 2.2.1, 3.5.3.1., and 4.2. The average values of the relative scores of firms similar to the maximum (minimum, first quartile, and so on) values of the relative scores, are not relatively meaningful and should not be suggested as the relative scores of firms, as logically illustrated in Sect. 4.2 as well.

In order to decrease the above shortcomings, Khezrimotlagh et al. (2013) proposed δ-Kourosh and Arash Method (δ-KAM) to bridge between technical efficiency and efficiency. Their proposed model is the same as Eq. 6.20, which is improved to Eq. 6.25 in this book, and will be improved again to cover several new topics.

The score of KAM is different from the scores of other models in the literature, and can be used as a fair judgment tool for ranking and benchmarking firms. The words 'Kourosh' and 'Arash' are also symbolic and referred to 'justice' and 'border' in ancient linguistic history of Persia.

From illustration in Sect. 6.3, while the value of delta is 0, the results of 0-KAM identify the firms which are technically efficient and technically inefficient, and should not be used to rank or benchmark firms. As the value of delta increases, the results of δ-KAM can be used to rank and benchmark firms, according to the value of delta and the introduced assumptions for weights/prices of input and output factors.

As is explained in Chaps. 1–6 the technical efficiency (as well as production function) depends upon the way of introducing the practical points and the efficiency depends upon the weights/worth/prices of the factors, and at least require the combination of the radiate, the convexity and the wholly dominant approaches to linearly estimate the efficiency scores of a set of homogenous firms. In the next sections, KAM is improved to measure the efficiency of firms with the least requirements to the radiate, the convexity and the wholly dominant approaches.

7.3.1 DEA Axioms

DEA construction can be used as a non-parametric tool for estimating the production function of a set of homogenous firms. In order to explain DEA, suppose that there are n firms, labeled $F_i\,(i = 1, 2, \ldots, n)$, and each firm has m input factors with the values $x_{ij}\,(j = 1, 2, \ldots, m)$ and p output factors with the values $y_{ik}\,(k = 1, 2, \ldots, p)$. Let's suppose that X_i is the vector of the input factors for ith firm, that is, $X_i = (x_{i1}, x_{i2}, \ldots, x_{im})$, and Y_i is the vector of the output factors for ith firm, that is, $Y_i = (y_{i1}, y_{i2}, \ldots, y_{ip})$, where $i = 1, 2, \ldots, n$. The corresponded PPS to these firms are generated by the following axioms, where $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_p)$:

Axiom 1. Each observed firm should be belonged to the PPS, that is, $\forall i\ (X_i, Y_i) \in$ PPS.

Axiom 2. If $(X, Y) \in$ PPS, then $(\lambda X, \lambda Y) \in$ PPS, for $\lambda \geq 0$, that is, for every point (X, Y) in the PPS, the same proportionate increase (decrease) in input factors results the same proportionate increase (decrease) in the output factors.

Axiom 3. Every point (X, Y) of the line-segment which connects each two points of the PPS should belong to the PPS, that is, if $(X', Y') \in$ PPS and $(X'', Y'') \in$ PPS, then $(X, Y) = [\lambda(X', Y') + (1 - \lambda)(X'', Y'')] \in$ PPS, for $\lambda \in [0, 1]$.

Axiom 4. Every point (X', Y') which has greater or equal values of the input factors with the same values of the output factors in comparison with at least one of the points in the PPS should belong to the PPS, that is, if $X' \geq X$ and $(X, Y) \in PPS$, then $(X', Y) \in$ PPS, (where $X \geq X' \equiv \forall j : x_j \geq x'_j$).

Axiom 5. Every point (X', Y') which has lesser or equal values of the output factors with the same values of the input factors in comparison with at least one of the points in the PPS should belonged the PPS, that is, if $Y' \leq Y$ and $(X, Y) \in$ PPS, then $(X, Y') \in$ PPS, (where $Y \leq Y' \equiv \forall k : y'_k \leq y_k$).

Axiom 6. The PPS is the intersection of all PPSs which have the above properties.

The above axioms are provided from Charnes et al. (1978). As illustrated in Chap. 1, the second axiom is the radiate approach, the third axiom is the convexity approach, and the fourth and fifth axioms are the wholly dominant approach. Therefore, the generated PPS is a CRS-PPS. In order to generate the CRS-PPS linearly, there is a need to apply the convexity approach for the observations without upper bound for λ, and after that apply the wholly dominant approach, as illustrated in Chap. 1. In other words, the CRS-PPS is a set of points, (X', Y'), with $m + p$ dimensions, that is, $\left(x'_1, x'_2, \ldots, x'_m, y'_1, y'_2, \ldots, y'_p\right)$, which have the following conditions $\sum_{i=1}^{n} \lambda_i x_{ij} \leq x'_j$, for $j = 1, 2, \ldots, m$, and $y'_k \leq \sum_{i=1}^{n} \lambda_i y_{ik}$, for $k = 1, 2, \ldots, p$, where $\lambda_i \geq 0$ for $i = 1, 2, .., n$.

The matrix view of m inequalities, $\sum_{i=1}^{n} \lambda_i x_{ij} \leq x'_j$, for $j = 1, 2, \ldots, m$, can be represented as follows:

$$[\lambda_1\ \lambda_2\ \dots\ \lambda_n]_{1\times n} \times \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}_{n\times m} \le [x'_1\ \ x'_2\ \ \dots\ \ x'_m]_{1\times m}. \tag{7.1}$$

Let's suppose that $X^j = [x_{1j} \quad x_{2j} \quad \dots \quad x_{nj}]^t{}_{1\times n}$, (that is, the transpose of j^{th} column in the above matrix of the input factors), which displays j^{th} values of the input factor of each firm, where $j = 1, 2, \dots, m$. In addition, assume that $\Lambda = [\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_n]_{1\times n}$, thus, the above equation can be rewritten as follows:

$$\Lambda \cdot X^j \le x'_j, \quad \text{for} \quad j = 1, 2, \dots, m. \tag{7.2}$$

The notation '·' in Eq. 7.2 represents *the inner product*, that is, $\Lambda \cdot X^j = \lambda_1 \times x_{1j} + \lambda_2 \times x_{2j} + \dots + \lambda_n \times x_{nj} = \sum_{i=1}^{n} \lambda_i x_{ij}$.

In addition, the matrix view of p inequalities, $y'_k \le \sum_{i=1}^{n} \lambda_i y_{ik}$, for $k = 1, 2, \dots, p$, can be represented as follows:

$$[\lambda_1 \quad \lambda_2 \quad \dots \quad \lambda_n]_{1\times n} \times \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1p} \\ y_{21} & y_{22} & \dots & y_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \dots & y_{np} \end{bmatrix}_{n\times p} \ge [y'_1 \quad y'_2 \quad \dots \quad y'_p]_{1\times p}.$$

Let's suppose that $Y^k = [y_{1k} \quad y_{2k} \quad \dots \quad y_{nk}]^t{}_{1\times n}$, (that is, the transpose of kth column in the above matrix of the output factors), which displays kth values of the input factor of each firm, where $k = 1, 2, \dots, p$. Therefore, the above equation can be rewritten as follows:

$$\Lambda \cdot Y^k \ge y'_k \quad \text{for} \quad k = 1, 2, \dots, p. \tag{7.3}$$

From Eqs. 7.2 and 7.3, the CRS-PPS, which is denoted by T_C, is given by ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$,and $k = 1, 2, \dots, p$):

$$T_C = \left\{ (X', Y') \in \mathbb{R}^{m+p} : \ \Lambda \cdot X^j \le x'_j, \ \ \Lambda \cdot Y^k \ge y'_k, \quad \lambda_i \ge 0, \quad \text{for } i, j, k \right\} \tag{7.4}$$

The frontier of the CRS-PPS is defined as a production function by CRS technology for the firms $F_i (i = 1, 2, \dots, n)$, and the firms on the frontier are introduced as technically efficient firms and the other firms are called technically inefficient.

Similar to Chap. 1, if the second axiom (the radiate approach) is removed from Axioms 1–6, the PPS is the VRS-PPS, that is, a PPS which is generated by the combination of the wholly dominant and the convexity approaches. This VRS-PPS, which is also denoted by T_V, was proposed by Banker et al. (1984), as follows ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$,$k = 1, 2, \dots, p$):

$$\left\{(X', Y') \in \mathbb{R}^{m+p} : \Lambda \cdot X^j \leq x'_j, \Lambda \cdot Y^k \geq y'_k, \sum\nolimits_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \text{for } i, j, k\right\}. \quad (7.5)$$

The frontier of the VRS-PPS is defined as a production function by VRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as VRS-technically efficient firms and the other firms are called VRS-technically inefficient.

When instead of the radiate approach, the outer radiate approach is used, the lower bound for λ_i is greater than 1, that is, $\lambda_i > 1$. The outer radiate approach can be used while the same proportionate increase in the input factors results the greater proportionate increase in the output factors. From Table 7.1, using the outer radiate approach instead of the radiate approach is called Increasing Returns to Scale (IRS) technology. If $\lambda_i \geq 1$, the technology is called Non-Decreasing Returns to Scale (NDRS). The frontier of the NDRS-PPS is defined as a production function by NDRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as NDRS-technically efficient firms and the other firms are called NDRS-technically inefficient (See, for instance, Zhu 2014).

If $\lambda_i < 1$, that is, using the inner radiate approach instead of the radiate approach, the same proportionate increase in the input factors results the lesser proportionate increase in the output factors. The inner radiate approach is used while the same proportionate increase in the input factors results in the greater proportionate increase in the output factors. As Table 7.1 displays, the combination of the wholly dominant, the convexity and the inner radiate approaches is called Decreasing Returns to Scale (DRS) technology, and if $\lambda_i \leq 1$, the technology is called Non-Increasing Returns to Scale (NIRS). The frontier of the NIRS-PPS is defined as a production function by NIRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as NIRS-technically efficient firms and the other firms are called NIRS-technically inefficient.

An example for IRS (NDRS) technology is the tax income on electricity usage, and for DRS (NIRS) technology is the revenue of publishing a newspaper according to circulation. In addition, the study about *Returns to Scale* (RS) is also important to merge small firms or divide large firms (See, for instance, Coelli et al. 2005).

It is quite possible that a real-life application could have several factors which each factor pursues one of the NDRS, NIRS, and CRS technologies. Nonetheless, the VRS technology is the intersection of all these technologies, and includes all the observations, which certainly provides an estimation of production function to measure VRS technical inefficiencies. The ratio of the technical efficiency over the ratio of VRS-technical efficiency is also called *scale efficiency* in economics (see Cooper et al. 2011).

In short, DEA axioms can be used to estimate the production function which only represents the concept of doing the job right (technical efficiency) with different approaches (technologies); but the provided axioms do not represent the concept of doing the job well (efficiency) and the concepts of allocative models are required (see Førsund et al. 1980). However, as it will be explained, KAM improves DEA to handle both the concepts of doing the job right (technical efficiency) and doing the job well (efficiency) by a single linear programming.

7.3.2 DEA Models

In Chap. 5, several linear programming models are introduced to measure the technical inefficiencies. The measurement which considers the possible decreasing of input factors without measuring the shortages of output factors is called *Input Orientation* (IO), and the measurement which considers the possible increasing of output factors without measuring the excesses of input factors is called *Output Orientation* (OO) in DEA (see also Toloo 2014). In other words, IO lets us measure the input technical inefficiencies without measuring the output technical inefficiencies, (that is, Form 2 by Eq. 5.26), OO lets us measure the output technical inefficiencies without measuring the input technical inefficiencies, (that is, Form 1 by Eq. 5.25), and none of these measurements provide a fair discrimination nor they calculate the technical efficiency completely. We also avoid using these measurements to benchmark firms, before calculating efficiency or ranking firms according to the concept of doing the job well.

The measurement which considers both possible decreasing of input factors and possible increasing of output factors is called *Non-Orientation* (NO) in DEA, similar to Form 3 by Eq. 5.28. In order to measure technical inefficiency and solve Eq. 5.28, we use 0-KAM (Eq. 6.31). If the VRS-technical inefficiency is necessary to calculate, we use 0-KAM (Eq. 6.25) when $\sum_{i=1}^{n} \lambda_i = 1$, and decompose technical inefficiency to VRS-technical inefficiency and non-VRS-technical inefficiency. As illustrated, inefficiency can be decomposed to technical inefficiency and non-technical inefficiency (allocative inefficiency). In other words, efficiency can be introduced by multiplying VRS-technical efficiency, non-VRS-technical efficiency (scale efficiency) and non-technical efficiency (allocative efficiency). The results of efficiency can be used for ranking and benchmarking firms, however, none of the decomposed parts of efficiency should independently be used to rank or benchmark firms.

As explained, the provided scores by Form 3 also cannot be used to discriminate firms or describe efficiency. In order to measure efficiency, similar to Forms 1–3, some linear programming proposed by Debreu (1951), Farrell (1957), Färe et al. (1985) and Tone (2002), called *Cost Efficiency* (CE) (Eq. 6.36), *Revenue Efficiency* (RE) (Eq. 6.38) and *Profit Efficiency* (PE) (Eqs. 6.40 and 6.41). Khezrimotlagh (2014) proved that the δ-KAM (Eq. 6.25) is equivalent with these models, (see Exercise 6.10–6.13), while the value of delta is large enough and the coefficients of the slacks are introduced as the available prices. Nonetheless, none of the CE, RE and PE models measure the efficiency of a set of homogenous firms completely, as the next section illustrates. In other words, a suitable model to measure the efficiency of firms should at least satisfy the introduced Types 1–6.

7.4 Conclusion

The meaning of technical efficiency, efficiency, effectiveness and productivity are discussed in this chapter. Productivity is a combination of effectiveness and efficiency. In the effectiveness measurement, the factors of each firm are compared with the desired goals of the firms, and in efficiency measurement, the firms' performances are compared to each other. The provided technical efficiency scores should not be used to rank firms; they have unfortunately been wrongly used in the literature of operations research for the last four decades. The discrimination between homogenous firms requires expert judgment either to introduce a set of weights for factors or to specify a measurement approximation to estimate the efficiency scores which are relatively meaningful.

7.5 Exercises

7.1. Explain the differences between efficiency and

 7.1.1. Technical efficiency.
 7.1.2. Effectiveness.
 7.1.3. Productivity.

7.2. Describe the following phrases:

 7.2.1. Allocative efficiency
 7.2.2. Scale efficiency

7.3. Prove that ERM and SBM are equivalent in that the lambda's values that are optimal for one are also optimal for the other. The ERM is given by the following equation.

$$\begin{aligned}
&\min \frac{(1/m)\sum_{j=1}^{m}\theta_j}{(1/p)\sum_{k=1}^{p}\varphi_k},\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n}\lambda_i x_{ij} \le \theta_j x_{lj}, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n}\lambda_i y_{ik} \ge \varphi_k y_{lk}, \quad \text{for} \quad k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&0 \le \theta_j \le 1, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&\varphi_k \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{7.6}$$

7.4. The following model is called super-efficiency model (Andersen and Petersen 1993).

$$\begin{aligned}
&\min \theta,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1, i\neq l}^{n}\lambda_i x_{ij} \le \theta_j x_{lj} \quad \text{for} \quad j = 1, 2, \ldots, m\\
&\textstyle\sum_{i=1, i\neq l}^{n}\lambda_i y_{ik} \ge y_{lk}, \quad \text{for} \quad k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&\theta \ge 0.
\end{aligned} \tag{7.7}$$

7.4.1. Describe Eq. 7.7.

7.4.2. Why the provided ranks for firms F_i's ($i = 1, 2, \ldots, n$) by Eq. 7.7 are not logically and relatively meaningful?

7.4.3. Give a counter example that Eq. 7.7 should not be used to identify outliers as well.

7.4.4. Run a simulation to show the results of BCC Super-efficiency is not even stronger as the result of BCC to rank DMUs.

Chapter 8
The Ratio of Output to Input Factors

8.1 Introduction

In the previous chapters, we provided transparent steps to learn the foundation of DEA to estimate the performance of a set of homogenous firms. In this chapter, we demonstrate mathematical properties to describe the natural relationships between the DEA frontier and the ratio of output to input factors.

8.2 Charnes, Cooper and Rhodes Model

Let's extend Eq. 3.24 in general for n DMUs A_i, for $i = 1, 2, \ldots, n$, in which each DMU has m positive input factors x_{ij}, for $j = 1, 2, \ldots, m$, and p positive output factors y_{ik}, for $k = 1, 2, \ldots, p$. Assume that A_l is evaluated, for $l = 1, 2, \ldots, n$, as Eq. 8.1 displays.

$$
\begin{aligned}
&\max \frac{\sum_{k=1}^{p} y_{lk} w_k^+}{\sum_{j=1}^{m} x_{lj} w_j^-}, \\
&\text{Subject to} \\
&\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \leq 1, \text{for } i = 1, 2, \ldots, n, \\
&w_j^+ \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m, \\
&w_k^- \geq 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.1}
$$

Charnes et al. (1978) proposed Eq. 8.1, and called it Charnes, Cooper and Rhodes' (CCR) model. The optimal value of CCR is at most 1 and we have the following definition:

D. Khezrimotlagh, Y. Chen, *Decision Making and Performance Evaluation Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 269, https://doi.org/10.1007/978-3-319-76345-3_8

Definition 8.1 A DMU is CCR-efficient if and only if the optimal value of CCR is equal to 1, for some positive optimal solutions w_j^{-*} and w_k^{+*}, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

As discussed in the previous chapters, if a DMU is CCR-efficient, the DMU has done the job right, and is technically efficient. Readers should note that if a DMU is CCR-efficient, this does not mean that the DMU has done the job well or the DMU is efficient.

Charnes et al. (1978) transformed CCR to equivalent linear programming models (Forms 1 and 2), using Charnes and Cooper's transformation (1962). They called these two forms as Output-Oriented (OO) CCR and Input Oriented (IO) CCR, respectively.

Equations 8.2 and 8.3 represent IO-CCR and OO-CCR in multiplier forms. Here the word "multiplier" refers to the weights (multipliers) w_j^+'s and w_k^-'s. As discussed in Chap. 3, IO-CCR only measures the possible decrease in the input factors radially, whereas OO-CCR only measures the possible increase in the output factors radially. In other words, all input factors are decreased with the same proportion, θ, by IO-CCR, where θ^* is the possible minimum value of θ and $\theta^* \in (0, 1]$. Similarly, all output factors are increased with the same proportion, φ, by OO-CCR, where φ^* is the possible maximum value of φ and $\varphi^* \in [1, +\infty)$

$$\begin{aligned}
&\theta_l^* = \max \textstyle\sum_{k=1}^{p} y_{lk} w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \geq 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{8.2}$$

$$\begin{aligned}
&\varphi_l^* = \min \textstyle\sum_{j=1}^{m} x_{lj} w_k^-,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{p} y_{lk} w_k^+ = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \geq 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{8.3}$$

Definition 8.2 A DMU is CCR-efficient if and only if $\theta^* = 1$, for some positive optimal solutions w_j^{-*} and w_k^{+*}, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Definition 8.3 A DMU is CCR-efficient if and only if $\varphi^* = 1$, for some positive optimal solutions w_j^{-*} and w_k^{+*}, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Theorem 8.1 *For every DMU,* $\theta^*\varphi^* = 1$.

Proof See Exercise 8.3. □

The dual linear programming models for IO-CCR and OO-CCR (Envelopment Forms) are as follows, respectively:

$$\begin{aligned}
&\theta_l^* = \min\ \theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}\theta_l, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}, \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.4}$$

$$\begin{aligned}
&\varphi_l^* = \max\ \varphi_l\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}\varphi_l, \text{for k} = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.5}$$

CCR can easily be solved using Microsoft Excel Solver software. For example, let's consider a data set used in Ali et al. (1995), as Table 8.1 represents. There are 11 DMUs in which each DMU consumes two input factors and produces two output factors.

In order to apply IO-CCR (Eq. 8.2) for data in Table 8.1, by the Microsoft Excel Solver software, the following steps can be followed.

1. Copy the 5 columns of Table 8.1 on an Excel sheet into cells A1:E12.
2. Label A14 as 'Index', and enter number 1 to B14.
3. Label A16 as "Weights" (reserve B16-E16 for changing cells).
4. Label F1 as "Constraints".

Table 8.1 Example of 11 DMUs with 4 factors

DMU	Input 1	Input 2	Output 1	Output 2
A01	40	30	160	100
A02	30	60	180	70
A03	93	40	170	60
A04	50	70	190	130
A05	80	30	180	120
A06	35	45	140	82
A07	105	75	120	90
A08	97	67	100	82
A09	100	50	140	40
A10	90	60	140	105
A11	98	65	140	50

5. Assign the following command into F2 '=Sumproduct(D2:E2,D$16:E$16)-Sumproduct(B2:C2,B$16:C$16)'.

   Note: the single quotation marks do not belong to the commands.

6. Copy F2 and then paste it into F3-F12.
7. Label E14 as "Constraint" and assign the following command into F14,

   '=Sumproduct(Index(B2:C12,B14,0),B16:C16)'

8. Label E18 as "Constraint" and assign the following command into F14,

   '=Sumproduct(Index(D2:E12,B14,0),D16:E16)'

9. Open 'Solver Parameters' window.
10. Assign 'F18' into 'Set Objective' and choose 'Max'.
11. Assign 'B16:E16' into 'By Changing Variable Cells'.
12. Click on 'Add' and assign 'F2:F12' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
13. Click on 'Add' and assign 'F14' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'. Then click on OK.
14. Tick 'Make Unconstrained Variables Non-Negative'.
15. Choose 'Simplex LP' from 'Select a Solving Method'.
16. Click on 'Solve'.
17. Save or 'Save As' your excel file with this type 'Excel macro-Enabled Workbook (*.xlsm)'.
18. Label H1 as 'Optimal Theta'.
19. Label I1-L1 as 'Optimal Weights' for I1, I2, O1 and O2, respectively.
20. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
21. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
22. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
23. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
24. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub' as Fig. 2.21 depicts.

```
Dim i As Integer
For i = 1 To 11
   Range("B14") = i
   SolverSolve Userfinish:=True
   Range("H" & i + 1) = Round(Range("F18"), 4)
   Range("B16:E16").Copy
   Range("I" & i + 1).Select
   Selection.PasteSpecial Paste:=xlPasteValues
Next i
```

Table 8.2 The results of IO-CCR for data in Table 8.1

DMU	θ^*	w_1^{-*}	w_2^{-*}	w_1^{+*}	w_2^{+*}
A01	1.0000	0.0238	0.0015	0.0000	0.0100
A02	1.0000	0.0156	0.0089	0.0056	0.0000
A03	0.7291	0.0021	0.0200	0.0043	0.0000
A04	1.0000	0.0183	0.0012	0.0000	0.0077
A05	1.0000	0.0042	0.0222	0.0000	0.0083
A06	0.9320	0.0273	0.0010	0.0014	0.0090
A07	0.3564	0.0020	0.0106	0.0000	0.0040
A08	0.3610	0.0022	0.0117	0.0000	0.0044
A09	0.4941	0.0018	0.0165	0.0035	0.0000
A10	0.5122	0.0024	0.0130	0.0000	0.0049
A11	0.3974	0.0014	0.0132	0.0028	0.0000

25. Close the 'Microsoft Visual Basic for Applications' window.
26. Click on the small rectangle which was automatically made on the Excel sheet by step 21.
27. The results are represented into cells H2:L12. Column H represents the CCR-scores for DMUs with four decimal digits, and columns I-L illustrate the optimal solutions for the weights, w_1^{-*}, w_2^{-*}, w_1^{+*}, and w_2^{+*}.

Table 8.2 illustrates the results of IO-CCR for data in Table 8.1. There are four DMUs which are CCR-efficient, and the rest of DMU are CCR-inefficient.

Note that, the scores in the first column of Table 8.2 are not relatively meaningful, as discussed in Sects. 2.3 and 4.3. As can be seen, the second column in Table 8.3 represents the relative scores of DMUs, according to the set of optimal weights for A01, and the third column in Table 8.3 illustrates the relative scores of DMUs according to the set of optimal weights for A02, and so on. The first row in Table 8.3 also shows that the CCR score for A01 is 1 for every set of weights in Table 8.2. The CCR scores are bold on the diameter of Table 8.3, to emphasize that the scores of CCR in Table 8.2 are not relatively meaningful. For instance, the relative scores for technically efficient DMUs A02 and A04 according to the optimal weights for A11 are 0.610 and 0.540, respectively.

In addition, the instructions to solve the envelopment form of IO-CCR for data in Table 8.1, by the Microsoft Excel Solver software, are as follows:

1. Copy the 5 columns of Table 8.1 on an Excel sheet into cells A1:E12.
2. Label A14 as 'Index', and enter number 1 to B14.
3. Label F1 as "Lambdas" (reserve F2-F12 for changing cells).
4. Label D14 as "Theta" (reserve E14 for the changing cell).
5. Label A16 as "Constraints".
6. Assign the following command into B16

   '=Sumproduct(B2:B12,$F2:$F12)'.

7. Copy B16 and then paste it into C16, D16 and E16.

Table 8.3 The relative scores of DMUs in Table 8.1

DMU	A01	A02	A03	A04	A05	A06	A07	A08	A09	A10	A11
A01	**1.000**	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
A02	0.867	**1.000**	0.610	0.867	0.400	1.000	0.400	0.400	0.610	0.400	0.610
A03	0.263	0.524	**0.729**	0.263	0.392	0.300	0.392	0.392	0.729	0.392	0.729
A04	1.000	0.754	0.540	**1.000**	0.614	1.000	0.614	0.614	0.540	0.614	0.540
A05	0.614	0.662	1.000	0.614	**1.000**	0.601	1.000	1.000	1.000	1.000	1.000
A06	0.907	0.824	0.615	0.907	0.596	**0.932**	0.596	0.596	0.615	0.596	0.615
A07	0.344	0.290	0.298	0.344	0.356	0.332	**0.356**	0.356	0.298	0.356	0.298
A08	0.339	0.264	0.277	0.339	0.361	0.323	0.361	**0.361**	0.277	0.361	0.277
A09	0.163	0.389	0.494	0.163	0.218	0.199	0.218	0.218	**0.494**	0.218	0.494
A10	0.469	0.402	0.431	0.469	0.512	0.453	0.512	0.512	0.431	**0.512**	0.431
A11	0.205	0.370	0.397	0.205	0.225	0.234	0.225	0.225	0.397	0.225	**0.397**

8. Assign the following command into B17.

   '=Index(B2:B12,$B14)*$E14'

9. Copy B17 and paste it into C17.
10. Assign the following command into D17.

    '=Index(D2:D12,$B14)'

11. Copy D17 and paste it into E17.
12. Open 'Solver Parameters' window.
13. Assign 'E14' into 'Set Objective' and choose 'Min'.
14. Assign 'E14, F2:F12' into 'By Changing Variable Cells'.
15. Click on 'Add' and assign 'B16:C16' into 'Cell Reference', then select '<=', and assign 'B17:C17' into 'Constraint'.
16. Click on 'Add' and assign 'D16:E16' into 'Cell Reference', then select '>=', and assign 'D17:E17' into 'Constraint'. Then click on OK.
17. Tick 'Make Unconstrained Variables Non-Negative'.
18. Choose 'Simplex LP' from 'Select a Solving Method'.
19. Click on 'Solve'.
20. Save or 'Save As' your excel file with this type 'Excel macro-Enabled Workbook (*.xlsm)'.
21. Label H1 as 'Optimal Theta'.
22. Label I1-S1 as 'Optimal Lambdas' for A01-A11, respectively.
23. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
24. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
25. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
26. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
27. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub' as Fig. 2.21 depicts.

```
Dim i As Integer
For i = 1 To 11
   Range("B14") = i
   SolverSolve Userfinish:=True
   Range("H" & i + 1) = Round(Range("E14"), 4)
   Range("F2:F12").Copy
   Range("I" & i + 1).Select
   Selection.PasteSpecial Transpose:=True
Next i
```

28. Close the 'Microsoft Visual Basic for Applications' window.
29. Click on the small rectangle which was automatically made on the Excel sheet by step 21.

Table 8.4 The optimal lambdas for data in Table 8.1

DMU	θ^*	λ^*_{A01}	λ^*_{A02}	λ^*_{A04}	λ^*_{A05}
A01	1.0000	1.0000			
A02	1.0000		1.0000		
A03	0.7291	0.2491			0.7230
A04	1.0000			1.0000	
A05	1.0000				1.0000
A06	0.9320	0.2875	0.2082	0.2975	
A07	0.3564	0.8465			0.0446
A08	0.3610	0.7370			0.0692
A09	0.4941	0.4118			0.4118
A10	0.5122	0.8963			0.1280
A11	0.3974	0.7483			0.1126

30. The results are represented into cells H2:S12. Column H represents the CCR-scores for DMUs with four decimal digits, and columns I-S illustrate the optimal solutions for the lambdas, $\lambda^*_{A01} - \lambda^*_{A11}$, as shown in Table 8.4. Note that positive lambdas are only shown in Table 8.4.

When a DMU lies on the frontier, that is, the DMU is CCR-efficient, the corresponded lambda for that DMU is 1 and the corresponded lambdas for the other DMUs are 0, as displayed in Table 8.4. When a DMU is CCR-inefficient, that is, the DMU does not lie on the frontier, CCR radially projects the DMU on the frontier. In this case, the corresponded positive values of λ^* represent the reference sets for that DMU. For instance, the reference sets for A03 are A01 and A05, because $\lambda^*_{A01} = 0.2491 > 0$ and $\lambda^*_{A05} = 0.7230 > 0$. In other words, CCR projects A03 on the line which connects A01 and A05. Similarly, the reference sets for A06 are A01, A02 and A04, because $\lambda^*_{A01} = 0.2875 > 0$, $\lambda^*_{A02} = 0.2082 > 0$ and $\lambda^*_{A04} = 0.2975 > 0$. In this case, CCR projects A06 on the plane which passes A01, A02 and A04.

8.3 Banker, Charnes and Cooper Model

Since CCR uses CRS technology, Banker et al. (1984) proposed CCR with VRS technology, and called the model Banker, Charnes and Cooper (BCC). Eqs. 8.6 and 8.7 are IO-BCC and OO-BCC, respectively.

$$
\begin{aligned}
&\theta_l^* = \max\left(\textstyle\sum_{k=1}^{p} y_{lk} w_k^+ + w\right),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = 1,\\
&\left(\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ + w\right) - \textstyle\sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.6}
$$

$$
\begin{aligned}
&\varphi_l^* = \min(\textstyle\sum_{j=1}^m x_{lj} w_j^- + w),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^p y_{lk} w_k^+ = 1,\\
&\textstyle\sum_{k=1}^p y_{ik} w_k^+ - (\sum_{j=1}^m x_{ij} w_j^- + w) \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0 \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{8.7}
$$

Similar to CCR, the optimal value of BCC is at most 1, and we have the following definitions:

Definition 8.4 A DMU is BCC-efficient if and only if $\theta^* = 1$ in Eq. 8.6, for some positive optimal solutions w_j^{-*} and w_k^{+*}, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Definition 8.5 A DMU is BCC-efficient if and only if $\varphi^* = 1$ in Eq. 8.7, for some positive optimal solutions w_j^{-*} and w_k^{+*}, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

The dual linear programming models for IO-BCC and OO-BCC (Envelopment Forms) are as follows, respectively:

$$
\begin{aligned}
&\theta_l^* = \min \theta_l\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^n x_{ij} \lambda_i \le x_{lj} \theta_l, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^n y_{ik} \lambda_i \ge y_{lk}, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^n \lambda_i = 1,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.8}
$$

$$
\begin{aligned}
&\varphi_l^* = \min \varphi_l\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^n x_{ij} \lambda_i \le x_{lj}, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^n y_{ik} \lambda_i \ge y_{lk} \varphi_l, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^n \lambda_i = 1,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.9}
$$

Since the generated PPS with VRS technology is a subset of the generated PPS with CRS, the BCC score for a DMU is not less than the CCR score for that DMU. In other words, A DMU can be BCC-efficient, but CCR-inefficient.

If the constraint $w \ge 0$ is added to the constraints of the multiplier form of IO-BCC (Eq. 8.6), we have the IO-CCR with the NDRS technology, and if the constraint $w \le 0$ is added to the constraints of the multiplier form of IO-BCC, we have the IO-CCR with the NIRS technology. Similarly, if the constraint $w \ge 0$ is added to the constraints of the multiplier form of OO-BCC (Eq. 8.7), we have the

OO-CCR with the NIRS technology, and if the constraint $w \leq 0$ is added to the constraints of the multiplier form of OO-BCC, we have the OO-CCR with the NDRS technology.

In addition, if the constraint $\sum_{i=1}^{n} \lambda_i \leq 1$ is added to the constraints of the envelopment form of IO-BCC (Eq. 8.8), we have the IO-CCR with the NIRS technology, and if the constraint $\sum_{i=1}^{n} \lambda_i \geq 1$ is added to the constraints of the envelopment form of IO-BCC, we have the IO-CCR with the NDRS technology. Similarly, if the constraint $\sum_{i=1}^{n} \lambda_i \leq 1$ is added to the constraints of the envelopment form of OO-BCC (Eq. 8.9), we have the OO-CCR with the NIRS technology, and if the constraint $\sum_{i=1}^{n} \lambda_i \geq 1$ is added to the constraints of the envelopment form of OO-BCC, we have the OO-CCR with the NDRS technology.

Now, suppose that $s_j^- = x_{lj}\theta - \sum_{i=1}^{n} x_{ij}\lambda_i$, for $j = 1, 2, \ldots, m$, is the excess of the jth input factor, and $s_k^+ = y_{lk} - \sum_{i=1}^{n} y_{ik}\lambda_i$, for $k = 1, 2, \ldots, p$, is the shortfall of the kth output factor.

Definition 8.5 (IO-CCR Phase II). A DMU is CCR-efficient if and only if $\theta^* = 1$ and $s_j^{-*} = 0$ and $s_k^{+*} = 0$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Definition 8.6 (OO-CCR Phase II). A DMU is CCR-efficient if and only if $\varphi^* = 1$ and $s_j^{-*} = 0$ and $s_k^{+*} = 0$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

The same definitions can be illustrated for IO-BCC Phase II and OO-BCC Phase II (see exercise 8.8).

8.4 CRS Output-Input Ratio Analysis

As discussed in the previous sections, a DMU labelled $l\,(l = 1, 2, .., n)$ is located on the CRS frontier, that is, DMU$_l$ is technically efficient or DMU$_l$ has 'done the job right', if the optimal value of θ for DMU$_l$ is 1. This means that

$$\max \frac{\dfrac{\sum_{k=1}^{p} y_{lk} w_k^+}{\sum_{j=1}^{m} x_{lj} w_j^-}}{\max\left\{\dfrac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} : i = 1, 2, \ldots, n\right\}} = 1. \tag{8.10}$$

That is, there exist $\tilde{w}_j^- \geq 0$ and $\tilde{w}_k^+ \geq 0$ such that

$$\frac{\sum_{k=1}^{p} \tilde{y}_{lk} w_k^+}{\sum_{j=1}^{m} \tilde{x}_{lj} w_j^-} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik} \tilde{w}_k^+}{\sum_{j=1}^{m} x_{ij} \tilde{w}_j^-} : i = 1, 2, \ldots, n\right\}. \tag{8.11}$$

Similarly, if there are exists $\tilde{w}_j^- \geq 0$ and $\tilde{w}_k^+ \geq 0$ such that Eq. 8.11 holds, DMU$_l$ is located on the CRS frontier, and we have the following theorem.

Theorem 8.2 *If there exist* $\tilde{w}_j^- \geq 0$, *for* $j = 1, 2, \ldots, m$, *and* $\tilde{w}_k^+ \geq 0$, *for* $k = 1, 2, \ldots, p$, *such that Eq. 8.11 holds,* DMU_l *is located on the CRS frontier.*

Proof See Exercise 8.7. □

Theorem 8.2 essentially allows a characterization of DMUs that lie on the CRS frontier as having the highest ratio of aggregated outputs to aggregated inputs using some selection of weights (Chen and Ali 2002). The theorem suggests the possibility of finding 'frontier DMUs' by selecting different combinations of input and output factors to determine a specific selection of weights.

Suppose that J is a subset of the indexes of input factors, that is, $J \subseteq \{1, 2, \ldots, m\}$, and $\tilde{w}_j^- = 1$, for $j \in J$, and $\tilde{w}_j^- = 0$, for $j \notin J$. Also assume that K is a subset of the indexes of output factors, that is, $K \subseteq \{1, 2, \ldots, p\}$, and $\tilde{w}_k^+ = 1$, for $k \in K$, and $\tilde{w}_k^+ = 0$, for $k \notin K$. Similar to Theorem 8.2, we have the following theorems.

Theorem 8.3 DMU_l *is located on the CRS frontier, if the following equation holds*

$$\frac{\sum_{k \in K} y_{lk}}{\sum_{j \in J} x_{lj}} = \max\left\{\frac{\sum_{k \in K} y_{ik}}{\sum_{j \in J} x_{ij}} : i = 1, 2, \ldots, n\right\}. \tag{8.12}$$

Proof The proof is simply concluded from Theorem 8.2. □

Theorem 8.4 DMU_l *is located on the CRS frontier, if the following equation holds*

$$\frac{\sum_{k=1}^{p} y_{lk}}{\sum_{j=1}^{m} x_{lj}} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik}}{\sum_{j=1}^{m} x_{ij}} : i = 1, 2, \ldots, n\right\}. \tag{8.13}$$

Proof The proof is simply concluded from Theorem 8.2. □

The outcomes of Theorems 8.2, 8.3, and 8.4 can be examined for data in Table 8.1. As Table 8.5 displays, let's calculate different combinations of input and output factors such as y_1/x_1, y_2/x_1, y_1/x_2, y_2/x_2, $(y_1 + y_2)/x_1$, $(y_1 + y_2)/x_2$, $y_1/(x_1 + x_2)$, $y_2/(x_1 + x_2)$ and $(y_1 + y_2)/(x_1 + x_2)$, where x_1, x_2, y_1 and y_2 are input 1, input 2, output 1 and output 2, respectively.

Each column in Tables 8.5 and 8.6 represent different combination of input and output factors. The maximum value for each combination is bolded. As can be seen, only DMUs A01, A02, A04 and A06 are located on the CRS frontier, as shown in Tables 8.2-8.4.

In this example, all DMUs on the CRS frontier are found by output–input ratios. However, in contrast with DEA, a performance measure based upon only the ratio of one single output to one single input fails to detect the entirety of performance regarding a set of input and output factors.

Table 8.5 Applying Theorems 8.2, 8.3 and 8.4 for data in Table 8.1

DMU	$\frac{y_1}{x_1}$	$\frac{y_2}{x_1}$	$\frac{y_1}{x_2}$	$\frac{y_2}{x_2}$	$\frac{y_1}{x_1+x_2}$	$\frac{y_2}{x_1+x_2}$	$\frac{y_1+y_2}{x_1}$	$\frac{y_1+y_2}{x_2}$	$\frac{y_1+y_2}{x_1+x_2}$
A01	4.00	5.33	2.50	3.33	**2.29**	**1.43**	6.50	8.67	**3.71**
A02	**6.00**	3.00	2.33	1.17	2.00	0.78	**8.33**	4.17	2.78
A03	1.83	4.25	0.65	1.50	1.28	0.45	2.47	5.75	1.73
A04	3.80	2.71	**2.60**	1.86	1.58	1.08	6.40	4.57	2.67
A05	2.25	**6.00**	1.50	**4.00**	1.64	1.09	3.75	**10.00**	2.73
A06	4.00	3.11	2.34	1.82	1.75	1.03	6.34	4.93	2.78
A07	1.14	1.60	0.86	1.20	0.67	0.50	2.00	2.80	1.17
A08	1.03	1.49	0.85	1.22	0.61	0.50	1.88	2.72	1.11
A09	1.40	2.80	0.40	0.80	0.93	0.27	1.80	3.60	1.20
A10	1.56	2.33	1.17	1.75	0.93	0.70	2.72	4.08	1.63
A11	1.43	2.15	0.51	0.77	0.86	0.31	1.94	2.92	1.17

Table 8.6 More combinations for data in Table 8.1

DMU	$\frac{y_1}{10x_1+x_2}$	$\frac{y_1}{x_1+10x_2}$	$\frac{10y_1+y_2}{x_1}$	$\frac{y_1+10y_2}{x_1}$	$\frac{10y_1+y_2}{x_1+x_2}$
A01	0.37	0.47	42.50	29.00	**24.29**
A02	**0.50**	0.29	**62.33**	29.33	20.78
A03	0.18	0.34	18.92	8.28	13.23
A04	0.33	0.25	40.60	**29.80**	16.92
A05	0.22	**0.47**	24.00	17.25	17.45
A06	0.35	0.29	42.34	27.43	18.53
A07	0.11	0.14	12.29	9.71	7.17
A08	0.10	0.13	11.15	9.48	6.60
A09	0.13	0.23	14.40	5.40	9.60
A10	0.15	0.20	16.72	13.22	10.03
A11	0.13	0.19	14.80	6.53	8.90

8.5 VRS Output-Input Ratio Analysis

Theorem 8.2 can be extended for VRS technology as well. If a DMU lies on the CRS frontier, the DMU lies on VRS frontier, but not vice versa. Consequently, the following theorems hold.

Theorem 8.4 If there are exists $\tilde{w}_j^- \geq 0$, for $j = 1, 2, \ldots, m$, and $\tilde{w}_k^+ \geq 0$, for $k = 1, 2, \ldots, p$, and $w \in \mathbb{R}$, such that Eq. 8.14 holds, DMU$_l$ is located on the VRS frontier.

$$\frac{\sum_{k=1}^{p} y_{lk}\tilde{w}_k^+ + w}{\sum_{j=1}^{m} x_{lj}\tilde{w}_j^-} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik}\tilde{w}_k^+ + w}{\sum_{j=1}^{m} x_{ij}\tilde{w}_j^-} : i = 1, 2, \ldots, n\right\}. \tag{8.14}$$

Proof The proof is simply concluded from Theorem 8.2. □

Theorem 8.5 *If there are exists* $\tilde{w}_j^- \geq 0$, *for* $j = 1, 2, \ldots, m$, *and* $\tilde{w}_k^+ \geq 0$, *for* $k = 1, 2, \ldots, p$, *and* $w \in \mathbb{R}$, *such that Eq.* 8.14 *holds,* DMU_l *is located on the VRS frontier.*

$$\frac{\sum_{j=1}^{m} x_{lj}\tilde{w}_j^- + w}{\sum_{k=1}^{p} y_{lk}\tilde{w}_k^+} = \min\left\{\frac{\sum_{j=1}^{m} x_{ij}\tilde{w}_j^- + w}{\sum_{k=1}^{p} y_{ik}\tilde{w}_k^+} : i = 1, 2, \ldots, n\right\}. \tag{8.15}$$

Proof The proof is simply concluded from Theorem 8.2. □

Now, suppose that $\tilde{w}_j^- = 1$, for $j \in J$, $\tilde{w}_k^+ = 0$, for all k, and $w = 1$. Thus we have.

Theorem 8.6 DMU_l *is located on the VRS frontier, if the following equation holds*

$$\sum\nolimits_{j \in J} x_{lj} = \min\left\{\sum\nolimits_{j \in J} x_{lj} =: i = 1, 2, \ldots, n\right\}. \tag{8.16}$$

Proof The proof is simply concluded from Theorem 8.2. □

Similarly, assume that $\tilde{w}_j^+ = 1$, for $k \in K$, $\tilde{w}_j^- = 0$, for all j, and $w = 1$. Thus we have.

Theorem 8.7 DMU_l *is located on the VRS frontier, if the following equation holds*

$$\sum\nolimits_{k \in K} y_{lk} = \max\left\{\sum\nolimits_{k \in K} y_{ik} : i = 1, 2, \ldots, n\right\}. \tag{8.17}$$

Proof The proof is simply concluded from Theorem 8.2. □

BCC can easily be solved by adding the command '=Sum(F2:F12)' into cell F13. After that, open 'Solver Parameters' window, and click on 'Add'. Assign 'F13' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'. The outcomes are illustrated in Table 8.7.

Five DMUs A01, A02, A04, A05, A06 are BCC-efficient. A06 is the only BCC-efficient DMU which is not CCR-efficient.

Table 8.8 shows different combinations of input and output factors such as, $(x_1 - 1)/y_2$, $(x_2 - 1)/y_2$ and $(2/100)x_1 + (1/150)x_2$, to exemplify Theorems 8.5 and 8.6. The minimum values in each column are bolded.

As can be seen, all BCC-efficient DMUs can be found by different combination of output–input ratios.

8.6 Conclusion

The relationship between ratio analysis and DEA efficiency is revealed in this section. DEA includes the basis of ratio analyses, that is, a unit with the highest rank, in respect to the ratio of a single output to a single input, dominates other units. It is also shown a deficiency of ratio analysis is that it fails to identify all types of dominating units, unlike DEA. In other words, a performance measure based on the

Table 8.7 The envelopment IO-BCC results for data in Table 8.1

DMU	θ^*	λ^*_{A01}	λ^*_{A02}	λ^*_{A04}	λ^*_{A05}	λ^*_{A06}
A01	1.0000	1.0000				
A02	1.0000		1.0000			
A03	0.7500	0.5000			0.5000	
A04	1.0000			1.0000		
A05	1.0000				1.0000	
A06	1.0000					1.0000
A07	0.4000	1.0000				
A08	0.4478	0.9142			0.0858	
A09	0.6000	1.0000				
A10	0.5303	0.7727		0.0455	0.1818	
A11	0.4615	1.0000				

Table 8.8 Applying Theorems 8.5 and 8.6 for data in Table 8.1

DMU	$(x_1 - 1)/y_2$	$(x_2 - 1)/y_1$	$0.02x_1 + 0.00\bar{6}x_2$
A01	0.3900	0.1813	**1.0000**
A02	0.4143	0.3278	**1.0000**
A03	1.5333	0.2294	2.1267
A04	**0.3769**	0.3632	1.4667
A05	0.6583	**0.1611**	1.8000
A06	0.4146	0.3143	**1.0000**
A07	1.1556	0.6167	2.6000
A08	1.1707	0.6600	2.3867
A09	2.4750	0.3500	2.3333
A10	0.8476	0.4214	2.2000
A11	1.9400	0.4571	2.3933

ratio of a single output to a single input fails to capture the entirety of performance with respect to a set of outputs and inputs.

8.7 Exercises

8.1. By a counterexample show that Theorem 8.1 is not necessary satisfied for BCC.
8.2. Prove that there exists a positive parameter t, in which the following model is equivalent with CCR.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = t,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&0 \le w_j^+ \le 1, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&0 \le w_k^- \le 1, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.18}
$$

8.3. Prove that the following model is equivalent with CCR.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&0 \le w_j^+ \le 1, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&0 \le w_k^- \le 1, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.19}
$$

8.4. Prove that the optimal solution of the following model is w_j^{-*}/θ^*, for $j = 1$, 2, …, m, and w_k^{+*}/θ^*, for $k = 1$, 2, …, p.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} \theta^* x_{lj} w_j^- = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} \theta^* x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.20}
$$

8.5. Prove that the optimal solution of the following model is w_j^{-*}/φ^*, for $j = 1$, 2, …, m, and w_k^{+*}/φ^*, for $k = 1$, 2, …, p.

$$
\begin{aligned}
&\max \textstyle\sum_{j=1}^{m} x_{lj} w_k^-,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{p} \varphi^* y_{lk} w_k^+ = 1,\\
&\textstyle\sum_{k=1}^{p} \varphi^* y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_k^+ \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, m,\\
&w_k^- \ge 1, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.21}
$$

8.6. Prove Theorem 8.1.
8.7. Prove Theorem 8.2.
8.8. Solve the following IO-BCC Phase II for data in Table 8.1, where θ_l^* can be found from Table 8.7.

$$\begin{aligned}
&\max \textstyle\sum_{j=1}^{m} s_j^- + \sum_{k=1}^{p} s_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = \theta_l^* x_{lj}, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^{n} \lambda_i = 1\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.22}$$

Chapter 9
Production Planning Problem

9.1 Introduction

Production in large firms with a centralized decision-making unit usually involves the contribution of several individual units. Supermarket chains or organizations with several workshops include more than one unit in which each unit contributes to a part of the entire production. For example, a sales plan is decided by a bank's cooperate management for its divisions regarding the number of credit cards to be issued and the number of loans to be processed. The central division defines a plan to optimize the average or overall production performance in the entire organization after planning. In this chapter, a production-planning problem which is regularly faced by the central decision-making units is illustrated. Two planning ideas, proposed by Du et al. (2010), are discussed to arrange new input and output plans for all specific units when demand deviations can be predicted.

9.2 Twenty Fast-Food restaurants

Assume a set of 20 fast food restaurants that are located in the city of Hefei, Anhui Province, China. These fast food restaurants belong to the same chain, with a central decision-making team of several members who supervise all branches' operations and make future sales plans for them, as well.

Table 9.1 illustrates the given data, where the restaurants are labeled R01–R20. These data are collected for a month, and the central decision-making team attempts to arrange input and sales (output) plans for all 20 fast food restaurants under the same chain for the next month in business.

There are two input factors for each restaurant, man-hour (10^3 h) and shop size (10^2 m^2), which are labeled I1 and I2, respectively. Here, man-hour means the labor force used within a certain period, and shop size means the total rental floor space of

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the restaurant that can be used for the purpose of serving customers. There are also five output factors, such as, the sales of meat dish, vegetable dish, soup, noodles and beverage, all in 10^3 serving unit, which are labeled O1–O5.

The demand changes for meat dish, vegetable dish, soup, noodles and beverage are forecasted as $\tilde{D}_1 = 3$, $\tilde{D}_2 = 2.4$, $\tilde{D}_3 = -3$, $\tilde{D}_4 = 6$, $\tilde{D}_5 = 3$ (10^3 serving) in the next business month, where a positive change represents an increase in demand while a negative change represents a decrease in demand. How should these 20 fast food restaurants be benchmarked to optimize the overall production performance of all restaurants after planning?

9.2.1 CCR Efficient Restaurants

The CCR model (See Eq. 8.2 and Form 1), for the 20 restaurants in Table 9.1 is consisted by Eq. 9.1.

Table 9.1 Example of 20 fast food restaurants

DMUs	I1	I2	O1	O2	O3	O4	O5
R01	3.2	2.0	2.24	2.46	1.22	3.12	0.96
R02	3.4	2.1	2.12	2.52	1.34	3.08	0.88
R03	3.1	1.8	2.08	2.25	1.05	2.85	0.74
R04	3.8	2.2	2.45	2.10	1.3	2.96	0.79
R05	4.2	2.6	2.80	2.78	1.42	3.48	1.05
R06	4.1	2.5	2.65	2.95	1.38	3.25	0.98
R07	3.8	2.3	2.60	2.24	1.15	3.18	0.95
R08	3.8	2.2	2.50	2.15	1.10	3.20	0.82
R09	2.9	1.6	2.10	2.04	0.98	2.88	0.72
R10	4.2	2.8	2.90	2.85	1.52	3.36	1.12
R11	3.4	2.1	2.60	2.45	1.36	3.32	0.82
R12	4.0	2.4	2.78	2.66	1.18	3.15	0.98
R13	3.8	2.6	2.84	2.38	1.25	3.29	0.85
R14	3.4	1.9	2.33	2.20	1.06	2.99	0.82
R15	2.8	1.6	2.00	2.18	1.96	2.84	0.71
R16	3.5	2.2	2.40	2.25	1.26	2.93	0.74
R17	4.2	2.5	2.68	2.50	1.46	3.22	0.92
R18	3.3	1.8	2.05	2.20	1.12	3.02	0.78
R19	3.6	1.9	2.00	2.16	1.02	2.89	0.74
R20	3.1	1.7	2.05	2.12	0.94	2.90	0.68

$$
\begin{aligned}
&\theta_l^* = \max\left(y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+ + y_{l4}w_4^+ + y_{l5}w_5^+\right),\\
&\text{Subject to}\\
&x_{l1}w_1^- + x_{l2}w_2^- = 1,\\
&\textstyle\sum_{k=1}^{5} y_{ik}w_k^+ - \sum_{j=1}^{2} x_{ij}w_j^- \le 0, \text{for } i = 1, 2, \ldots, 20,\\
&w_j^+ \ge 0, \text{for } j = 1, 2,\\
&w_k^- \ge 0, \text{for} k \ = 1, 2, \ldots, 5.
\end{aligned}
\tag{9.1}
$$

The following instructions solve Eq. 9.1 for data in Table 9.1.

1. Copy the 8 columns of Table 9.1 on an Excel sheet into cells A1:H21, as Fig. 9.1 depicts.
2. Label A23 as ‘Weights’, A25 as ‘Index’, D25 as ‘Constraint’, G25 as ‘Objective’, I1 as ‘Constraints’, J1 as ‘CCR Score’ and K1 as ‘Rank’.
3. Assign number 1 to B25.

H25 =SUMPRODUCT(INDEX(D2:H21,B25,0),D23:H23)

	A	B	C	D	E	F	G	H	I	J
1	DMUs	I1	I2	O1	O2	O3	O4	O5	Constraints	CCR
2	R01	3.2	2.0	2.24	2.46	1.22	3.12	0.96	0.000	1.0000
3	R02	3.4	2.1	2.12	2.52	1.34	3.08	0.88	-0.113	0.9564
4	R03	3.1	1.8	2.08	2.25	1.05	2.85	0.74	-0.088	0.9365
5	R04	3.8	2.2	2.45	2.10	1.30	2.96	0.79	-0.241	0.8724
6	R05	4.2	2.6	2.80	2.78	1.42	3.48	1.05	-0.200	0.9154
7	R06	4.1	2.5	2.65	2.95	1.38	3.25	0.98	-0.223	0.9253
8	R07	3.8	2.3	2.60	2.24	1.15	3.18	0.95	-0.152	0.9382
9	R08	3.8	2.2	2.50	2.15	1.10	3.20	0.82	-0.196	0.8880
10	R09	2.9	1.6	2.10	2.04	0.98	2.88	0.72	0.000	1.0000
11	R10	4.2	2.8	2.90	2.85	1.52	3.36	1.12	-0.259	0.9552
12	R11	3.4	2.1	2.60	2.45	1.36	3.32	0.82	-0.135	1.0000
13	R12	4.0	2.4	2.78	2.66	1.18	3.15	0.98	-0.182	0.9454
14	R13	3.8	2.6	2.84	2.38	1.25	3.29	0.85	-0.366	0.9773
15	R14	3.4	1.9	2.33	2.20	1.06	2.99	0.82	-0.066	0.9513
16	R15	2.8	1.6	2.00	2.18	1.96	2.84	0.71	-0.011	1.0000
17	R16	3.5	2.2	2.40	2.25	1.26	2.93	0.74	-0.281	0.8970
18	R17	4.2	2.5	2.68	2.50	1.46	3.22	0.92	-0.270	0.8642
19	R18	3.3	1.8	2.05	2.20	1.12	3.02	0.78	-0.043	0.9546
20	R19	3.6	1.9	2.00	2.16	1.02	2.89	0.74	-0.134	0.8648
21	R20	3.1	1.7	2.05	2.12	0.94	2.90	0.68	-0.078	0.9528
22										
23	Weights	0.0000	0.5000	0.0000	0.0000	0.0000	0.0926	0.7407		
24										
25	Index	1		Constraint	1.0000		Objective	1.0000		
26										

Fig. 9.1 Setting Excel sheet to solve Eq. 9.1

4. Assign the following command (without quotations mark) into I2,

   '=Sumproduct(D2:H2,D$23:H$23)-Sumproduct(B2:C2,B$23:C$23)'.

5. Copy I2 (by Ctrl + C), and paste it (by Ctrl + V) to I3–I21.
6. Assign the command '=Rank(J2,J$2:J$21,0)' into K2.
7. Copy K2 and then paste it to K3–K21.
8. Assign the command '=Sumproduct(Index(B2:C21,B25,0),B23:C23)' into E25.
9. Assign the command '=Sumproduct(Index(D2:H21,B25,0),D23:H23)' into H25.
10. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.2 illustrates.
11. Assign 'H25' into 'Set Objective' and choose 'Max'.
12. Assign 'B23:H23' into 'By Changing Variable Cells'.
13. Click on 'Add' and assign 'I2:I21' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
14. Click on 'Add' and assign 'E25' into 'Cell Reference', then select '=' and assign '1' into 'Constraint'. Then click on 'OK'.
15. Tick 'Make Unconstrained Variables Non-Negative'.

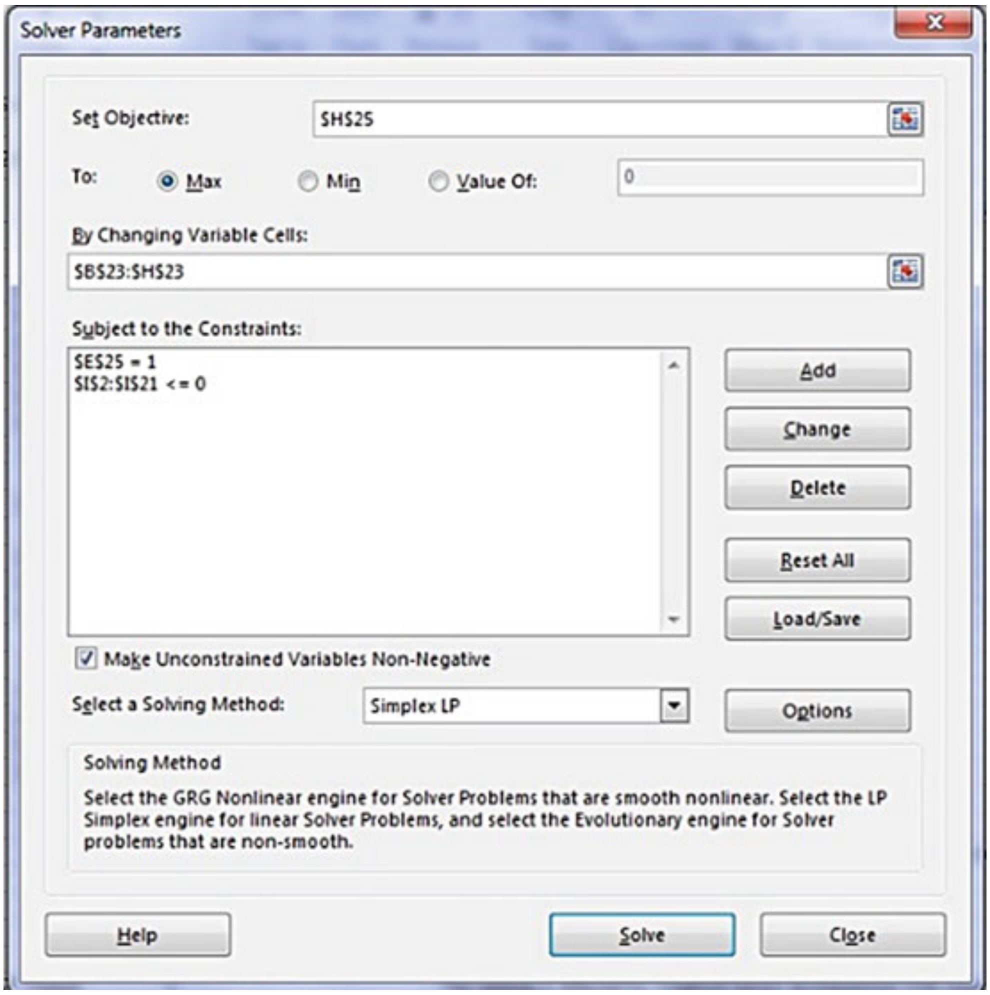


Fig. 9.2 Setting Solver to solve Eq. 9.1

16. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
17. Click on ‘Solve’ (Fig. 9.3).
18. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window (Fig. 9.4).

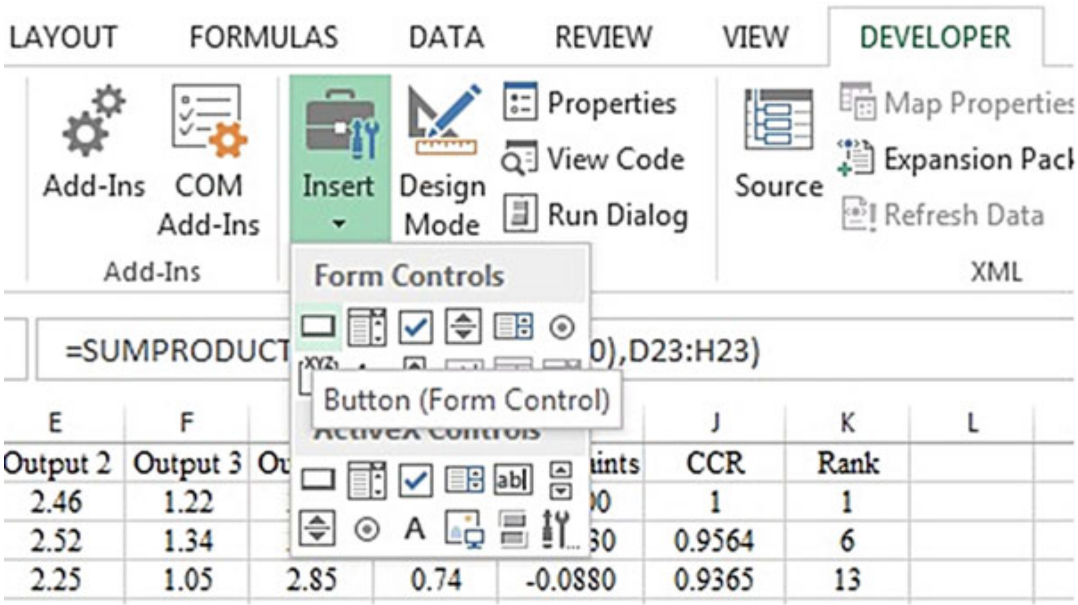


Fig. 9.3 The Form control menu

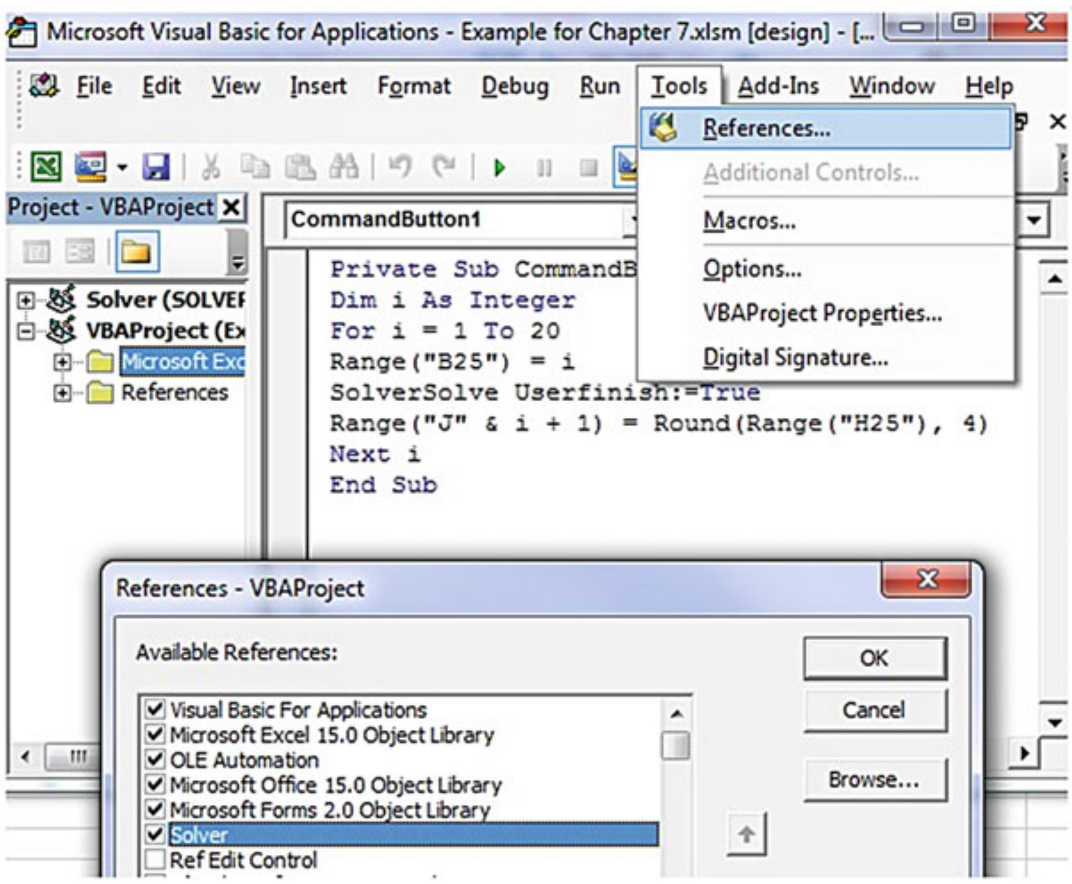


Fig. 9.4 Setting VBA to solve Eq. 9.1

19. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
20. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
21. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

```
Dim i As Integer
For i = 1 To 20
 Range("B25") = i
 SolverSolve Userfinish:=True
 Range("J" & i + 1) = Round(Range("H25"), 4)
Next i
```

22. Close the 'Microsoft Visual Basic for Applications' window.
23. Click on the small rectangle which was automatically made on the Excel sheet and created by step 19. The results are represented to cells J2:J21.

Table 9.2 illustrates the CCR-inefficient and CCR-efficient restaurants. There were four CCR-efficient restaurants, which are bolded in Table 9.2, and the rest of the restaurants were CCR-inefficient.

In the next sections, we construct two ideas to optimize overall performance of these 20 restaurants.

Table 9.2 The results of solving Eq. 9.1

DMUs	CCR Scores
1	**1.0000**
2	0.9564
3	0.9365
4	0.8724
5	0.9154
6	0.9253
7	0.9382
8	0.8880
9	**1.0000**
10	0.9552
11	**1.0000**
12	0.9454
13	0.9773
14	0.9513
15	**1.0000**
16	0.8970
17	0.8642
18	0.9546
19	0.8648
20	0.9528

9.3 Idea 1: Overall Production Performance

In this section, we theoretically construct Idea 1. First we assume that demands for all output factors are positive, that is, increase in demands.

Part 1 Suppose that a set of n DMUs, DMU$_i$, ($i = 1, 2, \ldots, n$) is given in which each DMU has m input factors, x_{ij}, ($j = 1, 2, \ldots, m$) and p output factors, y_{ik}, ($k = 1, 2, \ldots, p$). Assume that the upper demand change for output k ($k = 1, 2, \ldots, p$) in the next production season can be forecasted as $\tilde{D}_k$, where $\tilde{D}_k > 0$. As a result, with the same values of output factors for the production plan, we should have $\sum_{i=1}^{n} y_{ik} \leq \tilde{D}_k$, for $k = 1, 2, \ldots, p$. In order to meet the demand changes, the most preferred input-output plans should be determined for all DMUs.

For Idea 1, we choose a planning principle to maximize the CCR-efficiency of average input and output levels of the entire DMUs. This ensures that the average (overall) production capability for all DMUs to earn their highest potential after planning.

Similar to Eq. 5.24, we have the following model where DMU$_l$ is under evaluation.

$$
\begin{aligned}
&\max \frac{y_{l1}w_1^+ + y_{l2}w_2^+ + \cdots + y_{lp}w_p^+}{x_{l1}w_1^- + x_{l2}w_2^- + \cdots + x_{l\mathrm{m}}w_m^-},\\
&\text{Subject to}\\
&\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + \cdots + y_{ip}w_p^+}{x_{i1}w_1^- + x_{i2}w_2^- + \cdots + x_{i\mathrm{m}}w_m^-} \leq 1, \text{for}, i = 1, 2, \ldots, n,\\
&w_j^+ \geq 0, \text{for} j = 1, 2, \ldots, m,\\
&w_k^- \geq 0, \text{for} k = 1, 2, \ldots, p.
\end{aligned}
\tag{9.2}
$$

As illustrated in the previous chapters, CCR uses the CRS technology, so we suppose that for DMU$_i$, $i = 1, 2, \ldots, n$ the input and output factors are changed by the same proportion denoted by Δ_i, $i = 1, 2, \ldots, n$, where $\Delta_i \geq 0$.

Therefore, we assume that the new production plan for input and output factors of DMU$_i$, $i = 1, 2, \ldots, n$ is a DMU with the following factors $(x_{i1} + \Delta_i x_{i1})$, $(x_{i2} + \Delta_i x_{i2})$, $\ldots$, $(x_{im} + \Delta_i x_{im})$, $(y_{i1} + \Delta_i y_{i1})$, $(y_{i2} + \Delta_i y_{i2})$, $\ldots$, $(y_{ip} + \Delta_i y_{ip})$, where $\sum_{i=1}^{n} y_{ik}\Delta_i \leq \tilde{D}_k$, for $k = 1, 2, \ldots, p$.

Note that, $\tilde{D}_k$ is the upper demand change for output k ($k = 1, 2, \ldots, p$) in the next production season, and $(x_{ij} + \Delta_i x_{ij}) = (1 + \Delta_i)x_{ij}$, for $j = 1, 2, \ldots, m$, and $(y_{ik} + \Delta_i y_{ik}) = (1 + \Delta_i)y_{ik}$, for $k = 1, 2, \ldots, p$.

Equation 9.2 is not changed if we use the new input and output values for DMU$_i$, $i = 1, 2, \ldots, n$, because

$$\frac{(1+\Delta_i)y_{i1}w_1^+ + (1+\Delta_i)y_{i2}w_2^+ + \cdots + (1+\Delta_i)y_{ip}w_p^+}{(1+\Delta_i)x_{i1}w_1^- + (1+\Delta_i)x_{i2}w_2^- + \cdots + (1+\Delta_i)x_{\mathrm{im}}w_m^-}
= \frac{y_{i1}w_1^+ + y_{i2}w_2^+ + \cdots + y_{ip}w_p^+}{x_{i1}w_1^- + x_{i2}w_2^- + \cdots + x_{\mathrm{im}}w_m^-}. \tag{9.3}$$

Nonetheless, adding the constraints $\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k$, for $k = 1, 2, \ldots, p$, may change the optimal solutions, as the optimal solutions may not be unique.

The average input and output factors of all DMUs, denoted by $\bar{x}_j$ and $\bar{y}_k$, respectively, and are given by:

$$\begin{aligned}
\bar{x}_j &= \frac{(1+\Delta_1)x_{1j} + (1+\Delta_2)x_{2j} + \ldots + (1+\Delta_n)x_{nj}}{n}, \text{for } j = 1, 2, \ldots, m\& \\
\bar{y}_k &= \frac{(1+\Delta_1)y_{1k} + (1+\Delta_2)y_{2k} + \ldots + (1+\Delta_n)y_{nk}}{n}, \text{for } k = 1, 2, \ldots, p,
\end{aligned} \tag{9.4}$$

Therefore, we have $n + 1$ DMUs, that is, n DMUs (DMU_i, $i = 1, 2, \ldots, n$) plus the virtual DMU with the average input and output values. Equation 9.2 for this virtual DMU with the average input and output values, where the constraints $\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k$, for $k = 1, 2, \ldots, p$ are also added, is as follows:

$$\begin{aligned}
&\max \frac{\bar{y}_1 w_1^+ + \bar{y}_2 w_2^+ + \cdots + \bar{y}_{\mathrm{p}} w_p^+}{x_1 w_1^- + x_2 w_2^- + \cdots + x_{\mathrm{m}} w_m^-}, \\
&\text{Subject to} \\
&\frac{y_{i1}(1+\Delta_i)w_1^+ + y_{i2}(1+\Delta_i)w_2^+ + \cdots + y_{ip}(1+\Delta_i)w_p^+}{x_{i1}(1+\Delta_i)w_1^- + x_{i2}(1+\Delta_i)w_2^- + \cdots + x_{\mathrm{im}}(1+\Delta_i)w_m^-} \le 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\bar{y}_1 w_1^+ + \bar{y}_2 w_2^+ + \cdots + \bar{y}_{\mathrm{p}} w_p^+}{\bar{x}_1 w_1^- + \bar{x}_2 w_2^- + \cdots + \bar{x}_{\mathrm{m}} w_m^-} \le 1, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&w_j^+ \ge 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \ge 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{9.5}$$

The above equation yields that,

$$
\begin{aligned}
&\max \frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-}, \\
&\text{Subject to} \\
&\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \leq 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-} \leq 1, \\
&\sum_{i=1}^{n} y_{ik} \Delta_i \leq \tilde{D}_k, \text{for} k = 1, 2, \ldots, p, \\
&w_j^+ \geq 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \geq 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \geq 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.6}
$$

From Eq. 9.4 we have

$$
\frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-} = \frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)}. \tag{9.7}
$$

Thus, Eq. 9.6 is equal with the following equation.

$$
\begin{aligned}
&\max \frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)}, \\
&\text{Subject to} \\
&\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \leq 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)} \leq 1, \\
&\sum_{i=1}^{n} y_{ik} \Delta_i \leq \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&w_j^+ \geq 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \geq 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \geq 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.8}
$$

Let's assume that $\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i) = 1$, thus we have the following converted model.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \text{for } i = 1, 2, \ldots, n,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge 0, \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.9}
$$

As discussed in Sect. 6.2, we can define lower bound for the weights w_j^+ and w_k^-, that is, $w_j^+ \ge \varepsilon$ and $w_k^- \ge \varepsilon$, for $j = 1, 2, \ldots, m$ and for $k = 1, 2, \ldots, p$, respectively, where $\varepsilon > 0$. In this case, we have

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \text{for } i = 1, 2, \ldots, n,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge \varepsilon, \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge \varepsilon, \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.10}
$$

We can also set some restrictions on the changing proportions Δ_i, for $i = 1, 2, \ldots, n$ in Eq. 9.10, such as: $\Delta_i \ge \delta_l^{(i)}\Delta_l$, where $i \ne l$, $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, n$.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \quad \text{for} i = 1, 2, \ldots, n,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \quad \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i \ge \delta_l^{(i)}\Delta_l, \quad \text{for } i = 1, 2, \ldots, n \text{ and for } l = 1, 2, \ldots, n \ \text{ where } i \ne l.
\end{aligned}
\tag{9.11}
$$

The advantages of the restrictions $\Delta_i \geq \delta_l^{(i)} \Delta_l$ are to reveal the preferences of the central unit and to avoid the possibility that changes in the production plan might happen to a few DMUs only due to the nature of optimization.

Note that, Eqs. 9.10 and 9.11 are nonlinear programming. We later develop the models such that none of the local and global solutions from the non-linear models are critical. In addition, Eq. 9.11 is the same as Eq. 9.10 if $\delta_l^{(i)} = 0$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$.

Now, suppose that the data in Table 9.3 are given. There are six DMUs in which three DMUs are CCR-efficient and the other three DMUs are CCR-inefficient.

Assume that the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = 3$ in the next production season, that is, the situation of demand increases in the two outputs.

In order to apply Eq. 9.11 for data in Table 9.3, where $\delta_l^{(i)} = 0.2$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$, where $i \neq l$, we have the following instructions.

1. Copy the 5 columns of Table 9.3 on an Excel sheet into cells A1:E7, as Fig. 9.5 depicts.
2. Label A9 as 'Constraint', A11 as 'Weights', A12 as 'SumDelta', D9 as 'Objective', C14 as 'Constraints', C15 as 'Dtilda', F1 as 'Constraints', G1 as 'Deltas', H1 as 'Constraints', G9 as 'delta(*i*,*l*)', J1 as 'Target Input1', K1 as 'Target Input2', L1 as 'Target Output1' and M1 as 'Target Output2'.
3. Assign 0.2 into H9.
4. Assign the following command (without quotations mark) into B9,
   '=Sumproduct(B11:C11,B12:C12)'.
5. Assign the following command (without quotations mark) into E9,

   '=Sumproduct(D11:E11,D12:E12)'.

6. Assign the following command (without quotations mark) into B12,

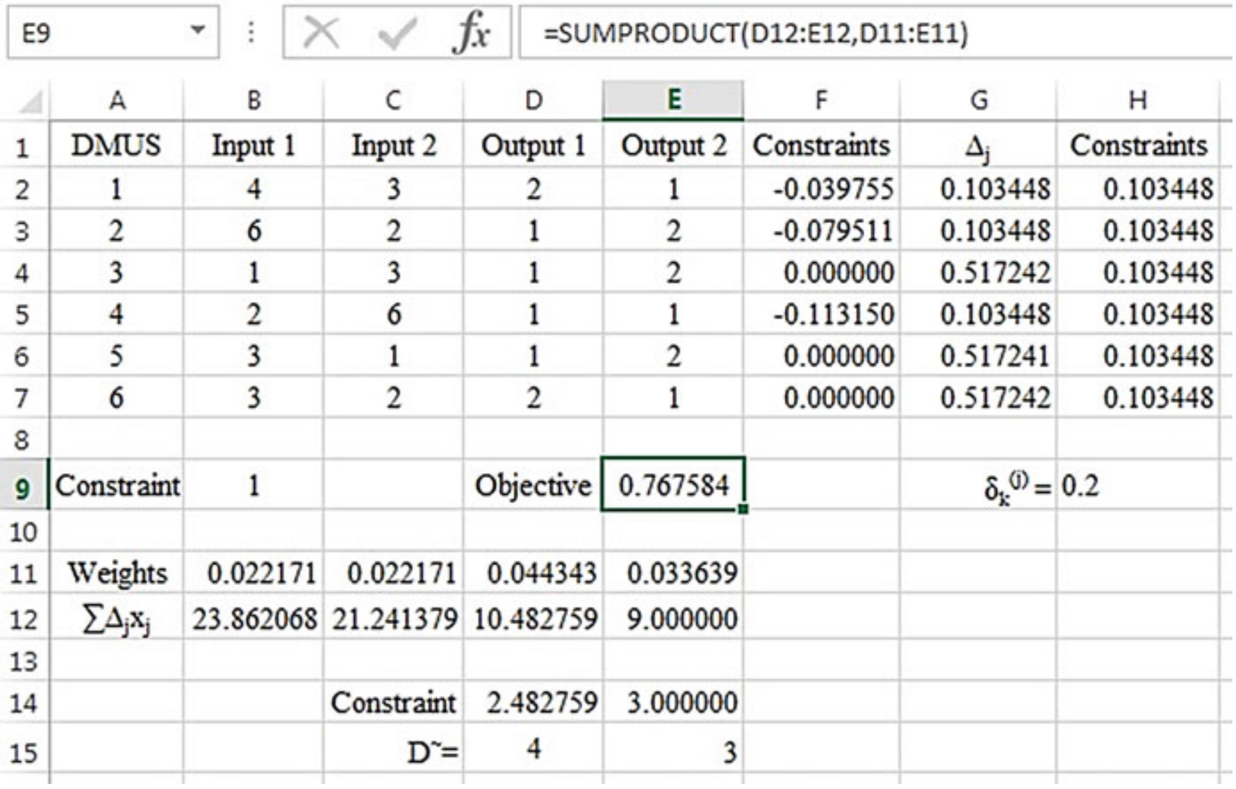
   '=Sumproduct(B2:B7,(1+$G2:$G7))'.

7. Copy B12 (by Ctrl + C), and paste it (by Ctrl + V) to C12, D12 and E12.
8. Assign the following command into D14,

   '=Sumproduct(D2:D7,$G2:$G7)'.

Table 9.3 Data of six DMUs with four factors

DMUS	Input 1	Input 2	Output 1	Output 2	CCR-Score
1	4	3	2	1	0.7368
2	6	2	1	2	0.5000
3	1	3	1	2	1.0000
4	2	6	1	1	0.5000
5	3	1	1	2	1.0000
6	3	2	2	1	1.0000



E9 =SUMPRODUCT(D12:E12,D11:E11)

	A	B	C	D	E	F	G	H
1	DMUS	Input 1	Input 2	Output 1	Output 2	Constraints	Δ_j	Constraints
2	1	4	3	2	1	-0.039755	0.103448	0.103448
3	2	6	2	1	2	-0.079511	0.103448	0.103448
4	3	1	3	1	2	0.000000	0.517242	0.103448
5	4	2	6	1	1	-0.113150	0.103448	0.103448
6	5	3	1	1	2	0.000000	0.517241	0.103448
7	6	3	2	2	1	0.000000	0.517242	0.103448
8								
9	Constraint	1		Objective	0.767584		$\delta_k^{(i)}$ =	0.2
10								
11	Weights	0.022171	0.022171	0.044343	0.033639			
12	$\sum\Delta_j x_j$	23.862068	21.241379	10.482759	9.000000			
13								
14			Constraint	2.482759	3.000000			
15			$\tilde{D}$=	4	3			

Fig. 9.5 Setting Excel sheet to solve Eq. 9.11

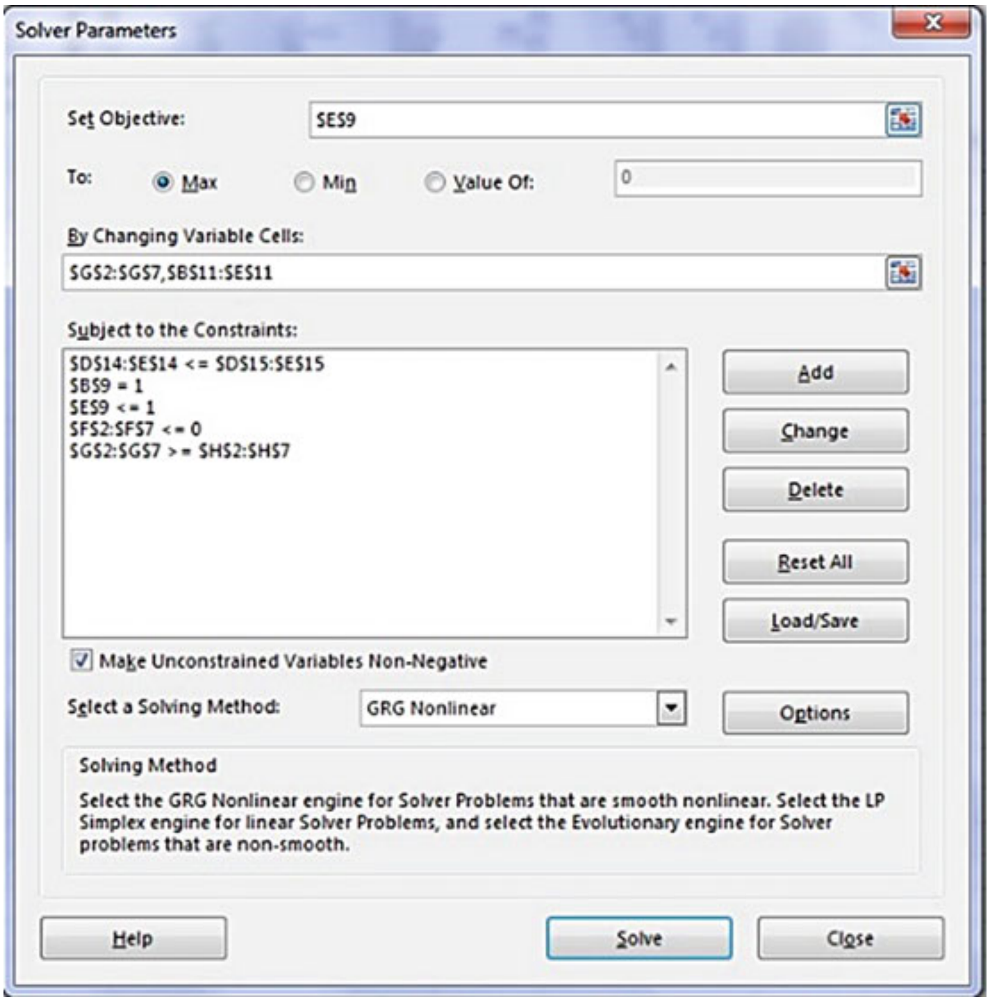


Fig. 9.6 Setting Solver to solve Eq. 9.11

9. Copy D14 and then paste it to E14.
10. Assign the following command into F2,

    '=Sumproduct(D2:E2,D$11:E$11)- Sumproduct(B2:C2,B$11:C$11)'.

11. Copy F2 and then paste it to F3-F7.
12. Assign the following command into H2,

    '=If(G2=Max(G$2:G$7),H$9*Small(G$2:G$7,5),H$9*Max(G$2:G$7))'.

13. Copy H2 and then paste it to H3-H7.
14. Assign the following command into J2,

    '=(1+$G2)*B2'.

15. Copy J2 and then paste it to J2-M7.
16. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.2 illustrates.
17. Assign 'E9' into 'Set Objective' and choose 'Max'.
18. Assign 'G2:G7, B11:E11' into 'By Changing Variable Cells'.
19. Click on 'Add' and assign 'D14:E14' into 'Cell Reference', then select '<=', and assign 'D15:E15' into 'Constraint'.
20. Click on 'Add' and assign 'B9' into 'Cell Reference', then select '=' and assign '1' into 'Constraint'. Then click on 'OK'.
21. Click on 'Add' and assign 'E9' into 'Cell Reference', then select '<=', and assign '1' into 'Constraint'.
22. Click on 'Add' and assign 'F2:F7' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
23. Click on 'Add' and assign 'G2:G7' into 'Cell Reference', then select '>=', and assign 'H2:H7' into 'Constraint'.
24. Tick 'Make Unconstrained Variables Non-Negative'.
25. Choose 'GRG Nonlinear' from 'Select a Solving Method'.
26. Click on 'Solve'.

Table 9.4 illustrates the CCR-inefficient and CCR-efficient restaurants. There were four CCR-efficient restaurants, which are bolded in Table 9.4, and the rest of the restaurants were CCR-inefficient.

Table 9.4 The results of solving Eq. 9.11

DMUs	Δ_i^*	$(1+\Delta_i^*)x_{i1}$	$(1+\Delta_i^*)x_{i2}$	$(1+\Delta_i^*)y_{i1}$	$(1+\Delta_i^*)y_{i2}$	CCR-Score
1	0.103447	4.413790	3.310342	2.206895	1.103447	0.736842
2	0.103447	6.620685	2.206895	1.103447	2.206895	0.500000
3	0.517242	1.517242	4.551725	1.517242	3.034483	1.000000
4	0.103447	2.206895	6.620685	1.103447	1.103447	0.499999
5	0.517242	4.551726	1.517242	1.517242	3.034484	1.000000
6	0.517242	4.551727	3.034485	3.034485	1.517242	1.000000

Table 9.5 The results of solving Eq. 9.10

DMUs	Δ_i^*	$(1+\Delta_i^*)x_{i1}$	$(1+\Delta_i^*)x_{i2}$	$(1+\Delta_i^*)y_{i1}$	$(1+\Delta_i^*)y_{i2}$	CCR-Score
1	0.000	4.000	3.000	2.000	1.000	0.737
2	0.000	6.000	2.000	1.000	2.000	0.500
3	0.667	1.667	5.000	1.667	3.333	1.000
4	0.000	2.000	6.000	1.000	1.000	0.500
5	0.000	3.000	1.000	1.000	2.000	1.000
6	1.667	8.000	5.333	5.333	2.667	1.000

To apply Eq. 9.10, we only need to assume that $\delta_l^{(i)} = 0$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$, that is, assign 0 into H9 and run Solver.

Part 2 Now, assume that demands for all output factors are negative, that is, a decrease in demands.

Since the demand decreases and can be 0 at least, thus Δ_i can be between 0 and 1, that is, $-1 \leq \Delta_i \leq 0$, which yields $0 \leq y_{ik} + \Delta_i y_{ik} \leq y_{ik}$. As a result, Eq. 9.11 can be as follows:

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \leq 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 1, \quad \text{for} i = 1,2,\ldots,n,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \leq \tilde{D}_k, \quad \text{and } k = 1,2,\ldots,p, \qquad (9.12)\\
&w_j^+ \geq \varepsilon, \quad \text{for } j = 1,2,\ldots,m,\\
&w_k^- \geq \varepsilon, \quad \text{for } k = 1,2,\ldots,p,\\
&\Delta_i \geq -1, \quad \text{for } i = 1,2,\ldots,n,\\
&\Delta_i \leq -\varepsilon', \quad \text{for } i = 1,2,\ldots,n,\\
&\Delta_i \geq \delta_l^{(i)}\Delta_l, \quad \text{for } i = 1,2,\ldots,n \text{ and for } l = 1,2,\ldots,n \text{ where } i \neq l.
\end{aligned}$$

Assume that the demand changes for Output 1 and Output 2 in Table 9.3 are predicated as $\tilde{D}_1 = -4, \tilde{D}_2 = -3$ and $\delta_l^{(i)} = 2$, in the next production season, that is, the situation of demand decreases in the two outputs. The results of applying Eq. 9.12, where $\varepsilon = 0$ and $\varepsilon' = 0.0001$ are illustrated in Table 9.6.

Note that, when $\Delta_i \leq 0$, for $i = 1, 2, \ldots, n$, then $\Delta_l \leq \delta_l^{(i)}\Delta_l \leq 0$, where $\delta_l^{(i)} \leq 1$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$. Thus, the constraints $\Delta_i \geq \delta_l^{(i)}\Delta_l$ yield that $\delta_l^{(i)} = 1$ and $\Delta_1 = \Delta_2 = \ldots = \Delta_n$. To avoid this, we assume that $\delta_l^{(i)} > 1$, for

Table 9.6 The results of solving Eq. 9.12

DMUs	Δ_i^*	$(1+\Delta_i^*)x_{i1}$	$(1+\Delta_i^*)x_{i2}$	$(1+\Delta_i^*)y_{i1}$	$(1+\Delta_i^*)y_{i2}$	CCR-Score
1	−0.4000	1.3333	1.0000	0.6667	0.3333	0.736842
2	−0.8000	2.0000	0.6667	0.3333	0.6667	0.500000
3	−0.4000	0.6667	2.0000	0.6667	1.3333	1.000000
4	−0.8000	0.6667	2.0000	0.3333	0.3333	0.499999
5	−0.4000	2.0000	0.6667	0.6667	1.3333	1.000000
6	−0.4000	2.0000	1.3333	1.3333	0.6667	1.000000

$i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$ where $i \neq l$, where $\tilde{D}_k < 0$, for $k = 1, 2, \ldots, p$. In addition, if $\Delta_l = 0$, for some $l = 1, 2, \ldots, n$, then $\Delta_i = 0$ for $i = 1, 2, \ldots, n$. Therefore, we consider a very small positive value ε' such as $\Delta_i \leq -\varepsilon'$, for $i = 1, 2, \ldots, n$.

Part 3 In real life applications, different directions can be planned for demand production changes. In other words, demand changes can be positive changes for some outputs, negative changes for some other outputs, and zero changes for the rest of outputs.

In order to deal with these multiple demand changes, we split all outputs into three groups, based upon their particular demand changing directions.

The three categories are denoted by O_p, that is, the sets of outputs with positive demand changes, O_n, that is, the sets of outputs with negative demand changes, and O_z, that is, the sets of outputs with zero demand changes. Thus, we have $O_p \cup O_n \cup O_z = \{1, 2, \ldots, p\}$. In addition, $\tilde{D}_k$ is positive for $k \in O_p$, $\tilde{D}_k$ is negative for $k \in O_n$ and $\tilde{D}_k$ is 0 for $k \in O_z$.

We now plan the upcoming production for all DMUs to fulfill the demand changes in two steps given by:

Step 1 *Considering the output changes in group O_p only*. Similar to Part 1, we suppose that the same positive proportional change, $\Delta_i^{(p)} \geq 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and output factors in O_p. After that, Eq. 9.11 is applied and the targets for the input and the output factors are considered as the data for the next step.

The related model for Step 1 is as follows:

$$\begin{aligned}
&\max \frac{\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik}}{\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)})},\\
&\text{Subject to}\\
&\frac{\sum_{k\in O_p} y_{ik}(1+\Delta_i^{(p)})w_k^+ + \sum_{k\notin O_p} y_{ik} w_k^+}{\sum_{j=1}^m x_{ij}(1+\Delta_i^{(p)})w_j^-} \le 1, \quad \text{and } i = 1,2,\ldots,n,\\
&\frac{\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik}}{\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)})} \le 1,\\
&\sum_{i=1}^n y_{ik}\Delta_i^{(p)} \le \tilde{D}_k, \text{for} k\in O_p,\\
&w_j^+ \ge \varepsilon, \text{for } j = 1,2,\ldots,m,\\
&w_k^- \ge \varepsilon, \quad \text{for } k = 1,2,\ldots,p,\\
&\Delta_i^{(p)} \ge 0, \quad \text{for } i = 1,2,\ldots,n.
\end{aligned} \tag{9.13}$$

Of course, we can define $\sum_{j=1}^m w_j^- \sum_{i=1}^n \left(1+\Delta_i^{(p)}\right)x_{ij} = 1$, and convert above fractional programming to the following non-linear programming.

$$\begin{aligned}
&\max \sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik},\\
&\text{Subject to}\\
&\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)}) = 1,\\
&\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik} \le 1,\\
&\sum_{k\in O_p} w_k^+ y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ y_{ik} \le \sum_{j=1}^m w_j^- x_{ij}(1+\Delta_i^{(p)}), \text{ for } i = 1,2,\ldots,n,\\
&\sum_{i=1}^n y_{ik}\Delta_i^{(p)} \le \tilde{D}_k, \quad \text{for } k\in O_p,\\
&w_j^+ \ge \varepsilon, \quad \text{for } j = 1,2,\ldots,m,\\
&w_k^- \ge \varepsilon, \quad \text{for } k = 1,2,\ldots,p,\\
&\Delta_i^{(p)} \ge 0, \text{for } i = 1,2,\ldots,n.
\end{aligned} \tag{9.14}$$

The targets for input and output factors of DMU_l ($l = 1, 2, \ldots, n$) by Eq. 9.14 are given by:

$$\begin{aligned}
x_{lj}^{*(1)} &= x_{lj}(1+\Delta_l^{(p)*}), \quad \text{for } j = 1,2,\ldots,m,\\
y_{lk}^{*(1)} &= y_{lk}(1+\Delta_l^{(p)*}) \quad \text{for } k\in O_p,\\
y_{lk}^{*(1)} &= y_{lk} \quad \text{for } k\in \{1,2,\ldots,p\} - O_p.
\end{aligned} \tag{9.15}$$

Step 2 *Considering the output changes in group $O_n$only*. Similar to Part 2, we suppose that the same negative proportional change, $-1 \leq \Delta_i^{(n)} \leq 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and output factors in O_n which are measured in Step 1. After that, Eq. 9.12 is applied and the new targets for the input and the output factors are proposed as the new production plans for all DMUs.

$$\begin{aligned}
&\max \sum\nolimits_{k \notin O_n} w_k^+ \sum\nolimits_{i=1}^n y_{ik}^{*(1)} + \sum\nolimits_{k \in O_n} w_k^+ \sum\nolimits_{i=1}^n y_{ik}^{*(1)} (1 + \Delta_i^{(n)}),\\
&\text{Subject to} \sum\nolimits_{j=1}^m w_j^- \sum\nolimits_{i=1}^n x_{ij}^{*(1)} (1 + \Delta_i^{(n)}) = 1,\\
&\sum\nolimits_{k \notin O_n} w_k^+ \sum\nolimits_{i=1}^n y_{ik}^{*(1)} + \sum\nolimits_{k \in O_n} w_k^+ \sum\nolimits_{i=1}^n y_{ik}^{*(1)} (1 + \Delta_i^{(n)}) \leq 1,\\
&\sum\nolimits_{k \notin O_n} y_{ik}^{*(1)} w_k^+ + \sum\nolimits_{k \in O_n} y_{ik}^{*(1)} (1 + \Delta_i^{(n)}) w_k^+ \leq \sum\nolimits_{j=1}^m x_{ij}^{*(1)} (1 + \Delta_i^{(n)}) w_j^-,\\
&\qquad\qquad\qquad \text{for } i = 1, 2, \ldots, n,\\
&\sum\nolimits_{i=1}^n y_{ik}^{*(1)} \Delta_i^{(n)} \leq \tilde{D}_k, \quad \text{for } k \in O_n,\\
&w_j^+ \geq \varepsilon, \quad \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \geq \varepsilon, \quad \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i^{(n)} \geq -1, \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i^{(n)} \leq -\varepsilon', \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{9.16}$$

The targets from Eq. 9.16 are given by:

$$\begin{aligned}
x_{lj}^{*(2)} &= x_{lj}^{*(1)} (1 + \Delta_l^{(n)*}), \text{for } j = 1, 2, \ldots, m,\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)} (1 + \Delta_l^{(n)*}), \text{for } k \in O_n,\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)}, \text{for } k \in \{1, 2, \ldots, p\} - O_n.
\end{aligned} \tag{9.17}$$

Or

$$\begin{aligned}
x_{lj}^{*(2)} &= x_{lj} (1 + \Delta_l^{(n)*})(1 + \Delta_l^{(p)*}), \quad \text{for } j = 1, 2, \ldots, m,\\
y_{lk}^{*(2)} &= y_{lk} (1 + \Delta_l^{(p)*}), \quad \text{for } k \in O_p\\
y_{lk}^{*(2)} &= y_{lk} (1 + \Delta_l^{(n)*}), \quad \text{for } k \in O_n\\
y_{lk}^{*(2)} &= y_{lk}, \quad \text{for } \ k \in O_z.
\end{aligned} \tag{9.18}$$

Now, assume that the demand changes for Output 1 and Output 2 in Table 9.3 are predicated as $\tilde{D}_1 = 4$, $\tilde{D}_2 = -3$, in the next production season, that is, the situation of demand increases in the first output and decreases in the second output. The results of applying Eq. 9.12, where $\varepsilon = 0$ and $\varepsilon' = 0.00001$ are illustrated in Tables 9.7 and 9.8, respectively.

Note that, from 'Option' in 'Solver parameters', we changed 'Population Size' to 200 in 'GRG Nonlinear' window and clicked on 'Use Multistart'. This option let solver run repeatedly and start from different starting points to find a better possible solution, and of course, it takes longer than a single run.

Table 9.7 The results of solving Eq. 9.12 where $\varepsilon = 0$

DMUs	$\Delta_i^{(p)*}$	$x_{i1}^{*(1)}$	$x_{i2}^{*(1)}$	$y_{i1}^{*(1)}$	$y_{i2}^{*(1)}$	CCR-Score
1	0.000000	4.000000	3.000000	2.000000	1.000000	0.736842
2	0.022879	6.137271	2.045757	1.022879	2.000000	1.000000
3	2.887869	3.887869	11.663606	3.887869	2.000000	1.000000
4	0.000000	2.000000	6.000000	1.000000	1.000000	0.971967
5	1.089253	6.267759	2.089253	2.089253	2.000000	1.000000
6	0.000000	3.000000	2.000000	2.000000	1.000000	1.000000

Table 9.8 The results of solving Eq. 9.12 where $\varepsilon = 0.00001$

DMUs	$\Delta_i^{(n)*}$	$x_{i1}^{*(2)}$	$x_{i2}^{*(2)}$	$y_{i1}^{*(2)}$	$y_{i2}^{*(2)}$	CCR-Score
1	−0.756565	0.973741	0.730306	2.000000	0.243435	1.000000
2	−0.282374	4.404266	1.468089	1.022879	1.435252	1.000000
3	−0.458243	2.106279	6.318838	3.887869	1.083514	1.000000
4	−0.757366	0.485267	1.455802	1.000000	0.242634	1.000000
5	−0.000010	6.267696	2.089232	2.089253	1.999980	1.000000
6	−0.010916	2.967251	1.978167	2.000000	0.989084	1.000000

It is also suggested that the restaurants are benchmarked before applying Idea 1 for demand changes. In this case, the restaurants will be CCR-efficient and, after panning, will have the same CCR-efficiency score as well.

9.4 Idea 2: Max-Min Output-Input

In this section, the three demand change situations, that is, positive, negative and zero demand changes, are incorporated into one single model. In other words, the same problem in production planning is considered, but with a different perspective from Idea 1.

The new plan is to maximize the total values of all output factors which are produced by all DMUs, and simultaneously to minimize the total values of all input factors which are consumed by all individuals. A DEA-based model is illustrated to benchmark all individuals (DMUs) within the original PPS with new input-output plans. The original PPS (See Eq. 4.4) is estimated by the input and output factors of all DMUs, which describes all technically feasible production plans.

Suppose that the forecasted demand change $\tilde{D}_k$, corresponded to the kth output factor, is given, for $k = 1, 2, \ldots, p$. In addition, assume that $\tilde{x}_{ij}$, for $j = 1, 2, \ldots, m$, and $\tilde{y}_{ik}$, for $k = 1, 2, \ldots, p$, are the planned targets for input and output factors of DMU$_i$, for $i = 1, 2, \ldots, n$.

The total values of the kth output targets of all DMUs should not be exceed than the total values of the original kth outputs of all DMUs plus the forecasted demand change $\tilde{D}_k$, that is,

$$\sum_{i=1}^{n} \tilde{y}_{ik} \leq \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p. \tag{9.19}$$

Each $\tilde{y}_{ik}$ can at least be 0, and since all demands for outputs can be changed positively or negatively, $\tilde{y}_{ik}$ can be either no more than y_{ik} or no less than y_{ik}, for $i = 1$, 2, …, n, and $k = 1$, 2, …, p. In other words,

$$\begin{aligned} \tilde{y}_{ik} &\leq y_{ik}, \ \text{if } \tilde{D}_k \leq 0, \\ \tilde{y}_{ik} &\geq y_{ik}, \ \text{if } \tilde{D}_k \geq 0. \end{aligned} \tag{9.20}$$

In order to maximize total output production and at the same time minimize total input consumption, the following multi-objective linear programming (MOLP) should be solved.

$$\begin{aligned}
&\max \sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}, \\
&\min \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}, \\
&\text{Subject to} \\
&\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \leq \tilde{x}_{lj}, \quad \text{for } l = 1, 2, \ldots, n, \text{and for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \leq \tilde{y}_{lk}, \quad \text{for } l = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p, \\
&\sum_{i=1}^{n} \tilde{y}_{ik} \leq \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i^{(l)} \geq 0, \quad \text{for } i = 1, 2, \ldots, n, \text{and for } l = 1, 2, \ldots, n, \\
&\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \tilde{y}_{ik} \leq y_{ik} \\ \tilde{y}_{ik} \geq 0 \end{array} \right. & \text{if } \tilde{D}_k \leq 0 \\ \tilde{y}_{ik} \geq y_{ik} & \text{if } \tilde{D}_k \geq 0. \end{array} \right. , \quad \text{for } i = 1, 2, \ldots, n \text{ and for } k = 1, 2, \ldots, p,
\end{aligned} \tag{9.21}$$

The unknown variables in Eq. 9.21 are $n \times n$ multipliers, $\lambda_i^{(l)}$ ($i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, n$), $n \times m$ input targets, $\tilde{x}_{ij}$ ($i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$), and $n \times p$ output targets, $\tilde{y}_{ik}$ ($i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$). The above MOLP can be rewritten as follows:

$$\begin{aligned}
&\max \sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik} - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}, \\
&\text{Subject to} \\
&\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \leq \tilde{x}_{lj}, \text{for } l = 1, 2, \ldots, n, \text{and for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \geq \tilde{y}_{lk}, \text{and } l = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p, \\
&\sum_{i=1}^{n} \tilde{y}_{ik} \leq \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i^{(l)} \geq 0, \text{for } i = 1, 2, \ldots, n, \text{and for } l = 1, 2, \ldots, n, \\
&\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \tilde{y}_{ik} \leq y_{ik} \\ \tilde{y}_{ik} \geq 0 \end{array} \right. & \text{if } \tilde{D}_k \leq 0 \\ \tilde{y}_{ik} \geq y_{ik} & \text{if } \tilde{D}_k \geq 0. \end{array} \right. , \quad \text{for } i = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p,
\end{aligned} \tag{9.22}$$

Equation 9.22 is a linear programming model which is equivalent with Eq. 9.21. The epsilon, ε, in the objective of Eq. 9.22 is also a very small positive real number.

From the first two sets of constraints, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \leq \tilde{x}_{lj}$, for $l = 1, 2, \ldots, n$, and for $j = 1, 2, \ldots, m$, $\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \geq \tilde{y}_{lk}$, for $l = 1, 2, \ldots, n$, and for $k = 1, 2, \ldots, p$, and the objective, that is, max $\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik} - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}$, we expect that $\tilde{x}_{ij}$ and $\tilde{y}_{ik}$ lie of the DEA frontier generated by DMUs. In other words, the planning results by Eq. 9.22 should completely improve the CCR-efficiency scores of all DMUs to one. We now prove this statement.

Let $\tilde{x}_{ij}^*$ and $\tilde{y}_{ik}^*$, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$, be the optimal solutions in Eq. 9.22 for $\tilde{x}_{ij}$ and $\tilde{y}_{ik}$, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$, respectively. Thus, the new input and output plan for DMU$_l$ ($l = 1, 2, \ldots, n$) is $\left(\tilde{x}_{l1}^*, \tilde{x}_{l2}^*, \ldots, \tilde{x}_{lm}^*, \tilde{y}_{l1}^*, \tilde{y}_{l2}^*, \ldots, \tilde{x}_{lp}^*\right)$. The CCR-efficiency of DMU$_l$ with this new plan, which can be simply called the new CCR-efficiency score for DMU$_l$, can be calculated by the following envelopment form of CCR, using the original PPS.

$$\begin{aligned}
&\min \theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \leq \tilde{x}_{lj}^*\theta_l, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \geq \tilde{y}_{lk}^*, \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, i = 1, 2, \ldots, n.
\end{aligned} \tag{9.23}$$

Now, the following theorem is proposed to prove this important property about the optimal targets from Eq. 9.22 which lie on the DEA-frontier generated by the DMUs. In other words, the new CCR-efficiency scores of all DMUs with planned targets measured by Eq. 9.22, can reach one when the original PPS is considered.

Theorem 9.1 In Eq. 9.23, $\theta_l^* = 1$.

Proof Let's suppose that an $\varepsilon > 0$ is given and $\tilde{x}_{ij}^*$, $\tilde{y}_{ik}^*$ and $\lambda_i^{(l)*}$ are optimal solutions for $\tilde{x}_{ij}$, $\tilde{y}_{ik}$ and $\lambda_i^{(l)}$ in Eq. 9.22, respectively, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$. Assume that π^* is the maximum value of the objective in Eq. 9.22, that is,

$$\pi^* = \sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}^*, \tag{9.24}$$

In order to prove that $\theta_l^* = 1$, we prove that the CCR score in Eq. 9.23 for none of DMUs are less than 1, that is, $\nexists l$ ($l = 1, 2, \ldots, n$): $\theta_l^* < 1$.

Suppose that there exist at least one DMU with the CCR score less than 1, that is, assume that $\exists l_0$ ($1 \leq l_0 \leq n$): $\theta_{l_0}^* < 1$.

Let the optimal solutions for the multipliers in Eq. 9.23 related to $\theta_{l_0}^*$ be $\lambda_i^{l_0*}$ ($i = 1, 2, \ldots, n$). Thus, from the constraints of Eq. 9.23 we have

$$\sum\nolimits_{i=1}^{n} \lambda_i^{l_0 *}\, x_{ij} \le \theta_{l_0}^{*} \tilde{x}_{l_0 j}^{*}, \quad \text{for } j = 1, 2, \ldots, m, \tag{9.25}$$

And

$$\sum\nolimits_{i=1}^{n} \lambda_i^{l_0 *}\, y_{ik} \ge \tilde{y}_{l_0 k}^{*}, \text{for } k = 1, 2, \ldots, p, \tag{9.26}$$

On the other hand, $\tilde{x}_{l_0 j}^{*}\theta_{l_0}^{*}, \tilde{x}_{ij}^{*}, \tilde{y}_{ik}^{*}, \lambda_i^{l_0 *}$ and $\lambda_i^{(l)*}$, for $i \neq l_0$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, is a feasible solution for Eq. 9.22. Thus, the objective of Eq. 9.22 for this feasible solution is given by:

$$\sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{p} \tilde{y}_{ik}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \Big(\tilde{x}_{l_0 j}^{*}\theta_{l_0}^{*} + \sum\nolimits_{\substack{i=1 \\ i \neq l_0}}^{n} \tilde{x}_{ij}^{*}\Big). \tag{9.27}$$

or

$$\sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{p} \tilde{y}_{ik}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*}\theta_{l_0}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \sum\nolimits_{\substack{i=1 \\ i \neq l_0}}^{n} \tilde{x}_{ij}^{*}. \tag{9.28}$$

From Eq. 9.29, we have

$$\tilde{x}_{l_0 j}^{*}\theta_{l_0}^{*} = \tilde{x}_{l_0 j}^{*} + \tilde{x}_{l_0 j}^{*}(\theta_{l_0}^{*} - 1) \tag{9.29}$$

$$\sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{p} \tilde{y}_{ik}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \left(\tilde{x}_{l_0 j}^{*} + \tilde{x}_{l_0 j}^{*}(\theta_{l_0}^{*} - 1)\right) - \varepsilon \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1,\, i \neq l_0}^{n} \tilde{x}_{ij}^{*}. \tag{9.30}$$

or

$$\sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{p} \tilde{y}_{ik}^{*} - \varepsilon\left(\theta_{l_0}^{*} - 1\right) \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1}^{n} \tilde{x}_{ij}^{*}. \tag{9.31}$$

or

$$\sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{p} \tilde{y}_{ik}^{*} - \varepsilon \sum\nolimits_{j=1}^{m} \sum\nolimits_{i=1}^{n} \tilde{x}_{ij}^{*} + \varepsilon\left(1 - \theta_{l_0}^{*}\right) \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*}. \tag{9.32}$$

or

$$\pi^{*} + \varepsilon\left(1 - \theta_{l_0}^{*}\right) \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*}. \tag{9.33}$$

Since $\theta_{l_0}^{*} < 1$, therefore $\varepsilon\left(1 - \theta_{l_0}^{*}\right) \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*} > 0$ and we have the following contradiction,

$$\pi^{*} < \pi^{*} + \varepsilon\left(1 - \theta_{l_0}^{*}\right) \sum\nolimits_{j=1}^{m} \tilde{x}_{l_0 j}^{*}. \tag{9.34}$$

Indeed, we assume that π^{*} is the maximum objective value in Eq. 9.22, but as can be seen in Eq. 9.34, we found a solution for Eq. 9.22, that is, $\theta_{l_0}^{*}\tilde{x}_{l_0 j}^{*}, \tilde{x}_{ij}^{*}, \tilde{y}_{ik}^{*}, \lambda_i^{l_0 *}$ and

$\lambda_i^{(l)*}$, for $i \neq l_0$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, which yields greater value for the objective of Eq. 9.22. Therefore, the assumption that there exists one DMU which has a CCR-score less than one is not valid.

Consequently the CCR- scores for all DMUs with new planned input and output factors by Eq. 9.22 are equal to 1, where the original PPS of DMUs are considered, and the proof is completed. □

The following instructions are proposed to see the results of Eq. 9.22 for data in Table 9.3, where the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = 3$ in the next production season.

1. Copy the 5 columns of Table 9.3 on an Excel sheet into cells A1:E7, as Fig. 9.7 depicts.
2. Label C9 as '$\tilde{D}$', G9 as 'Objective', J9 as 'Epsilon', B11 as 'Target Input 1', C11 as 'Target Input 2', D11 as 'Target Output 1', E11 as 'Target Output 2', F1-K1 as 'Lambdas 1-6', G11-J11 as 'Constraints', C19 as 'Total Target for Output', and C20 as 'Total Output + $\tilde{D}$', as Fig. 9.7 illustrates.
3. Assign 4 and 3 to D9 and E9, respectively.
4. Assign 0.000001 to K9.
5. Assign the following command (without quotations mark) into H9,

   '=Sum(D12:E17)-K9*Sum(B12:C17)'.

6. Assign '=Sum(D12:D17)' into D19.
7. Assign '=Sum(E12:E17)' into E19.
8. Assign '=Sum(D2:D7,D9)' into D20.

H9 =SUM(D12:E17)-K9*SUM(B12:C17)

	A	B	C	D	E	F	G	H	I	J	K
1	DMUS	Input 1	Input 2	Output 1	Output 2	λ_i^1	λ_i^2	λ_i^3	λ_i^4	λ_i^5	λ_i^6
2	1	4	3	2	1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	2	6	2	1	2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	3	1	3	1	2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
5	4	2	6	1	1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
6	5	3	1	1	2	0.000000	1.000000	1.666667	0.333333	1.000000	0.000000
7	6	3	2	2	1	1.000000	0.000000	1.666667	0.333333	0.000000	1.000000
8											
9			D~	4	3		Objective	24.0000		Epsilon	0.000001
10											
11		Tin 1	Tin 2	Tout 1	Tout 2		Constraints				
12		3	2	2	1		3	2	2	1	
13		3	1	1	2		3	1	1	2	
14		10	5	5	5		10	5	5	5	
15		2	1	1	1		2	1	1	1	
16		3	1	1	2		3	1	1	2	
17		3	2	2	1		3	2	2	1	
18											
19		Total target for Output		12	12						
20		Total Output + D~		12	12						
21											

Fig. 9.7 Setting Excel sheet to solve Eq. 9.22

9. Assign ‘=Sum(E2:E7,E9)’ into E20.
10. Assign the following command into G12,

    ‘= Sumproduct (B$2:B$7,$F2:$F7)’.

11. Copy G12, and paste it to H12, I12 and J12.
12. Assign the following command into G13,

    ‘= Sumproduct (B$2:B$7,$G2:$G7)’.

13. Copy G13, and paste it to H13, I13 and J13.
14. Assign the following command into G14,

    ‘= Sumproduct (B$2:B$7,$H2:$H7)’.

15. Copy G14, and paste it to H14, I14 and J14.
16. Assign the following command into G15,

    ‘= Sumproduct (B$2:B$7,$I2:$I7)’.

17. Copy G15, and paste it to H15, I15 and J15.
18. Assign the following command into G16,

    ‘= Sumproduct (B$2:B$7,$J2:$J7)’.

19. Copy G16, and paste it to H16, I16 and J16.
20. Assign the following command into G17,

    ‘= Sumproduct (B$2:B$7,$K2:$K7)’.

21. Copy G17, and paste it to H17, I17 and J17.
22. Open ‘Solver Parameters’ window, from ‘DATA’ in toolbar menu, as Fig. 9.8 illustrates.
23. Assign ‘H9’ into ‘Set Objective’ and choose ‘Max’.
24. Assign ‘F2:K7, B12:E17’ into ‘By Changing Variable Cells’.
25. Click on ‘Add’ and assign ‘G12:H17’ into ‘Cell Reference’, then select ‘<=’, and assign ‘B12:C17’ into ‘Constraint’.
26. Click on ‘Add’ and assign ‘I12:J17’ into ‘Cell Reference’, then select ‘>=’ and assign ‘D12:E17’ into ‘Constraint’. Then click on ‘OK’.
27. Click on ‘Add’ and assign ‘D12:D17’ into ‘Cell Reference’, then select ‘>=’, and assign ‘D2:D7’ into ‘Constraint’.
28. Click on ‘Add’ and assign ‘E12:E17’ into ‘Cell Reference’, then select ‘>=’, and assign ‘E2:E7’ into ‘Constraint’.
29. Click on ‘Add’ and assign ‘D19:E19’ into ‘Cell Reference’, then select ‘<=’, and assign ‘D20:E20’ into ‘Constraint’.
30. Tick ‘Make Unconstrained Variables Non-Negative’.
31. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
32. Click on ‘Solve’.

Table 9.9 illustrates the planned input and output factors. DMUs with these planned data are CCR-efficient and lie on the original DEA-frontier generated by

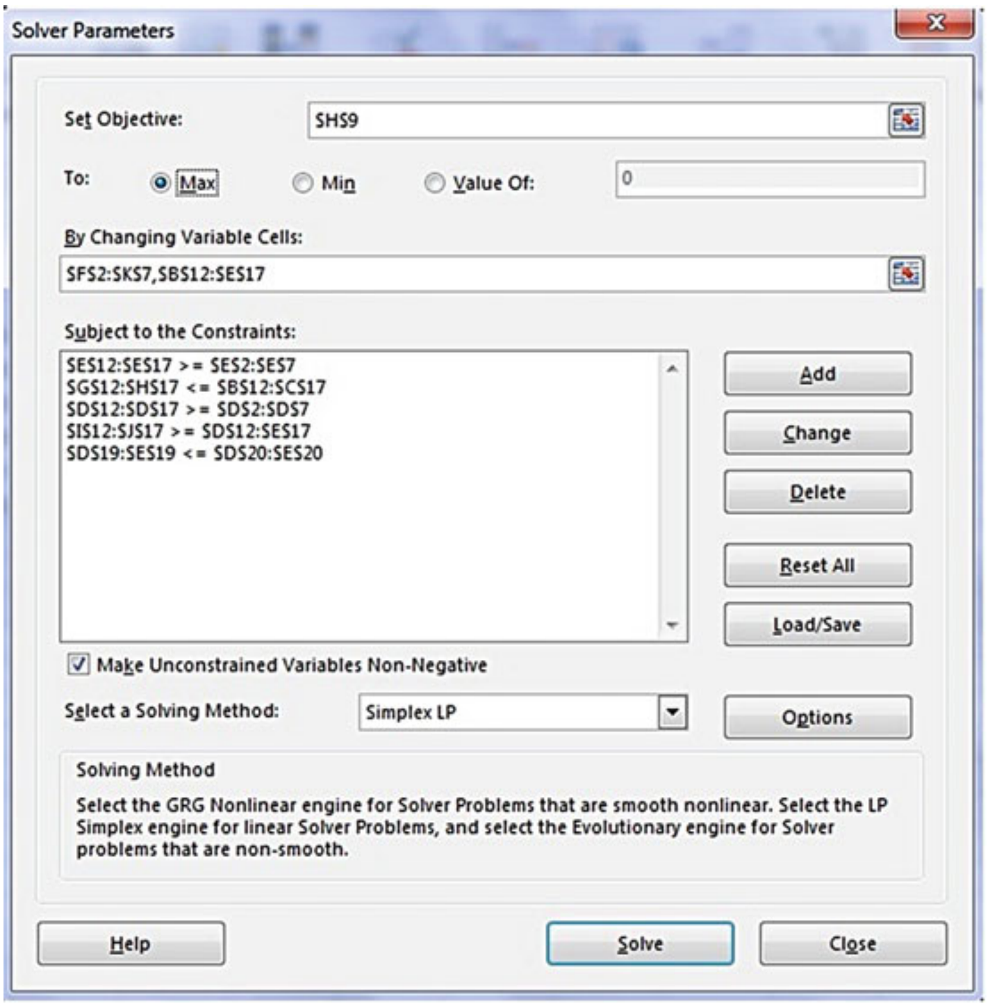


Fig. 9.8 Setting Solver to solve Eq. 9.22

Table 9.9 The results of solving Eq. 9.22

DMUs	$\tilde{x}^*_{i1}$	$\tilde{x}^*_{i2}$	$\tilde{y}^*_{i1}$	$\tilde{y}^*_{i2}$	CCR-Score
1	3	2	2	1	1.000000
2	3	1	1	2	1.000000
3	10	5	5	5	1.000000
4	2	1	1	1	1.000000
5	3	1	1	2	1.000000
6	3	2	2	1	1.000000

DMUs. It is also obvious that the DMUs with planned data in Table 9.9 lie on the PPS frontier which are generated by themselves.

If the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = -3$, the direction of inequality in Step 28 above should be changed to '<=', that is,

28. Click on 'Add' and assign 'E12:E17' into 'Cell Reference', then select '<=', and assign 'E2:E7' into 'Constraint'.

The results of the model while $\tilde{D}_1 = 4$ and $\tilde{D}_2 = -3$ are represented in Table 9.10.

Table 9.10 The results of solving Eq. 9.22 by known demands

DMUs	$\tilde{x}_{i1}^*$	$\tilde{x}_{i2}^*$	$\tilde{y}_{i1}^*$	$\tilde{y}_{i2}^*$	CCR-Score
1	3	2	2	1	1.000000
2	3	2	2	1	1.000000
3	6	4	4	2	1.000000
4	1.5	1	1	0.5	1.000000
5	1.5	1	1	0.5	1.000000
6	3	2	2	1	1.000000

9.5 Applying Ideas 1 and 2 for the Chain Restaurants

We now get back to Sect. 9.2 and apply both Ideas 1 and 2 for the chain restaurants. Suppose that the demand changes for meat dish, vegetable dish, soup, noodles and beverage are forecasted as $\tilde{D}_1 = 3,\ \tilde{D}_2 = 2.4,\ \tilde{D}_3 = -3,\ \tilde{D}_4 = 6,\ \tilde{D}_5 = 3$ (10^3 serving) in the next business month.

We first adapt the related models for these data, and after that we respectively formulize Step 1 of Idea 1, Step 2 of Idea 1, and CCR Envelopment form in three different excel sheets. We then write a macro (Visual Basic Procedure) to run all steps with just one click.

For the first step we have

$$
\begin{aligned}
&\max\left(\sum\nolimits_{k=1,\ k\neq 3}^{5} w_k^+ \sum\nolimits_{i=1}^{20} y_{ik}\left(1+\Delta_i^{(p)}\right)\right)+\left(w_3^+ \sum\nolimits_{i=1}^{20} y_{i3}\right),\\
&\text{Subject to}\\
&w_1^- \sum\nolimits_{i=1}^{20} x_{i1}\left(1+\Delta_i^{(p)}\right)+w_2^- \sum\nolimits_{i=1}^{20} x_{i2}\left(1+\Delta_i^{(p)}\right)=1,\\
&\sum\nolimits_{k=1,\ k\neq 3}^{5} w_k^+ \sum\nolimits_{i=1}^{20} y_{ik}\left(1+\Delta_i^{(p)}\right)+w_3^+ \sum\nolimits_{i=1}^{20} y_{i3}\leq 1,\\
&\left(\sum\nolimits_{k=1,\ k\neq 3}^{5} w_k^+ y_{ik}\left(1+\Delta_i^{(p)}\right)\right)+y_{i3}w_3^+ \leq \sum\nolimits_{j=1}^{2} w_j^- x_{ij}\left(1+\Delta_i^{(p)}\right),\\
&\qquad\qquad\qquad\qquad \text{for } i=1,2,\ldots,20,\\
&\sum\nolimits_{i=1}^{20} y_{i1}\Delta_i^{(p)} \leq 3,\\
&\sum\nolimits_{i=1}^{20} y_{i2}\Delta_i^{(p)} \leq 2.4,\\
&\sum\nolimits_{i=1}^{20} y_{i4}\Delta_i^{(p)} \leq 6,\\
&\sum\nolimits_{i=1}^{20} y_{i5}\Delta_i^{(p)} \leq 3,\\
&w_j^+ \geq \varepsilon,\ \text{for } j=1,2,\\
&w_k^- \geq \varepsilon,\ \text{for } k=1,2,\ldots,5,\\
&\Delta_i^{(p)} \geq 0,\ \text{for } i=1,2,\ldots,20.
\end{aligned}
\tag{9.35}
$$

The targets for DMU$_l$ ($l = 1, 2, \ldots, 20$) are:

$$\begin{aligned}
x_{l1}^{*(1)} &= x_{l1}(1+\Delta_l^{(p)*}),\\
x_{l2}^{*(1)} &= x_{l2}(1+\Delta_l^{(p)*}),\\
y_{l1}^{*(1)} &= y_{l1}(1+\Delta_l^{(p)*}),\\
y_{l2}^{*(1)} &= y_{l2}(1+\Delta_l^{(p)*}),\\
y_{l3}^{*(1)} &= y_{l3},\\
y_{l4}^{*(1)} &= y_{l4}(1+\Delta_l^{(p)*}),\\
y_{l5}^{*(1)} &= y_{l5}(1+\Delta_l^{(p)*}).
\end{aligned} \tag{9.36}$$

The objective in Eq. 9.35 can be written as follows, too. The same action can be applied for the constraints. In other words, when $\tilde{D}_k < 0$, we just need to subtract $w_k^+ \sum_{i=1}^{20} y_{ik}\Delta_i^{(p)}$ in the related equations.

$$\left(\sum_{k=1}^{5} w_k^+ \sum_{i=1}^{20} y_{ik}(1+\Delta_i^{(p)})\right) - (w_3^+ \sum_{i=1}^{20} y_{i3}\Delta_i^{(p)}). \tag{9.37}$$

For the second step, we consider the output changes in group $O_n = \{3\}$ only. Similarly, we suppose that the same negative proportional change, $-1 \le \Delta_i^{(n)} \le 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and the third output factors which are measured in Step 1.

Equation 9.12 represents the model in Step 2.

$$\begin{aligned}
&\max(\textstyle\sum_{k=1,\,k\neq 3}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}) + w_3^+ \sum_{i=1}^{5} y_{i3}^{*(1)}(1+\Delta_i^{(n)}),\\
&\text{Subject to}\\
&w_1^- \textstyle\sum_{i=1}^{5} x_{i1}^{*(1)}(1+\Delta_i^{(n)}) + w_2^- \sum_{i=1}^{5} x_{i2}^{*(1)}(1+\Delta_i^{(n)}) = 1,\\
&(\textstyle\sum_{k=1,\,k\neq 3}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}) + w_3^+ \sum_{i=1}^{n} y_{i3}^{*(1)}(1+\Delta_i^{(n)}) \le 1,\\
&(\textstyle\sum_{k=1,\,k\neq 3}^{5} y_{ik}^{*(1)} w_k^+) + w_3^+(1+\Delta_i^{(n)}) y_{i3}^{*(1)} \le \sum_{j=1}^{2} x_{ij}^{*(1)}(1+\Delta_i^{(n)}) w_j^-, \text{ for } i = 1,2,\ldots,n\\
&\textstyle\sum_{i=1}^{5} \Delta_i^{(n)} y_{i3}^{*(1)} \le \tilde{D}_3,\\
&w_j^+ \ge \varepsilon, \text{ for } j = 1,2,\\
&w_k^- \ge \varepsilon, \text{ for } k = 1,2,\ldots,5,\\
&\Delta_i^{(n)} \ge -1, \text{ for } i = 1,2,\ldots,20,\\
&\Delta_i^{(n)} \le -\varepsilon', \text{ for } i = 1,2,\ldots,20.
\end{aligned} \tag{9.38}$$

The targets from Eq. 9.16 are given by:

$$\begin{aligned}
x_{l1}^{*(2)} &= x_{l1}^{*(1)}(1+\Delta_l^{(n)*}),\\
x_{l2}^{*(2)} &= x_{l2}^{*(1)}(1+\Delta_l^{(n)*}),\\
y_{l1}^{*(2)} &= y_{l1}^{*(1)},\\
y_{l2}^{*(2)} &= y_{l2}^{*(1)},\\
y_{l3}^{*(2)} &= y_{l3}^{*(1)}(1+\Delta_l^{(n)*}),\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)},\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)}.
\end{aligned} \tag{9.39}$$

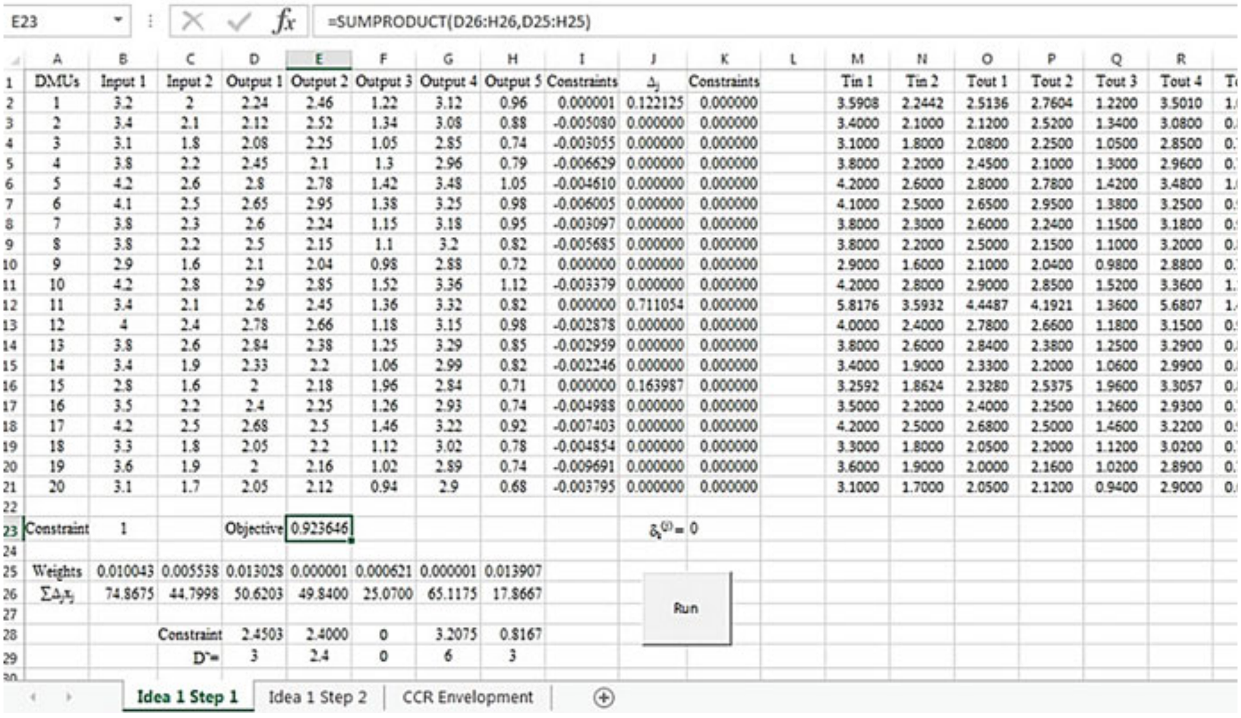
Or

$$\begin{aligned}
x_{l1}^{*(2)} &= x_{l1}(1+\Delta_l^{(n)*})(1+\Delta_l^{(p)*}),\\
x_{l2}^{*(2)} &= x_{l2}(1+\Delta_l^{(n)*})(1+\Delta_l^{(p)*})\\
y_{l1}^{*(2)} &= y_{l1}(1+\Delta_l^{(p)*}),\\
y_{l2}^{*(2)} &= y_{l2}(1+\Delta_l^{(p)*}),\\
y_{l3}^{*(2)} &= y_{l3}(1+\Delta_l^{(n)*}),\\
y_{l4}^{*(2)} &= y_{l4}(1+\Delta_l^{(p)*}),\\
y_{l5}^{*(2)} &= y_{l5}(1+\Delta_l^{(p)*}).
\end{aligned} \tag{9.40}$$

The objective in Eq. 9.38 can also be written as $\left(\sum_{k=1}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}\right) + y_{i3}^{*(1)}\Delta_i^{(n)} w_3^+$. The same action can be applied for the constraints in Eq. 9.38. In other words, when $\tilde{D}_k < 0$, we just need to add $w_k^+ \sum_{i=1}^{20} y_{ik}$ $\Delta_i^{(p)}$ in the related equations.

The following instructions illustrate the steps to run Eqs. 9.35 and 9.38 and CCR Envelopment to optimize the average production performance of all restaurants after planning using Idea 1.

1. Copy the 8 columns of Table 9.1 on an Excel sheet into cells A1:H21, as Fig. 9.9 illustrates.
2. Label A23 as 'Constraint, D23 as 'Objective', A25 as 'Weights', A26 as 'SumDeltasData', C28 as 'Constraints' C29 as '$\tilde{D}$', I1 as 'Constraints', J1 as 'Deltas', K1 as 'Constraint', M1 'Target Input 1', N1 as 'Target Input 2', O1 as 'Target Output 1', P1 as 'Target Output 2', P1 as 'Target Output 2', P1 as 'Target Output 2', and P1 as 'Target Output 2', as Fig. 9.9 represents.



E23 =SUMPRODUCT(D26:H26,D25:H25)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	DMUs	Input 1	Input 2	Output 1	Output 2	Output 3	Output 4	Output 5	Constraints	Δ_j	Constraints		Tin 1	Tin 2	Tout 1	Tout 2	Tout 3	Tout 4
2	1	3.2	2	2.24	2.46	1.22	3.12	0.96	0.000001	0.122125	0.000000		3.5908	2.2442	2.5136	2.7604	1.2200	3.5010
3	2	3.4	2.1	2.12	2.52	1.34	3.08	0.88	-0.005080	0.000000	0.000000		3.4000	2.1000	2.1200	2.5200	1.3400	3.0800
4	3	3.1	1.8	2.08	2.25	1.05	2.85	0.74	-0.003055	0.000000	0.000000		3.1000	1.8000	2.0800	2.2500	1.0500	2.8500
5	4	3.8	2.2	2.45	2.1	1.3	2.96	0.79	-0.006629	0.000000	0.000000		3.8000	2.2000	2.4500	2.1000	1.3000	2.9600
6	5	4.2	2.6	2.8	2.78	1.42	3.48	1.05	-0.004610	0.000000	0.000000		4.2000	2.6000	2.8000	2.7800	1.4200	3.4800
7	6	4.1	2.5	2.65	2.95	1.38	3.25	0.98	-0.006005	0.000000	0.000000		4.1000	2.5000	2.6500	2.9500	1.3800	3.2500
8	7	3.8	2.3	2.6	2.24	1.15	3.18	0.95	-0.003097	0.000000	0.000000		3.8000	2.3000	2.6000	2.2400	1.1500	3.1800
9	8	3.8	2.2	2.5	2.15	1.1	3.2	0.82	-0.005685	0.000000	0.000000		3.8000	2.2000	2.5000	2.1500	1.1000	3.2000
10	9	2.9	1.6	2.1	2.04	0.98	2.88	0.72	0.000000	0.000000	0.000000		2.9000	1.6000	2.1000	2.0400	0.9800	2.8800
11	10	4.2	2.8	2.9	2.85	1.52	3.36	1.12	-0.003379	0.000000	0.000000		4.2000	2.8000	2.9000	2.8500	1.5200	3.3600
12	11	3.4	2.1	2.6	2.45	1.36	3.32	0.82	0.000000	0.711054	0.000000		5.8176	3.5932	4.4487	4.1921	1.3600	5.6807
13	12	4	2.4	2.78	2.66	1.18	3.15	0.98	-0.002878	0.000000	0.000000		4.0000	2.4000	2.7800	2.6600	1.1800	3.1500
14	13	3.8	2.6	2.84	2.38	1.25	3.29	0.85	-0.002959	0.000000	0.000000		3.8000	2.6000	2.8400	2.3800	1.2500	3.2900
15	14	3.4	1.9	2.33	2.2	1.06	2.99	0.82	-0.002246	0.000000	0.000000		3.4000	1.9000	2.3300	2.2000	1.0600	2.9900
16	15	2.8	1.6	2	2.18	1.96	2.84	0.71	0.000000	0.163987	0.000000		3.2592	1.8624	2.3280	2.5375	1.9600	3.3057
17	16	3.5	2.2	2.4	2.25	1.26	2.93	0.74	-0.004988	0.000000	0.000000		3.5000	2.2000	2.4000	2.2500	1.2600	2.9300
18	17	4.2	2.5	2.68	2.5	1.46	3.22	0.92	-0.007403	0.000000	0.000000		4.2000	2.5000	2.6800	2.5000	1.4600	3.2200
19	18	3.3	1.8	2.05	2.2	1.12	3.02	0.78	-0.004854	0.000000	0.000000		3.3000	1.8000	2.0500	2.2000	1.1200	3.0200
20	19	3.6	1.9	2	2.16	1.02	2.89	0.74	-0.009691	0.000000	0.000000		3.6000	1.9000	2.0000	2.1600	1.0200	2.8900
21	20	3.1	1.7	2.05	2.12	0.94	2.9	0.68	-0.003795	0.000000	0.000000		3.1000	1.7000	2.0500	2.1200	0.9400	2.9000
22																		
23	Constraint	1		Objective	0.923646					$\delta_k^{(i)}=$	0							
24																		
25	Weights	0.010043	0.005538	0.013028	0.000001	0.000621	0.000001	0.013907										
26	$\Sigma\Delta_j x_j$	74.8675	44.7998	50.6203	49.8400	25.0700	65.1175	17.8667		Run								
27																		
28			Constraint	2.4503	2.4000	0	3.2075	0.8167										
29			$D^{*}=$	3	2.4	0	6	3										

Idea 1 Step 1 | Idea 1 Step 2 | CCR Envelopment

Fig. 9.9 Setting Excel sheet to solve Eqs. 9.35 and 9.38

3. Assign 3, 2.4, 0, 6 and 3 into D29-H29, respectively.
4. Assign 0 to K23.
5. Assign the following command (without quotations mark) into B23,

   ‘=Sumproduct(B26:C26,B25:C25)’.

6. Assign the following command into E23,

   ‘=Sumproduct(D26:H26,D25:H25)’.

7. Assign the following command into B26,

   ‘=Sumproduct(B2:B21,(1+$J$2:$J$21))’.

8. Copy B26, and paste it to C26.
9. Assign the following command into D26,

   ‘=If(D29>0,Sumproduct(D2:D21,(1+$J$2:$J$21)),Sum(D2:D21))’.

10. Copy D26, and paste it to E26-H26.
11. Assign the following command into D28,

    ‘=If(D29>0,Sumproduct(D2:D21,$J$2:$J$21),D29)’.

12. Copy D28, and paste it to E28-H28.
13. Assign the following command into I2,

    ‘=(Sumproduct(D2:H2,D$25:H$25)*(1+J2)-F2*J2*F$25)-Sumproduct(B2:C2,B$25:C$25)*(1+J2)’.

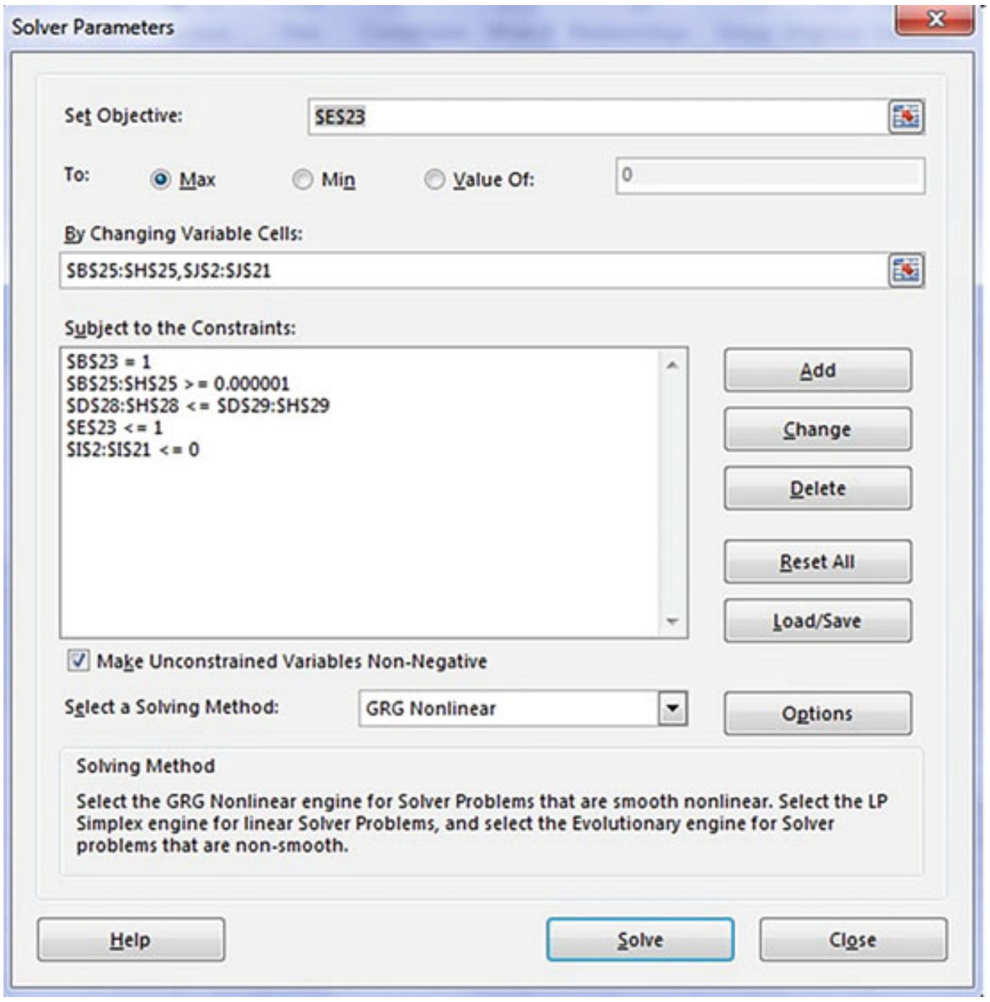


Fig. 9.10 Setting Solver to solve Eq. 9.35

14. Copy I2, and paste it to I3-I21.
15. Assign the following command into M2,

    '=(1+$J2)*B2'.

16. Copy M2, and paste it to M2 -N21.
17. Assign the following command into O2,

    '=If(D$29>0,(1+$J2)*D2,D2)'.

18. Copy O2, and paste it to O2-S21.
19. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.10 illustrates.
20. Assign 'E23' into 'Set Objective' and choose 'Max'.
21. Assign 'B25:H25, J2:J21' into 'By Changing Variable Cells'.
22. Click on 'Add' and assign 'B23' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'.
23. Click on 'Add' and assign 'B25:H25' into 'Cell Reference', then select '>=' and assign '0.000001' into 'Constraint'.
24. Click on 'Add' and assign 'D28:H28' into 'Cell Reference', then select '>=', and assign 'D29:H29' into 'Constraint'.

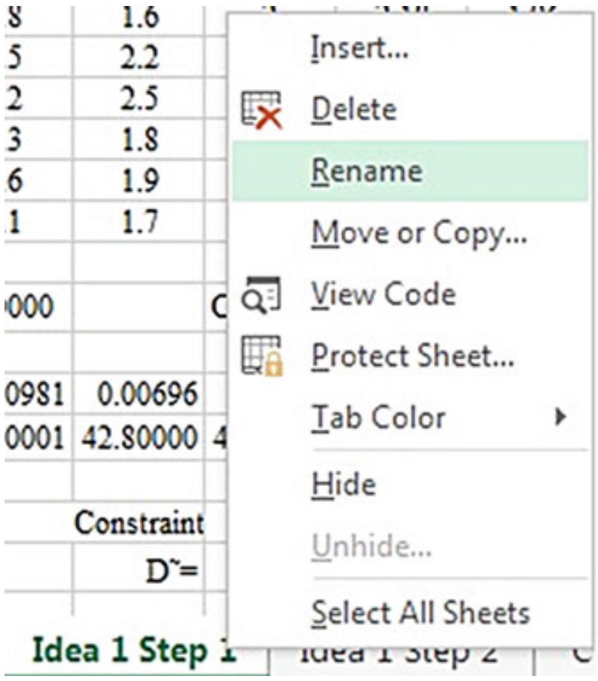


Fig. 9.11 Rename a worksheet

25. Click on ‘Add’ and assign ‘E23’ into ‘Cell Reference’, then select ‘<=’, and assign ‘1’ into ‘Constraint’.
26. Click on ‘Add’ and assign ‘I2:I21’ into ‘Cell Reference’, then select ‘<=’, and assign ‘0’ into ‘Constraint’.
27. Tick ‘Make Unconstrained Variables Non-Negative’.
28. Choose ‘GRG Nonlinear’ from ‘Select a Solving Method’.
29. Click on ‘Solve’.
30. Right click on the worksheet tab to rename the worksheet to ‘Idea 1 Step 1’, as Fig. 9.11 shows.
31. Right click on the worksheet tab again and select ‘Move or Copy . . .’, then tick ‘Create a copy’, as Fig. 9.12 depicts, and click on ‘OK’.
32. Rename the new sheet as ‘Idea 1 Step 2’.
33. Assign the following command into D26,

    ‘=If(D29<0,Sumproduct(D2:D21,(1+$J$2:$J$21)),Sum(D2:D21))’.

34. Copy D26, and paste it to E26-H26.
35. Assign the following command into D28,

    ‘=If(D29<0,Sumproduct(D2:D21,$J$2:$J$21),D29)’.

36. Copy D28, and paste it to E28-H28.
37. Assign the following command into I2,

    ‘=(Sumproduct(D2:H2,D$25:H$25)+F2*J2*F$25)-Sumproduct(B2:C2,B$25:C$25)*(1+J2)’.

38. Copy I2, and paste it to I3-I21.

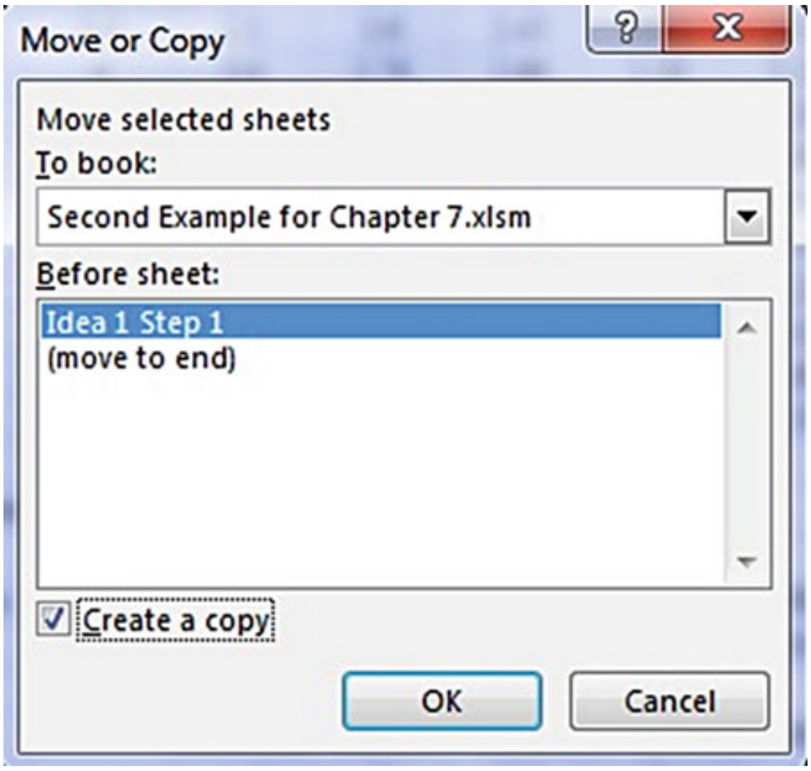


Fig. 9.12 Copy a worksheet

39. Assign the following command into O2,

    '=If(D$29<0,(1+$J2)*D2,D2)'.

40. Copy O2, and paste it to O2-S21.

The rest of constraints are not changed.

41. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.13 illustrates.
42. Click on 'Add' and assign 'J2:J21' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
43. Click on 'Add' and assign 'J2:J21' into 'Cell Reference', then select '>=', and assign '-1' into 'Constraint'.
44. Click on 'Solve'.

Now, we can either use our earlier CCR Multiplier model in Sect. 9.2 or make a new sheet to run CCR Envelopment model. Here we give the instructions to run the CCR Envelopment model.

45. Right click on the worksheet tab and select 'Move or Copy . . .', then tick 'Create a copy', and click on 'OK'.
46. Rename the new sheet as 'CCR Envelopment'.
47. Delete the formula in cells E23, I2-I21.
48. Assign 1 into B23 and
49. Label A23 as 'Index' and I1 as 'Lambda', as Fig. 9.14 shows.
50. Delete Rows 25-29.
51. Delete Column J-S.

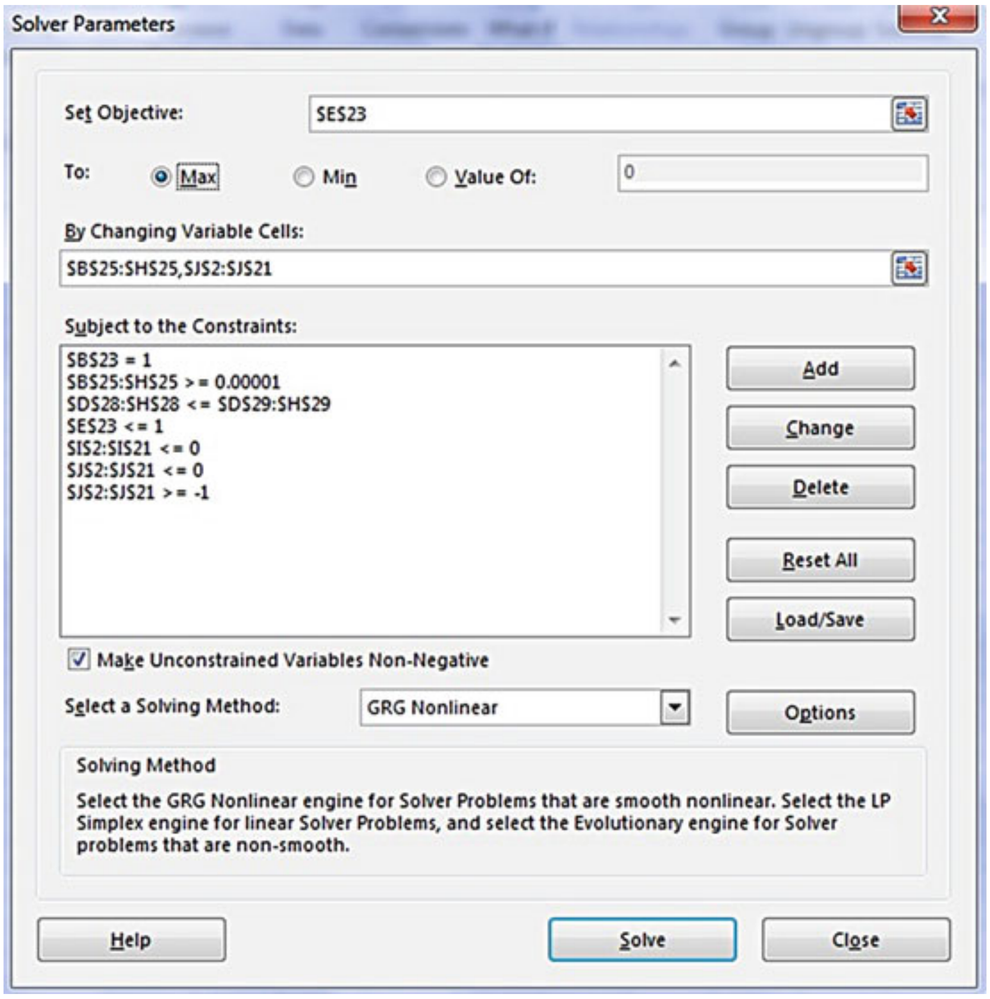


Fig. 9.13 Setting Solver to solve Eq. 9.38

52. Assign the following command into B25.

    '=Sumproduct(B2:B21,$I2:$I21)'.

53. Copy B25 and paste it into C25-H25.
54. Assign the following command into B26

    '=Index(B2:B21,$B23)*$E23'.

55. Copy B26 and paste it into C26.
56. Assign the following command into D26

    '=Index(B2:B21,$B23)'.

57. Copy D26 and paste it into E26-H26.
58. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.15 illustrates.
59. Click on 'Reset All'.
60. Assign 'E23' into 'Set Objective' and choose 'Min'.
61. Assign 'I2:I21, E23' into 'By Changing Variable Cells'.

B26 | =INDEX(B2:B21,$B$23)*$E$23

	A	B	C	D	E	F	G	H	I	J
1	DMUS	Input 1	Input 2	Output 1	Output 2	Output 3	Output 4	Output 5	Lambda	CCR
2	1	2.981372	1.863358	2.240001	2.460001	1.136648	3.120001	0.96	0.000000	1
3	2	2.845355	1.757425	2.12	2.52	1.121405	3.08	0.88	0.000000	1
4	3	2.781432	1.615025	2.08	2.25	0.942098	2.85	0.74	0.000000	1
5	4	3.163667	1.831597	2.45	2.1	1.082307	2.96	0.79	0.000000	1
6	5	3.584915	2.219233	2.8	2.78	1.212043	3.48	1.05	0.000000	1
7	6	3.407093	2.077496	2.65	2.95	1.146778	3.25	0.98	0.000000	1
8	7	3.345344	2.024813	2.6	2.24	1.012407	3.18	0.95	0.000000	1
9	8	3.2628	1.888989	2.5	2.15	0.944495	3.2	0.82	0.000000	1
10	9	2.87169	1.584381	2.100001	2.040001	0.970433	2.880001	0.72	0.000000	1
11	10	3.538109	2.358739	2.9	2.85	1.280458	3.36	1.12	0.000000	1
12	11	3.258673	2.01271	2.600001	2.450001	1.303469	3.320002	0.82	0.000000	1
13	12	3.531204	2.118722	2.78	2.66	1.041705	3.150001	0.98	0.000000	1
14	13	3.268234	2.23616	2.84	2.38	1.075077	3.29	0.85	0.000000	1
15	14	3.140074	1.754747	2.33	2.2	0.978964	2.99	0.82	0.000000	1
16	15	2.739333	1.565333	2.000001	2.180001	1.917532	2.840001	0.71	0.000000	1
17	16	2.943051	1.849918	2.4	2.25	1.059499	2.93	0.74	0.000000	1
18	17	3.439043	2.047049	2.68	2.5	1.195477	3.22	0.92	0.000000	1
19	18	2.907318	1.58581	2.05	2.2	0.986726	3.02	0.78	0.000000	1
20	19	2.859463	1.509161	2	2.16	0.810181	2.89	0.74	0.000000	1
21	20	2.810777	1.541394	2.05	2.12	0.8523	2.9	0.68	1.000000	1
22										
23	Index	20		Theta	1.00000					
24										
25		2.8108	1.5414	2.0500	2.1200	0.8523	2.9000	0.6800		
26		2.8108	1.5414	2.0500	2.1200	0.8523	2.9000	0.6800		

Fig. 9.14 Setting Excel sheet to solve CCR envelopment

62. Click on 'Add' and assign 'B25:C25' into 'Cell Reference', then select '<=', and assign 'B26:C26' into 'Constraint'.
63. Click on 'Add' and assign 'D25:H25' into 'Cell Reference', then select '>=', and assign 'D26:H26' into 'Constraint'.
64. Tick 'Make Unconstrained Variables Non-Negative'.
65. Choose 'Simplex LP' from 'Select a Solving Method'.
66. Click on 'Solve'.
67. Click on the worksheet labeled 'Idea 1 Step 1' to get back to the first step.
68. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
69. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
70. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
71. Inside of the 'Microsoft Visual Basic for Applications' window, write the commands in Fig. 9.16 between 'Sub Button1_Click ()' and 'End Sub'.

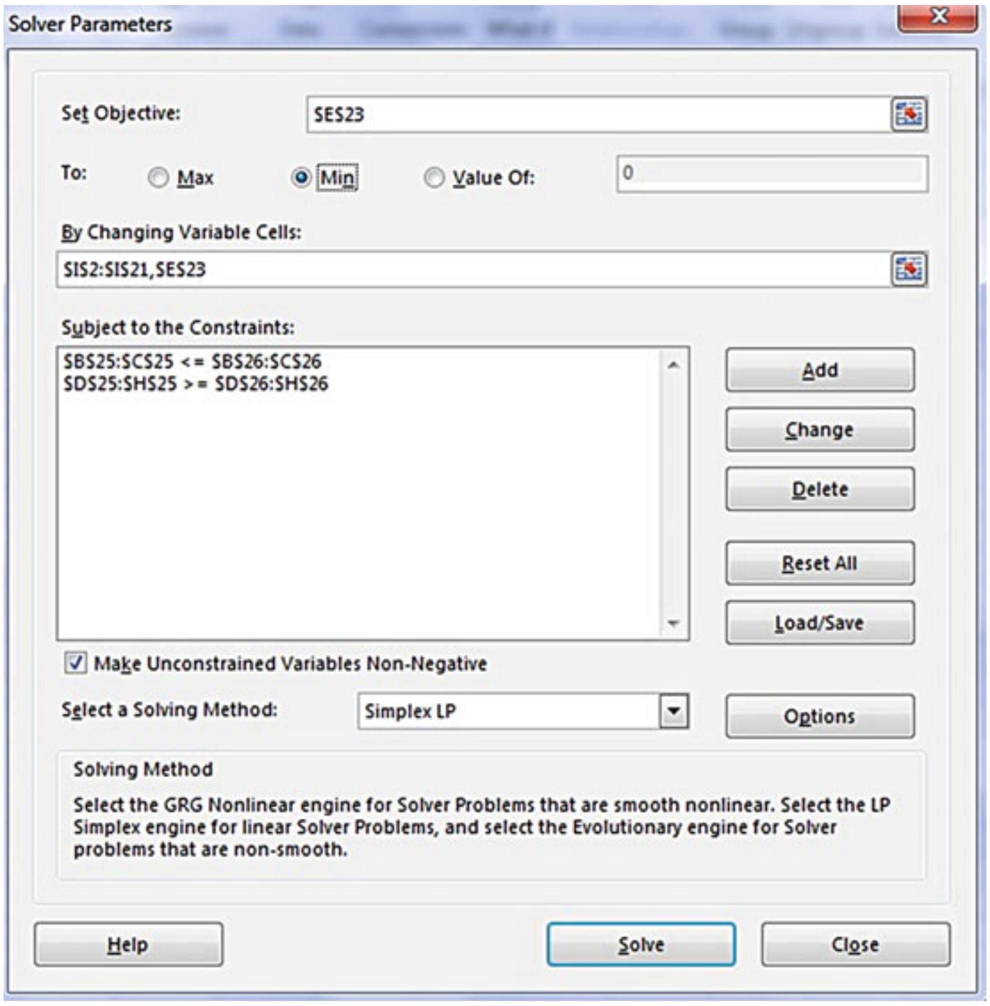


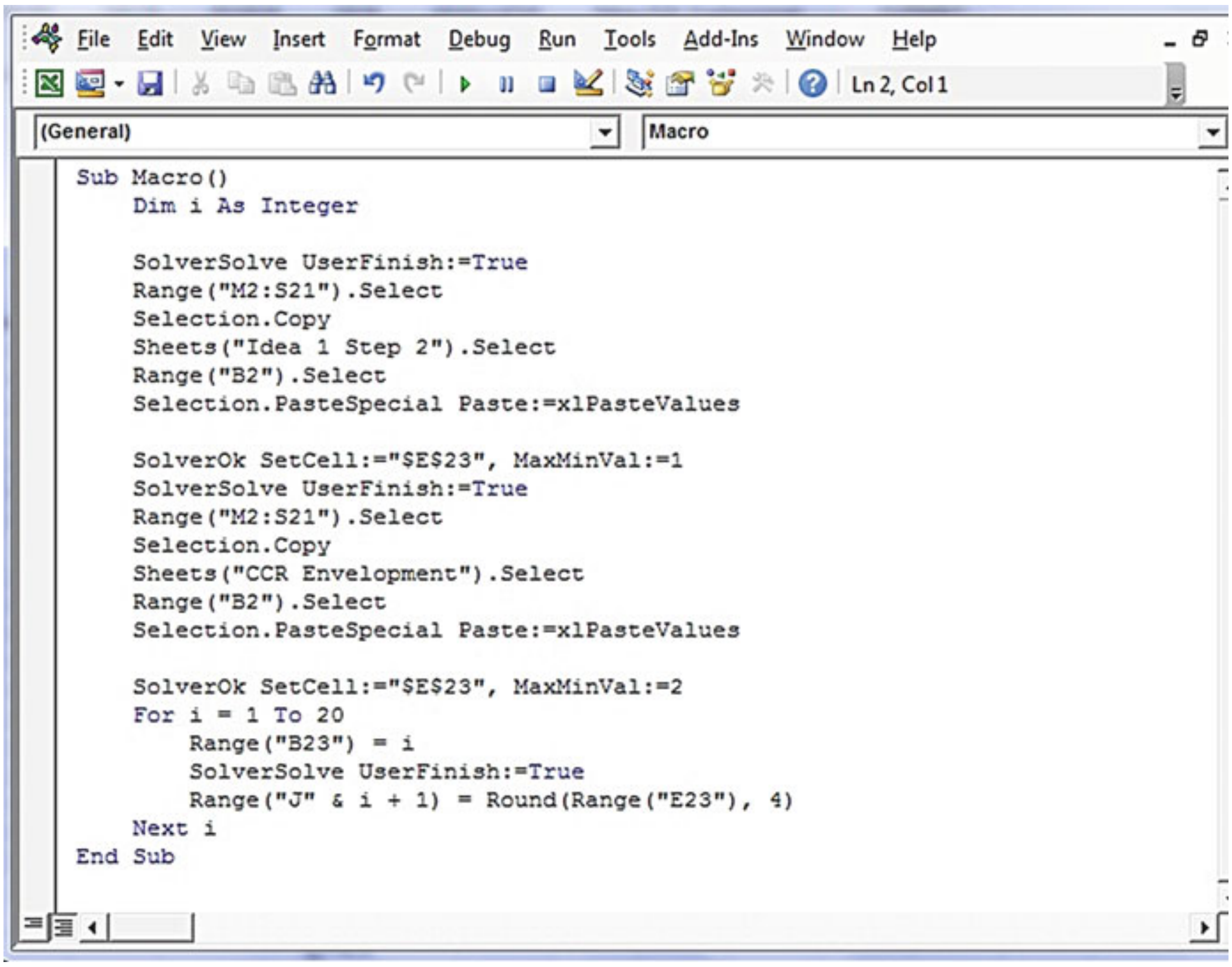
Fig. 9.15 Setting Solver to solve CCR envelopment

72. Close the 'Microsoft Visual Basic for Applications' window.
73. Click on the small rectangle which was automatically made on the Excel sheet and created by step 69.

The model in worksheet 'Idea 1 Step 1' is run and its results are copied into B2: H21 in worksheet 'Idea 1 Step 2'. The results of Step 2 are copied into B2:H21 in worksheet 'CCR Envelopment' and the model is run for every DMU. As can be seen with the planned input and output factors, the CCR scores of all DMUs are 1, as Fig. 9.14 illustrates.

Tables 9.11 demonstrates the results from Idea 1 Steps 1 and 2 as well as the results by CCR Envelopment model.

For Idea 2, we have the following model.




```
Sub Macro()
    Dim i As Integer

    SolverSolve UserFinish:=True
    Range("M2:S21").Select
    Selection.Copy
    Sheets("Idea 1 Step 2").Select
    Range("B2").Select
    Selection.PasteSpecial Paste:=xlPasteValues

    SolverOk SetCell:="$E$23", MaxMinVal:=1
    SolverSolve UserFinish:=True
    Range("M2:S21").Select
    Selection.Copy
    Sheets("CCR Envelopment").Select
    Range("B2").Select
    Selection.PasteSpecial Paste:=xlPasteValues

    SolverOk SetCell:="$E$23", MaxMinVal:=2
    For i = 1 To 20
        Range("B23") = i
        SolverSolve UserFinish:=True
        Range("J" & i + 1) = Round(Range("E23"), 4)
    Next i
End Sub
```

Fig. 9.16 Setting VBA to solve Eqs. 9.35 and 9.38

Table 9.11 The results of solving Eqs. 9.35 and 9.38

DMUs	$\Delta_i^{(p)*}$	$\Delta_i^{(n)*}$	$x_{i1}^{(2)*}$	$x_{i2}^{(2)*}$	$y_{i1}^{(2)*}$	$y_{i2}^{(2)*}$	$y_{i3}^{(2)*}$	$y_{i4}^{(2)*}$	$y_{i5}^{(2)*}$	CCR
R01	0.1221	−0.00015	3.59	2.24	2.51	2.76	1.22	3.50	1.08	1
R02	0.0000	−0.11761	3.00	1.85	2.12	2.52	1.18	3.08	0.88	1
R03	0.0000	−0.09680	2.80	1.63	2.08	2.25	0.95	2.85	0.74	1
R04	0.0000	−0.19112	3.07	1.78	2.45	2.10	1.05	2.96	0.79	1
R05	0.0000	−0.13069	3.65	2.26	2.80	2.78	1.23	3.48	1.05	1
R06	0.0000	−0.15299	3.47	2.12	2.65	2.95	1.17	3.25	0.98	1
R07	0.0000	−0.10483	3.40	2.06	2.60	2.24	1.03	3.18	0.95	1
R08	0.0000	−0.16284	3.18	1.84	2.50	2.15	0.92	3.20	0.82	1
R09	0.0000	0.00000	2.90	1.60	2.10	2.04	0.98	2.88	0.72	1
R10	0.0000	−0.14870	3.58	2.38	2.90	2.85	1.29	3.36	1.12	1
R11	0.7111	−0.11155	5.17	3.19	4.45	4.19	1.21	5.68	1.40	1
R12	0.0000	−0.10804	3.57	2.14	2.78	2.66	1.05	3.15	0.98	1
R13	0.0000	−0.24644	2.86	1.96	2.84	2.38	0.94	3.29	0.85	1
R14	0.0000	−0.05181	3.22	1.80	2.33	2.20	1.01	2.99	0.82	1
R15	0.1640	−0.02256	3.19	1.82	2.33	2.54	1.92	3.31	0.83	1
R16	0.0000	−0.23045	2.69	1.69	2.40	2.25	0.97	2.93	0.74	1
R17	0.0000	−0.18755	3.41	2.03	2.68	2.50	1.19	3.22	0.92	1
R18	0.0000	−0.06396	3.09	1.68	2.05	2.20	1.05	3.02	0.78	1
R19	0.0000	−0.15245	3.05	1.61	2.00	2.16	0.86	2.89	0.74	1
R20	0.0000	−0.10014	2.79	1.53	2.05	2.12	0.85	2.90	0.68	1

$$
\begin{aligned}
&\max \left(\textstyle\sum_{i=1}^{20}\ \sum_{k=1}^{5} \tilde{y}_{ik}\right) - \varepsilon \textstyle\sum_{i=1}^{20} \left(\tilde{x}_{i1} + \tilde{x}_{i2}\right),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{20} x_{i1}\lambda_i^{(l)} \le \tilde{x}_{l1}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{20} x_{i2}\lambda_i^{(l)} \le \tilde{x}_{l2}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} y_{i1}\lambda_i^{(l)} \ge \tilde{y}_{l1}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} y_{i2}\lambda_i^{(l)} \ge \tilde{y}_{l2}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} y_{i3}\lambda_i^{(l)} \ge \tilde{y}_{l3}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} y_{i4}\lambda_i^{(l)} \ge \tilde{y}_{l4}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} y_{i5}\lambda_i^{(l)} \ge \tilde{y}_{l5}, \ \ \text{for } l = 1, 2, \ldots, 20,\\
&\textstyle\sum_{i=1}^{n} \tilde{y}_{i1} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_1,\\
&\textstyle\sum_{i=1}^{n} \tilde{y}_{i2} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_2,\\
&\textstyle\sum_{i=1}^{n} \tilde{y}_{i3} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_3,\\
&\textstyle\sum_{i=1}^{n} \tilde{y}_{i4} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_4,\\
&\textstyle\sum_{i=1}^{n} \tilde{y}_{i5} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_5,\\
&\lambda_i^{(l)} \ge 0, \ \ \text{for } i = 1, 2, \ldots, 20, \ \ \text{and } \ \text{for } l = 1, 2, \ldots, 20,\\
&\tilde{y}_{i1} \ge y_{i1}, \ \ \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i2} \ge y_{i2}, \ \ \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i3} \le y_{i3}, \ \ \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i3} \ge 0, \ \ \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i4} \ge y_{i4}, \ \ \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i5} \ge y_{i5}, \ \ \text{for } i = 1, 2, \ldots, 20.
\end{aligned}
\tag{9.41}
$$

Equation 9.41 has 540 decision variables or changing cells and 286 constraints, whereas the limit for the number of changing variables and constraints of the Standard Microsoft Excel Solver are at most 200 and 100, respectively. To deal with this limitation, we use Frontline Solver Products which has a limit of 2000 decision variables.

The second to eighth columns in Table 9.12 illustrate the planned input and output factors by Idea 2 and the ninth column represents the CCR scores for the restaurants. As can be seen, the CCR score is equal to 1 for each restaurant. The results in table 9.12 are suggested by the plan to maximize the total values of all output factors which are produced by all restaurants, and simultaneously to minimize the total values of all input factors which are consumed by all restaurants.

Table 9.12 The results of applying Idea 2

DMUs	$x_{i1}^{(2)*}$	$x_{i2}^{(2)*}$	$y_{i1}^{(2)*}$	$y_{i2}^{(2)*}$	$y_{i3}^{(2)*}$	$y_{i4}^{(2)*}$	$y_{i5}^{(2)*}$	CCR
R01	3.520	2.178	2.471	2.460	1.049	3.310	1.041	1
R02	3.713	2.296	2.607	2.520	1.248	3.488	1.096	1
R03	3.157	1.946	2.219	2.251	0.994	2.955	0.928	1
R04	3.484	2.053	2.480	2.297	1.244	3.316	0.957	1
R05	3.974	2.417	2.804	2.784	1.420	3.744	1.145	1
R06	4.473	2.679	3.170	2.950	1.303	4.261	1.260	1
R07	3.723	2.117	2.675	2.453	1.037	3.523	0.969	1
R08	3.566	2.031	2.562	2.342	0.890	3.290	0.930	1
R09	3.119	1.891	2.203	2.194	0.957	2.889	0.895	1
R10	4.122	2.512	2.907	2.961	1.246	3.936	1.191	1
R11	3.771	2.132	2.714	2.608	1.248	3.516	0.973	1
R12	4.016	2.259	2.894	2.760	1.156	3.857	1.027	1
R13	4.207	2.563	2.967	2.794	0.965	4.116	1.215	1
R14	3.312	1.979	2.349	2.340	0.989	3.113	0.929	1
R15	3.040	1.822	2.153	2.191	0.970	2.841	0.857	1
R16	3.393	1.930	2.438	2.368	1.181	3.227	0.883	1
R17	3.845	2.184	2.764	2.692	1.271	3.669	0.999	1
R18	3.300	2.009	2.328	2.355	1.107	3.021	0.952	1
R19	3.131	1.893	2.213	2.240	0.936	2.899	0.894	1
R20	3.187	1.926	2.252	2.281	0.860	2.937	0.910	1

In this application, both planning designs lead to a technical efficiency score equal to one for all newly expanded restaurants input–output plans, which determines the reasonability of both methods. Note that, the second input, shop size, is a non-controllable input in real-life (see Sect. 12.3). This means that the rental floor space of the restaurants cannot be changed in a common sense. To fit the real condition more suitably, one can basically apply the introduced two Ideas 1 and 2 into such application where the shop size input unchanged.

9.6 Conclusion

In this chapter, two approaches are introduced which allow making future production plans when demand changes can be predicted in a centralized decision-making situation. The first plan is to optimize the overall or average production performance in the whole organization after planning using CCR efficiency scores. The second plan is to maximize the total output productions and simultaneously minimize the total input consumptions in the entire organization. All the individual DMUs are supposed to be able to modify their output productions and input usages. The approaches are exemplified with a simple numerical example and a real world data set. The production possibility set is characterized to indicate the production plans that are technically feasible. After that, the set is estimated by the observed performances under consideration and supposed to be unchanged during a planning period. The planning results from this method for all observations, lie on the original empirical production frontier with a CCR score of 1.

9.7 Exercises

9.1. Are 'the original PPS generated by observed DMUs' and 'the PPS generated by the DMUs with planned data', by Ideas 1 and 2, the same? Why?

9.2. Improve the VBA procedures to create the new worksheets, set Solver, and run Eqs. 9.35 and 9.38 and CCR Envelopment to optimize the average production performance of all restaurants in Table 9.1 after planning using Idea 1.

9.3. Using variable returns to scale and given data in Table 9.1

 9.3.1. Apply Idea 1.
 9.3.2. Apply Idea 2.

Chapter 10
Context-Dependent DEA

10.1 Introduction

In this chapter, the context-dependent DEA is discussed. Since a product can appear attractive in comparison with a contextual of less attractive or unattractive alternatives, the performance of firms can be influenced by the context. For an example, twenty-three Tokyo public libraries are considered and a context-dependent DEA proposed by Chen et al. (2005) is discussed. The attractiveness of each library on a particular performance level in comparison with other libraries are measured. Libraries are classified on several empirical efficient frontiers, where each frontier is used to evaluate the attractiveness. The performance of the technically efficient libraries changes as the technically inefficient libraries change their performance. The context-dependent DEA also represents another view to differentiate the performance of efficient DMUs. When DMUs in a particular level are observed as having the same performance, the attractiveness measure lets us discriminate the "equal performance" based upon the third option or the same particular evaluation context. We also develop the VBA procedure to measure the attractiveness with just one click.

10.2 Context-Dependent

As discussed in the previous chapters, adding or excluding a technically inefficient DMU or a set of technically inefficient DMUs does not change the technical efficiencies of the existing DMUs or the estimated production frontier. The technical inefficiency scores change only if the estimated production frontier is changed. In other words, the performance of DMUs by DEA approach completely depends on the estimated production frontier, where the performance of technically efficient

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DMUs is not influenced by the performance of technically inefficient DMUs. Nevertheless, in some real-life applications, consumer choices can be influenced by the context as well as presence of technically inefficient DMUs. Thus, to obtain the attractiveness within the context-dependent, CCR is modified to a situation where the performance is calculated with respect to a specific evaluation context.

Suppose that there are n DMUs A_i, for $i = 1, 2, \ldots, n$, in which each DMU has m positive input factors x_{ij}, for $j = 1, 2, \ldots, m$, and p positive output factors y_{ik}, for $k = 1, 2, \ldots, p$. Assume that A_l is evaluated, for $l = 1, 2, \ldots, n$. The input oriented envelopment form of CCR is given by:

$$\begin{aligned}
&\theta_l^* = \min\ \theta_l, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}, \quad \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,
\end{aligned} \tag{10.1}$$

After solving the CCR model, a non-empty set of technically efficient DMUs is obtained. Suppose that the observed set of DMUs is shown by $I^1 = \{A_i : i = 1, 2, \ldots, n\}$ and the set of DMUs with CCR scores of 1 from I^1 is shown by $L^1 = \left\{A_l \in I^1 : \theta_l^* = 1\right\}$. This first empirical production frontier generated by DMUs in I^1 provides an evaluation context to measure the attractiveness. The set L^1 is called 'Level 1' empirical production frontier.

Now, we exclude the DMUs with the CCR scores equal to 1 from the observed DMUs to find 'Level 2', the second empirical production frontier. In other words, the second set of DMUs is $I^2 = I^1 - L^1$, and the set L^2 is defined as the set of DMUs in I^2 with CCR scores equal to 1, that is, $L^2 = \left\{A_l \in I^2 : \theta_l^* = 1\right\}$.

Consequently, the set I^{t+1} represents the $t + 1$th set of DMUs after excluding the set of L^t from I^t, that is, $I^{t+1} = I^t - L^t$, where L^t indicates 'Level t', the t^{th} empirical production frontier. The generating of these levels is stopped when $I^t = \varnothing$. In other words, we have the following algorithm to identify the levels:

Step 1: Set $t = 1$, and I^1 as the set of observed DMUs.
Step 2: Evaluate the set I^t by Eq. 10.1 to obtain Level t technically efficient DMUs, or Level t production frontier, L^t.
Step 2. Exclude DMUs in the set L^t from I^t to obtain I^{t+1}.
Step 3: If $I^{t+1} = \varnothing$, go to Step 5.
Step 4: Set $t = t + 1$ and go to Step 2.
Step 5: Stop the algorithm.

From the above illustrations, the evaluation contexts are found by subdividing a set of DMUs into several levels of empirical production frontiers. Each production frontier offers an evaluation context for measuring the attractiveness. For example, Level 2 production frontier, generated by L^2, is considered as the evaluation context for measuring the attractiveness of the DMUs in L^1. In addition, the presence, absence, or the shape of the Level 2 production frontier affects the attractiveness of DMUs on the level 1 production frontier.

The introduced nest-sets of DMUs have the following properties:

(i) $I^1 = \cup_{t=1}^{s} L^t$, where $L^t \cap L^{t'} = \varnothing$ for $t \neq t'$, where $L^{s+1} = \varnothing$.

(ii) The DMUs in $L^{t'}$ are dominated by the DMUs in L^t or a linear combination of DMUs in L^t, where $t' \geq t$.

(iii) Each DMU in set L^t is technically efficient with respect to the DMUs in set $L^{t'}$ for all $t' \geq t$.

(iv) Each DMU in set L^t is technically inefficient with respect to the DMUs in set $L^{t'}$ for all $t' \leq t$.

As a result, each DMU belongs to one of the calculated levels. DMUs in L^t, that is, the DMUs which lie on level t production frontier, are attractive to themselves. They are least attractive in comparison with DMUs in L^{t-1}, and most attractive in comparison with DMUs in L^{t+1}.

After classifying the levels and partitioning DMUs, we can measure the performance of each DMU with respect to each level of production frontier. Eq. 10.2 illustrates the input oriented CCR model to measure the attractiveness of DMU_l in set L^t based upon the empirical production frontier in Level t', that is, the frontier which is made by DMUs in $L^{t'}$. In other words, x_{lj}^t and y_{lk}^t are the input and output factors of DMU_l in set L^t, and $x_{ij}^{t'}$ and $y_{ik}^{t'}$ are the input and output factors of DMUs in set $L^{t'}$, for $j = 1, 2, \ldots, m$, for $k = 1, 2, \ldots, p$ and $i = 1, 2 \ldots, n_{t'}$ where $n_{t'}$ is the number of DMUs in set $L^{t'}$.

$$
\begin{aligned}
&\theta_l^{*t} = \min\ \theta_l^t,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n_{t'}} x_{ij}^{t'}\lambda_i \leq x_{lj}^t\theta_l^t,\quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n_{t'}} y_{ik}^{t'}\lambda_i \geq y_{lk}^t,\quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0,\quad \text{for } i = 1, 2, \ldots, n.,
\end{aligned}
\tag{10.2}
$$

If $t < t'$, then $\theta_l^{*t} > 1$, because in this case, DMUs in $L^{t'}$ are dominated by a linear combination of DMUs in L^t. It is obvious that when $t = t'$, then $\theta_l^{*t} = 1$, because the DMUs in L^t are technically efficient and lie on the Level t empirical production frontier.

Definition 10.1 The score, θ_l^{*t}, measured for DMU_l from set L^t by Eq. 10.2 with respect to the generated PPS by DMUs in set $L^{t'}$, is called the input oriented t-degree attractiveness of DMU_l from level t'.

The bigger θ_l^{*t} represents the more attractive DMU_l, for $l = 1, 2, \ldots, n_t$ in comparison with DMUs in L^t. Eq. 10.2 determines the attractiveness score for DMU_l when outputs are fixed at their current levels. Similarly, we can have the output-oriented version of the context-dependent DEA to determine the attractiveness score for DMU_l when inputs are fixed at their current levels.

As explained in Chaps. 4 and 8, in constant retunes to scale technology, $\theta^* = 1/\varphi^*$, and the measured attractiveness in the input oriented model can easily be calculated in the output-oriented model as well. Nonetheless, if the variable returns to scale technology is considered, the related models should be solved separately. Eq. 10.3 represent the output-oriented CCR model to measure the t-degree attractiveness of DMU_l from set L^t according to the generated PPS by DMUs in set $L^{t'}$.

$$\begin{aligned}
&\varphi_l^{*t} = \max \varphi_l^t, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n_{t'}} x_{ij}^{t'} \lambda_i \leq x_{lj}^t, \quad \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n_{t'}} y_{ik}^{t'} \lambda_i \geq y_{lk}^t \varphi_l^t, \quad \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n.,
\end{aligned} \tag{10.3}$$

Definition 10.2 The $1/\varphi_l^{*t}$, where φ_l^{*t} is the measured score for DMU_l from set L^t by Eq. 10.2 with respect to the generated PPS by DMUs in set $L^{t'}$, is called *the output-oriented t-degree attractiveness* of DMU_l from level t'.

The larger the value of $1/\varphi_l^{*t}$, the more attractive DMU_l is, because this makes itself more distinctive from the evaluation context. We are also able to rank the DMUs in L^t based upon their attractiveness scores and identify the best one. In the next section, an application of the context-dependent is discussed.

10.3 An Example of Twenty-Three Public Libraries in Tokyo

In this section, we apply the illustrated context-dependent model in the previous section to measure the attractiveness of 23 public libraries in Tokyo (Cooper et al. 2007).

Table 10.1 shows the data for the 23 public libraries in the Tokyo Metropolitan Area. The input factors are floor area in $1000m^2$, the number of books in 1000 unit measurement, the number of staff in 1000 unit measurement, and the population in 1000 unit measurement. The output factor are the number of registered residents in 1000 unit measurement and the number of borrowed books in 1000 unit measurement.

In order to measure the attractiveness of these libraries, we need to solve Eq. 10.2 several times to find the levels and after that measure the attractiveness score of each DMU based upon each level. In this example, more than 20 times Eq. 10.2 should be solved. The previous way of solving models requires programming in 20 different sheets, which is not user-friendly. Thus, we develop the VBA programming to solve all process with just one click. The following instructions illustrate how to measure the attractiveness of the 23 libraries in Table 10.1 by Microsoft Excel Solver with one click only.

Table 10.1 Data of twenty three libraries in Tokyo

DMUs	Area	Books	Staff	Population	Regist.	Borrow.
1	2249	163,523	26	49,196	5561	105,321
2	4617	338,671	30	78,599	18,106	314,682
3	3873	281,655	51	176,381	16,498	542,349
4	5541	400,993	78	189,397	30,810	847,872
5	11,381	363,116	69	192,235	57,279	758,704
6	10,086	541,658	114	194,091	66,137	1,438,746
7	5434	508,141	61	228,535	35,295	839,597
8	7524	338,804	74	238,691	33,188	540,821
9	5077	511,467	84	267,385	65,391	1,562,274
10	7029	393,815	68	277,402	41,197	978,117
11	11,121	509,682	96	330,609	47,032	930,437
12	7072	527,457	92	332,609	56,064	1,345,185
13	9348	601,594	127	356,504	69,536	1,164,801
14	7781	528,799	96	365,844	37,467	1,348,588
15	6235	394,158	77	389,894	57,727	1,100,779
16	10,593	515,624	101	417,513	46,160	1,070,488
17	10,866	566,708	118	503,914	102,967	1,707,645
18	6500	467,617	74	517,318	47,236	1,223,026
19	11,469	768,484	103	537,746	84,510	2,299,694
20	10,868	669,996	107	590,601	69,576	1,901,465
21	10,717	844,949	120	622,550	89,401	1,909,698
22	19,716	1,258,981	242	660,164	97,941	3,055,193
23	10,888	1,148,863	202	808,369	191,166	4,096,300

1. Copy the 7 columns of Table 10.1 on an Excel sheet into cells A1:G24, as Fig. 10.1 depicts.
2. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
3. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
4. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
5. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK' (Fig. 10.2).
6. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

	A	B	C	D	E	F	G	H	I
1	No.	Area	Books	Staff	Population	Regist.	Borrow.		Lambdas
2	1	2249	163523	26	49196	5561	105321		0.000
3	2	4617	338671	30	78599	18106	314682		0.000
4	3	3873	281655	51	176381	16498	542349		0.000
5	4	5541	400993	78	189397	30810	847872		0.000
6	5	11381	363116	69	192235	57279	758704		0.031
7	6	10086	541658	114	194091	66137	1438746		0.042
8	7	5434	508141	61	228535	35295	839597		0.000
9	8	7524	338804	74	238691	33188	540821		0.000
10	9	5077	511467	84	267385	65391	1562274		0.000
11	10	7029	393815	68	277402	41197	978117		0.000
12	11	11121	509682	96	330609	47032	930437		0.000
13	12	7072	527457	92	332609	56064	1345185		0.000
14	13	9348	601594	127	356504	69536	1164801		0.000
15	14	7781	528799	96	365844	37467	1348588		0.000
16	15	6235	394158	77	389894	57727	1100779		0.000
17	16	10593	515624	101	417513	46160	1070488		0.000
18	17	10866	566708	118	503914	102967	1707645		0.000
19	18	6500	467617	74	517318	47236	1223026		0.000
20	19	11469	768484	103	537746	84510	2299694		0.000
21	20	10868	669996	107	590601	69576	1901465		0.000
22	21	10717	844949	120	622550	89401	1909698		0.000
23	22	19716	1258981	242	660164	97941	3055193		0.000
24	23	10888	1148863	202	808369	191166	4096300		0.005
25									
26	Index	1			Theta	0.372764			
27									
28		838.347	40087.253	8.001	18338.51126	5561	105321		
29		838.347	60955.532	9.692	18338.51126	5561	105321		
30									
31									

Fig. 10.1 Copying data into an Excel sheet

```
Dim i, j, k, t, r As Integer
Dim Store(1 To 10), Level1() As Variant

Range("B26") = 1

Range("B28:G28").Formula = "=SUMPRODUCT(B2:B24,$I2:$I24)"
Range("B29:E29").Formula = "=INDEX(B2:B24,$B26)*$F26"
Range("F29:G29").Formula = "=INDEX(F2:F24,$B26)"

SolverReset
SolverAdd CellRef:="$B$28:$E$28", relation:=1, _
    Formulatext:="$B$29:$E$29"
```

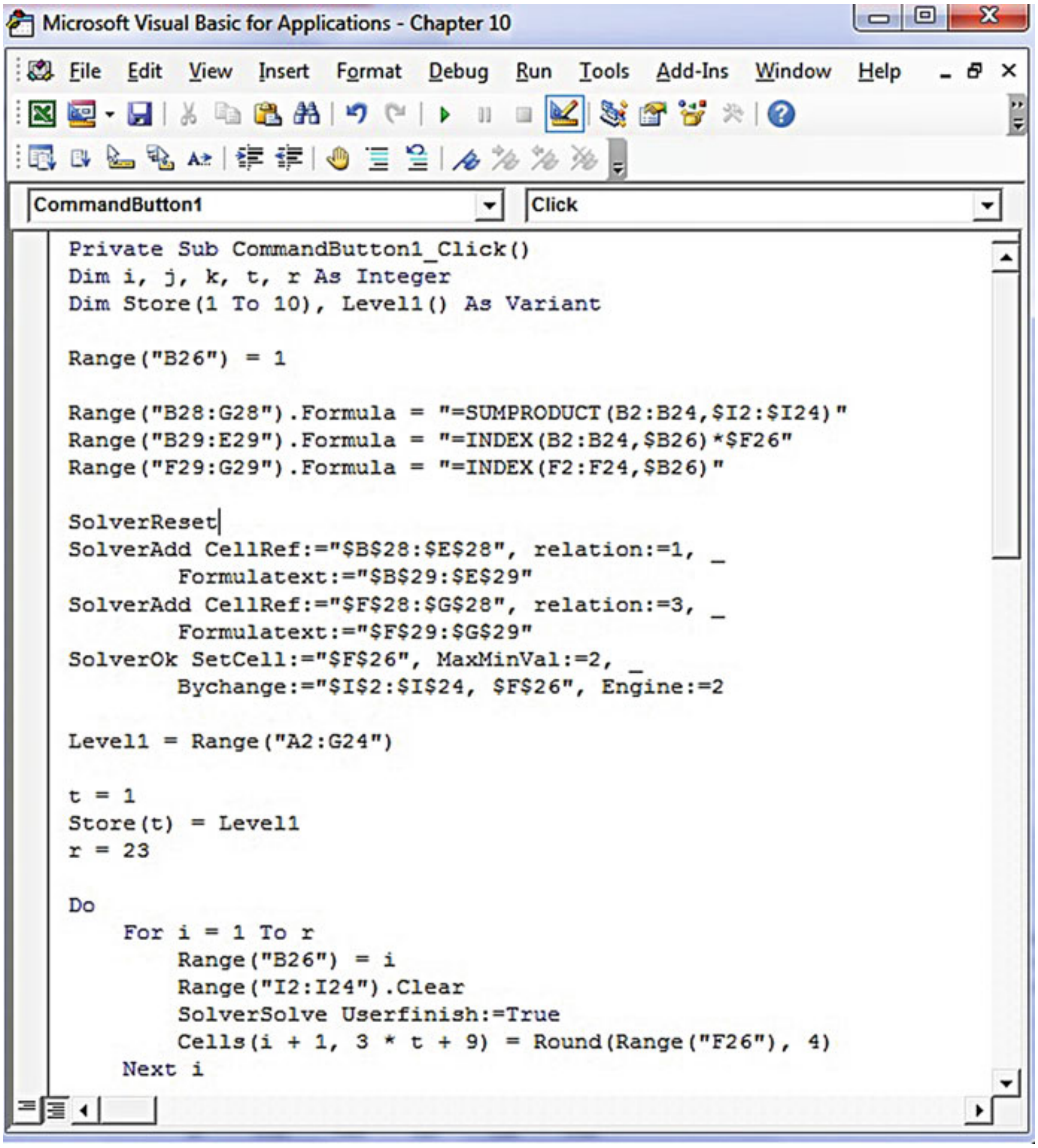


Fig. 10.2 Setting VBA for context-dependent with one click

```
SolverAdd CellRef:="$F$28:$G$28", relation:=3, _
        Formulatext:="$F$29:$G$29"
SolverOk SetCell:="$F$26", MaxMinVal:=2, _
        Bychange:="$I$2:$I$24, $F$26", Engine:=2
Level1 = Range("A2:G24")

t = 1
Store(t) = Level1
r = 23

Do
    For i = 1 To r
```

```
        Range("B26") = i
        Range("I2:I24").Clear
        SolverSolve Userfinish:=True
        Cells(i + 1, 3 * t + 9) = Round(Range("F26"), 4)
    Next i

    j = 0
    Level1 = Store(t)

    For i = 1 To r
        If Cells(i + 1, 3 * t + 9) <> 1 Then
            j = j + 1
            For k = 1 To 7
                Level1(j, k) = Level1(i, k)
            Next k
        Else
            Cells(i + 1, 3 * t + 8) = Level1(i, 1)
        End If
    Next i

    For i = j + 1 To r
        For k = 1 To 7
            Level1(i, k) = ""
        Next k
    Next i

    r = j
    Range("A2:G24") = Level1
    Store(t + 1) = Level1
    t = t + 1

Loop Until r = 0

Range("B29:E29").Formula = "=B31*$F26"
Range("F29:G29").Formula = "=F31"

For s = 1 To t - 1
    Level1 = Store(1)
    For r = 1 To t - 1
        Range("A2:G24") = Store(r)
        j = 1
        For i = 1 To 23
            If Cells(i + 1, 3 * s + 8) <> "" Then
                For k = 1 To 7
                    Cells(31, k) = Level1(Cells(i + 1, 3 * s + 8), k)
                Next k
                SolverSolve Userfinish:=True
                Cells(j + 26, 7 * (s - 1) + r + 11) = Round(Range("F26"), 4)
                Cells(j + 26, 7 * (s - 1) + 11) = Cells(i + 1, 3 * s + 8)
                j = j + 1
            End If
        Next i
```

```
    Next r
Next s

Range("I2:I24").Clear
Range("A2:G24") = Store(1)
```

7. Close the ‘Microsoft Visual Basic for Applications’ window.
8. Click on the small rectangle which was automatically made on the Excel sheet and created by step 19. The results are represented to cells K2:AR34.

From the above procedure, DMUs are partitioned in cells K2:X24. The DMUs in Level 1 are shown in cells K2:K24, where the corresponding scores of DMUs in Level 1 are presented in cells L2:L24, as demonstrated in columns 2 and 3 of Table 10.2. Thus, L^1 is $\{5, 6, 9, 17, 19, 23\}$.

In order to find the DMUs in Level 2, the DMUs in Level 1 are eliminated from the observed DMUs. The DMUs in Level 2 are shown in cells N2:

Table 10.2 Classifying DMUs and indicating the levels

	Level 1		Level 2		Level 3		Level 4		Level 5	
DMUs	No.	Score	No.	Score	No.	Score	No.	Score	No.	Score
1		0.373		0.539		0.732		0.799	1	1.000
2		0.792	2	1.000		0.835	3	1.000		
3		0.573		0.757	7	1.000	8	1.000		
4		0.719	4	1.000		0.936	11	1.000		
5	5	1.000		0.934	10	1.000	16	1.000		
6	6	1.000		0.788		0.952				
7		0.697		0.955	14	1.000				
8		0.580		0.813		0.856				
9	9	1.000	12	1.000						
10		0.705	13	1.000						
11		0.569		0.973						
12		0.758	15	1.000						
13		0.867		0.795						
14		0.722	18	1.000						
15		0.844	20	1.000						
16		0.582	21	1.000						
17	17	1.000	22	1.000						
18		0.787								
19	19	1.000								
20		0.849								
21		0.787								
22		0.785								
23	23	1.000								

N18, where the corresponding scores of DMUs in Level 1 are presented in cells O2:O18, as demonstrated in columns 4 and 5 of Table 10.2. The DMUs in Level 2 are $L^2 = \{2, 4, 12, 13, 15, 18, 20, 21, 22\}$.

The same step is followed for DMUs in Level 3 and 4. The last partition is a set of one DMU, that is, $L^5 = \{1\}$. The first library, DMU_1 is the least attractive library among the other libraries in Tokyo.

Tables 10.3, 10.4, 10.5, 10.6, 10.7, 10.8, 10.9, 10.10 and 10.11 demonstrate the attractiveness scores of each DMU in Level t in comparison with the generated PPS by DMUs in Level t'. For example, columns 2–6 in Table 10.3 illustrate the attractiveness scores for each DMU in set L^1 according to generated PPS by DMUs in sets L^1, L^2, L^3, L^4 and L^5.

The ranks of DMUs in $L^1 = \{5, 6, 9, 17, 19, 23\}$ in comparison with the contexts in Levels 1–5 are also represented in Table 10.4. From the results in Tables 10.3 and 10.4, the library numbered 23, is the best library among the observed libraries, followed by libraries 9 and 6. As can be seen in Table 10.4, the ranking of DMUs is

Table 10.3 Attractiveness for the libraries in level 1 vs. levels 1–5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
5	1.000	1.507	1.961	2.095	4.639
6	1.000	1.786	2.206	2.583	4.124
9	1.000	1.627	1.989	2.961	6.571
17	1.000	1.310	1.737	2.149	5.343
19	1.000	1.295	1.566	2.103	5.512
23	1.000	2.049	2.703	4.017	8.034

Table 10.4 Attractiveness' ranks for the libraries in Table 10.3

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
5	1	4	4	6	5
6	1	2	2	3	6
9	1	3	3	2	2
17	1	5	5	4	4
19	1	6	6	5	3
23	1	1	1	1	1

Table 10.5 Attractiveness for the libraries in level 2 vs. levels 1–5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
2	0.792	1.000	1.492	1.645	2.822
4	0.719	1.000	1.217	1.496	3.283
12	0.758	1.000	1.290	1.823	4.062
13	0.867	1.000	1.296	1.688	3.399
15	0.844	1.000	1.548	2.109	4.336
18	0.787	1.000	1.239	1.678	4.080
20	0.849	1.000	1.247	1.676	4.406
21	0.787	1.000	1.335	1.908	3.929
22	0.785	1.000	1.257	1.516	3.768

Table 10.6 Attractiveness' ranks for the libraries in Table 10.5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
2	4	1	2	7	9
4	9	1	9	9	8
12	8	1	5	3	4
13	1	1	4	4	7
15	3	1	1	1	2
18	6	1	8	5	3
20	2	1	7	6	1
21	5	1	3	2	5
22	7	1	6	8	6

Table 10.7 Attractiveness for the libraries in level 3 vs. levels 1–5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
7	0.697	0.934	1.000	1.493	3.398
10	0.705	0.955	1.000	1.357	3.856
14	0.722	0.973	1.000	1.322	3.960

Table 10.8 Attractiveness' ranks for the libraries in Table 10.7

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
7	3	3	1	1	3
10	2	2	1	2	2
14	1	1	1	3	1

Table 10.9 Attractiveness for the libraries in level 4 vs. levels 1–5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
3	0.573	0.757	0.835	1.000	2.990
8	0.580	0.788	0.936	1.000	2.880
11	0.569	0.813	0.952	1.000	2.834
16	0.582	0.795	0.856	1.000	3.223

Table 10.10 Attractiveness' ranks for the libraries in Table 10.9

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
3	3	4	4	1	2
8	2	3	2	1	3
11	4	1	1	1	4
16	1	2	3	1	1

Table 10.11 Attractiveness for the libraries in level 1 vs. levels 1–5

DMUs	Level 1	Level 2	Level 3	Level 4	Level 5
1	0.373	0.539	0.732	0.799	1.000

changed as the evaluation context is changed. In other words, the performance of a library is dependent on the evaluation context and background. For instance, when the evaluation context is L^4, the rank of DMU_5 is 6 among DMUs in L^1, whereas when the evaluation context is L^2, the rank of DMU_5 is 4 among DMUs in L^1.

The same illustration can be discussed for Tables 10.5–10.11. The scores in Table 10.11 represent how DMU_1 is not attractive with respect to a different evaluation context.

We can also calculate the average attractiveness scores of each DMUs among the Levels 1–5, as shown in Table 10.12. The rank of each DMU is also shown in the last column. As discussed, DMU_{23} is the best library followed by DMU_9 and DMU_6. The least average attractiveness scores belong to DMU_1, DMU_3 and DMU_{11}. The interesting result to measure the average is that DMU_{10} in Level 3 has a larger average attractiveness in comparison with DMU_2 in Level 2. As a results, the performance evaluation of a library can be different where the evaluation context is changed.

Note that in the above procedure, the command "relation:=1" means "<=", "relation:=2" means "=", and "relation:=3" means ">=". In addition, "MaxMinVal:=1" means maximization, and "MaxMinVal:=2" means minimization. These information are needed in Exercises 10.1 and 10.2.

Table 10.12 Average attractiveness score

No.	Average attractiveness score	Rank
1	0.6886	23
2	1.5501	16
3	1.2310	22
4	1.5428	17
5	2.2401	6
6	2.3399	3
7	1.5042	18
8	1.2370	20
9	2.8295	2
10	1.5746	15
11	1.2336	21
12	1.7866	10
13	1.6500	13
14	1.5952	14
15	1.9674	7
16	1.2912	19
17	2.3078	4
18	1.7568	11
19	2.2951	5
20	1.8356	8
21	1.7919	9
22	1.6652	12
23	3.5605	1

10.4 Conclusion

In this chapter, a DEA-based context-dependent model is presented. The context-dependent captures situations where the performance of DMUs depends on the absence or presence of a third option, and is called attractiveness. The attractiveness of a set of public libraries in Tokyo is measured with respect to different measured evaluation contexts. Unlike the traditional first-level of empirical production frontier, different strata of empirical production frontiers are measured step by step, and considered as evaluation contexts. The context-dependent performance depends on both technically inefficient and efficient DMUs. Such change makes DEA more flexible and allows DEA to globally and locally detect better options. In particular, the attractiveness measure can be used to identify DMUs that have outstanding performance, and to differentiate the performance of DEA technically efficient DMUs.

10.5 Exercises

10.1 Update the procedure in Sect. 10.3 to measure the attractiveness scores of libraries by CCR output-oriented.

10.2 Use the BCC input oriented to partition the libraries, and after that use BCC output-oriented to measure the attractiveness of each library with respect to different evaluation context.
Note that "changing cells" in Solver should be changed for each level, when the variable returns to scale technology is considered.

10.3 Use the BCC output-oriented to partition the libraries, and after that use BCC input oriented to measure the attractiveness of each library with respect to different evaluation context.

10.4 Compare the results in Exercises 10.2 and 10.3. Are the outcomes the same? In other words, are the output degree attractiveness is the same as the input degree attractiveness for each libraries, when variable returns to scale technology is used?

10.5 Use SBM to partition the libraries, and to measure the attractiveness of each library with respect to different evaluation context.

10.6 Use Eq. 3.31 to partition the libraries, and to measure the attractiveness of each library with respect to different evaluation context.

Chapter 11
Efficiency Change Over Different Times

11.1 Introduction

In the literature of macroeconomics and the business economic press, there is a strong interest in the variation of efficiency over different times (Ray 2004). There are two methods to address this issue, such as, the Tornqvist and the Fisher productivity indexes. These two indexes use the price information of input and output factors without requiring the production technology of firms. Caves et al. (1982) introduced an index called the Malmquist efficiency index to construct a production frontier to represent the production technology of firms. Färe et al. (1992) combined ideas on the Farrell measurement and Caves et al. (1982) efficiency measurement to construct a Malmquist efficiency index directly from the input and output factors using DEA. The Malmquist efficiency index can be decomposed into two components. One component is to measure the technical change and the other component is to measure the frontier shift. In this chapter, we first illustrate the basic Malmquist index. After that, we illustrate the works of Chen and Ali (2004) to provide an extension to the Malmquist index by analyzing the above two components with an example of computer industry. Finally, a non-linear Malmquist index, proposed by Chen (2003), is discussed.

11.2 The Basic Theory on the Malmquist Index

In this section we illustrate the basic knowledge to measure the Malmquist efficiency index by a simple example. Suppose that there are five homogenous firms in which each firm has one single input and one single output factor over two different time periods, as data in Table 11.1 represents. For instance, A^1 represents the firm A in the first period and A^2 represents the same firm in the second period.

D. Khezrimotlagh, Y. Chen, *Decision Making and Performance Evaluation Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 269, https://doi.org/10.1007/978-3-319-76345-3_11

Table 11.1 Example of four DMUs

Firm	Input	Output	Firm	Input	Output
A^1	5.00	4.00	A^2	2.00	4.00
B^1	7.00	2.00	B^2	7.00	4.00
C^1	3.00	2.00	C^2	3.00	4.00
D^1	5.00	2.00	D^2	5.00	2.00

Table 11.2 Efficiency of the four DMUs

Firm	Efficiency	Firm	Output
A^1	4/5	A^2	4/2
B^1	2/7	B^2	4/7
C^1	2/3	C^2	4/3
D^1	2/5	D^2	2/5

The efficiency of each firm is measured by Eq. 1.2, that is, output/input. Table 11.2 shows the efficiency of each firm corresponding to the first and the second periods.

Four different relative efficiency scores can be assigned for a firm. One of the relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the first period with the efficiency scores of all firms in the first period. We denote this relative efficiency score with $D^1_{A^1}$.

As discussed in Sect. 1.5 this relative efficiency score is measured by the ratio of the efficiency score of A in the first period to the maximum efficiency scores of firms A–D in the first period. In other words,

$$D^1_{A^1} = \frac{4/5}{\max\left\{\frac{4}{5}, \frac{2}{7}, \frac{2}{3}, \frac{2}{5}\right\}} = \frac{4/5}{4/5} = 1. \tag{11.1}$$

This relative efficiency score can also be measured by sketching the straight line which passes the origin and A^1, as Fig. 11.1 illustrates the relative efficiency score of A^1 is 1, as A^1 lies on the frontier or since the ratio of TA^1 to $TA^1 = 1$. In addition, the relative efficiency score of B^1 is the ratio of HP to HB^1.

Another relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the first period with the efficiency scores of all firms in the second period. We denote this relative efficiency score with $D^2_{A^1}$. Similarly, this relative efficiency score is measured by the ratio of the efficiency score of A in the first period to the maximum efficiency scores of firms A-D in the second period. In other words,

$$D^2_{A^1} = \frac{4/5}{\max\left\{\frac{4}{2}, \frac{4}{7}, \frac{4}{3}, \frac{2}{5}\right\}} = \frac{4/5}{4/2} = 0.4. \tag{11.2}$$

As Fig. 11.2 represents, the relative efficiency score of A^1 is measured by the ratio of TR to TA^1, and so on for the other firms B-D.

The third relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the second period with the efficiency scores of all

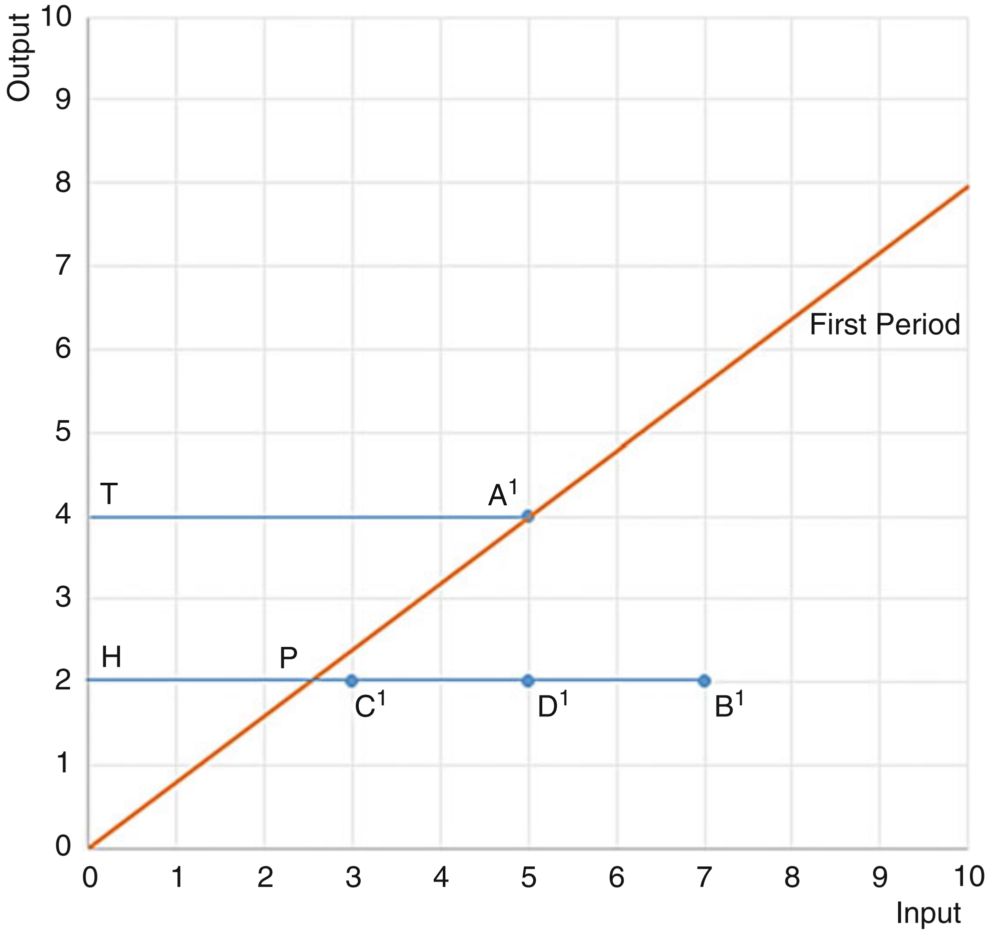


Fig. 11.1 Measuring the first component

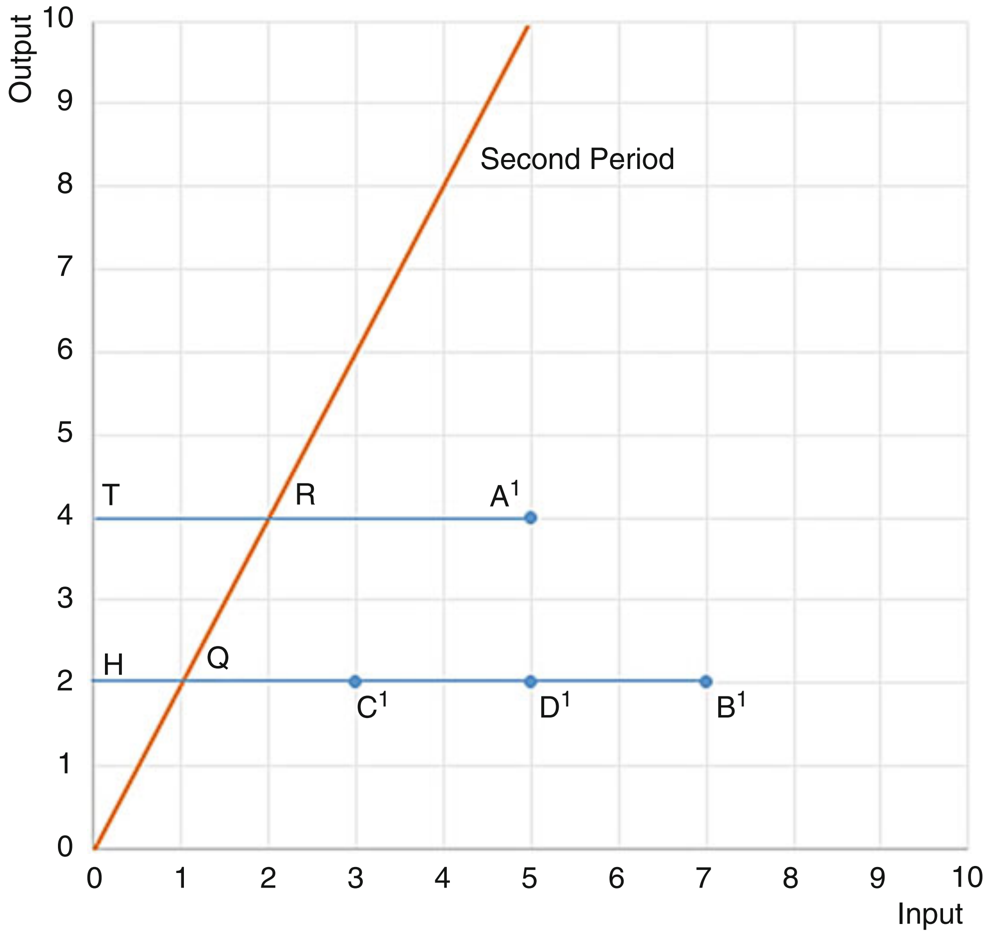


Fig. 11.2 Measuring the second component

firms in the first period. We denote this relative efficiency score with $D^1_{A^2}$. Correspondingly, this relative efficiency score is measured by the ratio of the efficiency score of A in the second period to the maximum efficiency scores of firms A-D in the first period. In other words,

$$D^1_{A^2} = \frac{4/2}{\max\left\{\frac{4}{5}, \frac{2}{7}, \frac{2}{3}, \frac{2}{5}\right\}} = \frac{4/2}{4/5} = 2.5. \tag{11.3}$$

Figure 11.3 illustrates the frontier generated by firms A^1-D^1. The firms B^2 and D^2 are under the line which passes the origin and A^1 and the firms A^2 and C^2 are above the line. The relative efficiency scores of A^2 and C^2, which are measured by the ratio of TA^1 to TA^2 and TA^1 to TC^2, respectively. Both of these scores are greater than 1, as they are above the frontier. Similarly, the relative efficiency scores of B^2 and D^2 are less than 1 and they are measured by the ratio of TA^1 to TB^2 and HP to HD^2, respectively.

The last relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the second period with the efficiency scores of all firms in the second period, as Fig. 11.4 depicts. We denote this relative efficiency score with $D^2_{A^2}$.

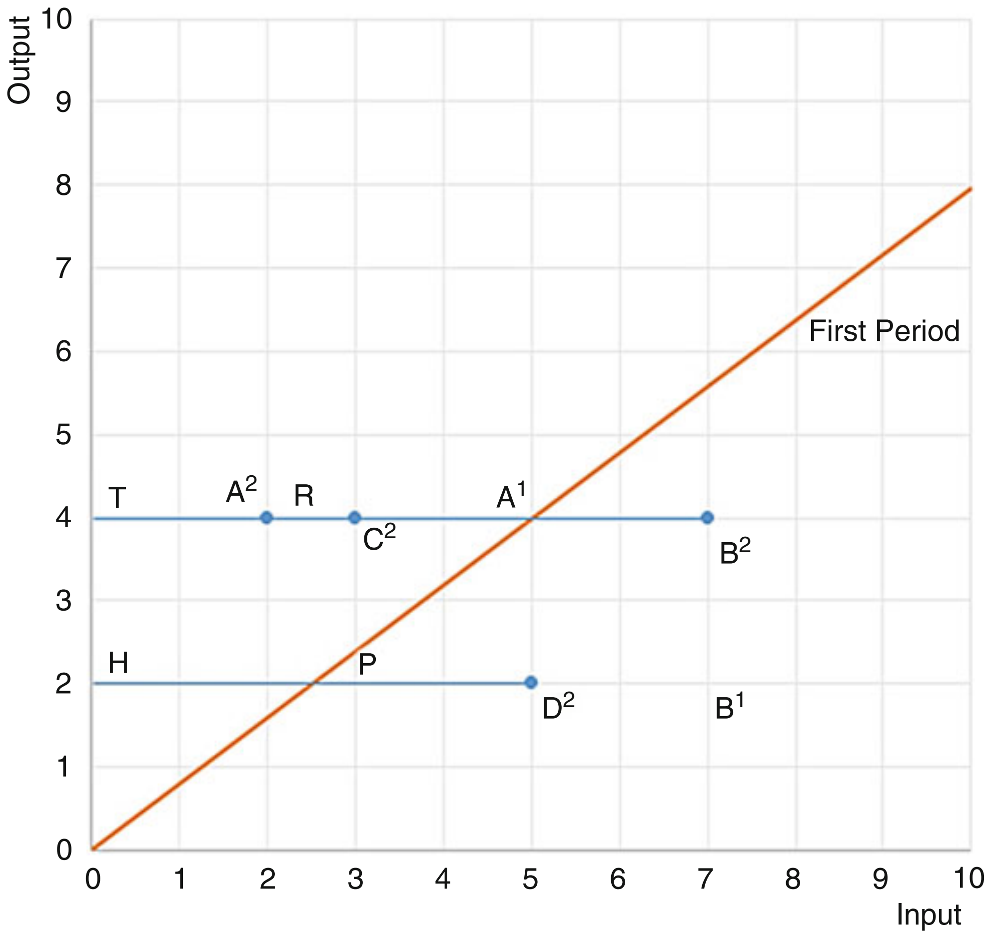


Fig. 11.3 Measuring the third component

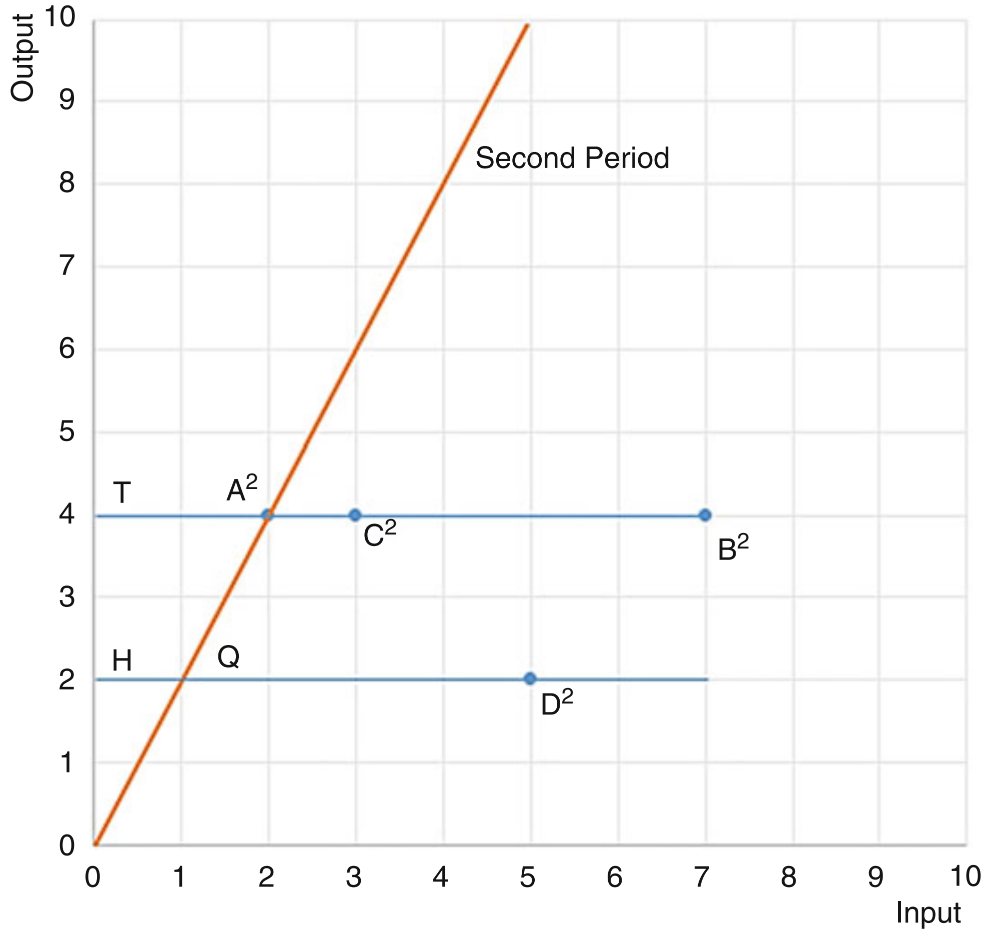


Fig. 11.4 Measuring the fourth component

Table 11.3 The measured four components

Firm	D_1^1	D_1^2	D_2^1	D_2^2
A	1.00	0.40	2.50	1.00
B	0.36	0.14	0.71	0.29
C	0.83	0.33	1.67	0.67
D	0.50	0.20	0.50	0.20

This relative efficiency score is measured by the ratio of the efficiency score of A in the second period to the maximum efficiency scores of firms A-D in the second period. In other words,

$$D_{A^2}^2 = \frac{4/2}{\max\left\{\frac{4}{2}, \frac{4}{7}, \frac{4}{3}, \frac{2}{5}\right\}} = \frac{4/2}{4/2} = 1. \tag{11.4}$$

Table 11.3 represents the relative efficiency scores of firms A-D regarding each category and time period.

Here D_2^1 denotes the efficiency score of firms in the second category related to the first period. For instance, D_2^1 for firm A refers to Eq. 11.3, that is, $D_{A^2}^1$.

The Malmquist efficiency index is defined as follows

Table 11.4 The Malmquist efficiency index of firms A-D

Firm	*M*
A	2.50
B	2.00
C	2.00
D	1.00

$$M_A = \sqrt{\frac{D^1_{A^2} \times D^2_{A^2}}{D^1_{A^1} \times D^2_{A^1}}}. \tag{11.5}$$

Equation 11.5 demonstrates the geometric mean of $D^1_{A^2}/D^1_{A^1}$ and $D^2_{A^2}/D^2_{A^1}$. Table 11.4 shows the DEA Malmquist efficiency index for firms A-D.

The Malmquist efficiency index in Eq. 11.5 can also be addressed using the CCR Envelopment model. Note that CCR-efficiency radially calculates the technical efficiency scores of firms which are the same as the efficiency scores of firms where the firms have only one single input and one single output factors. These four different CCR Envelopment models given by Models 11.6–11.9.

The first model, Model 11.6 measures the CCR efficiency score of A^1 in the generated PPS by A^1- D^1. Consequently, Models 11.7–11.9 measure the CCR efficiency scores of A^1 in the generated PPS by A^2- D^2, the CCR efficiency scores of A^2 in the generated PPS by A^1- D^1, and the CCR efficiency scores of A^2 in the generated PPS by A^2- D^2.

$$\begin{aligned}
&D^1_{A^1} = \min\ \theta,\\
&\text{Subject to}\\
&x_{A^1}\lambda_1 + x_{B^1}\lambda_2 + x_{C^1}\lambda_3 + x_{D^1}\lambda_4 \le x_{A^1}\theta,\\
&y_{A^1}\lambda_1 + y_{B^1}\lambda_2 + y_{C^1}\lambda_3 + y_{D^1}\lambda_4 \ge y_{A^1},\\
&\lambda_i \ge 0,\quad \text{for } i = 1, 2, 3, 4
\end{aligned} \tag{11.6}$$

$$\begin{aligned}
&D^2_{A^1} = \min\ \theta,\\
&\text{Subject to}\\
&x_{A^2}\lambda_1 + x_{B^2}\lambda_2 + x_{C^2}\lambda_3 + x_{D^2}\lambda_4 \le x_{A^1}\theta,\\
&y_{A^2}\lambda_1 + y_{B^2}\lambda_2 + y_{C^2}\lambda_3 + y_{D^2}\lambda_4 \ge y_{A^1},\\
&\lambda_i \ge 0,\quad \text{for } i = 1, 2, 3, 4.
\end{aligned} \tag{11.7}$$

$$\begin{aligned}
&D^1_{A^2} = \min\ \theta,\\
&\text{Subject to}\\
&x_{A^1}\lambda_1 + x_{B^1}\lambda_2 + x_{C^1}\lambda_3 + x_{D^1}\lambda_4 \le x_{A^2}\theta,\\
&y_{A^1}\lambda_1 + y_{B^1}\lambda_2 + y_{C^1}\lambda_3 + y_{D^1}\lambda_4 \ge y_{A^2},\\
&\lambda_i \ge 0,\quad \text{for } i = 1, 2, 3, 4
\end{aligned} \tag{11.8}$$

Fig. 11.5 Coping data in excel

	A	B	C
1	Firm	Input	Output
2	A^1	5.00	4.00
3	B^1	7.00	2.00
4	C^1	3.00	2.00
5	D^1	5.00	2.00
6	A^2	2.00	4.00
7	B^2	7.00	4.00
8	C^2	3.00	4.00
9	D^2	5.00	2.00

$$
\begin{aligned}
&D^2_{A^2} = \min \theta, \\
&\text{Subject to} \\
&x_{A^2}\lambda_1 + x_{B^2}\lambda_2 + x_{C^2}\lambda_3 + x_{D^2}\lambda_4 \le x_{A^2}\theta, \\
&y_{A^2}\lambda_1 + y_{B^2}\lambda_2 + y_{C^2}\lambda_3 + y_{D^2}\lambda_4 \ge y_{A^2}, \\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, 3, 4
\end{aligned}
\tag{11.9}
$$

We now formulize an Excel sheet to solve the above four models with one single click, as the following instructions illustrate.

1. Copy data of Table 7.1 on an Excel sheet into cells A1:C9, as Fig. 11.5 illustrates.
2. Label D1 as 'Lambdas', E1 as 'Firm', F1 as 'D_1^1', G1 as 'D_1^2', H1 as 'D_2^1', I1 as 'D_2^2', J1 as 'Malmquist Index', A11 as 'Index of Firm', D11 as 'Theta', A13 as Models, B13-C13 as 'Input Constraints', E13-F13 as 'Output Constraints', A19 as 'Index of Model', and A21 as 'Selected Model' (Fig. 11.6).
3. Assign number 1 to B11 and B19.
4. Assign the following command into B14 and B16,

   '=Sumproduct(B2:B5,D2:D5)'.

5. Assign the following command into B15 and B17,

   '=Sumproduct(B6:B9,D6:D9)'.

6. Assign the following command into C14 and C15,

   '=Index(B2:B5,B11)*E11'.

	A	B	C	D	E	F	G	H	I	J
1	Firm	Input	Output	Lambdas	Firm	D_1^1	D_1^2	D_2^1	D_2^2	M
2	A^1	5.00	4.00	0.00	A	1.00	0.40	2.50	1.00	2.50
3	B^1	7.00	2.00	0.00	B	0.36	0.14	0.71	0.29	2.00
4	C^1	3.00	2.00	0.00	C	0.83	0.33	1.67	0.67	2.00
5	D^1	5.00	2.00	0.00	D	0.50	0.20	0.50	0.20	1.00
6	A^2	2.00	4.00	0.50						
7	B^2	7.00	4.00	0.00						
8	C^2	3.00	4.00	0.00			Run			
9	D^2	5.00	2.00	0.00						
10										
11	Index Firm	4		Theta	0.2					
12										
13	Models	Input Constraints			Output Constraints					
14	1	0	1		0	2				
15	2	1	1		2	2				
16	3	0	1		0	2				
17	4	1	1		2	2				
18										
19	Index Model	4								
20										
21	Selected Model	1	1		2	2				

Fig. 11.6 Setting Excel to solve Eqs. 11.6, 11.7, 11.8 and 11.9

7. Assign the following command into C16 and C17,

   '=Index(B6:B9,B11)*E11'.

8. Assign the following command into E14 and E16,

   '=Sumproduct(C2:C5,D2:D5)'.

9. Assign the following command into E15 and E17,

   '=Sumproduct(C6:C9,D6:D9)'.

10. Assign the following command into F14 and F15,

    '=Index(C2:C5,B11)'.

11. Assign the following command into F16 and F17,

    '=Index(C6:C9,B11)'.

12. Assign the following command into B21,

    '=Index(B14:B17,$B19)'.

13. Copy B21 and paste it to C21, E21 and F21.
14. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.7 illustrates.

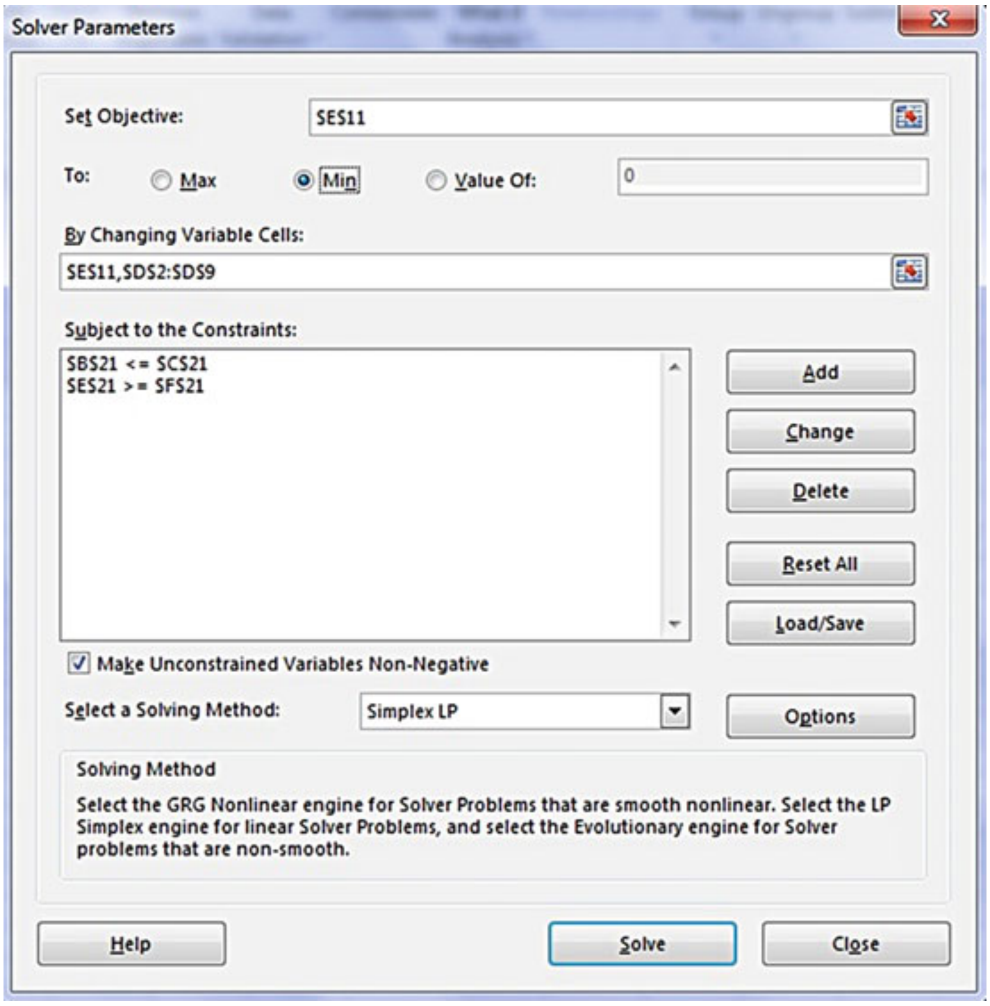


Fig. 11.7 Setting solver to solve Eqs. 11.6, 11.7, 11.8 and 11.9

15. Assign 'E11' into 'Set Objective' and choose 'Min'.
16. Assign 'E11, D2:D9' into 'By Changing Variable Cells'.
17. Click on 'Add' and assign 'B21' into 'Cell Reference', then select '<=', and assign 'C21' into 'Constraint'.
18. Click on 'Add' and assign 'E21' into 'Cell Reference', then select '>=' and assign 'F21' into 'Constraint'. Then click on 'OK'.
19. Tick 'Make Unconstrained Variables Non-Negative'.
20. Choose 'Simplex LP' from 'Select a Solving Method'.
21. Click on 'Solve'.
22. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
23. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
24. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
25. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub', as Fig. 11.8 shows.

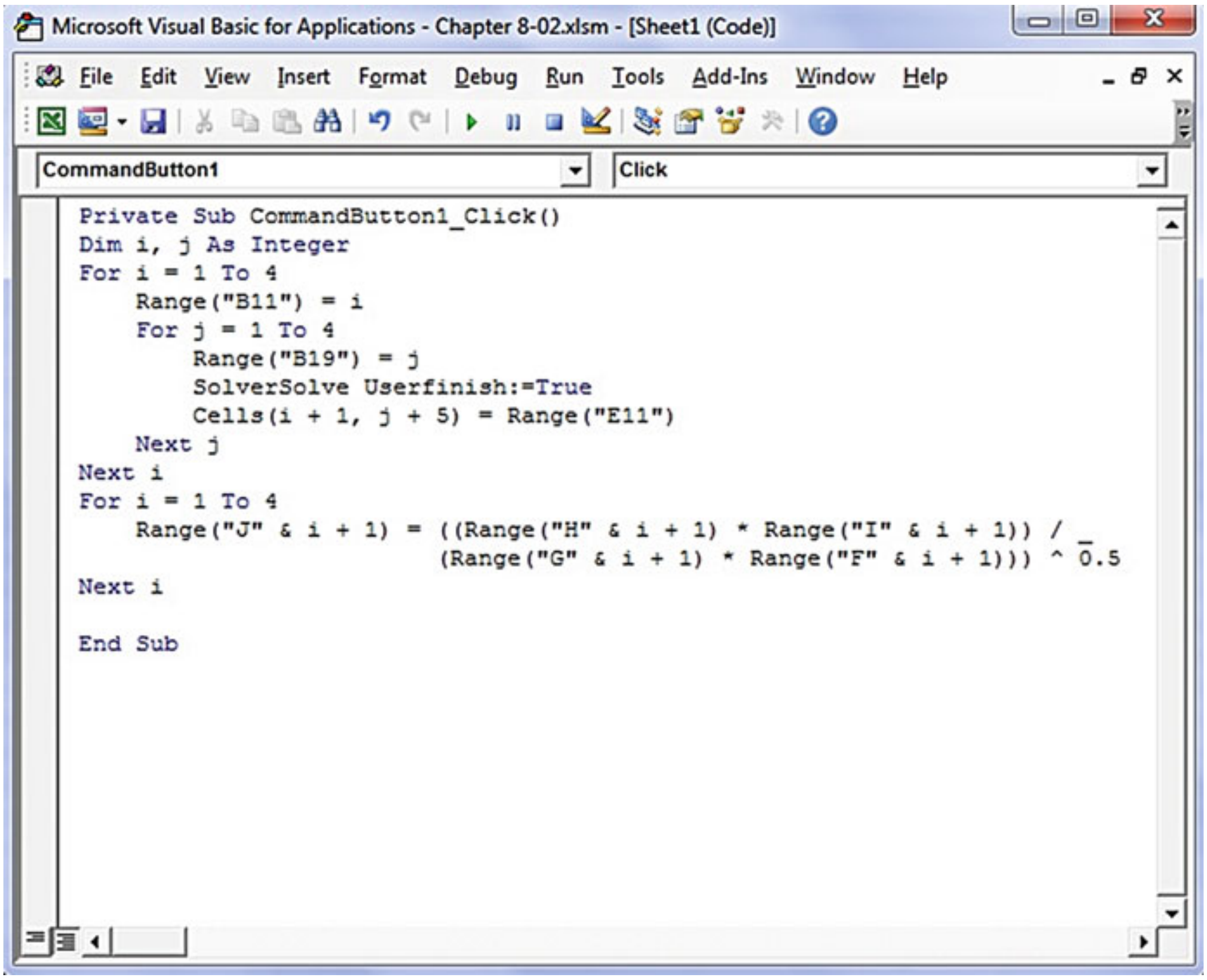


Fig. 11.8 Setting VBA to solve Eqs. 11.6, 11.7, 11.8 and 11.9

```
Dim i, j As Integer
For i = 1 To 4
   Range("B11") = i
   For j = 1 To 4
      Range("A19") = j
      SolverSolve Userfinish:=True
      Cells(i + 1, j + 5) = Range("E11")
   Next j
Next i
For i = 1 To 4
Range("J" & i + 1) = ((Range("G" & i + 1) * Range("I" & i + 1)) / (Range("F" & i + 1)
* Range("H" & i + 1))) ^ 0.5
Next i
```

26. Close the ‘Microsoft Visual Basic for Applications’ window.
27. Click on the small rectangle which was automatically made on the Excel sheet and created by step 23. The results are represented to cells F2:J5.

In general case, the four CCR Envelopment models, where DMU$_l$ ($l = 1, 2, \ldots, n$) is evaluated, are given by:

$$
\begin{aligned}
&D_{l^t}^{t} = \min\theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}^{t}\lambda_i \le x_{lj}^{t}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}^{t}\lambda_i \ge y_{lk}^{t}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.10}
$$

$$
\begin{aligned}
&D_{l^t}^{t+1} = \min\theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}^{t+1}\lambda_i \le x_{lj}^{t}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}^{t+1}\lambda_i \ge y_{lk}^{t}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,
\end{aligned}
\tag{11.11}
$$

$$
\begin{aligned}
&D_{l^{t+1}}^{t} = \min\theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}^{t}\lambda_i \le x_{lj}^{t+1}\theta_l, \quad \text{for } j = 1, 2, \ldots, m\\
&\textstyle\sum_{i=1}^{n} y_{ik}^{t}\lambda_i \ge y_{lk}^{t+1}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.12}
$$

$$
\begin{aligned}
&D_{l^{t+1}}^{t+1} = \min\theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}^{t+1}\lambda_i \le x_{lj}^{t+1}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}^{t+1}\lambda_i \ge y_{lk}^{t+1}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.13}
$$

In models 11.10–11.13, the index t shows the first time period and $t + 1$ shows the second time period. The Malmquist CCR-Efficiency index is also given by

$$
M_l = \sqrt{\frac{D_{l^{t+1}}^{t} \times D_{l^{t+1}}^{t+1}}{D_{l^t}^{t} \times D_{l^t}^{t+1}}}. \tag{11.14}
$$

Färe et al. (1992) defined that if $M_l > 1$ the efficiency gain; if $M_l < 1$ indicates efficiency loss; and if $M_l = 1$ means no change in efficiency from the first time t to the second time $t + 1$.

11.3 The Component of the Malmquist Index

Färe et al. (1992) decomposed the Malmquist efficiency index into two different components as follows:

$$M_l = \left(\frac{D^{t+1}_{l^{t+1}}}{D^t_{l^t}}\right)\left(\frac{D^t_{l^t} \times D^t_{l^{t+1}}}{D^{t+1}_{l^t} \times D^{t+1}_{l^{t+1}}}\right)^{1/2}. \tag{11.15}$$

The first fraction, $D^{t+1}_{l^{t+1}}/D^t_{l^t}$, is the component that measures the change in the technical efficiencies and the second component measures the production technology. If the value of the second component is greater (smaller) than 1, this indicates that we have a positive (negative) shift or a technical progress. If the value of the second component is equal to 1, this means that we have no shift in production technology.

In this section, the two components are examined to reveal sources and designs of efficiency change that are hidden by the aggregated nature of the discussed Malmquist index. Indeed, it is discussed that more information can be obtained from each of the Malmquist components.

Note that, when the prices (weights) of input and output factors are available, the allocative efficiency in the characterization of *strategy shift* as well as scale efficiency should also be considered, because a better *strategy* choice of a firm reflects considerations of all technical, scale and allocative efficiencies.

Let's call the first component in the Malmquist efficiency index in Eq. 11.15 the Technical Efficiency Change (TEC) for DMU_l from time t to $t+1$, and denoted by TEC_l, as shown in Eq. 11.16.

$$TEC_l = \frac{D^{t+1}_{l^{t+1}}}{D^t_{l^t}} \tag{11.16}$$

It is obvious that TEC_l can be greater than 1, less than 1 or equal to 1. In other words,

$$\frac{D^{t+1}_{l^{t+1}}}{D^t_{l^t}} > 1 \text{ or } \frac{D^{t+1}_{l^{t+1}}}{D^t_{l^t}} < 1 \text{ or } \frac{D^{t+1}_{l^{t+1}}}{D^t_{l^t}} = 1 \tag{11.17}$$

When the value TEC_l is greater than 1, this means that the radial distance of DMU_l in time $t+1$ to the production frontier in time $t+1$ is closer than the radial distance of DMU_l in time t to the production frontier in time t. When the value TEC_l is less than 1, this means that the radial distance of DMU_l in time $t+1$ to the production frontier in time $t+1$ is greater than the radial distance of DMU_l in time t to the production frontier in time t. When the value TEC_l is equal to 1, this means that DMU_l in time $t+1$ is as close to the production frontier in time $t+1$ as DMU_l in time t to the production frontier in time t. Of course, it is a very rare situation to have $TEC_l = 1$, except when DMU_l is equally efficient in both time t and $t+1$.

We also call the second component in the Malmquist efficiency index in Eq. 11.15 as the Efficiency Frontier Change (EFC) for DMU_l from time t to $t+1$, as shown in Eq. 11.18.

$$EFC_l = \left(\frac{D^t_{l^t}}{D^{t+1}_{l^t}} \times \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} \right)^{1/2}. \tag{11.18}$$

As a result, the Malmquist efficiency index is the TEC_l index times to the EFC_l index, that is,

$$M_l = TEC_l \times EFC_l. \tag{11.19}$$

As can be seen, the Malmquist efficiency index only shows the average of efficiency change, which is oversimplified or over-aggregated.

The EFC_l index can also be greater than 1, less than 1 or equal to 1. When the EFC_l value is greater than 1, this means a positive shift in DMU$_l$ or a technical progress in DMU$_l$. If the EFC_l value is less than 1, this means a negative shift in DMU$_l$ or a technical regress in DMU$_l$. There is no shift in production technology if the EFC_l value is equal to 1.

The EFC_l index can be described as an average aggregated change in production technology of DMU$_l$ from time period t to $t+1$. It can also be decomposed into two different fractions, and each fraction can also be greater than 1, less than 1 or equal to 1, as Eqs. 11.20 and 11.21 represent.

$$\frac{D^t_{l^t}}{D^{t+1}_{l^t}} > 1 \quad \text{or} \quad \frac{D^t_{l^t}}{D^{t+1}_{l^t}} < 1 \quad \text{or} \quad \frac{D^t_{l^t}}{D^{t+1}_{l^t}} = 1 \tag{11.20}$$

$$\frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} > 1 \quad \text{or} \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} < 1 \quad \text{or} \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} = 1 \tag{11.21}$$

Since the frontier from time to time can have a downward shift in one region and an upward shift in another, the average frontier shift index, EFC_l, oversimplifies or over-aggregates the frontier shift. This can lead to the omission of some very important managerial information. To describe this issue, suppose that we have three DMUs, A_1^t, A_2^t and A_3^t in which each A_i^t $(i = 1, 2, 3)$ has two input factors at time period t, as Fig. 11.9 shows.

As depicted in Fig. 11.10, let's select A_1^t and assume that the production frontier from time period t is shifted to time period $t+1$.

Figure 11.11 illustrates three different locations that DMU A_1 may have in time period $t+1$, such as, A_1^{t+1}, A_2^{t+1} or A_3^{t+1}.

For example, assume that A_1^t in time period t will have a location at A_3^{t+1} in time period $t+1$. The technical efficiencies of A_1^t and A_3^{t+1} in time t and $t+1$ are calculated as follows:

$$D^t_{A_1^t} = \frac{OP}{OA_1^t}, \tag{11.22}$$

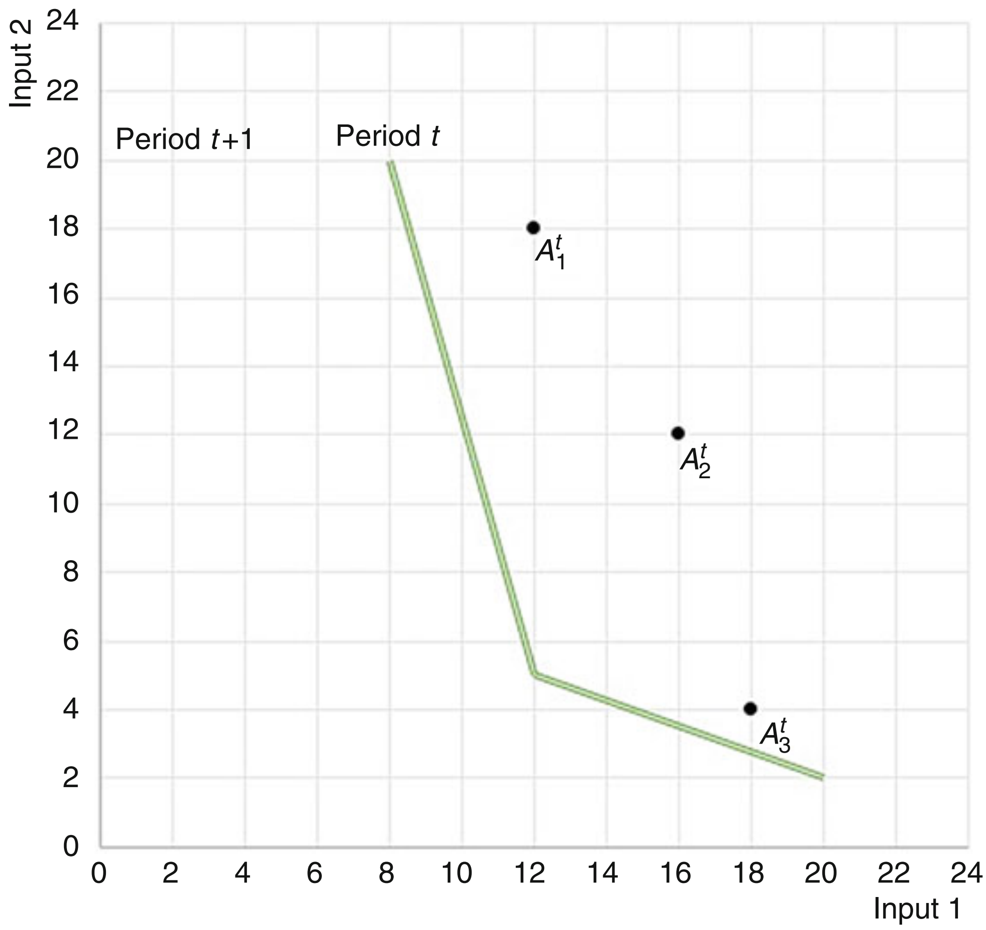


Fig. 11.9 Example of three DMUs

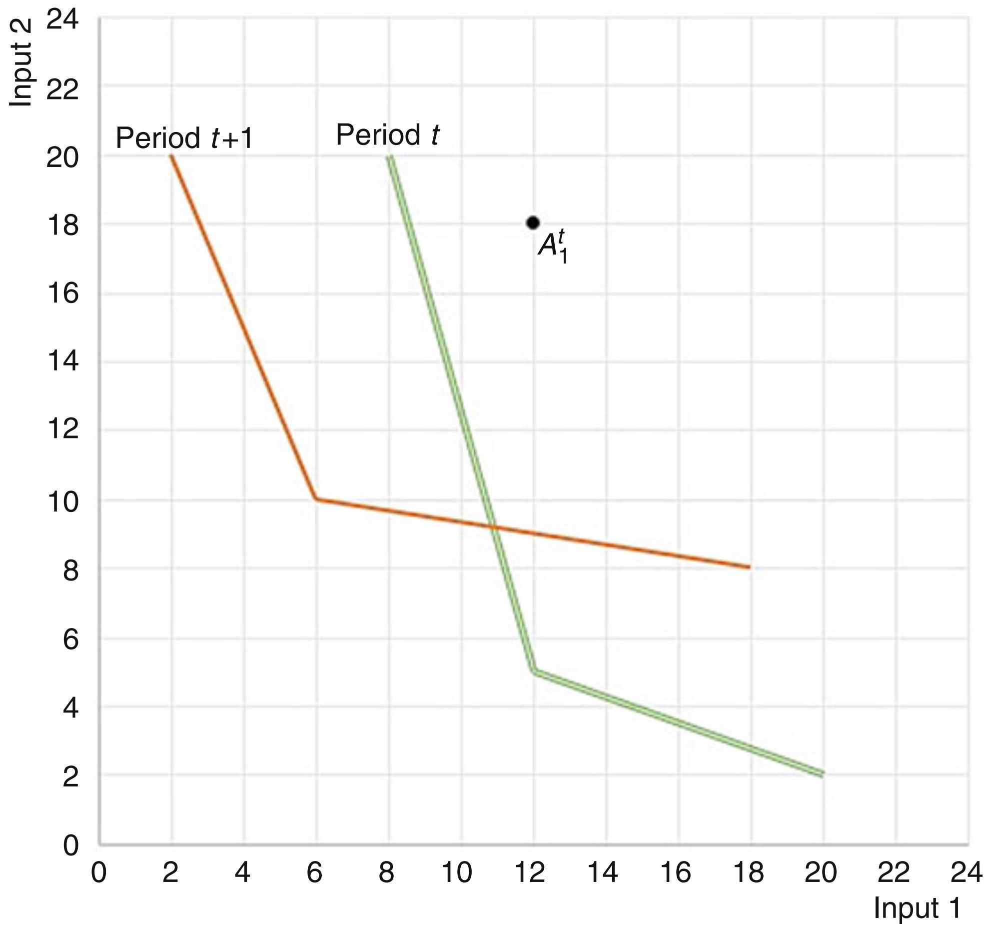


Fig. 11.10 Frontiers in two period time

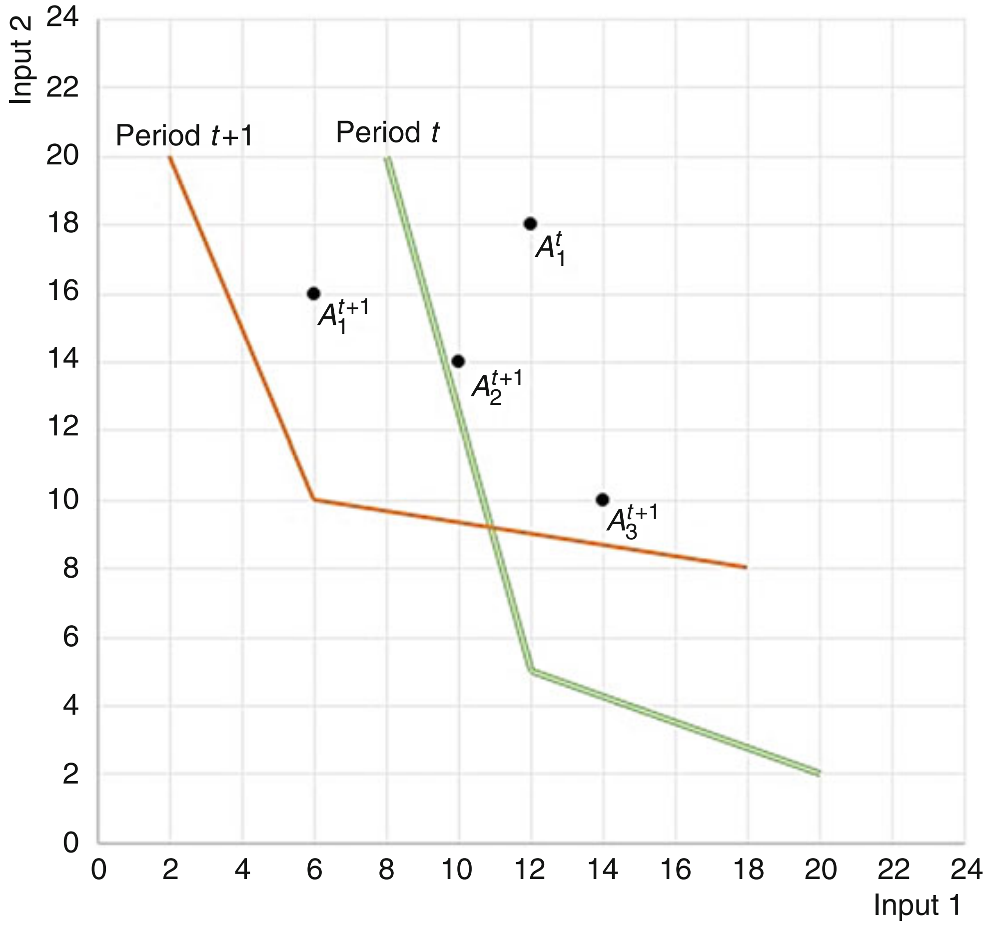


Fig. 11.11 Possible locations for movement

$$D_{A_1^t}^{t+1} = \frac{OQ}{OA_1^t}, \tag{11.23}$$

$$D_{A_3^{t+1}}^{t} = \frac{OS}{OA_1^t}, \tag{11.24}$$

$$D_{A_3^{t+1}}^{t+1} = \frac{OR}{OA_1^t}. \tag{11.25}$$

From Fig. 11.12, it is clear that $D_{A_1^t}^t > D_{A_1^t}^{t+1}$ and $D_{A_3^{t+1}}^t < D_{A_3^{t+1}}^{t+1}$. Thus,

$$\frac{D_{A_1^t}^t}{D_{A_1^t}^{t+1}} > 1 \quad \& \quad \frac{D_{A_3^{t+1}}^t}{D_{A_3^{t+1}}^{t+1}} < 1. \tag{11.26}$$

From the inequalities in Eq. 11.26, we cannot directly conclude whether $EFC_l > 1$ or $EFC_l < 1$. However, if A_1^t have a location at A_2^{t+1} (or A_1^{t+1}) in time period $t+1$, we will have the following inequalities which result that $EFC_{A_1^t} > 1$.

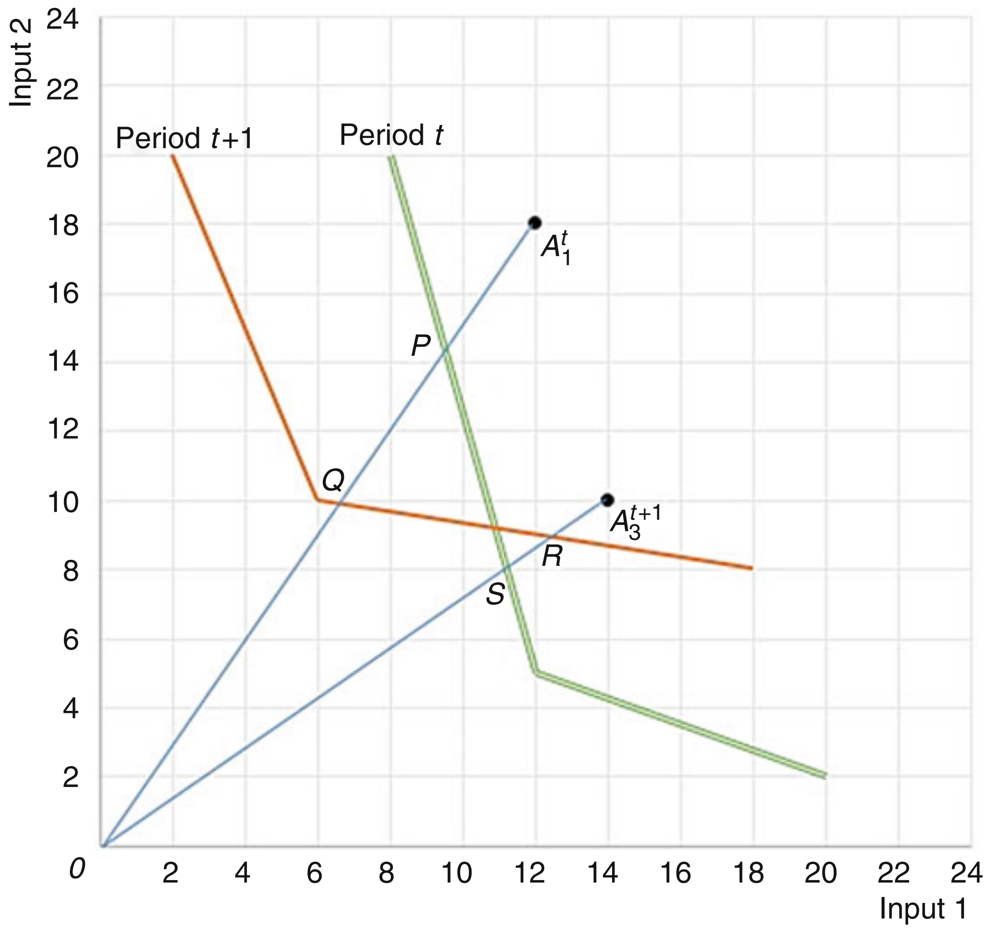


Fig. 11.12 Measuring the radial scores

Indeed, both A_1^t and A_2^{t+1} are closer to the production frontier in time period t than the production frontier in time period $t+1$, as Fig. 11.13 illustrates. Therefore, we have the following inequalities which yields that $EFC_{A_1^t} > 1$.

$$\frac{D_{A_1^t}^t}{D_{A_1^t}^{t+1}} > 1 \quad \& \quad \frac{D_{A_2^{t+1}}^t}{D_{A_2^{t+1}}^{t+1}} > 1. \tag{11.27}$$

In Fig. 11.14, a downward (upward) shift in the production frontier, that is, shift towards (away) the origin, from period t to period $t+1$ represents a positive (negative) shift or indicates a production technology progress (decline). The above illustration can also be discussed for A_2^t and A_3^t. Each one of the three DMUs, A_1^t, A_2^t and A_3^t may find one of the locations A_1^{t+1}, A_2^{t+1} and A_3^{t+1} in time period $t+1$. Thus, nine different inequalities can be considered, as the tree diagram in Fig. 11.15 represents.

There are 5 branches of the 9 branches in Fig. 11.15 that do not have a certain answer as to whether the value of EFC_l is less or greater than 1. Overall four different cases can be seen in Fig. 11.15.

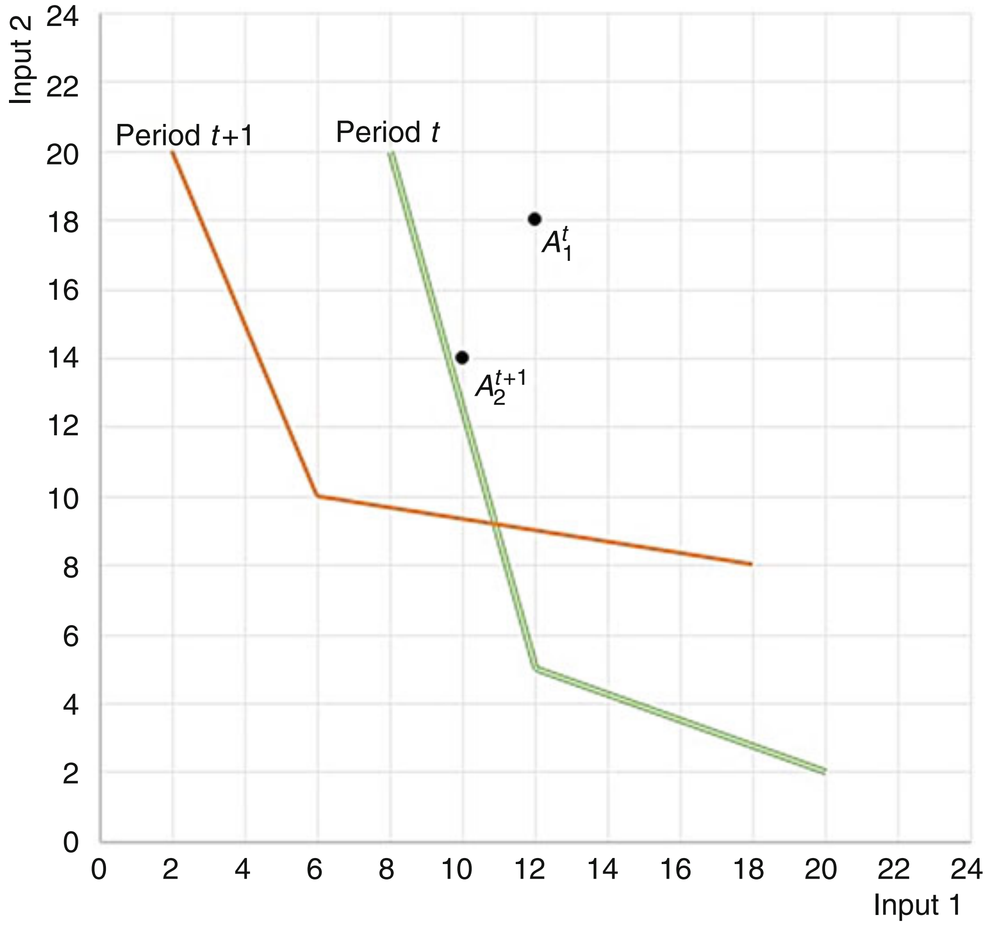


Fig. 11.13 Locations closer to time period t

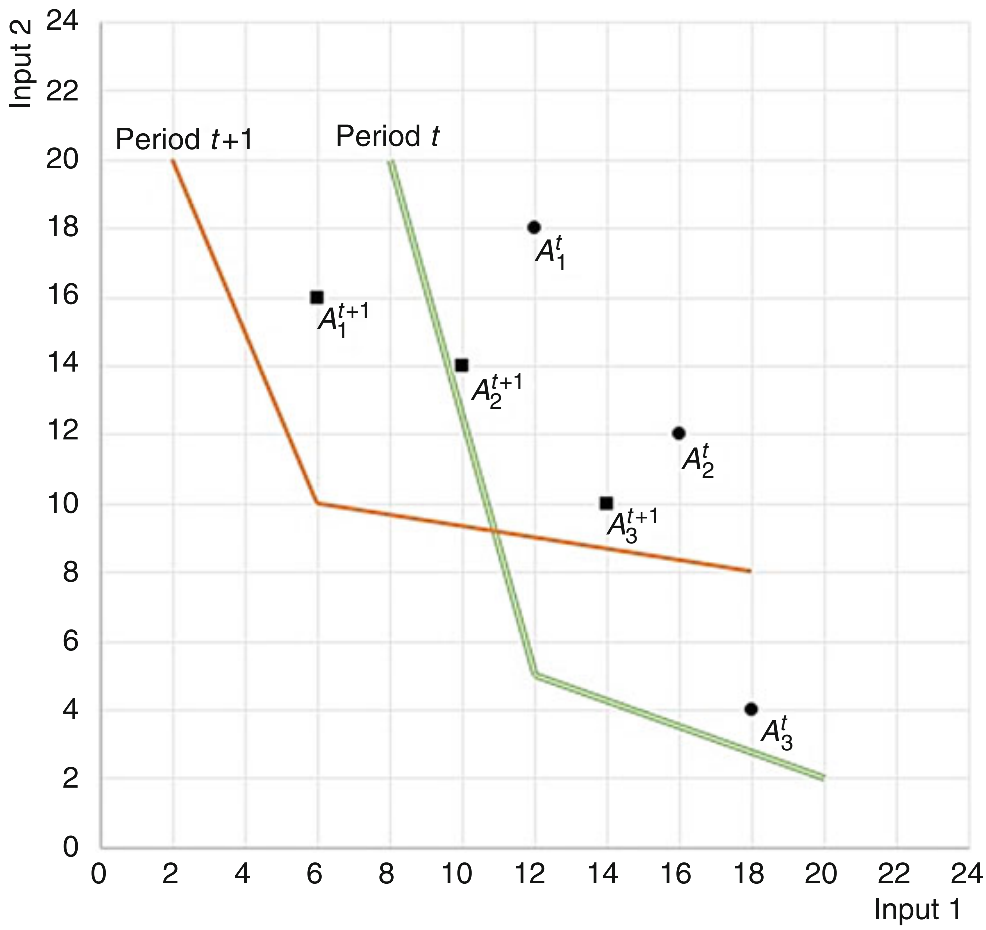


Fig. 11.14 Downward (upward) shift in the frontiers

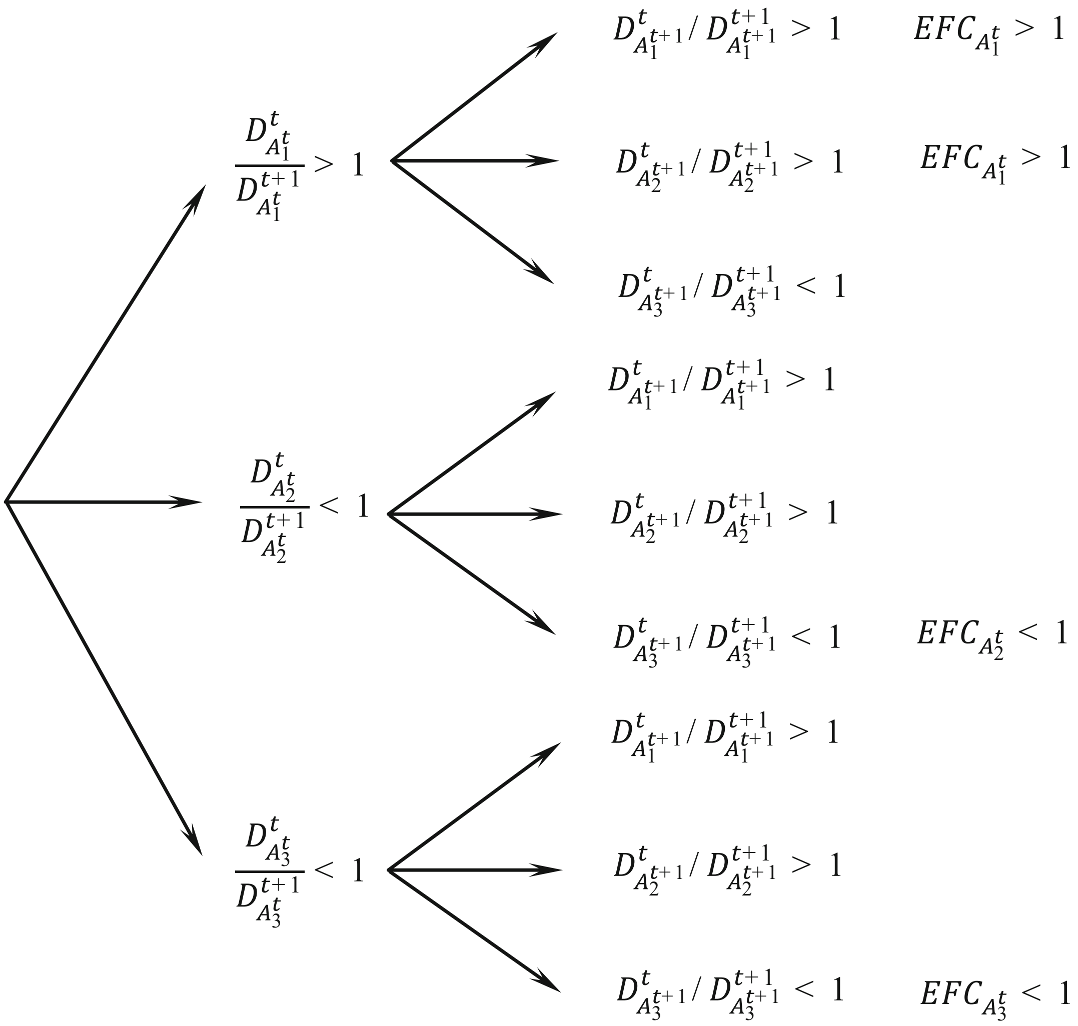


Fig. 11.15 Nine different inequalities for the Malmquist index

Case (a) both inequalities are greater than 1, that is,

$$\frac{D^t_{l^t}}{D^{t+1}_{l^t}} > 1 \quad \& \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} > 1. \tag{11.28}$$

In this case, the value of EFC_l is larger than 1, and indicates that DMU_l is moved onto a side which has a positive shift as well as progresses in the production technology of DMU_l.

Case (b) both inequalities are less than 1, that is,

$$\frac{D^t_{l^t}}{D^{t+1}_{l^t}} < 1 \quad \& \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} < 1. \tag{11.29}$$

In this case, the value of EFC_l is less than 1, and indicates that DMU_l is moved onto a side which has a negative shift as well as declines in the production technology of DMU_l.

Case (c) The first inequality is less than 1 and the second inequality is greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} < 1 \quad \& \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} > 1. \tag{11.30}$$

In this case, the value of EFC_l can be larger or less than 1, and indicates that the production technology of DMU_l projected from a negative shift side towards a positive shift side. In other words, there is a tradeoff change between the two inputs. The change obtained from the positive shift side is greater than that of the negative shift side, if EFC_l is greater than 1, that is, $EFC_l > 1$. This shows that the production technology of DMU_l improves on average. The change obtained from the negative shift side is greater than that of the positive shift side, if EFC_l is less than 1, that is, $EFC_l < 1$. This shows that the production technology of DMU_l drops on average. The production technology of DMU_l is not changed on average, if $EFC_l = 1$.

Case (d) The second inequality is less than 1 and the first inequality is greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} > 1 \quad \& \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} < 1. \tag{11.31}$$

Similarly, the value of EFC_l can be less or larger than 1. This indicates that the production technology of DMU_l is projected from a positive shift side towards a negative shift side. As a result, the change obtained from the positive shift side is less than that of the negative shift side, if EFC_l is less than 1, that is, $EFC_l < 1$. This shows that the production technology of DMU_l declines on average. The change obtained from the positive shift side is greater than that of the negative shift side if EFC_l is greater than 1, that is, $EFC_l > 1$. This shows that the production technology of DMU_l progresses on average. The production technology of DMU_l remains the same on average, if $EFC_l = 1$.

As can be seen, a DMU changes its strategy if cases (c) or (d) occurs, that is, a tradeoff change between the two inputs occur. From a productivity perspective, case (c) is more favorable than case (d) for a DMU.

Now, the geometric mean of all EFC_l values, $l = 1, 2, \ldots, n$, in an industry provides an estimation of the productive frontier change in the industry, and is called the Malmquist Productive Frontier Shift (MPFS) index for that industry. In other words,

$$MPFS = \sqrt[n]{\prod_{l=1}^{n} EFC_l}. \tag{11.32}$$

From the above discussions, there are three main cases for the industry view, that is, considering all EFC_l values where $l = 1, 2, \ldots, n$.

Case (A) all inequalities for $l = 1, 2, \ldots, n$ are greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} > 1, \text{for } l = 1, 2, \ldots, n, \\ \& \\ \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} > 1, \text{for } l = 1, 2, \ldots, n. \tag{11.33}$$

Since the value of EFC_l is larger than 1, for $l = 1, 2, \ldots, n$, thus *MPFS* is greater than 1. This shows a pure positive shift of the entire production frontier, and indicates that the production technology of the industry progresses.

Case (B) all inequalities for $l = 1, 2, \ldots, n$ are less than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} < 1, \text{for } l = 1, 2, \ldots, n, \\ \& \\ \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} < 1, \text{for } l = 1, 2, \ldots, n \tag{11.34}$$

Since the value of EFC_l is less than 1, for $l = 1, 2, \ldots, n$, thus *MPFS* is less than 1. This shows a pure negative shift of the entire production frontier, and indicates that the production technology of the industry declines.

Case (C) Some of the inequalities are less than 1 and some are greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} < 1, \text{for some } l = 1, 2, \ldots, n, \\ \& \\ \frac{D_{l^t}^t}{D_{l^t}^{t+1}} > 1, \text{for some } l = 1, 2, \ldots, n, \\ \& \\ \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} < 1, \text{for some } l = 1, 2, \ldots, n, \\ \& \\ \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} > 1, \text{for some } l = 1, 2, \ldots, n. \tag{11.35}$$

In this case, the frontier shift is neither pure positive nor pure negative. As a result, there is a cross-frontier shift. If $MPFS > 1$, average the industry production technology progresses on average, and if $MPFS < 1$ the industry production technology declines.

In Eq. 11.19, the Malmquist efficiency index is defined as $TEC_l \times EFC_l$, for $l = 1, 2, \ldots, n$. The TEC_l index is $D_{l^{t+1}}^{t+1}/D_{l^t}^t$ and the EFC_l index is also the geometry mean of $D_{l^t}^t/D_{l^t}^{t+1}$ and $D_{l^{t+1}}^t/D_{l^{t+1}}^{t+1}$, for $l = 1, 2, \ldots, n$. Similarly, the components of the Malmquist efficiency index can be analyzed, as Fig. 11.16 illustrates.

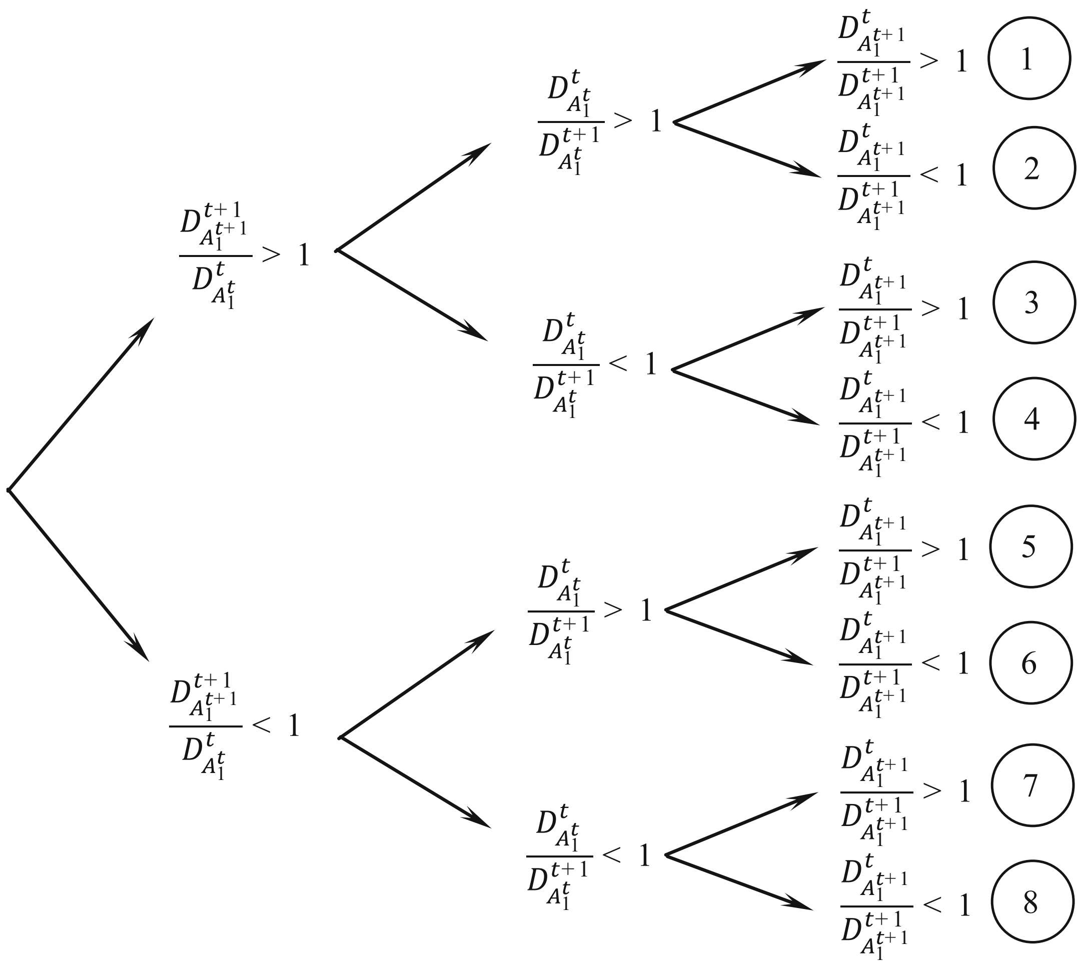


Fig. 11.16 Positive (negative) production shifts or technical changes

The tree diagram in Fig. 11.16 shows the different cases for the Malmquist efficiency index for A_1. The certain answer to find out whether the Malmquist efficiency index for A_1 is greater or less than 1, that is, $M_{A_1} > 1$ or $M_{A_1} < 1$, is obtained from Cases 1 and 8, given by

$$\frac{D^{t+1}_{A_1^{t+1}}}{D^{t}_{A_1^{t}}} > 1 \quad \& \quad \frac{D^{t}_{A_1^{t}}}{D^{t+1}_{A_1^{t}}} > 1 \quad \& \quad \frac{D^{t}_{A_1^{t+1}}}{D^{t+1}_{A_1^{t+1}}} > 1. \tag{11.36}$$

$$\frac{D^{t+1}_{A_1^{t+1}}}{D^{t}_{A_1^{t}}} < 1 \quad \& \quad \frac{D^{t}_{A_1^{t}}}{D^{t+1}_{A_1^{t}}} < 1 \quad \& \quad \frac{D^{t}_{A_1^{t+1}}}{D^{t+1}_{A_1^{t+1}}} < 1. \tag{11.37}$$

For the rest of cases, the efficiency change is mixed either with a positive or a negative production technology shift and an increase or a decrease in technical efficiency. Thus, it is essential to investigate the indications of each individual component.

Case 1 This case is the best efficiency improvement scenario. As can be seen, the efficiency change is related to a positive production frontier shift as well as development in technical efficiency.

Case 2 In this case, the production technology changes from a positive shift side to negative shift side which shows that A_1 has an unfavorable policy change, as explained in case (d). The joint effect of the production technology growth and the technical efficiency improvement occurs, if $M_{A_1} > 1$, that is, the efficiency gains. If $M_{A_1} < 1$, the efficiency loss is from the joint effect of the production technology regress and the technical efficiency improvement.

Case 3 This case is one of the favorable situations, because if $M_{A_1} > 1$, the Malmquist efficiency gain is from an efficiency progress and a production technology shift from negative shift sector to positive shift sector of the frontier. This indicates that A_1 has a favorable policy shift as well as technical efficiency improvement with regard to a positively shift frontier.

In the other hand, if $M_{A_1} < 1$, an extreme condition occurs. Indeed, only the decline in the production technology causes the efficiency loss. The production technology also changes from a negative shift side towards a positive shift side, indicating a favorable strategy adjustment. As a result, the combined Malmquist efficiency score may provide misleading information in this case.

Case 4 This case says that there is progress in the technical efficiency, that is, A_1 is closer to its production frontier in time period $t + 1$ than to its production frontier in time period t. However, the performance of A_1 is related to a negative production frontier shift. An improvement in the efficiency occurs if the negative frontier shift cannot take in the progress in the technical efficiency.

Case 5 This case says that the performance of A_1 is related to a positive production frontier shift, but there is a decline in the technical efficiency, that is, A_1 is farer to its production frontier in time period $t + 1$ than to its production frontier in time period t. An improvement in the efficiency occurs if the positive frontier shift can take in the decline in the technical efficiency.

Case 6 This case is one of the least favorable conditions, because if $M_{A_1} < 1$, the Malmquist efficiency decrease is due to an efficiency decline and a production technology change from a positive shift sector of the frontier to a negative shift sector of the frontier. This indicates that A_1 has an unfavorable policy shift and loses the technical efficiency with regard to a negative shift frontier.

On the other hand, if $M_{A_1} > 1$, an unfavorable strategy change occurs. Indeed, only the increase in the production technology causes the efficiency gain. The production technology also changes from a positive shift side towards a negative shift side. As a result, the combined Malmquist efficiency score in this case may provide misleading information, as well.

Case 7 In this case, A_1 has a favorable policy change, because the production technology changes from a negative shift side to a positive shift side. If $M_{A_1} > 1$, the efficiency gain is from the joint effects of the average production technology growth and the technical efficiency drop. If $M_{A_1} < 1$, the efficiency loss is from the joint effects of the average production technology regress and the technical efficiency decrease.

Case 8 Without a doubt, this case is the worst scenario for A_1. The efficiency decline is associated with a negative production frontier shift and a drop in the technical efficiency.

As discussed earlier, if the prices are available or the isocost line is known, the interpretation could be changed. For instance, if the technology shift yields an input value with a relatively low price in Case 6, a change could possibly yield to lower cost. Thus, similar to differences between efficiency and technical efficiency, the above favorable or unfavorable policy change is based upon the isoquant changes only. In order to fully characterize the strategy change, the price information should be considered to study overall efficiency or allocative efficiency.

11.4 A Computer Industry Example

The DEA Malmquist efficiency index has been used in many real-life applications and there is an extensive form of applications that uses the DEA Malmquist efficiency index. For example, efficiency growths in Swedish hospitals, deregulation's effects on Spanish saving banks, variations in agricultural efficiency in 18 developing countries, an experiential study of the catch-up hypothesis for a group of high and low income countries, telecommunications efficiency, a Swedish eye-care facility delivery, production machinery catch-up and invention in 74 countries, and so on (Chen and Ali 2004). As can be seen, the DEA Malmquist efficiency index has confirmed itself to be an exceptional tool for measuring the efficiency change of DMUs.

In this section, a set of Fortune Global 500 Computer and Office Equipment companies from 1991 to 1997 which was examined by Chen and Ali (2004) is discussed. They estimated whether or not the strategy shifts of companies were favorable. There were eight companies, such as APPLE, CANON, COMPAQ, DIGITAL, FUJITSU, HP, IBM, and RICOH. Four factors were selected for each company, three input factors and one output factor. The three input factors were assets, shareholder's equity and the number of employees. The single output factor was revenue. In their study, they did not have data for APPLE in 1997, since Apple did not appear on the list in 1997.

We illustrated in Sect. 11.1 how to calculate the Malmquist efficiency index with one single click by using Microsoft Excel Solver. Here we discuss the outcomes. Table 11.5 illustrates the technical efficiency scores of companies in 1991–1997.

APPLE was the only CCR technically efficient company in each period between 1991 and 1996. COMPAQ was also CCR technically efficient between 1991 and 1997. The last row in Table 11.5 also represents the average of CCR efficiency scores of companies.

Table 11.6 displays the Malmquist technical efficiency changes for each company as well as the average of technical efficiency changes for all companies from 1991 to 1997. There was no improvement in technical efficiency of APPLE from 1991 to 1996, but it does not mean that APPLE was not the best practice, as APPLE lies on the

Table 11.5 The CCR efficiency scores of companies (1991–1997)

Company	1991	1992	1993	1994	1995	1996	1997
APPLE	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	–
CANON	0.7462	0.8358	0.6493	0.6200	0.7280	0.5815	0.8218
COMPAQ	0.7484	0.9012	1.0000	1.0000	1.0000	1.0000	1.0000
DIGITAL	0.6540	0.7039	0.8261	1.0000	1.0000	0.8452	1.0000
FUJITSU	0.7209	0.8926	0.7154	0.6537	0.9310	0.8786	1.0000
HP	0.6725	0.7147	0.7394	0.7322	0.7199	0.7567	0.8967
IBM	0.4850	0.7210	0.8072	0.6988	0.8306	0.7349	1.0000
RICOH	0.7873	0.8283	0.6671	0.6006	0.7877	0.7173	0.7815
Average	0.7268	0.8247	0.8005	0.7882	0.8746	0.8143	0.9286

Table 11.6 The Malmquist technical efficiency changes for companies

Company	91 → 92	92 → 93	93 → 94	94 → 95	95 → 96	96 → 97
APPLE	1.0000	1.0000	1.0000	1.0000	1.0000	–
CANON	1.1200	0.7768	0.9549	1.1742	0.7986	1.4133
COMPAQ	1.2041	1.1096	1.0000	1.0000	1.0000	1.0000
DIGITAL	1.0762	1.1735	1.2105	1.0000	0.8451	1.1831
FUJITSU	1.2382	0.8014	0.9137	1.4242	0.9437	1.1381
HP	1.0626	1.0345	0.9902	0.9832	1.0511	1.1851
IBM	1.4865	1.1195	0.8657	1.1885	0.8848	1.3606
RICOH	1.0520	0.8053	0.9004	1.3114	0.9106	1.0895
Average	1.1466	0.9656	0.9748	1.1352	0.9292	1.1957

estimated production frontier in time period t and time period $t + 1$. Conversely, a score greater than 1 in Table 11.6 only indicates that there was progress in the CCR technical efficiency score. But, this does not necessarily mean that there was an improvement in performance. For example, the technical efficiency change for DIGITAL is greater than 1 from 1991 to 1992, 1992 to 1993 and 1993 to 1994. These scores do not indicate that the performance of DIGITAL in technical efficiency progress is better than the performance of APPLE in technical efficiency progress.

Both progress and decline in Malmquist technical efficiency changes can be seen for the companies CANON, DIGITAL FUJITSU, HP, IBM and RICOH. No Malmquist technical efficiency decline existed for APPLE and COMPAQ companies. The average of Malmquist technical efficiency change of all companies progresses from 1991 to 1992 by 14.7%, from 1994 to 1995 by 13.5% and from 1996 to 1997 by 19.6%, respectively. In addition, the average of Malmquist technical efficiency change of all companies declines from 1992 to 1993 by 3.4%, from 1993 to 1994 by 2.5% and from 1995 to 1996 by 7.1%, respectively. It is also interesting to note that none of the companies had a decline in their technical efficiency scores in time period 1996–1997.

Table 11.7 also represents the Malmquist frontier shift for each company from time period t to time period $t + 1$ as well as the geometrical mean of the scores in each shift.

Table 11.7 The Malmquist frontier shift for companies

Company	91 → 92	92 → 93	93 → 94	94 → 95	95 → 96	96 → 97
APPLE	0.9969	1.0931	1.0428	0.9608	1.1367	–
CANON	0.9072	1.2151	0.9907	0.9007	1.2482	0.7783
COMPAQ	1.0901	1.3155	1.0509	1.0107	1.1373	0.9600
DIGITAL	0.9291	1.0724	1.0625	0.9159	1.2366	0.9072
FUJITSU	0.9072	1.2151	1.0087	0.8897	1.2394	0.9921
HP	0.9291	0.9974	1.0628	0.9790	1.0225	0.8331
IBM	0.9072	1.2151	0.9991	0.8963	1.2449	0.8542
RICOH	0.9072	1.2151	1.0050	0.8917	1.2386	1.0124
Average	0.9449	1.1632	1.0274	0.9296	1.1854	0.9017

Table 11.8 Malmquist technology shift from periods 91–93

Time	91 → 92		92 → 93	
Company	I	II	I	II
APPLE	0.9704	0.9772	1.0609	1.0831
CANON	0.9628	0.9622	1.0889	1.0894
COMPAQ	0.9830	0.9880	1.0862	1.1055
DIGITAL	0.9475	0.9475	0.9984	1.0445
FUJITSU	0.9501	0.9520	1.0941	1.0938
HP	0.9548	0.9557	1.0254	1.0378
IBM	0.9687	0.9654	1.0913	1.1052
RICOH	0.9507	0.9544	1.0956	1.0964

From Table 11.7, on average, there was 5.5% decline in the industry technology frontier from 1991 to 1992, 7.0% decline from 1994 to 1995, and declined 10% decline from 1996 to 1997, respectively. There was also 16.3% progress from 1992 to 1993, 3.03% progress from 1993 to 1994, and 18.5% progress from 1995 to 1996.

In addition, from 1991 to 1992, all the companies had a negative shift in the technology frontier, except COMPAQ. From 1992 to 1993, except HP, all the companies had a positive technology frontier shift. From 1993 to 1994, six of the companies had a positive technology frontier shift, and the other two companies had a negative technology frontier shift. From 1994 to 1995, all the companies had a negative technology frontier shift, except COMPAQ. From 1995 to 1996, all the companies had a positive technology frontier shift. And from 1996 to 1997 except RICOH, all the companies show a negative technology frontier shift in the technology frontier. Note that, we could also employ a modified method where period t technology is created from input and output factors of all companies in all periods before period t and period t itself. In this case, the technologies in place in earlier periods are also remembered, and will remain available to adopt with the current period.

Tables 11.8, 11.9 and 11.10 illustrates the Malmquist component shifts in technology frontier based upon the ratios discussed in Fig. 11.16. Here, (I) and (II) represents the ratios, $D_{l^t}^t/D_{l^t}^{t+1}$ and $D_{l^{t+1}}^t/D_{l^{t+1}}^{t+1}$, respectively.

Table 11.9 Malmquist technology shift from periods 93–95

Time	93 → 94		94 → 95	
Company	I	II	I	II
APPLE	1.0178	1.0980	0.9057	1.0608
CANON	0.9131	0.9600	1.1034	0.9915
COMPAQ	1.0629	1.0650	1.0318	1.0346
DIGITAL	1.0817	0.7299	1.0175	0.9613
FUJITSU	0.7971	0.8414	1.2172	0.9827
HP	1.0750	1.0768	0.9178	1.0450
IBM	0.8333	0.8972	1.1559	0.9878
RICOH	0.8040	0.8686	1.1852	0.9820

Table 11.10 Malmquist technology shift from periods 95–97

Time	95 → 96		96 → 97	
Company	I	II	I	II
APPLE	1.1304	1.1832	–	–
CANON	1.3576	1.1398	0.6008	0.6035
COMPAQ	1.1586	1.0550	0.8334	0.7610
DIGITAL	1.6186	1.1435	0.6029	0.5993
FUJITSU	1.6045	1.1594	0.5925	0.6039
HP	1.1381	1.1382	0.6305	0.6421
IBM	1.4777	1.1548	0.5732	0.5715
RICOH	1.6178	1.1583	0.5955	0.5957

As can be seen, the industry technology frontier has a pure negative shift from 1991 to 1992 and 1996 to 1997. The industry technology frontier has a cross-frontier shift from 1992 to 1993, 1993 to 1994, and 1994 to 1995. The industry technology frontier has a pure positive shift from 1995 to 1996 only.

For example, the technology frontier change at the company level indicates that for COMPAQ, the two ratios, $D_{t'}^{t}/D_{t'}^{t+1}$ and $D_{t'+1}^{t}/D_{t'+1}^{t+1}$, are all larger than 1 from 1994 to 1995. This shows that COMPAQ has a consistent operations strategy from 1994 to 1995. Other companies in this period of time, had a move between two sides, which indicates that the companies had a variation in their operations strategy. For instance, the technology frontier for Apple and HP changes from a negative shift side towards a positive shift side. This change means that there is a favorable strategy change for these two companies.

Table 11.11 represents the Malmquist efficiency indexes for the companies as well as the average Malmquist efficiency index for all companies. From the table, the efficiency of computer industry improves on average from time period t to $t+1$ by 8.2%, 12.3%, 0.16%, 4.5%, 9.7% and 7.1%, for $t = 1991, 1992, \ldots,$ and 1996, respectively.

As illustrated earlier, the average Malmquist efficiency variation is the product of the geometry mean of the technical efficiency change and the technology frontier shift. In order to analyze the results in Table 11.11, we should refer to Tables 11.6, 11.7, 11.8, 11.9 and 11.10. Based upon these tables, for instance, the average

Table 11.11 The Malmquist efficiency indexes for the companies

Company	91 → 92	92 → 93	93 → 94	94 → 95	95 → 96	96 → 97
APPLE	0.9969	1.0930	1.0428	0.9608	1.1367	–
CANON	1.0161	0.9439	0.9460	1.0576	0.9969	1.1000
COMPAQ	1.3127	1.4597	1.0509	1.0107	1.1373	0.9600
DIGITAL	1.0000	1.2585	1.2863	0.9159	1.0451	1.0734
FUJITSU	1.1233	0.9738	0.9217	1.2672	1.1696	1.1292
HP	0.9872	1.0319	1.0525	0.9625	1.0748	0.9874
IBM	1.3340	1.3603	0.8649	1.0653	1.1016	1.1624
RICOH	0.9544	0.9785	0.9049	1.1693	1.1280	1.1031
Average	1.0819	1.1233	1.0015	1.0457	1.0974	1.0714

Table 11.12 The Malmquist components in periods 91–93

Time	91 → 92				92 → 93			
Company/ratio	I	II	III	IV	I	II	III	IV
APPLE	=1	<1	<1	<1	=1	>1	>1	>1
CANON	>1	<1	<1	>1	<1	>1	>1	<1
COMPAQ	>1	<1	<1	>1	>1	>1	>1	>1
DIGITAL	>1	<1	<1	<1	>1	<1	>1	>1
FUJITSU	>1	<1	<1	>1	<1	>1	>1	<1
HP	>1	<1	<1	>1	>1	>1	>1	>1
IBM	>1	<1	<1	>1	>1	>1	>1	>1
RICOH	>1	<1	<1	<1	<1	>1	>1	<1

efficiency gain from 1991 to 1992 is a joint effect of an improvement in the technical efficiency change on average and a negative shift in the technology frontier on average. Therefore, the progress in the technical efficiency is the only source of the efficiency gain from 1991 to 1992.

The average efficiency progress from 1992 to 1993 is a mutual effect of an average drop in the technical efficiency and an average positive shift in the technology frontier. As a result, an improvement in the technology frontier shift is the only source of the efficiency progress.

The average efficiency from 1993 to 1994 slightly gains, due to the joint effect of an average drop in the technical efficiency and an average positive shift in the technology frontier. Consequently, a slight efficiency progress is due to a positive shift in the technology frontier. The same analysis can be made for time periods 1994 to 1995, 1995 to 1996, and 1996 to 1997.

Tables 11.12, 11.13 and 11.14 illustrates the components of the Malmquist efficiency indexes associated with the efficiency change for each company from 1991 to 1997. In the tables, ratios I, II, III and IV indicate the ratios $D_{l^{t+1}}^{t+1}/D_{l^t}^{t}$, $D_{l^t}^{t}/D_{l^t}^{t+1}$, $D_{l^{t+1}}^{t}/D_{l^{t+1}}^{t+1}$ and M_l, respectively.

There are two cases which need specific consideration in the previous tables. The first case is that DIGITAL had the efficiency gain ($M_l > 1$) from 1992 to 1993 and

Table 11.13 The Malmquist components in periods 93–95

Time	93 → 94				94 → 95			
Company/ratio	I	II	III	IV	I	II	III	IV
APPLE	=1	>1	>1	>1	=1	<1	>1	<1
CANON	>1	<1	<1	<1	>1	>1	<1	>1
COMPAQ	=1	>1	>1	>1	=1	>1	>1	>1
DIGITAL	>1	>1	<1	>1	=1	>1	<1	<1
FUJITSU	>1	<1	<1	<1	>1	>1	<1	>1
HP	>1	>1	>1	>1	>1	<1	>1	<1
IBM	>1	<1	<1	<1	>1	>1	<1	>1
RICOH	>1	<1	<1	<1	>1	>1	<1	>1

Table 11.14 The Malmquist components in periods 95–97

Time	95 → 96				96 → 97			
Company/ratio	I	II	III	IV	I	II	III	IV
APPLE	=1	>1	>1	>1	–	–	–	–
CANON	<1	>1	>1	<1	>1	<1	<1	<1
COMPAQ	=1	>1	>1	>1	=1	<1	<1	<1
DIGITAL	<1	>1	>1	<1	>1	<1	<1	<1
FUJITSU	<1	>1	>1	<1	>1	<1	<1	>1
HP	<1	>1	>1	>1	>1	<1	<1	<1
IBM	<1	>1	>1	<1	>1	<1	<1	>1
RICOH	<1	>1	>1	<1	>1	<1	<1	<1

from 1993 to 1994. Nevertheless, the technology frontier of DIGITAL changes from a negative shift side towards a positive shift side from 1992 to 1993; while from 1993 to 1994, the technology frontier of DIGITAL changes from a positive shift side towards a negative shift side. As can be seen, the cause for the efficiency gain is different which should be considered for interpretation.

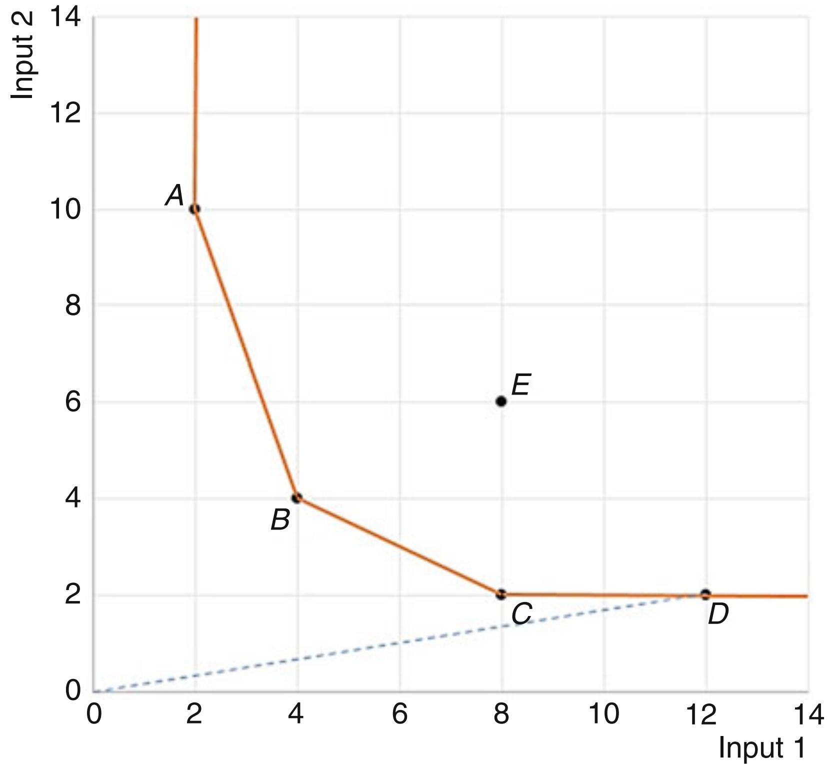
The second case is that HP had an efficiency loss ($M_l < 1$) from 1994 to 1995. The only reason that causes the efficiency loss is the average technology frontier decline. Nevertheless, the technology of HP changes from a negative shift side towards a positive shift side, indicating a favorable policy change.

The above discussion shows that more insights for the efficiency changes can be obtained by analyzing the Malmquist efficiency components.

11.5 A Non-radial Malmquist Efficiency Index

In this section, the Malmquist radial efficiency index is extended into a Malmquist non-radial efficiency index, proposed by Chen (2003). The new index incorporates with the preferences of decision makers over performance progress, and measures all inefficiencies which might be represented by non-zero slacks. The radial approach does not measure possible slacks, as shown in Fig. 11.17.

Fig. 11.17 Weak technical efficiency



There are five DMUs in Fig. 11.17 in which each DMU has two input factors. Note that the output factor in this case is considered as a single constant value such as 1. DMUs A-D are on the estimated production frontier, thus their CCR scores are 1. As can be seen, D is dominated by C, because D has two units more than C in the first input and the same amount in the second input. In other words, there is a slack in the first input of D, that is, $s_1^- = 2$, which cannot be measured by the radial approach. In other words, the CCR-efficiency is measured by a radial approach and possible non-zero input or output slacks. The introduced radial Malmquist efficiency index in the previous section is also based upon the CCR scores only. Disregarding non-zero input (output) slacks in input (output) oriented model clearly cannot completely characterize the efficiency change. The previous radial Malmquist efficiency index also fails to corporate the preference over the performance progress of individual input and output factors by decision-makers. DEA analysis with incorporation of value judgment is vital in applications to avoid incorrect results and false implications. We can, of course, use the ADD DEA model to measure the possible slacks after measuring the CCR scores. This technique is called the Two-Phase CCR model, shown in Eq. 11.38.

$$
\begin{aligned}
&\min \theta_l - \varepsilon\left(\textstyle\sum_{j=1}^{m} s_{lj}^- + \sum_{k=1}^{p} s_{lk}^+\right),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_{lj}^- = x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_{lk}^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned}
\tag{11.38}
$$

The above model is two models which are combined into one single model. In order to solve Eq. 11.38, one should first solve Eq. 11.39 to calculate θ_l^*, for $l = 1$, 2, …, n, and after that solve Eq. 11.40 to calculate the optimal slacks.

$$
\begin{aligned}
&\min \theta_l,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}, \quad \text{for } k = 1, 2, \ldots, p\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.39}
$$

$$
\begin{aligned}
&\max \textstyle\sum_{j=1}^{m} s_{lj}^- + \sum_{k=1}^{p} s_{lk}^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_{lj}^- = x_{lj}\theta_l^*, \quad \text{for } j = 1, 2, \ldots, m\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_{lk}^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \ge 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \ge 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned}
\tag{11.40}
$$

After that, for an ε value, which is a very small positive real number, the score for is estimated by the Two-Phase CCR, given by:

$$
\theta_l^* - \varepsilon\left(\sum_{j=1}^{m} s_{lj}^{-*} + \sum_{k=1}^{p} s_{lk}^{+*}\right).
$$

We can also use SBM or KAM to fully measure the inefficiencies in both input and output orientations, but we leave it as exercises. Here we use an improvement of the Russell Measure (RM) proposed by Zhu (1996) to incorporate the preferences of decision makers over performance progress as well as measuring all inefficiencies which might be represented by non-zero slacks. The use of this non-radial Malmquist efficiency index correctly measures the efficiency changes while the radial approach may deliver distorted information.

Suppose that there are n DMUs, A_i, $i = 1, 2, \ldots, n$, in which each DMU has m input factors, x_{ij}, $j = 1, 2, \ldots, m$, and p output factors, y_{ik}, $k = 1, 2, \ldots, p$. Also assume that α_j, $j = 1, 2, \ldots, m$, are the user-specified weights to indicate preferences over the input progresses. The following input oriented DEA model was established by Zhu (1996).

$$
\begin{aligned}
&\min \left(\textstyle\sum_{j=1}^{m} \alpha_j\theta_{lj}\right) / \textstyle\sum_{j=1}^{m} \alpha_j,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}\theta_{lj}, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\theta_{lj} \text{ free},\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.41}
$$

The optimal value of θ_{lj} can be less, equal or greater than 1, for $j = 1, 2, \ldots, m$. In addition, the equality occurs in the corresponded input constraints for the optimal solutions, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i^* = x_{lj}\theta_{lj}^*$, for $j = 1, 2, \ldots, m$. Because, if $\sum_{i=1}^{n} x_{ij}\lambda_i^* < x_{lj}\theta_{lj}^*$, for some $j = 1, 2, \ldots, m$, this means θ_{lj}^* is not the optimal (minimum) value, as there will be a smaller value θ_{lj} such that $\theta_{lj} < \theta_{lj}^*$ which can result in equality. Therefore, there is no positive input slack exists in the optimal solutions of Eqs. 11.41 and 11.42, that is, $s_{lj}^{-*} = 0$, for $j = 1, 2, \ldots, m$, and the following theorem is concluded:

Theorem 11.1 Any optimal solution in Eqs. 11.41 and 11.42 will always have all input slacks equal to zero.

In order to show that the objective in Eqs. 11.41 and 11.42 is between 0 and 1, we have the following theorem.

Theorem 11.2 The objective in Eqs. 11.41 and 11.42 is less than or equal to 1.

Proof We first find the dual linear programming for Eq. 11.24. For such an aim, suppose the weights w_j^- and w_k^+ are used to sum up the constraints, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. Thus, we have the following equations, respectively:

$$\begin{cases} \sum_{j=1}^{m} w_j^- (\sum_{i=1}^{n} x_{ij}\lambda_i) \le \sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{k=1}^{p} y_{ik} w_k^+ (\sum_{i=1}^{n} \lambda_i) \ge \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.42}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij} w_j^- \lambda_i \ge -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{k=1}^{p} \sum_{i=1}^{n} y_{ik} w_k^+ \lambda_i \ge \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.43}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} w_j^- \lambda_i \ge -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{i=1}^{n} \sum_{k=1}^{p} y_{ik} w_k^+ \lambda_i \ge \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.44}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} x_{ij} w_j^- \ge -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{p} y_{ik} w_k^+ \ge \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.45}$$

$\Leftrightarrow$

$$\sum_{i=1}^{n} \lambda_i (\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^-) \ge \sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \tag{11.46}$$

$\Leftrightarrow$

$$\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj} \ge \sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{i=1}^{n} \lambda_i (\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^-), \tag{11.47}$$

Assume that $\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0$, for $i = 1, 2, \ldots, n$, and also $\alpha_j / \sum_{j=1}^{m} \alpha_j = x_{lj} w_j^-$, for $j = 1, 2, \ldots, m$. Therefore, we have the following dual linear programming

$$\begin{aligned}
&\max \sum_{k=1}^{p} y_{lk} w_k^+, \\
&\text{Subject to} \\
&\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 0, \quad \text{for } i = 1, 2, \ldots, n, \\
&x_{lj} w_j^- = \alpha_j / \sum_{j=1}^{m} \alpha_j, \quad \text{for } j = 1, 2, \ldots, m, \\
&w_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{11.48}$$

Now, since

$$\sum_{j=1}^{m} x_{lj} w_j^- = \sum_{j=1}^{m} \frac{\alpha_j}{\sum_{j=1}^{m} \alpha_j} = 1. \tag{11.49}$$

Thus, the constraint $\sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{j=1}^{m} x_{lj} w_j^- \leq 0$, yields that $\sum_{k=1}^{p} w_k^+ y_{lk} \leq 1$, and the proof is completed. □

Model 11.42 measures the technical efficiency of DMU$_l$ ($l = 1, 2, \ldots, n$) under weights α_j, for $j = 1, 2, \ldots, m$, and determines a preferred empirical production frontier. If $\alpha_j = 0$, or some $j = 1, 2, \ldots, m$, we set the corresponding θ_{lj} equal to 1. When the greater value for a weight α_j ($j = 1, 2, \ldots, m$) is considered DMUs give higher priority to reduce the corresponding input value. The value of α_j can be selected differently based upon the user judgment. For example α_j can be defined as $1/x_{lj}$ or $x_{lj}/\sum_{j=1}^{n} x_{lj}$.

Let's consider the DMUs in Fig. 11.17. DMU E is an inefficient DMU. Figure 11.18 depicts the targets for E by applying the radial CCR and non-radial Model 11.42. The target for E by CCR is $x_{1E}^* = 4.8$ and $x_{2E}^* = 3.6$, whereas the target for E by Model 11.42 is B, where $\alpha_1 = 1$ and $\alpha_2 = 1$.

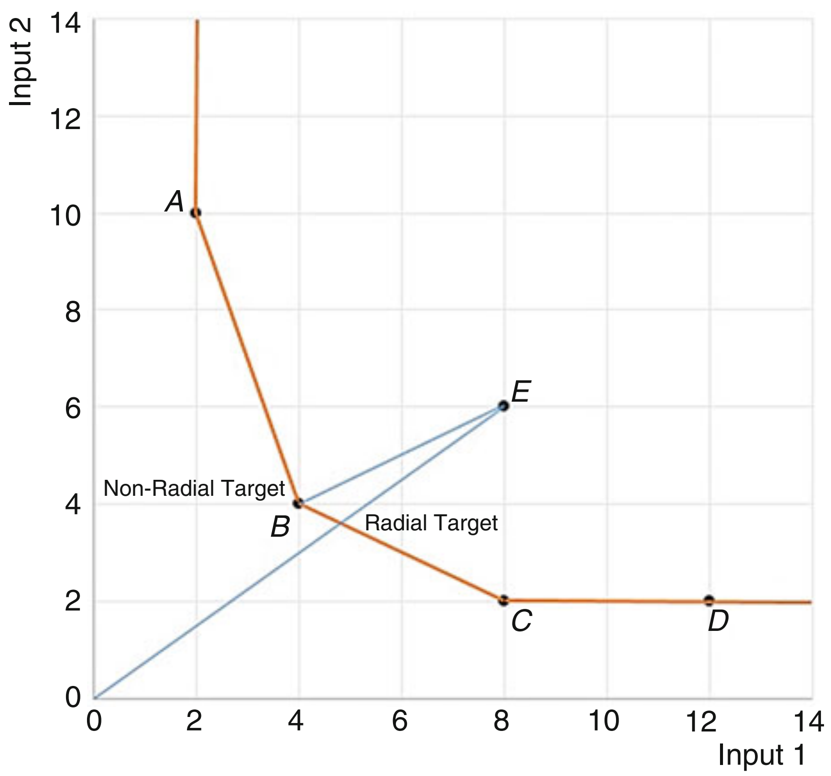


Fig. 11.18 Radial and non-radial targets

We now introduce the four models to measure the non-radial Malmquist efficiency index which is incorporated with the preference over the individual input improvements, and does not allow the existence of non-zero input slacks. Equations 11.50, 11.51, 11.52 and 11.53 show these four models.

The non-radial input oriented Model 11.50 measures the technical efficiency score of DMU$_l$ (l=1,2,...,n) in time period t according to the generated PPS of all DMUs in time period t.

$$\begin{aligned} &D_{l'}^{t} = \min\left(\textstyle\sum_{j=1}^{m} \alpha_j \theta_{l'j}^{t}\right) / \textstyle\sum_{j=1}^{m} \alpha_j, \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^{n} x_{ij}^{t} \lambda_i \le x_{l'j}^{t} \theta_{l'j}^{t}, \quad \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^{n} y_{ik}^{t} \lambda_i \ge y_{l'k}^{t}, \quad \text{for } k = 1, 2, \ldots, p, \\ &\theta_{l'j}^{t} \text{ free}, \\ &\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{11.50}$$

The non-radial input oriented Model 11.51 calculates the technical efficiency score of DMU$_l$ (l=1,2,...,n) in time period t according to the generated PPS of all DMUs in time period $t + 1$.

$$\begin{aligned} &D_{l'}^{t+1} = \min\left(\textstyle\sum_{j=1}^{m} \alpha_j \theta_{l'j}^{t+1}\right) / \textstyle\sum_{j=1}^{m} \alpha_j, \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^{n} x_{ij}^{t+1} \lambda_i \le x_{l'j}^{t} \theta_{l'j}^{t+1}, \quad \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^{n} y_{ik}^{t+1} \lambda_i \ge y_{l'k}^{t}, \quad \text{for } k = 1, 2, \ldots, p, \\ &\theta_{l'j}^{t+1} \text{ free}, \\ &\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{11.51}$$

The non-radial input oriented Model 11.52 measures the technical efficiency score of DMU$_l$ (l=1,2,...,n) in time period $t + 1$ according to the generated PPS of all DMUs in time period t.

$$\begin{aligned} &D_{l^{t+1}}^{t} = \min\left(\textstyle\sum_{j=1}^{m} \alpha_j \theta_{l^{t+1}j}^{t}\right) / \textstyle\sum_{j=1}^{m} \alpha_j, \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^{n} x_{ij}^{t} \lambda_i \le x_{l^{t+1}j}^{t} \theta_{l^{t+1}j}^{t}, \quad \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^{n} y_{ik}^{t} \lambda_i \ge y_{l^{t+1}k}^{t}, \quad \text{for } k = 1, 2, \ldots, p, \\ &\theta_{l^{t+1}j}^{t} \text{ free}, \\ &\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{11.52}$$

The non-radial input oriented Model 11.53 calculates the technical efficiency score of DMU$_l$ (l=1,2,...,n) in time period $t + 1$ according to the generated PPS of all DMUs in time period $t + 1$.

$$\begin{aligned}
&D_{l^{t+1}}^{t+1} = \min\left(\sum_{j=1}^{m} \alpha_j \theta_{l^{t+1}j}^{t+1}\right) / \sum_{j=1}^{m} \alpha_j, \\
&\text{Subject to} \\
&\sum_{i=1}^{n} x_{ij}^{t+1} \lambda_i \le x_{l^{t+1}j}^{t} \theta_{l^{t+1}j}^{t+1}, \text{for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}^{t+1} \lambda_i \ge y_{l^{t+1}k}^{t}, \text{for } k = 1, 2, \ldots, p, \\
&\theta_{l^{t+1}j}^{t+1} \text{ free}, \\
&\lambda_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{11.53}$$

Equation 11.54 represents the non-radial input oriented Malmquist efficiency index

$$ME_l = \frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}} \times \left(\frac{D_{l^t}^{t}}{D_{l^t}^{t+1}} \times \frac{D_{l^{t+1}}^{t}}{D_{l^{t+1}}^{t+1}} \right)^{1/2}. \tag{11.54}$$

The first term in Eq. 11.54, that is, $D_{l^{t+1}}^{t+1} / D_{l^t}^{t}$, calculates the weighted non-radial input oriented efficiency change, and the second term, that is, $\left(D_{l^t}^{t} / D_{l^t}^{t+1}\right)\left(D_{l^{t+1}}^{t} / D_{l^{t+1}}^{t+1}\right)$ calculates the frontier shift in preferred empirical production function. It is also possible that the two different empirical production frontiers (created in two different time periods) have intersections and some sides (facets) shift backwards and some forwards. In other words, the sides of empirical production frontier may not all shift in one direction. In such a situation, the movement of empirical production frontier is DMU-specific, that is, the Malmquist efficiency index calculates the performance of a specific DMU in terms of the movement of its referent DMUs.

11.6 An Example of Three Major Chinese Industries

In this section, the radial and non-radial models are applied to calculate the efficiency change and the impact of economic development plans on efficiency changes of three Chinese major industries: (1) Textiles, (2) Chemicals and (3) metallurgy during five-year-plan in four different periods, from 1966 to 1985.

Table 11.15 illustrates the five DMUs and the different time periods. Each DMU is defined as a year in the first period from 1966 to 1970. The year after 5 years corresponding to the first period, that is, 1971–1975, 1976–1980, and 1981–1985 are

Table 11.15 An example of four five-year plan periods

DMUs\Period	$t = 1$	$t = 2$	$t = 3$	$t = 4$
DMU1	1966	1971	1976	1981
DMU2	1967	1972	1977	1982
DMU3	1968	1973	1978	1983
DMU4	1969	1974	1979	1984
DMU5	1970	1975	1980	1985

defined as the same DMU1-DMU5 in second, third and fourth periods. For example, DMU 1 is the year 1966 in the first period, the year 1971 in the second period, the year 1976 in the third period and the year 1981 in the fourth period. In other words, the first years in all five-year-plan periods are considered as the same DMU, but in different time periods.

Note that a series of 5-year plans were launched by the Chinese government since the year 1953. Here, $t = 1$ refers to the third five-year-plan period, and $t = 2$ refers to the fourth five-year-plan period, and so on. In other words, the period 1966–1985 refers to four five-year-plan periods during which the third five-year-plan period came on stream during 1966–1970. In each five-year plan, some economic development targets and plans were arranged. Thus, it is meaningful to review the efficiency change between two sequential five-year-plan periods, in order to measure the impact of economic development plans.

In order to characterize an industry in which is labor intensive, the textile industry is considered. The chemical industry is considered to characterize an industry which is capital intensive.

In addition, the metallurgy industry is considered to characterize an industry which both labor intensive and capital intensive. These selections allow us to determine the use of the discussed Malmquist efficiency indexes.

Tables 11.16, 11.17 and 11.18 illustrates the data for the three major Chinese industries: (1) Textiles, (2) Chemicals and (3) metallurgy from 1966 to 1985. Each

Table 11.16 The data of textiles industry

DMUs/periods	Year	Capital	Labor	AGIOV
T_1^1	1966	48,002	106,910	145,163
T_2^1	1967	38,135	107,047	121,057
T_3^1	1968	40,584	111,428	135,572
T_4^1	1969	50,454	115,222	177,849
T_5^1	1970	49,218	122,428	183,630
T_1^2	1971	43,604	127,393	160,291
T_2^2	1972	44,677	131,991	161,853
T_3^2	1973	50,958	133,555	181,968
T_4^2	1974	53,419	134,456	188,066
T_5^2	1975	59,430	135,642	206,317
T_1^3	1976	57,914	139,195	196,584
T_2^3	1977	56,410	145,303	187,317
T_3^3	1978	70,558	125,627	229,308
T_4^3	1979	73,542	134,303	258,529
T_5^3	1980	87,180	149,533	313,734
T_1^4	1981	105,123	166,794	361,155
T_2^4	1982	122,385	194,801	347,229
T_3^4	1983	143,098	202,171	369,631
T_4^4	1984	276,615	218,937	465,235
T_5^4	1985	311,977	214,961	492,053

Table 11.17 The data of chemicals industry

DMUs/periods	Year	Capital	Labor	AGIOV
C_1^1	1966	57,785	68,800	116,884
C_2^1	1967	47,570	68,959	100,354
C_3^1	1968	46,565	71,665	102,307
C_4^1	1969	64,768	77,788	160,948
C_5^1	1970	67,488	85,612	171,774
C_1^2	1971	77,468	94,896	195,428
C_2^2	1972	79,225	100,775	197,665
C_3^2	1973	84,557	103,336	209,838
C_4^2	1974	87,820	105,591	219,256
C_5^2	1975	92,777	110,636	232,676
C_1^3	1976	83,593	114,661	209,057
C_2^3	1977	87,818	124,392	214,105
C_3^3	1978	107,463	125,328	272,935
C_4^3	1979	117,348	126,285	278,142
C_5^3	1980	125,644	144,818	286,499
C_1^4	1981	143,419	153,201	309,717
C_2^4	1982	160,309	159,691	338,989
C_3^4	1983	173,788	160,238	354,515
C_4^4	1984	188,013	177,426	373,221
C_5^4	1985	222,236	169,310	410,720

DMU has two input factors, Capital and Labor, and a single output, Annual Gross Industrial Output Value (AGIOV). These factors were considered to be consistent with the previous researchers and are enough to run the proposed models. For each industry, the single output factor, AGIOV, and the input factor, Capital, are measured in 10,000 RMB1 by the 1980 official prices. The labor factor is the number of workers and staff in the corresponding industry.

The selected input and output factors for these three industries are the used key measures by the Chinese government in assessing the industrial performance. The data are also gathered from the Yearbook of China's 40 Years which is issued by the Chinese Statistical Bureau.

This is obvious and completely reasonable that DEA is sensitive to variable selection. Indeed, adding/removing a factor or a DMU definitely should change the results, as DEA is naturally consistent with the real-life situation. DEA estimates the production frontier based upon the observed DMUs while there are no guesses or any known relationships between factors. It is almost always impossible to find causation from the observed data, thus, it is not realistic to assume that the estimated production frontier by DEA should be unique or not sensible to variable selection. In other words, managers usually consider the factors that might be related in measuring the performance of a set of firms. The strong logic of DEA compares each pair of

Table 11.18 The data of metallurgy industry

DMUs/periods	Year	Capital	Labor	AGIOV
M_1^1	1966	26,504	24,974	51,060
M_2^1	1967	27,400	25,603	50,087
M_3^1	1968	29,940	27,430	46,259
M_4^1	1969	38,374	29,174	59,358
M_5^1	1970	31,720	37,580	64,959
M_1^2	1971	36,777	27,152	70,446
M_2^2	1972	30,634	30,754	89,592
M_3^2	1973	39,781	35,151	112,588
M_4^2	1974	39,988	32,170	121,013
M_5^2	1975	49,216	31,933	132,093
M_1^3	1976	42,909	33,849	109,393
M_2^3	1977	47,857	36,506	119,411
M_3^3	1978	68,538	36,183	171,228
M_4^3	1979	78,746	35,363	185,703
M_5^3	1980	86,504	37,403	196,998
M_1^4	1981	89,003	39,412	199,055
M_2^4	1982	94,149	40,005	208,897
M_3^4	1983	103,925	45,650	226,187
M_4^4	1984	108,083	41,094	228,109
M_5^4	1985	125,269	41,146	235,981

firms based on the selected observed factors only. Even if the real production frontier is known, it is possible that none of the DMUs lie on that real production frontier, but DEA does focus on reality and considers the best practices as references for inefficient DMUs.

Of course, it is good to establish a priori of the existence of an association between the inputs and outputs. In this example, there is a strong association between the two input factors and the output factor in Tables 11.16, 11.17 and 11.18.

The input oriented CCR model is used in this example, as success in meeting physical targets is important for the Chinese government and managers were rewarded from this view.

The following instructions show the steps to measure the components of the Malmquist efficiency index for the textile industry in the first and second periods.

1. Copy data from 1966 to 1975 in Table 11.16 on an Excel sheet into cells A1: D11, as Fig. 11.19 illustrates.
2. Label E1 as 'Lambdas', F1 as 'Firm', G1 as 'D_1^1', H1 as 'D_1^2', I1 as 'D_2^1', J1 as 'D_2^2', K1 as 'Malmquist Index', A13 as 'Theta', G13 as 'Index 1', G14 as 'Index 2', A16 as 'Model 1', A17 as 'Model 2', A18 as 'Model 3', A19 as 'Model 4', and A21 as 'Selected Model' (Fig. 11.20).
3. Assign number 1 to H13 and H14.

Fig. 11.19 Copying data in Table 11.16 in an excel sheet

	A	B	C	D
1	Year \ Textiles	Capital	Labor	AGIOV
2	1966	48002	106910	145163
3	1967	38135	107047	121057
4	1968	40584	111428	135572
5	1969	50454	115222	177849
6	1970	49218	122428	183630
7	1971	43604	127393	160291
8	1972	44677	131991	161853
9	1973	50958	133555	181968
10	1974	53419	134456	188066
11	1975	59430	135642	206317

F16 =INDEX(B2:B6,$H12)*$B13

	A	B	C	D	E	F	G	H	I	J	K
1	Year \ Textiles	Capital	Labor	AGIOV	Lambda	CCR-Scores	D_1^1	D_1^2	D_2^1	D_2^2	M
2	1966	48002	106910	145163		DMU1					
3	1967	38135	107047	121057		DMU2					
4	1968	40584	111428	135572		DMU3					
5	1969	50454	115222	177849		DMU4					
6	1970	49218	122428	183630		DMU5					
7	1971	43604	127393	160291							
8	1972	44677	131991	161853							
9	1973	50958	133555	181968		Run					
10	1974	53419	134456	188066							
11	1975	59430	135642	206317							
12							Index 1	1			
13	Theta	1					Index 2	1			
14											
15											
16	1	0	0	0		48002	106910	145163			
17	2	0	0	0		48002	106910	145163			
18	3	0	0	0		43604	127393	160291			
19	4	0	0	0		43604	127393	160291			
20											
21	Selected	0	0	0		48002	106910	145163			

Fig. 11.20 Setting Excel sheet for data in Table 11.16

4. Assign the following command into B16 and B18,

   '=Sumproduct(B2:B6,$E2:$E6)'.

5. Copy B16 and paste it into C16 and D16.
6. Copy B18 and paste it into C18 and D18.

7. Assign the following command into B17 and B19,

   '=Sumproduct(B7:B11,$E7:$E11)'.

8. Copy B17 and paste it into C17 and D17.
9. Copy B19 and paste it into C19 and D19.
10. Assign the following command into F16 and F17,

    '=Index(B2:B6,$H12)*$B13'.

11. Copy F16 and paste it into G16.
12. Copy F17 and paste it into G17.
13. Assign the following command into F18 and F19,

    '=Index(B7:B11,$H12)*$B13'.

14. Copy F18 and paste it into G18.
15. Copy F19 and paste it into G19.
16. Assign the following command into H16 and H17,

    '=Index(D2:D6,$H12)'.

17. Assign the following command into H18 and H19,

    '=Index(D7:D11,$H12)'.

18. Assign the following command into B21,

    '=Index(B16:B19,$H13)'.

19. Copy B21 and paste it to C21, D21, F21, G21 and H21.
20. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.21 illustrates.
21. Assign 'B13' into 'Set Objective' and choose 'Min'.
22. Assign 'E2:E11, B13' into 'By Changing Variable Cells'.
23. Click on 'Add' and assign 'B21:C21' into 'Cell Reference', then select '<=', and assign 'F21:G21' into 'Constraint'.
24. Click on 'Add' and assign 'D21' into 'Cell Reference', then select '>=' and assign 'H21' into 'Constraint'. Then click on 'OK'.
25. Tick 'Make Unconstrained Variables Non-Negative'.
26. Choose 'Simplex LP' from 'Select a Solving Method'.
27. Click on 'Solve'.
28. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
29. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
30. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.

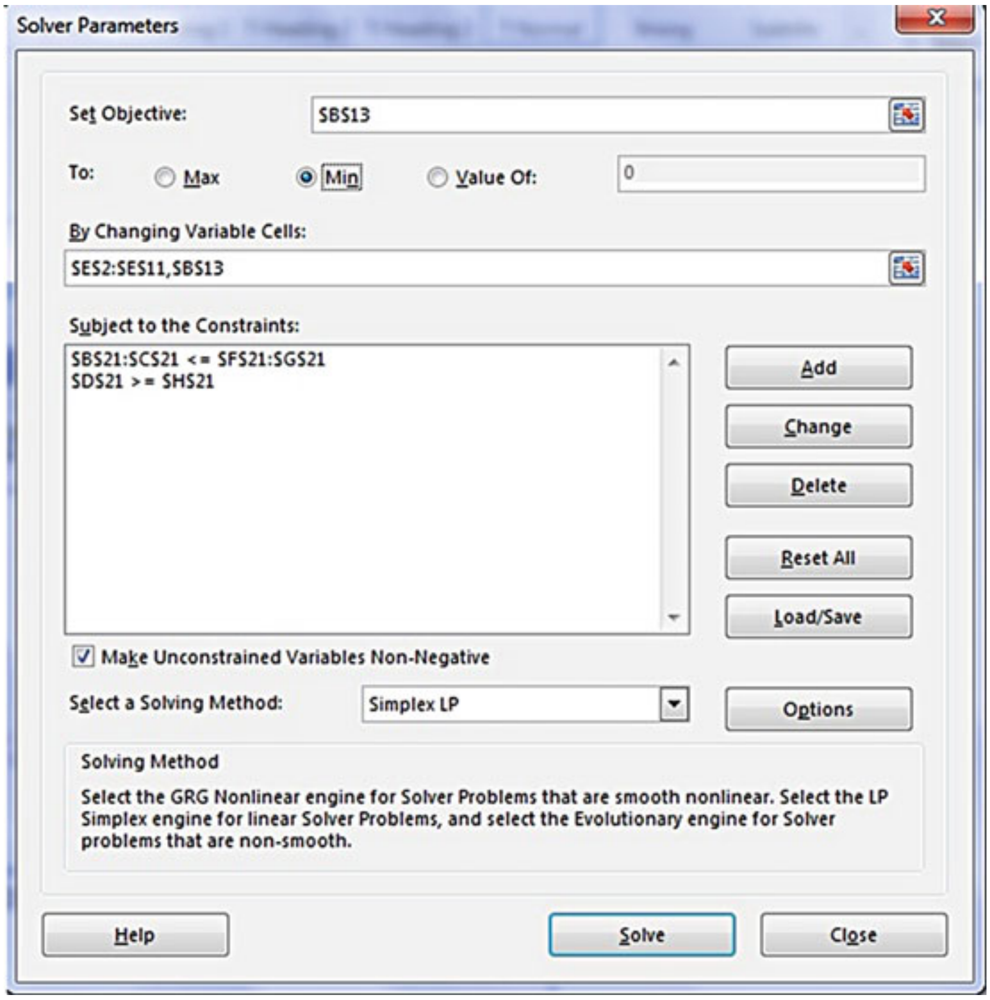


Fig. 11.21 Setting solver for data in Table 11.16

31. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’, as Fig. 11.22 shows.

```
Dim i, j As Integer
For i = 1 To 5
   Range("H12") = i
   For j = 1 To 4
      Range("H13") = j
      SolverSolve Userfinish:=True
      Cells(i + 1, j + 6) = Range("B13")
   Next j
Next i
For i = 1 To 5
   Range("K" & i + 1) = ((Range("I" & i + 1) * Range("J" & i + 1)) / _
                  (Range("H" & i + 1) * Range("G" & i + 1))) ^ 0.5
Next i
```

32. Close the ‘Microsoft Visual Basic for Applications’ window.

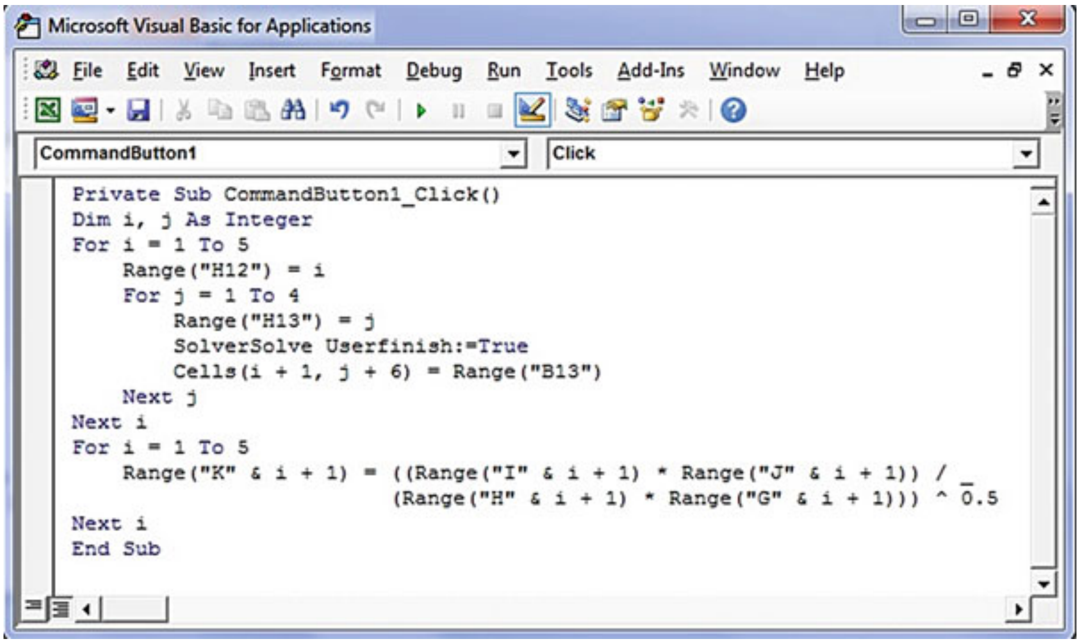
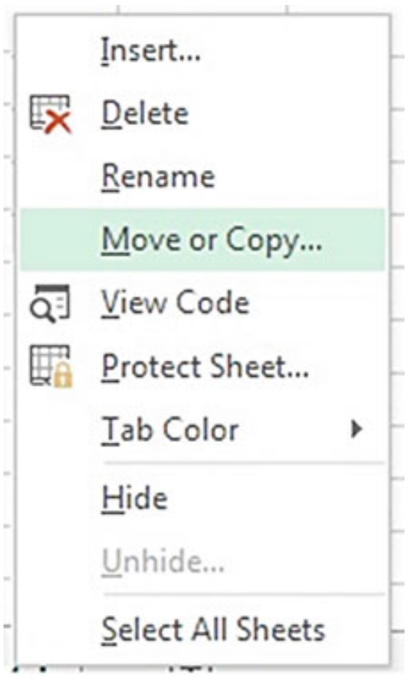


Fig. 11.22 Setting VBA for data in Table 11.16

Fig. 11.23 make a copy of a worksheet



33. Click on the small rectangle which was automatically made on the Excel sheet and created by step 29. The results are represented to cells G2:K6.
34. Right click on the worksheet tab and select ‘Move or Copy . . .’, as Fig. 11.23 shows.
35. Select ‘Create a copy’, as Fig. 11.24 depicts, and click on ‘OK’.
36. Label A14 as ‘Alpha’, and E13 as ‘Objective’, as Fig. 11.25 shows.
37. Assign 0.3 and 0.7 into B14 and C14.

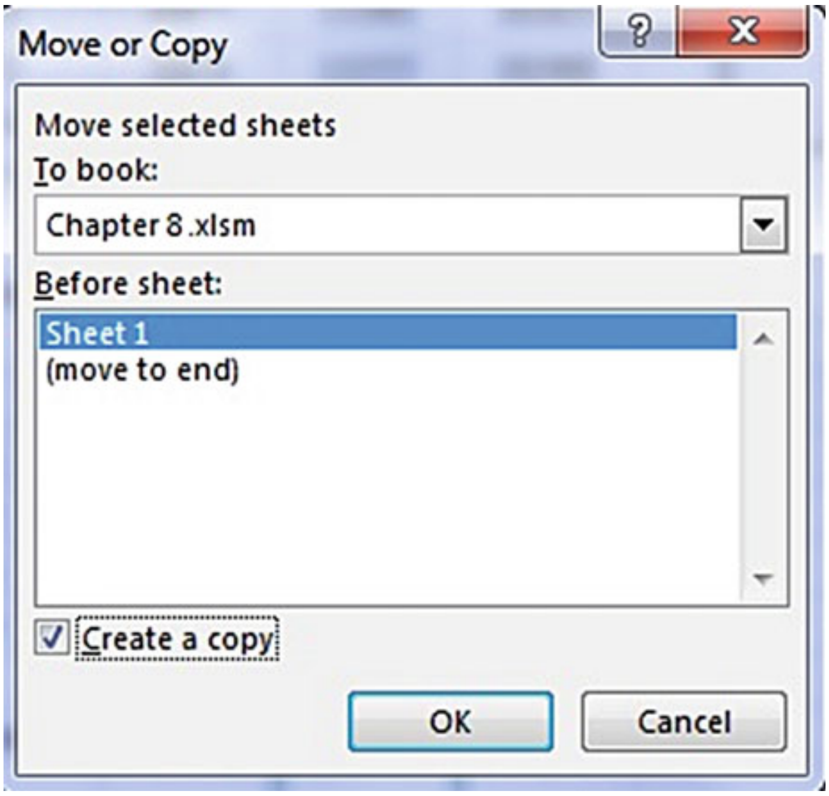
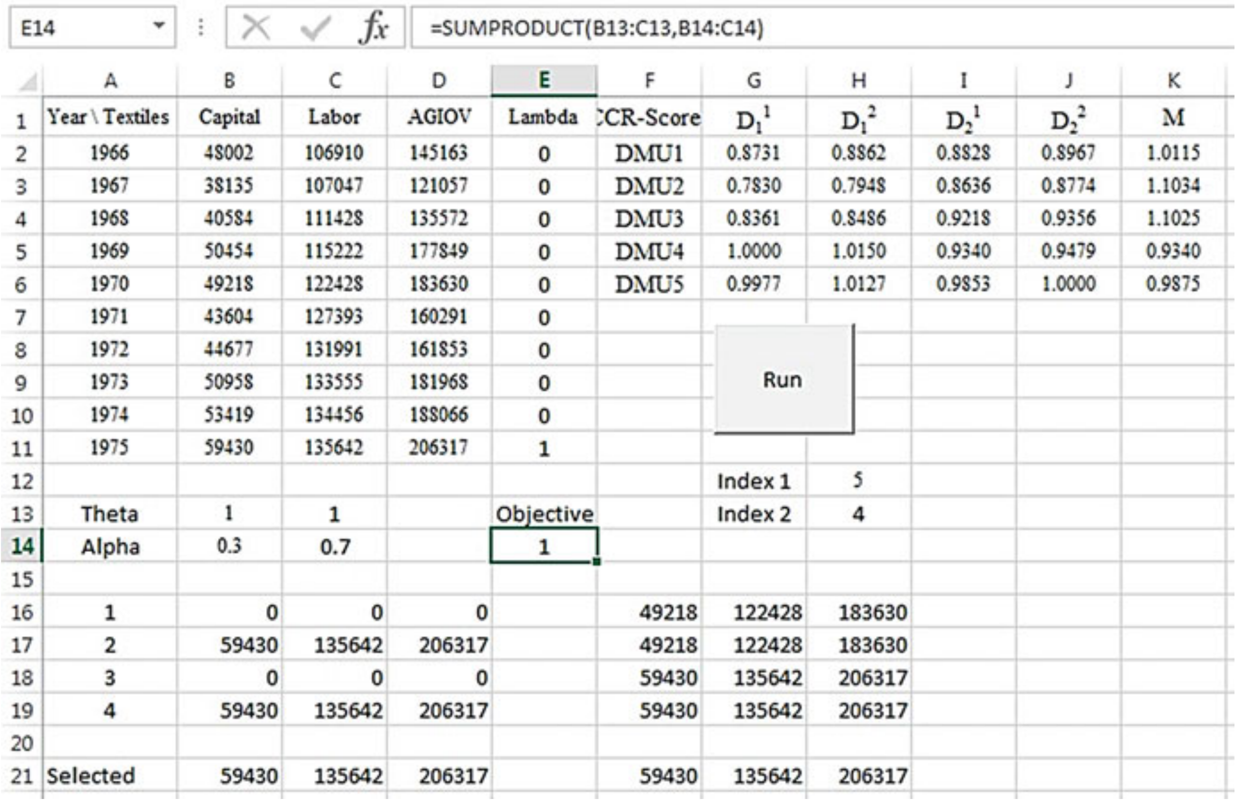


Fig. 11.24 Create a copy of a worksheet



E14 =SUMPRODUCT(B13:C13,B14:C14)

	A	B	C	D	E	F	G	H	I	J	K
1	Year \ Textiles	Capital	Labor	AGIOV	Lambda	CCR-Score	D_1^1	D_1^2	D_2^1	D_2^2	M
2	1966	48002	106910	145163	0	DMU1	0.8731	0.8862	0.8828	0.8967	1.0115
3	1967	38135	107047	121057	0	DMU2	0.7830	0.7948	0.8636	0.8774	1.1034
4	1968	40584	111428	135572	0	DMU3	0.8361	0.8486	0.9218	0.9356	1.1025
5	1969	50454	115222	177849	0	DMU4	1.0000	1.0150	0.9340	0.9479	0.9340
6	1970	49218	122428	183630	0	DMU5	0.9977	1.0127	0.9853	1.0000	0.9875
7	1971	43604	127393	160291	0						
8	1972	44677	131991	161853	0						
9	1973	50958	133555	181968	0		Run				
10	1974	53419	134456	188066	0						
11	1975	59430	135642	206317	1						
12							Index 1	5			
13	Theta	1	1		Objective		Index 2	4			
14	Alpha	0.3	0.7		1						
15											
16	1	0	0	0		49218	122428	183630			
17	2	59430	135642	206317		49218	122428	183630			
18	3	0	0	0		59430	135642	206317			
19	4	59430	135642	206317		59430	135642	206317			
20											
21	Selected	59430	135642	206317		59430	135642	206317			

Fig. 11.25 Setting Excel for taking the alpha values

38. Assign the following command into G16 and G17,

    '=Index(C2:C6,$H12)*C13'.

39. Assign the following command into B21,

    '=Index(C7:C11,$H12)*C13'.

40. Assign the following command into E14,

    '=Sumproduct(B13:C13,B14:C14)'.

41. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.26 illustrates.
42. Assign 'E14' into 'Set Objective' and choose 'Min'.
43. Assign 'E2:E11, B13:C13' into 'By Changing Variable Cells'.
44. Click on 'Solve'.
45. Right click on the small rectangle which was automatically made on the Excel sheet and created by step 29, and click on 'Assign Macro...' (Fig. 11.27).

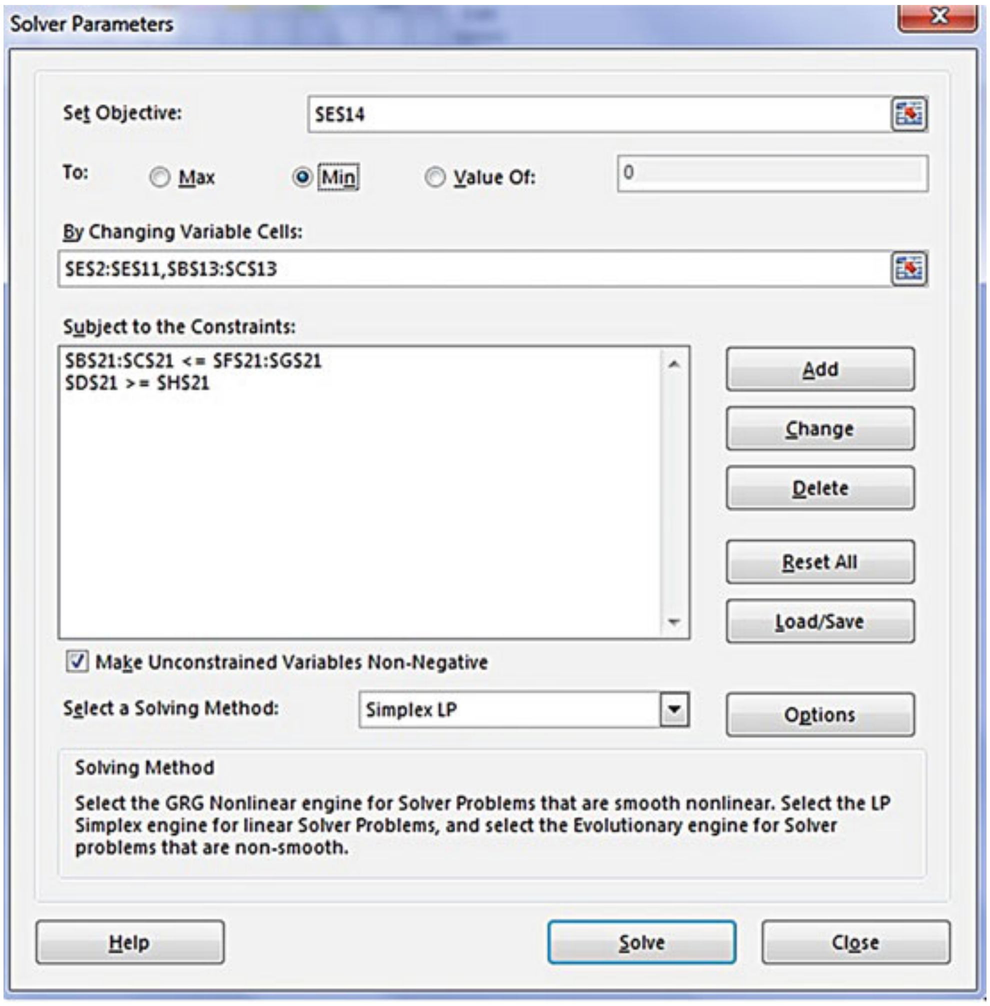


Fig. 11.26 Setting VBA for program in Fig. 11.25

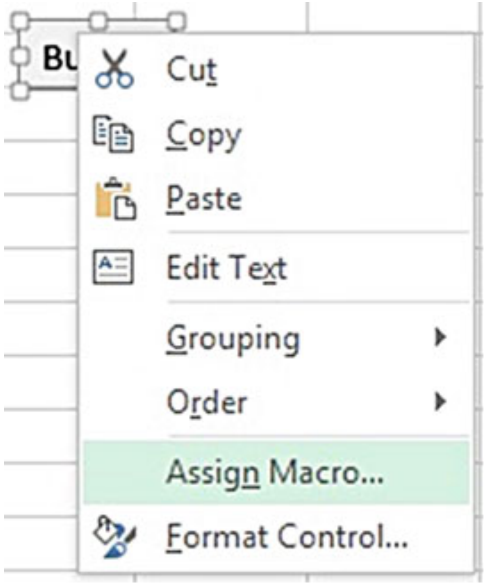


Fig. 11.27 Assign a macro to a worksheet

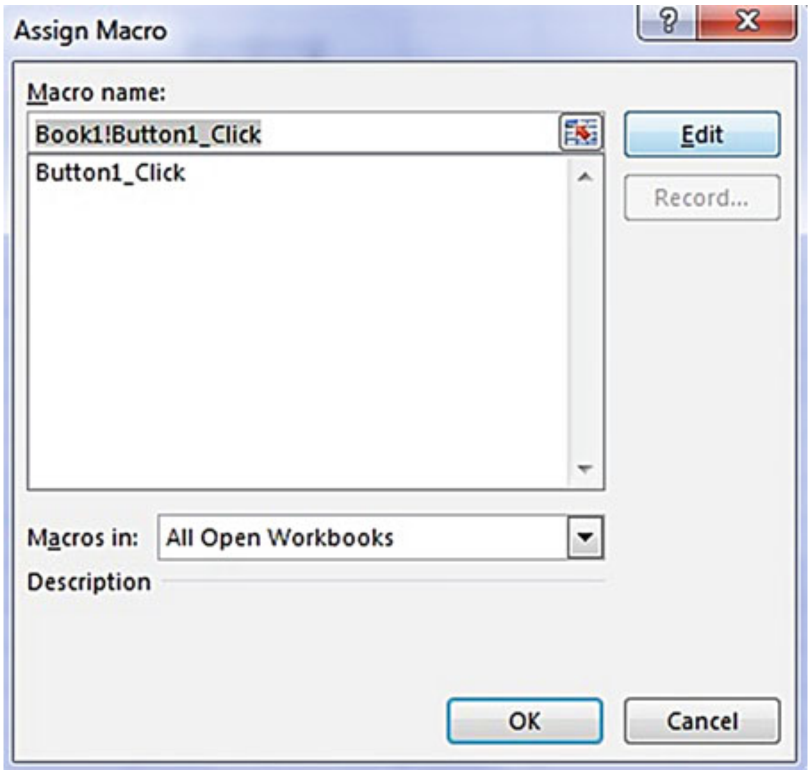


Fig. 11.28 Edit a macro in a worksheet

46. Click on 'Edit', as shown in Fig. 11.28, and instead 'Range("B13")' in the command 'Cells(i + 1, j + 6) = Range("B13")', write Range("E14").

In other words, inside of the 'Microsoft Visual Basic for Applications' window, the following commands between 'Sub Button1_Click ()' and 'End Sub', should be written.

```
Dim i, j As Integer
For i = 1 To 5
   Range("H12") = i
   For j = 1 To 4
      Range("H13") = j
      SolverSolve Userfinish:=True
      Cells(i + 1, j + 6) = Range("E14")
   Next j
Next i
For i = 1 To 5
   Range("K" & i + 1) = ((Range("I" & i + 1) * Range("J" & i + 1)) / _
                 (Range("H" & i + 1) * Range("G" & i + 1))) ^ 0.5
Next i
```

47. Close the ‘Microsoft Visual Basic for Applications’ window.
48. Click on the small rectangle. The results are represented to cells G2:K6.

Tables 11.19, 11.20, 11.21, 11.22, 11.23, 11.24, 11.25, 11.26, 11.27, 11.28, 11.29, 11.30, 11.31, 11.32, 11.33, 11.34, 11.35 and 11.36 illustrates the results of the CCR radial model to measure the Malmquist efficiency indexes for the three Chinese industries during four different time periods from 1966 to 1985.

The first two tables, Tables 11.19 and 11.20, represent the result of the model for the textile industry from the first period to the second period. Table 11.19 illustrates

Table 11.19 Textile radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
T_1	0.8797	0.8927	0.9853	1.0000	1.1201
T_2	0.8508	0.8722	0.9710	0.9855	1.1355
T_3	0.8954	0.9229	0.9571	0.9975	1.0749
T_4	1.0000	1.0153	0.9436	0.9927	0.9605
T_5	1.0000	1.0548	0.9854	1.0000	0.9666

Table 11.20 Textile radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$
T_1	1.1368	0.9854	0.9853	0.9854
T_2	1.1583	0.9755	0.9853	0.9804
T_3	1.1141	0.9702	0.9595	0.9648
T_4	0.9927	0.9850	0.9506	0.9676
T_5	1.0000	0.9481	0.9854	0.9666

Table 11.21 Textile radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
T_1	1.0000	1.0215	0.9670	0.9432	0.9449
T_2	0.9855	1.0067	0.9313	0.9227	0.9307
T_3	0.9975	0.9923	1.2000	0.9031	1.0464
T_4	0.9927	0.9783	1.2656	0.9769	1.1283
T_5	1.0000	0.9647	1.3794	1.0000	1.1958

Table 11.22 Textile radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
T_1	0.9432	0.9790	1.0252	1.0018
T_2	0.9363	0.9790	1.0093	0.9940
T_3	0.9053	1.0053	1.3288	1.1558
T_4	0.9841	1.0147	1.2955	1.1465
T_5	1.0000	1.0366	1.3794	1.1958

Table 11.23 Textile radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
T_1	0.9432	0.9880	1.0320	1.0000	1.0523
T_2	0.9227	0.9666	0.8496	0.8258	0.8869
T_3	0.9031	0.9460	0.8714	0.8398	0.9256
T_4	0.9769	1.0232	1.0128	0.9400	0.9759
T_5	1.0000	1.0475	1.0910	1.0000	1.0206

Table 11.24 Textile radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
T_1	1.0602	0.9547	1.0320	0.9926
T_2	0.8950	0.9547	1.0287	0.9910
T_3	0.9300	0.9547	1.0376	0.9953
T_4	0.9622	0.9547	1.0775	1.0142
T_5	1.0000	0.9547	1.0910	1.0206

Table 11.25 Chemicals radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
C_1	0.8211	0.8078	1.0065	1.0000	1.2319
C_2	0.8288	0.8363	0.9802	0.9890	1.1827
C_3	0.8632	0.8709	0.9911	0.9842	1.1391
C_4	1.0000	0.9893	1.0042	0.9937	1.0043
C_5	1.0000	1.0089	1.0164	1.0000	1.0037

Table 11.26 Chemicals radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$
C_1	1.2179	1.0164	1.0065	1.0115
C_2	1.1933	0.9911	0.9911	0.9911
C_3	1.1402	0.9911	1.0070	0.9990
C_4	0.9937	1.0108	1.0105	1.0107
C_5	1.0000	0.9911	1.0164	1.0037

Table 11.27 Chemicals radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
C_1	1.0000	0.9933	0.9914	0.9847	0.9914
C_2	0.9890	0.9824	0.9664	0.9599	0.9772
C_3	0.9842	0.9771	1.0355	1.0000	1.0377
C_4	0.9937	0.9830	1.0473	1.0000	1.0354
C_5	1.0000	0.9874	0.9407	0.9070	0.9295

Table 11.28 Chemicals radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
C_1	0.9847	1.0068	1.0068	1.0068
C_2	0.9706	1.0068	1.0068	1.0068
C_3	1.0160	1.0073	1.0355	1.0213
C_4	1.0063	1.0109	1.0473	1.0289
C_5	0.9070	1.0127	1.0372	1.0249

Table 11.29 Chemicals radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
C_1	0.9847	1.1581	0.9179	1.0000	0.8972
C_2	0.9599	1.1290	0.9638	1.0000	0.9430
C_3	1.0000	1.1761	1.0045	1.0000	0.9242
C_4	1.0000	1.0976	0.9551	0.9628	0.9153
C_5	0.9070	1.0559	1.1014	1.0000	1.0724

Table 11.30 Chemicals radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
C_1	1.0156	0.8503	0.9179	0.8834
C_2	1.0417	0.8503	0.9638	0.9053
C_3	1.0000	0.8503	1.0045	0.9242
C_4	0.9628	0.9111	0.9920	0.9507
C_5	1.1026	0.8590	1.1014	0.9727

Table 11.31 Metallurgy radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
M_1	0.8414	1.2713	0.6627	1.0000	0.7871
M_2	0.8755	1.3262	0.8474	1.0000	0.8543
M_3	1.0000	1.5385	0.7458	0.9025	0.6615
M_4	0.8290	1.2752	0.6790	0.8014	0.7175
M_5	1.0000	1.5324	0.5477	0.6898	0.4965

Table 11.32 Metallurgy radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$
M_1	1.1885	0.6618	0.6627	0.6623
M_2	1.1422	0.6601	0.8474	0.7479
M_3	0.9025	0.6500	0.8264	0.7329
M_4	0.9667	0.6501	0.8474	0.7422
M_5	0.6898	0.6526	0.7939	0.7198

Table 11.33 Metallurgy radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
M_1	1.0000	1.2456	0.8715	1.0000	0.8365
M_2	1.0000	1.2726	0.8563	0.9880	0.8153
M_3	0.9025	1.1201	0.5920	0.6829	0.6324
M_4	0.8014	1.0198	0.5253	0.6154	0.6290
M_5	0.6898	0.8225	0.5187	0.6136	0.7490

Table 11.34 Metallurgy radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
M_1	1.0000	0.8028	0.8715	0.8365
M_2	0.9880	0.7858	0.8666	0.8252
M_3	0.7567	0.8057	0.8669	0.8358
M_4	0.7680	0.7858	0.8536	0.8190
M_5	0.8895	0.8387	0.8454	0.8420

Table 11.35 Metallurgy radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
M_1	1.0000	1.7814	0.6399	1.0000	0.5993
M_2	0.9880	1.7226	0.6189	0.9618	0.5914
M_3	0.6829	1.1922	0.6523	1.0000	0.8950
M_4	0.6154	1.0141	0.5822	0.8926	0.9125
M_5	0.6136	0.9764	0.5635	0.8639	0.9014

Table 11.36 Metallurgy radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
M_1	1.0000	0.5613	0.6399	0.5993
M_2	0.9735	0.5735	0.6435	0.6075
M_3	1.4643	0.5728	0.6523	0.6113
M_4	1.4504	0.6068	0.6523	0.6291
M_5	1.4080	0.6284	0.6523	0.6402

the scores of CCR by Models 11.10–11.13 as well as the Malmquist efficiency indexes for each DMU. Table 11.20 also shows the components of the Malmquist efficiency indexes.

From Table 11.20, the ratio D_2^2/D_1^1 represents the value of technical efficiency change which is greater than 1 for all DMUs except DMU 4. Thus, on the average, the radial distance of DMUs in the second time period to the production frontier in the second time period is closer than the radial distance of DMUs in the first time period to the production frontier in the first period time. In contrast, the production frontier change value is less than 1 for all DMUs. This indicates a negative shift and technical regress occurred. On average, we have Case 4, which is represented in Fig. 11.16, for the textile industry from the first period to the second period. DMU 4 had the worst case, that is, Case 8, indicating decline in both technical efficiency and frontier changes. The Malmquist efficiency indexes in Table 11.19 show performance progress for the first three DMUs and regress for the last two DMUs, from the first period to the second period.

From the second time period to the third time period, the ratio D_2^2/D_1^1 is less than 1 for all DMUs except DMU 5, as shown in Tables 11.21 and 11.22. This indicates that, on the average, the radial distance of DMUs in the second time period to the production frontier in the second time period is closer than the radial distance of DMUs in the third time period to the production frontier in the third period time. In addition, the production frontier change value is greater than 1 for all DMUs and we have Case a, that is, DMUs moved onto a side which has positive shift and progress in the production technology.

Overall, technical progress/regress happened after each five-year-plan period in the textile industry. The same illustrations can be written for the rest of the information in Tables 11.16, 11.17, 11.18, 11.19, 11.20, 11.21, 11.22, 11.23, 11.24, 11.25, 11.26, 11.27, 11.28, 11.29, 11.30, 11.31, 11.32, 11.33, 11.34, 11.35 and 11.36.

Now, assume that $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$ for the textile industry, $\alpha_1 = 0.7$ and $\alpha_2 = 0.3$ for the chemical industry, and $\alpha_1 = 0.5$ and $\alpha_2 = 0.5$ for metallurgy. By identifying these weights, we suppose that (i) it is more important to decrease the amount of labor in the textile industry when we plan to improve the performance of the textile industry, because the industry is labor intensive; (ii) it is more important to decrease the amount of capital in the chemical industry when we plan to improve the performance of the chemical industry, because the industry is capital intensive.

In the metallurgy industry, no preference over the two inputs is given. In other words, the metallurgy industry is labor intensive and capital intensive, because the two inputs are equally important.

The results of applying the non-radial Models 11.50–11.53 are illustrated in Tables 11.37, 11.38, 11.39, 11.40, 11.41, 11.42, 11.43, 11.44, 11.45, 11.46, 11.47, 11.48, 11.49, 11.50, 11.51, 11.52, 11.53, and 11.54. Similarly, the average efficiency change along with the average technical efficiency change and the average production frontier movement are illustrated for each industry.

Table 11.37 Textile non-radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
T_1	0.8731	0.8862	0.8828	0.8967	1.0115
T_2	0.7830	0.7948	0.8636	0.8774	1.1034
T_3	0.8361	0.8486	0.9218	0.9356	1.1025
T_4	1.0000	1.0150	0.9340	0.9479	0.9340
T_5	0.9977	1.0127	0.9853	1.0000	0.9875

Table 11.38 Textile non-radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1 D_2^1/\left(D_1^2 D_2^2\right)\right)^{1/2}$
T_1	1.0270	0.9853	0.98s45	0.9849
T_2	1.1205	0.9852	0.9843	0.9847
T_3	1.1191	0.9852	0.9852	0.9852
T_4	0.9479	0.9853	0.9852	0.9853
T_5	1.0023	0.9852	0.9853	0.9853

Table 11.39 Textile non-radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
T_1	0.8967	0.7262	0.9433	0.7542	1.0452
T_2	0.8774	0.7111	0.8802	0.7069	0.9987
T_3	0.9356	0.7523	1.1209	0.8799	1.1838
T_4	0.9479	0.7602	1.1897	0.9353	1.2426
T_5	1.0000	0.7969	1.2765	1.0000	1.2657

Table 11.40 Textile non-radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
T_1	0.8410	1.2347	1.2508	1.2427
T_2	0.8057	1.2338	1.2452	1.2395
T_3	0.9405	1.2437	1.2738	1.2587
T_4	0.9867	1.2470	1.2720	1.2594
T_5	1.0000	1.2549	1.2765	1.2657

Table 11.41 Textile non-radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
T_1	0.7542	0.7530	1.0088	1.0000	1.3329
T_2	0.7069	0.7067	0.8312	0.8240	1.1709
T_3	0.8799	0.8739	0.8253	0.8166	0.9362
T_4	0.9353	0.9293	0.8492	0.8338	0.9026
T_5	1.0000	0.9925	0.8952	0.8777	0.8897

Table 11.42 Textile non-radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
T_1	1.3260	1.0016	1.0088	1.0052
T_2	1.1656	1.0003	1.0088	1.0045
T_3	0.9281	1.0069	1.0107	1.0088
T_4	0.8915	1.0065	1.0184	1.0124
T_5	0.8777	1.0075	1.0199	1.0137

Table 11.43 Chemicals non-radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
C_1	0.8103	0.8069	1.0017	0.9979	1.2364
C_2	0.7978	0.7964	0.9794	0.9762	1.2267
C_3	0.8177	0.8169	0.9861	0.9823	1.2043
C_4	0.9928	0.9888	0.9971	0.9931	1.0044
C_5	1.0000	0.9966	1.0042	1.0000	1.0038

Table 11.44 Chemicals non-radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$
C_1	1.2315	1.0042	1.0038	1.0040
C_2	1.2236	1.0017	1.0033	1.0025
C_3	1.2013	1.0010	1.0039	1.0024
C_4	1.0003	1.0041	1.0041	1.0041
C_5	1.0000	1.0034	1.0042	1.0038

Table 11.45 Chemicals non-radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
C_1	0.9979	0.9790	0.9581	0.9404	0.9604
C_2	0.9762	0.9578	0.9260	0.9091	0.9488
C_3	0.9823	0.9637	1.0196	1.0000	1.0378
C_4	0.9931	0.9742	0.9758	0.9567	0.9823
C_5	1.0000	0.9809	0.9187	0.9010	0.9186

Table 11.46 Chemicals non-radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
C_1	0.9424	1.0193	1.0188	1.0191
C_2	0.9312	1.0191	1.0187	1.0189
C_3	1.0180	1.0193	1.0196	1.0194
C_4	0.9634	1.0194	1.0199	1.0197
C_5	0.9010	1.0195	1.0196	1.0195

Table 11.47 Chemicals non-radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
C_1	0.9404	1.0812	0.8737	1.0000	0.9269
C_2	0.9091	1.0457	0.8752	1.0000	0.9595
C_3	1.0000	1.1464	0.8670	0.9880	0.8644
C_4	0.9567	1.0951	0.8369	0.9544	0.8731
C_5	0.9010	1.0327	0.8435	0.9546	0.9303

Table 11.48 Chemicals non-radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
C_1	1.0633	0.8698	0.8737	0.8717
C_2	1.1000	0.8693	0.8752	0.8723
C_3	0.9880	0.8723	0.8776	0.8749
C_4	0.9976	0.8736	0.8769	0.8752
C_5	1.0595	0.8725	0.8836	0.8780

Table 11.49 Metallurgy non-radial performance from 1st period to 2nd

DMUs	D_1^1	D_1^2	D_2^1	D_2^2	M
M_1	1.0000	0.5901	1.1316	0.6613	1.1262
M_2	0.9529	0.5621	1.4715	0.8704	1.5465
M_3	0.8134	0.4794	1.5178	0.8933	1.8647
M_4	0.8990	0.5260	1.7054	1.0000	1.8990
M_5	0.9542	0.5681	1.7082	0.9933	1.7691

Table 11.50 Metallurgy non-radial Malmquist index from 1st period to 2nd

DMUs	D_2^2/D_1^1	D_1^1/D_1^2	D_2^1/D_2^2	$\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$
M_1	0.6613	1.6947	1.7111	1.7029
M_2	0.9135	1.6953	1.6905	1.6929
M_3	1.0983	1.6966	1.6991	1.6978
M_4	1.1123	1.7092	1.7054	1.7073
M_5	1.0409	1.6796	1.7198	1.6996

Table 11.51 Metallurgy non-radial performance from 2nd period to 3rd

DMUs	D_2^2	D_2^3	D_3^2	D_3^3	M
M_1	0.6613	0.6532	0.8508	0.8482	1.2926
M_2	0.8704	0.8931	0.8470	0.8405	0.9570
M_3	0.8933	0.9048	1.0374	0.9803	1.1216
M_4	1.0000	0.9998	1.0741	1.0000	1.0365
M_5	0.9933	0.9629	1.0609	0.9843	1.0449

Table 11.52 Metallurgy non-radial Malmquist index from 2nd period to 3rd

DMUs	D_3^3/D_2^2	D_2^2/D_2^3	D_3^2/D_3^3	$\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$
M_1	1.2826	1.0125	1.0030	1.0078
M_2	0.9656	0.9746	1.0078	0.9911
M_3	1.0973	0.9873	1.0583	1.0222
M_4	1.0000	1.0002	1.0741	1.0365
M_5	0.9910	1.0315	1.0778	1.0544

Table 11.53 Metallurgy non-radial performance from 3rd period to 4th

DMUs	D_3^3	D_3^4	D_4^3	D_4^4	M
M_1	0.8840	0.9551	0.9848	1.1200	1.1200
M_2	0.8755	0.9676	0.9960	1.1444	1.1444
M_3	1.0161	0.9332	0.9619	0.9493	0.9493
M_4	1.0317	0.9760	1.0000	0.9726	0.9726
M_5	1.0139	0.9455	0.9629	0.9551	0.9551

Table 11.54 Metallurgy non-radial Malmquist index from 3rd period to 4th

DMUs	D_4^4/D_3^3	D_3^3/D_3^4	D_4^3/D_4^4	$\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$
M_1	1.1610	0.9596	0.9698	0.9647
M_2	1.1851	0.9600	0.9715	0.9657
M_3	0.9813	0.9647	0.9702	0.9674
M_4	1.0000	0.9693	0.9760	0.9726
M_5	0.9782	0.9708	0.9819	0.9763

When a larger weight is defined on the labor input factor in the textile industry by the non-radial model, on average the efficiency improvement can be seen in each time period, whereas there was a decline in efficiency from the third to fourth period by the radial model. In other words, the efficiency improvement from the first period to the second period was 23% on average, from the second period to the third period was 15% on average, and from third period to fourth period was 5% on average. In addition, the non-radial model results that, on average, the technical progress and positive frontier shift happened after each five-year-plan period in the textile industry.

On the other hand, when a larger weight is defined on the capital input in the chemical industry, the results are almost the same on average. Nevertheless, while the radial model results that on average the efficiency declined in each five-year-plan period in the metallurgy industry, the non-radial results opposite. This means, on average there was efficiency progress in each five-year-plan period in the metallurgy industry. The improvement in efficiency was on average 64% from the first period to the second period, 9% from the second period to the third period, and 3% from the third to the fourth period. This is due to the fact that the radial efficiency index only considers the proportional changes of all inputs and ignores the non-zero slacks.

The newly defined non-radial Malmquist efficiency index represents that on average efficiency improvement happened in each industry from the third to the fourth five-year-plan period, while the radial Malmquist efficiency index represents the opposite outcomes. In fact, the Chinese industrial growth was especially rapid in the early 1960s, except for a dip early in the 1966–1976 Cultural Revolution period. This shows that the non-radial Malmquist efficiency index delivers outcomes in consistent with the variations during the 1978–1983 economic reform period.

11.7 Conclusion

In this chapter, the DEA Malmquist productivity approach is discussed. Each individual component of Malmquist productivity index is analyzed, and show that the analyses are essential to indicate the performance of a firm specifically. The Malmquist approach is extended to identify the strategic change of DMUs in a particular period, and whether or not the strategic change is favorable. The DEA Malmquist efficiency index can be employed to assess the technology and productivity shifts result from economic development plans. A non-radial Malmquist efficiency index and its prose are also illustrated to incorporate the preference over the performance progress and to measure the slacks. The methods are exemplified for an example of the three major Chinese industries whose industrial activities constitute important components of China's five-year economic development planning efforts.

11.8 Exercises

11.1. Develop the Malmquist Efficiency index using CCR output-oriented model.

11.2. Develop an output-oriented non-radial Malmquist efficiency index, when input factors are fixed at their current levels; that is, using CCR output-oriented.

11.3. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.1.

11.4. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.16.

11.5. Develop the Malmquist Efficiency index using SBM.

11.6. Write a VBA procedure to run the models in Exercise 11.5 for the data in Table 11.1.

11.7. Develop the Malmquist efficiency index using one set of weights models, such as Eqs. 6.1 and 6.2.

11.8. Develop the Malmquist Efficiency index using Eq. 6.31.

11.9. Write a VBA procedure to run the models in Exercise 11.9 for the data in Table 11.1.

11.10. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.1.

Chapter 12
Delta Neighborhood Extension

12.1 Introduction

In this chapter, the introduced mathematical model in Chap. 6 is improved to fairly rank firms in various conditions, and the chapter is finished with several outcomes of the model.

12.2 The Delta KAM

Suppose that there are n firms, labeled F_i ($i = 1, 2, \ldots, n$), and each firm has m input factors with the values x_{ij} ($j = 1, 2, \ldots, m$) and p output factors with the values y_{ik} ($k = 1, 2, \ldots, p$). Assume that the weights/prices or the approximation of the relationships between input and output factors are W_j^- and W_k^+, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, respectively. Suppose that, V_j^- and V_k^+ are defined as Eq. 12.1, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$:

$$V_j^- = \frac{W_j^-}{\sum_{j=1}^m W_j^- x_{lj}} \quad \& \quad V_k^+ = \frac{W_k^+}{\sum_{k=1}^p W_k^+ y_{lk}}. \tag{12.1}$$

Assume that the delta vector, with bolded notation $\boldsymbol{\Delta}$, is given by $\boldsymbol{\Delta} = \left(\delta_1^-, \delta_2^-, \ldots, \delta_m^-, \delta_1^+, \delta_2^+, \ldots, \delta_p^+\right)$, to introduce a delta neighborhood of firm F_l, ($l = 1, 2, \ldots, n$). The components of delta vector are introduced by Eq. 12.2, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

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$$\delta_j^- = \delta / W_j^- \quad \text{and} \quad \delta_k^+ = \delta / W_k^+. \tag{12.2}$$

The value of delta has the same meaning for each factor, when Eq. 12.2 is considered. If the components of delta vector are defined by Eq. 12.3, where $\delta \in [0, +\infty)$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, the $\boldsymbol{\Delta}$-KAM is the same as Eq. 5.25, where $m = 4$ and $p = 3$.

$$\delta_j^- = \delta \times x_{lj} \quad \text{and} \quad \delta_k^+ = \delta \times y_{lk}. \tag{12.3}$$

There are a lot of ways to introduce the components of delta vector, according to the aim of discrimination. For instance, δ_j^- and δ_k^+ can be introduced as Eq. 12.4, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$. In this case, the components of delta vector are commensurate with the corresponded input and output factors, but are not changed from one firm to another.

$$\delta_j^- = \delta \times \text{ave}_{1 \le i \le n} x_{ij} \quad \text{and} \quad \delta_k^+ = \delta \times \text{ave}_{1 \le i \le n} y_{ik}. \tag{12.4}$$

It is also possible to introduce one (or more) of the components of delta vector as 0, regarding the purpose of discrimination. For example, the components of delta can be introduced by $\delta_j^- = \delta \times x_{lj}$ and $\delta_k^+ = 0$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, which let's consider the errors in input factors only.

Equation 12.5 illustrates the $\boldsymbol{\Delta}$-KAM, when the performance of F_l is measured ($l = 1, 2, \ldots, n$).

$$\begin{aligned}
&\min \frac{\sum_{j=1}^{m} V_j^- (x_{lj} + \delta_j^- - s_j^-)}{\sum_{k=1}^{p} V_k^+ (y_{lk} - \delta_k^+ + s_k^+)}, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta_j^-, \ \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta_k^+, \ \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \ge 0, \ \text{for } i = 1, 2, \ldots, n, \\
&s_j^- \ge 0, \ \text{for } j = 1, 2, \ldots, m, \\
&s_k^+ \ge 0, \ \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.5}$$

The $\boldsymbol{\Delta}$-KAM (Eq. 12.5) can linearly be solved by Eq. 12.6. In order to solve the model by Microsoft Excel Solver software, similar instructions to solve Eqs. 5.31 and 5.32 can be used.

$$
\begin{aligned}
&\min\left[\sum_{j=1}^{m} V_j^-\left(tx_{lj}+t\delta_j^- - s_j^-\right)\right],\\
&\text{Subject to}\\
&\left[\sum_{k=1}^{p} V_k^+\left(y_{lk}t-\delta_k^+ t+s_k^+\right)\right]=1,\\
&\sum_{i=1}^{n} x_{ij}\lambda_i+s_j^- = x_{lj}t+\delta_j^- t,\ \text{ for } j=1,2,\ \ldots,m,\\
&\sum_{i=1}^{n} y_{ik}\lambda_i-s_k^+ = y_{lk}t-\delta_k^+ t,\ \text{ for } k=1,2,\ \ldots,p,\\
&\lambda_i\ge 0,\ \text{ for } i=1,2,\ \ldots,n,\\
&s_j^-\ge 0,\ \text{ for } j=1,2,\ \ldots,m,\\
&s_k^+\ge 0,\ \text{ for } k=1,2,\ \ldots,p,\\
&t>0.
\end{aligned}
\tag{12.6}
$$

The score of KAM represents that the efficiency score of firm l, that is, $\sum_{k=1}^{p} W_k^+ y_{lk}/\sum_{j=1}^{m} W_j^- x_{lj}$, is compared with the efficiency score of a point on the estimated production function, that is, $\sum_{k=1}^{p} W_k^+\left(y_{lk}-\delta_k^+ + s_k^+\right)/\sum_{j=1}^{m} W_j^-\left(x_{lj}+\delta_j^- - s_j^-\right)$, such that, the ratio of $\sum_{k=1}^{p} W_k^+ y_{lk}/\sum_{j=1}^{m} W_j^- x_{lj}$ to $\sum_{k=1}^{p} W_k^+\left(y_{lk}-\delta_k^+ + s_k^+\right)/\sum_{j=1}^{m} W_j^-\left(x_{lj}+\delta_j^- - s_j^-\right)$ becomes minimum. Since the efficiency of firm l is a constant value, KAM finds a point on the estimated production function which has an equal or greater efficiency score in comparison with that of firm l, regarding the value of delta and introduced W_j^- and W_k^+.

The target for firm l ($l = 1, 2, \ldots, n$) from Eq. 12.6, which lies on the estimated production function, and has a greater (or equal) ratio of the linear combination of output factors to the linear combination of input factors, regarding the value of delta and introduced W_j^- and W_k^+, is given by:

$$
\begin{aligned}
x_{lj}^* &= x_{lj}+\delta_j^- - s_j^{-*}/t^*,\ \text{ for } j=1,2,\ \ldots,m,\\
y_{lk}^* &= y_{lk}-\delta_k^+ + s_k^{+*}/t^*,\ \text{ for } k=1,2,\ \ldots,p,
\end{aligned}
\tag{12.7}
$$

The dual linear programming of Eq. 12.6 is also given by Eq. 12.8.

$$
\begin{aligned}
&\max\tau,\\
&\left(\sum_{k=1}^{p} V_k^+\left(y_{lk}-\delta_k^+\right)\right)\tau+\sum_{j=1}^{m}\left(x_{lj}+\delta_j^-\right)w_j^- - \sum_{\mathrm{k}=1}^{\mathrm{p}}\left(y_{lk}-\delta_k^+\right)w_k^+ = \sum_{j=1}^{m} V_j^-\left(x_{lj}+\delta_j^-\right),\\
&\sum_{k=1}^{p} y_{ik}w_k^+ - \sum_{j=1}^{m} x_{ij}w_j^- \le 0,\ \text{for } i=1,2,\ldots,n,\\
&w_j^- \ge V_j^- \ \text{for } j=1,2,\ldots,m,\\
&w_k^+ \ge \tau V_k^+ \ \text{for } k=1,2,\ldots,p.
\end{aligned}
\tag{12.8}
$$

Suppose that the component of delta vector is introduced by Eq. 12.2. Please note we usually use the notation δ-KAM instead of $\boldsymbol{\Delta}$-KAM in this book. When $\delta = 0$, the 0-KAM measures the technical efficiency of firms, and divides the firms into two categories similar to DEA models. If the score of 0-KAM is less than 1 for a firm, that firm is technically inefficient, and if the score of 0-KAM is 1, the firm is technically efficient. However, the scores of 0-KAM (similar to DEA models) should neither be used to rank firms, nor the proposed targets can be used to benchmark firms.

The δ-KAM is SBM (Eq. 5.26), where $\delta = 0$, $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$. When $\delta > 0$, and $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$, we express '*the δ-KAM with SBM approach*'. Note that, the SBM approach does not satisfy Theorem 4.1 in order to measure efficiency, but can be used to introduce the technically efficient firms. When $\delta > 0$, and $W_j^- = 1/\min\{x_{lj} : x_{lj} \neq 0\}$ and $W_k^+ = 1/\min\{y_{lk} : y_{lk} \neq 0\}$, we state '*the δ-KAM with minimum approach*'.

Suppose that W_j^- and W_k^+ are given as the available costs of input and output factors, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. We can decrease the discrimination power of δ-KAM, as Eqs. 12.9, 12.10 represent, to illustrate the lack of CF and RF (or PF) measurements, respectively, (see also Exercises 6.10–6.15).

$$\begin{aligned}
&\min \textstyle\sum_{j=1}^{m} V_j^-(x_{lj} + \delta_j^- - s_j^-),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta_j^-, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.9}$$

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} V_k^+(y_{lk} - \delta_k^+ + s_k^+),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta_k^+, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.10}$$

In other words, none of Eqs. 12.9 and 12.10 provide a fair measure to discrimination the efficiency of firms. A suitable model should at least satisfy the introduced Types 1–6. For instance, Eq. 12.9 does not satisfy Type 4, and Eq. 12.10 does not satisfy Type 3.

As explained in the previous chapters, a firm, F_l, partially dominates another firm, $F_{l'}$, if and only if, the value of Eq. 12.11 for F_l is greater than that of $F_{l'}$, where l and l' belongs to $\{1, 2, \ldots, n\}$.

$$\frac{\sum_{k=1}^{p} W_k^+ y_k}{\sum_{j=1}^{m} W_j^- x_j} = \frac{W_k^+ y_1 + W_k^+ y_2 + \ldots + W_k^+ y_p}{W_j^- x_1 + W_j^- x_2 + \ldots + W_j^- x_m}. \tag{12.11}$$

The six introduced types which increase the value of Eq. 12.11 are expressed by:

Type 1: Decreasing the value of denominator in Eq. 12.11 when the value of numerator is fixed or increased.

Type 2: Increasing the value of numerator in Eq. 12.11 when the value of denominator is fixed or decreased.

Type 3: If the rate of increasing the value of numerator in Eq. 12.11 is greater than the rate of increasing the value of denominator.

Type 4: If the rate of decreasing the value of numerator in Eq. 12.11 is greater than the rate of decreasing the value of denominator.

Type 5: The value of denominator in Eq. 12.11 is decreased (increased) by: (1) decreasing (increasing) the value of one or more of input factors, or (2) increasing (decreasing) a small value of one or more of input factors and decreasing (increasing) a large value of one or more of other input factors.

Type 6: The value of numerator in Eq. 12.11 is increased (decreased) by: (1) increasing (decreasing) the value of one or more of output factors, or (2) decreasing (increasing) a small value of one or more of output factors and increasing (decreasing) a large value of one or more of other output factors.

In short, Eq. 12.5 shows that KAM compares the efficiency of a firm with the efficiency of the points on the estimated production function, and finds the best target for the firm, regarding the value of delta and introduced weights. The discrimination power of KAM is greater than CF, RF and PF models, and KAM provides a fair measure to assess the inefficiencies of the firms. In the next sections, KAM is improved and the optimum of delta is also measured.

12.3 KAM and Uncontrollable Factors

In the airport example in Chap. 3, the runway is an input factor which has the standard area, according to the documents of International Civil Aviation Organization (ICAO). The area of runway should not be less than the standard value, and depends on the aircrafts which want to land and take off from that runway. Because of the safety of passengers, decreasing the length/area of a runway is not suggested. This kind of factor is called *non-controllable* or *uncontrollable* factor (Banker and Morey 1986; Charnes et al. 1987; Cooper et al. 2007). In other words, an uncontrollable factor may not be controlled by managers, although, it may affect the performance of firms.

For instance, suppose that j^{th} input factor ($j = 1, 2, \ldots, m$) is an uncontrollable factor and the efficiency of F_l is measured ($l = 1, 2, \ldots, n$). If this uncontrollable factor should not be decreased and increased, the corresponded linear combination of this j^{th} input factor of firms in Eq. 12.2, (that is, $\sum_{i=1}^{n} x_{ij}\lambda_i$) should be equal to the corresponded value of j^{th} factor of firm l (that is, x_{lj}), that is, $\sum_{i=1}^{n} x_{ij}\lambda_i = x_{lj}$, (where $l = 1, 2, \ldots, n$). Nonetheless, it is valuable to examine a delta neighborhood of an uncontrollable factor, in order to measure the effect of such restriction on other factors. For such an aim, the constraint $s_j^- \leq \delta_j^-$ (or $s_j^- \leq t\delta_j^-$) can be added to the constraints in Eq. 12.5 (Eq. 12.6).

The term δ_j^- is a value which is added to j^{th} input factor of F_l, to introduce a neighborhood of this factor, thus the linear combination of this uncontrollable factor of firms is at least equal to x_{lj}, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i = x_{lj} + \delta_j^- - s_j^- \geq x_{lj}$. Therefore, the optimal value of j^{th} input factor is not decreased, but it may slightly be increased according to value of δ_j^-.

For the case that the optimal value of j^{th} input factor should not also be increased, the value of delta can be considered very small, such that, the value of δ_j^- is quite negligible. For instance, if j^{th} input factor is measured with three decimal digits, the negligible error can be less than 0.005. Similar discussion can be illustrated for an uncontrollable output factor as well.

Now, suppose that J_u is a subset of input factor indexes, $\{1, 2, \ldots, m\}$, corresponded to the uncontrollable input factors, and K_u is a subset of output factor indexes, $\{1, 2, \ldots, p\}$, corresponded to the uncontrollable output factors. The δ-KAM is given by Eq. 12.12.

$$
\begin{aligned}
&\min \left[\sum_{j=1}^{m} V_j^-\left(tx_{lj} + t\delta_j^- - s_j^-\right)\right],\\
&\text{Subject to}\\
&\left[\sum_{k=1}^{p} V_k^+\left(y_{lk}t - \delta_k^+ t + s_k^+\right)\right] = 1,\\
&\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}t + \delta_j^- t, \ \text{ for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}t - \delta_k^+ t, \ \text{ for } k = 1, 2, \ldots, p,\\
&s_j^- \leq \delta_j^- t \ \text{ for } j \in J_u,\\
&s_k^+ \geq \delta_k^+ t \ \text{ for } k \in K_u,\\
&s_j^- \geq 0 \ \text{ for } j = 1, 2, \ldots, m,\\
&s_k^+ \geq 0 \ \text{ for } k = 1, 2, \ldots, p,\\
&t > 0.
\end{aligned}
\tag{12.12}
$$

For an example of uncontrollable factor, suppose that fourth input factor (runway) is uncontrollable in the airport example in Chap. 5. Assume that the same conditions to solve Eq. 5.31 in Chap. 5 are considered, that is, $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, $W_k^+ = 1/y_{lk}$, for $k = 1, 2, 3$, where δ is 0, 0.0001, 0.01 and 0.1, and the component of delta vectors are introduced by Eq. 12.3. Since the weights are introduced by the inverse of data similar to SBM, the approach is the SBM approach and the results are only used to express the methodology.

When $\delta = 0$, in Eq. 12.12, the corresponded constraints for fourth input factor of firm F_l, (that is, $s_4^- \leq \delta x_{l4}t$, and $\sum_{i=1}^{8} x_{i4}\lambda_i + s_4^- = (1+\delta)x_{l4}t$) are equal to zero, that

Table 12.1 The δ-KAM scores where $s_4^- \leq t\delta x_{14}$

Delta	A	B	C	D	E	F	G	H
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.0001	0.99983	0.99997	0.99924	0.98588	0.99258	0.99993	0.99990	0.99999
0.01	0.98307	0.99747	0.92921	0.63590	0.54534	0.99274	0.98991	0.99857
0.1	0.85672	0.97644	0.55163	0.50990	0.36364	0.93414	0.91129	0.98565

is, $s_4^- = \delta x_{l4}t = 0$, and $\sum_{i=1}^{8} x_{i4}\lambda_i = x_{l4}t$. As Table 12.1 illustrates, the 0-KAM by SBM approach represents that all airports A-H are technically efficient. In other words, this simple restriction on fourth factor of airports yields that airports C and E become technically efficient.

As the value of delta is increased, (for instance, when δ is 0.0001, 0.01 and 0.1), δ-KAM discriminates the airports according to the introduced value of delta and the SBM approach.

If the uncontrollable factor cannot be decreased and increased, the constraint $s_4^- \leq \delta x_{l4}t$ can be replaced by $s_4^- = \delta x_{l4}t$. The results of δ-KAM for this assumption are illustrated in Table 12.1, which can be compared with the results in Table 6.10 as well. In this case, the value of fourth factor for airport number $l$is not changed. Indeed, the targets for firm l ($l = 1, 2, \ldots, n$) can be measured from Eq. 12.7. Since $s_4^- = \delta_j^- t = \delta x_{l4}t$, for instance, when $\delta = 0.0001$, Table 12.2 represents that the suggested target for fourth input factor of airport number l is not changed, ($l = 1, 2, \ldots, 8$).

The suggested targets in Table 12.2 (or the targets by Eq. 12.5) lie on the frontier of feasible area, and have better performance in comparison with the real data in Table 5.1, regarding the assumptions of discrimination. In other words, 0.0001-KAM not only discriminates the airports according to the introduced errors, but it also benchmarks all technically efficient airports, and suggests how they can regulate their factors according to Types 1–6, in order to improve their performances, (regarding the introduced assumptions). For instance, 0.0001-KAM says that A should decrease the area of airport and increase the numbers of flights and passengers, even if the area of apron and terminal are increased, or the amount of cargo decreased, according to the value of delta and the SBM approach.

Now, suppose that $s_4^- \leq \delta x_{l4}t$ is only added to Eq. 5.31, the corresponded results to Tables 12.1 and 6.10 are displayed in Table 12.3. The constraint, $s_4^- \leq \delta x_{l4}t$, does not let the value of fourth factor decrease, but it may be increased in order to find a better situation on the production frontier, according to the SBM approach.

Table 12.4 represents the targets of KAM when $\delta = 0.0001$. According to the table, A should seriously improve the numbers of flights and passengers, even if the input factors are increased and the value of cargo is decreased. In other words, the inefficiency of A is due to the small numbers of flights and passengers, according to the data of other airports. All of these assessments are approximations and do not mean that it is impossible to increase the number of flights of airport A without increasing the values of its input factors, but at the same time, express the way of increasing the efficiency of A.

Table 12.2 The 10^{-4}-KAM targets where $s_4^- \leq t\delta x_{14}$

Airport	Area	Apron	Terminal	Runway	Flights	Passengers	Cargo
A	1187.28	325,581.90	48,948.45	353,610.00	44,770.19	5,080,228.95	66,765.60
B	553.30	230,417.98	40,786.56	348,120.00	51,816.17	5,187,091.37	20,175.81
C	418.46	45,103.30	12,980.00	269,955.00	23,914.76	1,258,627.24	3681.02
D	778.22	123,710.40	23,155.00	395,730.00	35,883.90	2,441589.64	17,250.27
E	297.08	33,000.00	8800.00	192,330.00	16,583.00	865,955.98	2825.11
F	497.53	69300.00	23,630.14	389,115.00	40,918.80	2,218,202.93	7181.15
G	486.49	51,931.00	10,230.00	268,995.00	19,494.13	1,047,313.36	5212.20
H	1357.68	483,620.85	73,294.86	421,305.00	116,237.70	10,746,024.81	46,368.88

Table 12.3 The δ-KAM scores where $s_4^- \leq t\delta x_{14}$

Delta	A	B	C	D	E	F	G	H
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.0001	0.99980	0.99997	0.99924	0.98468	0.99258	0.99993	0.99990	0.99997
0.01	0.98056	0.99747	0.92921	0.63590	0.54534	0.99274	0.98991	0.99739
0.1	0.84096	0.97644	0.55163	0.50990	0.36364	0.93414	0.91129	0.97558

The same illustration can be discussed for other airports as well. In addition, instead of SBM approach, the average measurement approach can also be used, that is, $W_j^- = 1/\text{ave}\{x_{ij} : i = 1, 2, \ldots, 8\}$, for j=1, 2, 3, 4, $W_k^- = 1/\text{ave}\{y_{ik} : i = 1, 2, \ldots, 8\}$, for k = 1, 2, 3, and so on for any other interested approaches introduced by expert judgment.

12.4 KAM and the Production Tape

In Chaps. 1 and 2, several approaches are proposed to introduce practical points from a set of homogenous firms. The largest feasible area, which is linearly generated, is called the CRS-PPS or T_C which is introduced by Eq. 7.4. Now, from the outcomes of the δ-KAM, the feasible area can be extended corresponded to the value of delta. In other words, when the practical points are generated by the wholly dominant, the convexity and the radiate approaches, a delta neighborhood of the feasible area can be practical as well. The three introduced approaches may not be suitable to apply for the points in the delta neighborhood of the feasible area, but an exact data can rarely be measured in real life applications, and even if data are exact, this technique is useful to recuperate the estimated production function from linear approaches. Indeed, it is hard to prove that a production function does not have any curves, as can be seen in Fig. 12.1.

Assume that the blue curve in Fig. 12.1 represents the exact production function for firms A-F. The CRS technology suggests the frontier of T_C, which is generated by applying the radiate and the convexity approaches for the observed data at the first step, and at the second step by applying the wholly dominant approach, as proved by Theorems 2.4–2.7.

After introducing the frontier of T_C, a delta neighborhood can linearly be added to T_C, by shifting the frontier toward the directions which improve the factors, regarding the value of delta. This extension of T_C is also the smallest set to recuperate T_C, regarding the value of delta.

Suppose that a T_C is given. There is always a δ>0 to introduce an extension of T_C, as Eq. 12.13 represents.

The delta in Eq. 12.13 is a real value, and can also be introduced as a vector with $m + p$ components, such as, $\boldsymbol{\Delta} = \left(\delta_1^-, \delta_2^-, \ldots, \delta_m^-, \delta_1^+, \delta_2^+, \ldots, \delta_p^+\right) \in \mathbb{R}_+^{m+p}$, to introduce $T_C^{+\Delta}$. For instance, the components of delta vector in δ-KAM by the SBM

Table 12.4 The 10^{-4}-KAM targets where $s_4^- \leq t\delta x_{14}$

Airport	Area	Apron	Terminal	Runway	Flights	Passengers	Cargo
A	1200.00	304,211.34	45,604.56	353,615.42	30,723.69	4,032,130.88	74,176.58
B	503.05	213,745.69	38,780.01	348,120.00	46,879.94	4,783,523.97	19,051.13
C	799.62	41,007.10	11,801.18	269,955.00	15,616.31	1,040,185.66	1589.09
D	1031.88	112,475.25	21,052.11	395,769.57	39,867.01	1,769,654.15	5046.57
E	997.61	30,003.00	8000.80	192,330.00	4949.97	430,744.19	1573.84
F	478.02	63,006.30	23,000.63	389,115.00	41,087.83	2,165,624.63	5415.77
G	481.01	47,214.72	9300.93	268,995.00	19,010.48	971,389.00	3827.39
H	1346.13	503,280.12	76,370.76	421,340.40	129,140.08	11,708,951.44	39,571.86

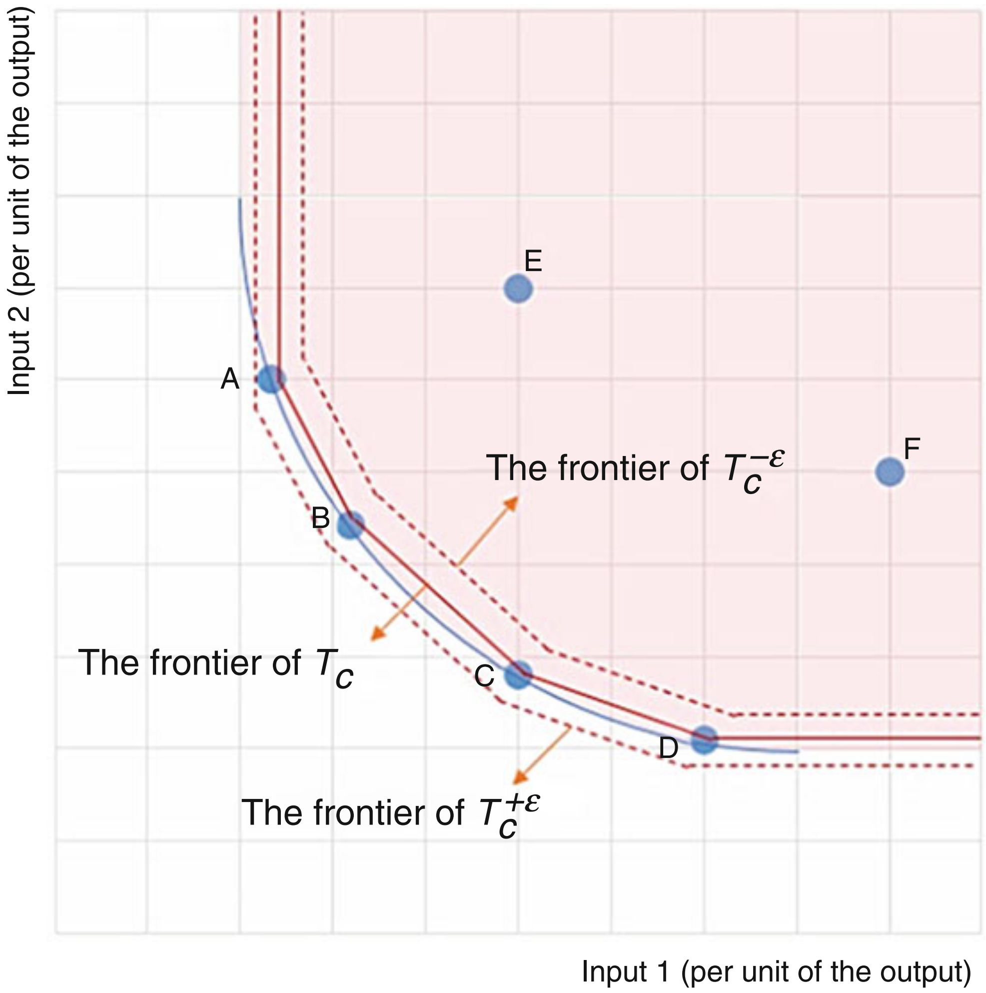

Fig. 12.1 The production tape

approach when firm F_l is evaluated are $\delta_{lj}^{-} = \delta \times x_{lj}$, for $j = 1, 2, \ldots, m$, and $\delta_{lk}^{+} = \delta \times y_{ik}$, for $k = 1, 2, \ldots, p$.

$$T_C^{+\delta} = \left\{ (X', Y') \in \mathbb{R}^{m+p} : \ \Lambda \cdot X^j - \delta \leq x_j', \ \Lambda \cdot Y^k + \delta \geq y_k', \ \lambda_i \geq 0, \right. \\ \left. \text{for } i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k = 1, 2, \ldots, p \right\}. \tag{12.13}$$

Similarly, the generated feasible area by CRS technology can be contracted regarding the value of delta, as Eq. 12.14 represents.

$$T_C^{-\delta} = \left\{ (X', Y') \in \mathbb{R}^{m+p} : \Lambda \cdot X^j + \delta \leq x_j', \ \Lambda \cdot Y^k - \delta \geq y_k', \ \lambda_i \geq 0, \right. \\ \left. \text{for } i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k = 1, 2, \ldots, p \right\}. \tag{12.14}$$

Then, the *production tape* can be defined by $T_C^{+\delta} - T_C^{-\delta} = TT_C^{\delta}$. When the targets of δ-KAM are measured by Eq. 12.7, the targets are on the frontier of $T_C^0 = T_C$, then the targets can be transferred to the frontier of $T_C^{+\delta}$ by Eq. 12.15.

$$\begin{aligned} x_{lj}^* &= x_{lj} - s_j^{-*}/t^*, \quad \text{for } j = 1, 2, \ldots, m, \\ y_{lk}^* &= y_{lk} + s_k^{+*}/t^*, \quad \text{for } k = 1, 2, \ldots, p. \end{aligned} \tag{12.15}$$

It is possible that the values of input factors for the suggested targets in Eq. 12.15 become negative. Thus, the constraints $tx_{lj} - s_j^- \geq 0$, for $j = 1, 2, \ldots, m$, are added to Eq. 12.6 to guarantee positive values for the suggested targets of input factors. We may also look for the minimum value of $\sum_{j=1}^{4} V_j^- \left(x_{lj} - s_j^- \right)$ over $\sum_{k=1}^{3} V_k^+ \left(y_{lk} + s_k^+ \right)$ subject to the constraints of Eq. 12.5, and the constraints $x_{lj} - s_j^- \geq 0$, for $j = 1, 2, \ldots, m$, for a given $\delta > 0$.

The same illustration can be discussed for the targets which are transferred to the frontier of $T_C^{-\delta}$, as Eq. 12.16 represents.

$$\begin{aligned} x_{lj}^* &= x_{lj} + 2\delta_j^- - s_j^{-*}/t^* \quad \text{for } j = 1, 2, \ldots, m, \\ y_{lk}^* &= y_{lk} - 2\delta_k^+ + s_k^{+*}/t^* \quad \text{for } k = 1, 2, \ldots, p. \end{aligned} \tag{12.16}$$

Since the values of output factors for the suggested targets in Eq. 12.16 can be negative, the constraints $ty_{lk} - 2t\delta_k^+ + s_k^+ \geq 0$, for $k = 1, 2, \ldots, p$, should be added to Eq. 12.6 to guarantee positive values for the suggested targets of output factors. As a result, in order to introduce the production tape, Eq. 12.17 should be solved for a given $\delta > 0$.

$$\begin{aligned}
&\min \left[\sum_{j=1}^{m} V_j^- \left(x_{lj}t + \delta_j^- t - s_j^- \right) \right], \\
&\left[\sum_{k=1}^{p} V_k^+ \left(y_{lk}t - \delta_k^+ t + s_k^+ \right) \right] = 1, \\
&\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}t + \delta_j^- t, \quad \text{for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}t - \delta_k^+ t, \quad \text{for } k = 1, 2, \ldots, p, \\
&x_{lj}t - s_{lj}^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m, \\
&y_{lk}t - 2\delta_{lk}^+ t + s_{lk}^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n, \\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m, \\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p, \\
&t > 0.
\end{aligned} \tag{12.17}$$

In order to introduce a production tape, the delta value should be small. We can also measure the optimum value of δ such that at least two corresponded components to the input (output) factors of the $T_C^{+\delta}$'s frontier ($T_C^{-\delta}$'s frontier) become positive. (How?)

12.5 An Improvement of KAM

Firms may have a request that the values of a factor should be in an introduced range or interval, for instance, due to standardization or the intention of firms. Sometimes a manager plans (or is asked) to decrease (increase) a certain amount of an input (output) factor in a period of time. A manager may only able to improve or control a special amount of a factor. All these situations can affect the efficiency (productivity) measurement and assessing the performance of a set of homogenous firms. Indeed, the provided relative scores to discriminate, to rank and to benchmark firms should satisfy such restrictions and desired requests as well. Thus, KAM is improved to handle such purposes.

Suppose that the $\boldsymbol{\Delta}$-KAM by Eq. 12.5 is given. Thus, the $\boldsymbol{\Delta}$-KAM targets for firm number l are $x_{lj} + \delta_{lj}^{-} - s_{lj}^{-}$ and $y_{lk} - \delta_{lk}^{+} + s_{lk}^{+}$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$. Assume that the target of j^{th} input factor should belong to $[a_j, b_j]$ and the target of k^{th} output factor should belong to $[c_k, d_k]$, where a_j, b_j, c_k and d_k are non-negative real values, $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. Equation 12.18 illustrates the constraints which should be added to the constraints of Eq. 12.5 to satisfy the purpose of discrimination.

$$\begin{aligned} & a_j \le x_{lj} + \delta_{lj}^{-} - s_{lj}^{-}, \ \text{ for } j = 1, 2, \ldots, m, \\ & x_{lj} + \delta_{lj}^{-} - s_{lj}^{-} \le b_j, \ \text{ for } j = 1, 2, \ldots, m, \\ & c_k \le y_{lk} - \delta_{lk}^{+} + s_{lk}^{+}, \ \text{ for } k = 1, 2, \ldots, p, \\ & y_{lk} - \delta_{lk}^{+} + s_{lk}^{+} \le d_k, \ \text{ for } k = 1, 2, \ldots, p. \end{aligned} \tag{12.18}$$

The values of a_j, b_j, c_k and d_k should reasonably be selected in order to have a feasible area, regarding the components of delta vector. These variables should also be commensurate with the units of corresponded factors. In order to decrease the number of introduced variables by expert judgment and make sure there is always a feasible area as well as simplifying the model and improving the applicability of the model, let's assume that the components of delta vector are exchangeable and at least one of the firms satisfies the introduced ranges, $[a_j, b_j]$ and $[c_k, d_k]$, for $j = 1, 2\ldots$, m and $k = 1, 2, \ldots, p$. Thus, the optimal values of the delta components can also be measured and Eq. 12.4 can be extended as Eq. 12.19.

$$
\begin{aligned}
& KA^*_{\Delta_l} = \min \frac{\sum_{j=1}^{m} V_j^-(x_{lj} + \delta_{lj}^- - s_{lj}^-)}{\sum_{k=1}^{p} V_k^+(y_{lk} - \delta_{lk}^+ + s_{lk}^+)}, \\
& \text{Subject to} \\
& \textstyle\sum_{i=1}^{n} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj} + \delta_{lj}^-, \ \text{for } j = 1, 2, \ldots, m, \\
& \textstyle\sum_{i=1}^{n} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk} - \delta_{lk}^+, \ \text{for } k = 1, 2, \ldots, p, \\
& a_j \leq x_{lj} + \delta_{lj}^- - s_{lj}^-, \ \text{for } j = 1, 2, \ldots, m, \\
& x_{lj} + \delta_{lj}^- - s_{lj}^- \leq b_j, \ \text{for } j = 1, 2, \ldots, m, \\
& c_k \leq y_{lk} - \delta_{lk}^+ + s_{lk}^+, \ \text{for } k = 1, 2, \ldots, p, \\
& y_{lk} - \delta_{lk}^+ + s_{lk}^+ \leq d_k, \ \text{for } k = 1, 2, \ldots, p, \\
& \lambda_{li} \geq 0, \ \text{for } i = 1, 2, \ldots, n, \\
& s_{lj}^- \geq 0, \ \text{for } j = 1, 2, \ldots, m, \\
& \delta_{lj}^- \geq 0, \ \text{for } j = 1, 2, \ldots, m, \\
& s_{lk}^+ \geq 0, \ \text{for } k = 1, 2, \ldots, p, \\
& \delta_{lk}^+ \geq 0 \ \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.19}
$$

If there are no restrictions for the target values, the variables a_j, b_j, c_k and d_k can be introduced as $a_j = 0$ and $b_j = M_j$, for $j = 1, 2, \ldots, m$, and $c_k = 0$ and $d_k = N_k$, for $k = 1, 2, \ldots, p$. Here the meaning of M_j and N_k are the suitable large numbers according to data or the used software. The variable can also be different from one firm to another as long as the homogeneity of firms is satisfied.

The optimal values for the components of delta vector may be different from one firm to another; however, the provided scores by Eq. 12.19 for F_i's ($i = 1, 2\ldots, n$) are relatively meaningful, regarding to a vector of delta that its corresponded components are the maximum values of the components of the measured optimal delta vectors of F_i's ($i = 1, 2\ldots, n$).

The components of delta vector can also be introduced by Eq. 12.2. In this case, the δ^*-KAM is given by Eq. 12.20, where $\delta^* = \max\left\{\delta_l^* : l = 1, 2, \ldots, n\right\}$.

$$KA^*_{\delta^*_l} = \min \frac{\sum_{j=1}^m W_j^-(x_{lj} + \delta_l/W_j^- - s_{lj}^-)/\sum_{j=1}^m W_j^- x_{lj}}{\sum_{k=1}^p W_k^+(y_{lk} - \delta_l/W_k^+ + s_{lk}^+)/\sum_{k=1}^p W_k^+ y_{lk}},$$

$$\begin{aligned}
&\text{Subject to}\\
&\textstyle\sum_{i=1}^n x_{ij}\lambda_{li} + s_{lj}^- = x_{lj} + \delta_l/W_j^-, \text{ for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^n y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk} - \delta_l/W_k^+, \text{ for } k = 1, 2, \ldots, p,\\
&x_{lj} + \delta_l/W_j^- - s_{lj}^- \le b_j, \text{ for } j = 1, 2, \ldots, m,\\
&a_j \le x_{lj} + \delta_l/W_j^- - s_{lj}^-, \text{ for } j = 1, 2, \ldots, m,\\
&c_k \le y_{lk} - \delta_l/W_k^+ + s_{lk}^+, \text{ for } k = 1, 2, \ldots, p,\\
&\lambda_{li} \ge 0, \text{ for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \ge 0, \text{ for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \ge 0, \text{ for } k = 1, 2, \ldots, p,\\
&\delta_l \ge 0.
\end{aligned} \tag{12.20}$$

The score of δ-KAM is non-increasing, that is, $KA^*_{\delta_1} \ge KA^*_{\delta_2}$ if $\delta_1 \le \delta_2$. Moreover, $KA^*_{\delta^*_l} = KA^*_{\delta}$, for $\delta \ge \delta^*_l$. The δ^*-KAM scores are relatively meaningful and depend on the introduced W_j^-, for $j = 1, 2\ldots, m$, and W_k^+, for $k = 1, 2\ldots, p$.

12.5.1 Solving KAM for a Request

Suppose that managers in the airport example can only regulate at about 10% of the factors' values, where $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. In other words, assume that $a_j = x_{lj} - 10\ \%\ x_{lj}$ and $b_j = x_{lj} + 10\ \%\ x_{lj}$, for $j = 1, 2, 3, 4$, and $c_k = y_{lk} - 10\ \%\ y_{lk}$ and $d_k = y_{lk} + 10\ \%\ y_{lk}$, for $k = 1, 2, 3$.

From such assumption, the feasible area for the airports may vary from one airport to another. Nonetheless, KAM compares the efficiency scores of the points in the estimated feasible area with the efficiency score of the evaluated airport, and finds the best target for that airport, regarding the assumptions of the example.

On the other hand, Eq. 12.20 can linearly be solved by Eq. 12.21.

$$
\begin{aligned}
& KA^*_{\delta_l^*} = \min \textstyle\sum_{j=1}^m W_j^-(x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-)/\sum_{j=1}^m W_j^- x_{lj}, \\
& \text{Subject to} \\
& \textstyle\sum_{i=1}^n x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-,\ \text{for } j = 1, 2, \ldots, m, \\
& \textstyle\sum_{i=1}^n y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l - \delta_l/W_k^+,\ \text{for } k = 1, 2, \ldots, p, \\
& a_j t_l \le x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-,\ \text{for } j = 1, 2, \ldots, m, \\
& x_{lj}t_l + \delta_l/W_j^- - s_{lj}^- \le b_j t_l,\ \text{for } j = 1, 2, \ldots, m, \\
& c_k t_l \le y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+,\ \text{for } k = 1, 2, \ldots, p, \\
& y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+ \le d_k t_l,\ \text{for } k = 1, 2, \ldots, p, \\
& \lambda_{li} \ge 0,\ \text{for } i = 1, 2, \ldots, n, \\
& s_{lj}^- \ge 0,\ \text{for } j = 1, 2, \ldots, m, \\
& s_{lk}^+ \ge 0,\ \text{for } k = 1, 2, \ldots, p, \\
& \delta_l \ge 0, \\
& t_l \ge 0.
\end{aligned}
\tag{12.21}
$$

The following instructions illustrate how to solve Eq. 12.21 for the airports in Table 5.1, by the Microsoft Excel Solver 2013 software.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.1 depicts.
2. Label B11 as 'Index1', E11 as 'Range', H11 as 'Delta', B13 as 'Ws', B15 as 'Constraint', E15 as 't', H15 as 'Objective', B17 as 'Left side', B18 as 'Slacks', B19 as 'Right side', B20 as 'Deltas', B22 as 'Targets', B24 as 'Lower bounds', B25 as 'Upper bounds', J1 as 'Lambdas', K1 as 'Solver Code', L1 as 'KAM score', and M1 as 'Delta'.
3. Assign number 1 to C11, and 0.1 to F11.
4. Assign number 10 to C13, 1 to D13, 100 to E13, 10 to F13, 100 to G13, 1 to H13, and 10 to I13.
5. Assign the following command to C15

   '=Sumproduct(G13:I13,G19:I19 + G18:I18)/Sumproduct(G13:I13,Index(G2:I9,C11,0))'.

6. Assign the following command to I15

   '=Sumproduct(C13:F13,C19:F19-C18:F18)/Sumproduct(C13:F13,Index (C2:F9,C11,0))'.

7. Assign the command '=Sumproduct(C2:C9,$J2:$J9) + C18' to C17. Then, copy C17 and paste it to D17, E17 and F17.
8. Assign the command '=Sumproduct(G2:G9,$J2:$J9)-G18' to G17. Then, copy G17 and paste it to H17 and I17.
9. Assign '=$F15*Index(C2:C9,$C11) + C20' to C19. Then, copy C19 and paste it to D19, E19 and F19.

10. Assign ‘=$F15*Index(G2:G9,$C11)-G20’ to G19. Then, copy G19 and paste it to H19 and I19.
11. Assign ‘=$I11/C13’ to C20. Then copy C20 and paste it to D20-I20.
12. Assign ‘=C19-C18’ to C22. Then copy C22 and paste it to D22-F22.
13. Assign ‘=G19 + G18’ to G22. Then copy G22 and paste it to H22 and I22.
14. Assign ‘=(1-$F11)*Index(C2:C9,$C11)*$F15’ to C24. Then copy C24 and paste it to D24-I24.
15. Assign ‘=(1 + $F11)*Index(C2:C9,$C11)*$F15’ to C25. Then copy C25 and paste it to D25-I25.
16. Open ‘Solver Parameters’ window from ‘DATA’ in Excel toolbar.
17. Assign ‘I15’ into ‘Set Objective’ and choose ‘Min’.
18. Assign ‘J2:J9, C18:I18, F15, I11’, into ‘By Changing Variable Cells’.
19. Click on ‘Add’ and assign ‘C15’ into ‘Cell Reference’, then select ‘=’, and assign ‘1’ into ‘Constraint’.
20. Click on ‘Add’ and assign ‘C17:I17’ into ‘Cell Reference’, then select ‘=’, and assign ‘C19:I19’ into ‘Constraint’.
21. Click on ‘Add’ and assign ‘C22:I22’ into ‘Cell Reference’, then select ‘>=’, and assign ‘C24:I24’ into ‘Constraint’.
22. Click on ‘Add’ and assign ‘C22:I22’ into ‘Cell Reference’, then select ‘<=’, and assign ‘C25:I25’ into ‘Constraint’. Then click on ‘OK’.
23. Tick ‘Make Unconstrained Variables Non-Negative’.
24. Choose ‘Simplex LP’ from ‘Select a Solving Method’ and then ‘Solve’.
25. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window.
26. Click on the first icon, ‘Button (Form Control)’, and then click on a place in the Excel sheet.
27. In the open window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
28. From the toolbar menu, click on ‘Tools> References...>’ and make sure ‘Solver’ is ticked, and then ‘OK’.
29. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’.

```
Dim i As Integer
For i = 1 To 8
   Range("C11") = i
   Range("K" & i + 1) = SolverSolve(Userfinish:=True)
   Range("L" & i + 1) = Range("I15")
   Range("M" & i + 1) = Range("I11")/ Range("F15")
Next i
Range("M10") = WorksheetFunction.Max(Range("M2:M9"))
```

30. Close the ‘Microsoft Visual Basic for Applications’ window.
31. Click on the small rectangle which was automatically made in the Excel sheet in Step 27.

Table 12.5 Returns values by Microsoft Excel Solver 2013

Solver	Description
0	Constraints and optimality conditions are satisfied and Solver has found a solution.
1	Constraints are satisfied, but Solver has converged to the current solution.
2	Constraints are satisfied, but Solver cannot improve the current solution.
3	Solver is stopped chosen because of the maximum repetition's limit was reached.
4	The set cell values do not converge.
5	Solver could not find a feasible solution.
6	Solver is stopped at user's request.
7	The required linearity conditions for Solver are not satisfied.
8	The problem is too large and Solver cannot handle it.
9	Solver encountered an error value in a target or a constraint cell.
10	Solver is stopped chosen because of the maximum times' limit has been reached.
11	There is not enough memory available to solve the problem.

Table 12.6 The outcomes of Eq. 12.21 for the introduced assumptions

N	Airport	Solver Code	KAM Score	Delta values
1	A	0	0.9103466	74184.00000
2	B	0	0.8839073	74257.83920
3	C	0	0.9294655	60481.71351
4	D	0	0.9323123	149631.38643
5	E	0	0.8416030	605.56693
6	F	0	0.9417391	165675.79624
7	G	0	0.8920495	93000.00000
8	H	0	1.0000000	0.00000

32. The corresponded Solver code, KAM score, and Delta for each airport are represented into cells K2:M9, respectively, and the maximum value of delta is measured in M10.

The Solver code illustrates whether the Microsoft Excel Solver 2013 software was able to find an optimal solution for Eq. 12.21, as Table 12.5 displays. When the Solver code is 0, it means that the Solver finds an optimal solution which optimally satisfies all the constraints and conditions.

Table 12.6 also expresses the scores of KAM according to the measured delta and the assumptions of this example, as well as the Solver codes which are 0 for all airports and display that Solver is found the optimal solution for the objective when all the constraints are optimally satisfied.

The corresponded targets to the results in Table 12.6 are also illustrated in Table 12.7.

When the prices for the factors are $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, and managers can only regulate at about 10% of the factors' values, KAM robustly measures the inefficiency of each airport according to Table 12.6 and suggests how the efficiency of each airport can be improved, according to the results in Table 12.7.

Table 12.7 The targets of airports from Eq. 12.21

Airport	Area	Apron	Terminal	Runway	Flights	Passengers	Cargo
A	1118.45	292,337.26	43,874.67	330,923.65	33,777.70	4,148,385.19	66,765.60
B	553.30	201,919.60	36,177.97	313,308.00	51,562.50	4,708,862.16	17,145.00
C	720.00	43,476.53	11,253.40	247,903.15	16,222.70	979,485.29	1745.70
D	936.90	111,542.31	20,509.11	356,157.00	38,374.69	1,856,032.66	5410.90
E	901.80	30,605.57	7713.14	174,652.28	5375.70	470,771.40	1702.56
F	454.19	69300.00	22,190.56	350,203.50	39,431.24	2,220,238.62	5848.43
G	462.64	51,931.00	10,230.00	242,095.50	20,037.49	1,068,444.30	3824.83
H	1346.00	503,274.00	76,370.00	421,305.00	129,153.00	11,709,741.00	39,556.00

Table 12.8 The outcomes of Eq. 12.21 by SBM approach

N	Airport	Solver Code	KAM Score	Delta values
1	A	0	0.9386553	0.10000
2	B	0	0.9523211	0.10000
3	C	0	0.9101377	0.00000
4	D	0	0.8880952	0.09307
5	E	0	0.8624157	0.01806
6	F	0	0.9536431	0.10000
7	G	0	0.9113718	0.10000
8	H	0	0.9891776	0.10000

Instead of the introduced prices for each factor, for instance, the SBM approach can be applied. Table 12.8 illustrates the results of KAM when SBM approach is applied.

12.5.2 A Specified Request for KAM

In the previous example, every airport has different restrictions. Now, suppose that the managers of airports want to regulate the factors of airports according to the following conditions: (1) the area of airport should not be more than 1200 Hectares, (2) the number of passengers should be greater than 10,000,000 people, and (3) the amount of cargo should at least be 30,000 metric tons. Assume that $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. Is there any feasible area for such purposes? What are the inefficiencies of the airport according to these conditions?

In order to answer these questions, Eq. 12.22 should be solved.

$$
\begin{aligned}
&KA^*_{\delta_l^*} = \min\left[\sum\nolimits_{j=1}^{4} W_j^-\left(x_{lj}t_l + \delta_l/W_{lj}^- - s_{lj}^-\right)\right] / \sum\nolimits_{j=1}^{4} W_j^- x_{lj},\\
&\text{Subject to}\\
&\left[\sum\nolimits_{k=1}^{3} W_k^+\left(y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+\right)\right] / \sum\nolimits_{k=1}^{3} W_k^+ y_{lk} = 1,\\
&\sum\nolimits_{i=1}^{8} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-, \ \text{ for } j = 1,2,3,4,\\
&\sum\nolimits_{i=1}^{8} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l + \delta_l/W_k^+, \ \text{ for } k = 1,2,3,\\
&x_{l1}t_l + \delta_l/W_1^- - s_{l1}^- \le 10000t_l,\\
&10000000t_l \le y_{l2}t_l - \delta_l/W_2^+ + s_{l2}^+,\\
&50000t_l \le y_{l3}t_l - \delta_l/W_3^+ + s_{l3}^+,\\
&\lambda_{li} \ge 0, \ \text{ for } i = 1,2,\ldots,8,\\
&s_{lj}^- \ge 0, \ \text{ for } j = 1,2,3,4,\\
&s_{lk}^+ \ge 0, \ \text{ for } k = 1,2,3,\\
&\delta_l > 0,\\
&t_l > 0.
\end{aligned}
\tag{12.22}
$$

Table 12.9 The results of Eq. 12.22

N	Airport	Solver Code	KAM Score	Delta values
1	A	0	0.4608393	1,961,920.50863
2	B	0	0.6301526	2,644,120.50799
3	C	0	0.3292076	5,341,920.50548
4	D	0	0.4619512	4,416,920.50634
5	E	0	0.1667772	5,721,920.50512
6	F	0	0.4997589	4,221,920.50652
7	G	0	0.3917765	5,591,920.50525
8	H	0	1.0000000	1,885,764.84456

The results are represented in Table 12.9. According to the Solver codes, the feasible area is available and the KAM score for each airport is optimally measured. The measured target for all airports is also displayed in Table 12.10.

As can be seen, H still has the best performance to satisfy the purpose of this example. However, the relative scores of other airports drop sharply down. For instance, KAM measures that the ratio of the efficiency of A to the efficiency of the measured target is 0.4608393 to 1. The value 0.4608393 shows the inefficiency of A in comparison with the measured target in Table 12.9. Thus, A should decrease 12.21% of the area of airport, increase 41.29% of the area of apron, increase 43.02% of the area of terminal and so on, to satisfy the conditions of this example, as Table 12.11 illustrates. The same illustration can be expressed for the other airports, as well.

12.6 KAM Scores and Decomposition of Inefficiency

Suppose that there are n firms, labeled F_i ($i = 1, 2, \ldots, n$), and each firm has m input factors with the values x_{ij} ($j = 1, 2, \ldots, m$) and p output factors with the values y_{ik} ($k = 1, 2, \ldots, p$). Assume that the weights/prices or the approximation of the relationships between input and output factors are W_j^- and W_k^+, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, respectively. The δ-KAM is given by Eq. 12.23 to measure the efficiency score of F_l's ($l = 1, 2, \ldots, n$) and optimize the value of delta.

$$
\begin{aligned}
&KA^*_{\delta_l^*} = \min \textstyle\sum_{j=1}^m W_j^-\left(x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-\right)/\textstyle\sum_{j=1}^m W_j^- x_{lj},\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^p W_k^+\left(y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+\right)/\textstyle\sum_{k=1}^p W_k^+ y_{lk} = 1,\\
&\textstyle\sum_{i=1}^n x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-, \ \text{ for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^n y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l - \delta_l/W_k^+, \ \text{ for } k = 1, 2, \ldots, p,\\
&\lambda_{li} \geq 0, \ \text{ for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \geq 0, \ \text{ for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \geq 0, \ \text{ for } k = 1, 2, \ldots, p,\\
&\delta_l \geq 0,\\
&t_l \geq 0.
\end{aligned}
\tag{12.23}
$$

Table 12.10 The targets of airports from Eq. 12.22

Airport	Area	Apron	Terminal	Runway	Flights	Passengers	Cargo
A-H	1149.47	429,790.89	65,219.21	359,790.20	110,295.35	10,000,000.00	33,780.42

Table 12.11 The rate of regulating the factors by Eq. 12.22

Airport	Area (%)	Apron (%)	Terminal (%)	Runway (%)	Flights (%)	Passengers (%)	Cargo (%)
A	−4.21	41.29	43.02	1.75	259.19	148.09	−54.46
B	128.52	101.09	68.19	3.35	135.30	109.07	77.33
C	43.68	948.19	452.71	33.28	606.66	861.57	2028.57
D	10.42	282.16	209.83	−9.08	176.63	473.22	586.73
E	14.72	1332.64	715.24	87.07	2156.91	2236.59	2046.15
F	140.47	582.21	183.56	−7.54	168.44	361.77	523.95
G	138.98	810.38	601.28	33.75	480.20	929.53	782.92
H	−14.60	−14.60	−14.60	−14.60	−14.60	−14.60	−14.60

The scores of δ^*-KAM, where $\delta^* = \max\left\{\delta_l^* : l = 1, 2, \ldots, n\right\}$, are the relative efficiency scores of F_i's ($i = 1, 2, \ldots, n$) and can absolutely be used to rank a set of homogenous firms.

As illustrated in Sect. 6.4.3, KAM represents the minimum value of Eq. 5.23, and compares the efficiency of a firm with the efficiency of the best possible location in the estimated feasible area, that is, KAM measures all the inefficiency of the firm, and its scores are relatively meaningful.

Even if the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ (or $\sum_{i=1}^{n} \lambda_{li} \geq t$ or $\sum_{i=1}^{n} \lambda_{li} \leq t$) is added to the constraints of Eq. 12.23, the scores of KAM are the same. This is due to this fact that, all the observed firms are in the estimated feasible area by Eq. 12.23, so KAM compares all the firms to each other and regulates the factors to find the best location of the feasible area. Therefore, even if the observed firms are only considered (that is, a discrete set with n points) the scores of KAM are the same as when the CRS technology is applied by Eq. 12.23. This phenomenon lets us completely decompose the inefficiency of firms with the scores of KAM.

If the delta value is equal to zero, that is, $\delta = 0$, the Technical Efficiency (TE) scores of the firms are measured. If the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ is added to Eq. 12.23, when $\delta = 0$, the Variable Returns to Scale-Technical Efficiency (VRS-TE) scores are measured. The Non-Technical Efficiency (Non-TE) scores (allocative efficiency scores) are measured by the efficiency scores of the firms over the TE scores, and the Non-Variable Returns to Scale-Technical Efficiency (Non-VRS-TE) scores (scale efficiency scores) of the firms are measured by the ratio of TE scores to VRS-TE scores. All of these scores belong to interval [0,1]. Indeed, the score of δ^*-KAM is not greater than the score of 0-KAM (TE score), and the score of TE is not greater than the score of VRS-TE.

For example, suppose that the airports in Table 5.1 is given, where $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. Table 12.12 illustrates the efficiency, TE, Non-TE, VRS-TE, and Non-VRS-TE scores of the airports, according to the introduced W_j^-'s and W_k^+'s.

None of the Non-TE, TE, VRS-TE, and Non-VRS-TE scores can be used to rank the airports, and they only represent the reasons of inefficiency. Only the

Table 12.12 The decomposition of efficiency scores of the airports

N	Airport	Efficiency	Non-TE	TE	VRS-TE	Non-VRS-TE
1	A	0.4608393	0.4608393	1.0000000	1.0000000	1.0000000
2	B	0.6301526	0.6301526	1.0000000	1.0000000	1.0000000
3	C	0.3292076	0.6754831	0.4873662	1.0000000	0.4873662
4	D	0.4619512	0.4619512	1.0000000	1.0000000	1.0000000
5	E	0.1667772	1.0000000	0.1667772	1.0000000	0.1667772
6	F	0.4997589	0.4997589	1.0000000	1.0000000	1.0000000
7	G	0.3917765	0.3917765	1.0000000	1.0000000	1.0000000
8	H	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

efficiency scores in the third column of Table 12.8 should be used to rank the airports for $\delta \geq \delta^*$, (see also Table 5.2).

From Table 12.12, the inefficiency of C and E are due to the scale inefficiency (Non-VRS-TE), whereas the inefficiency of A, B, D, F and G are because of the allocative inefficiency (Non-TE) (see Khezrimotlagh et al. 2012b and Mohsenpour et al. 2013).

If extra restrictions are added to the model, they may change the efficiency scores of firms, and the measured outcomes. In this situation, the inefficiency by the extra restrictions should also be measured. For instance, when it is supposed that managers in the airport example can only regulate at about 10% value of each factor, KAM can only compare an airport with the points in the area that the corresponded components have 10% lesser or greater values than that of the airport.

In order to elucidate this statement, suppose that the divisions in the petroleum example in Chap. 2 are selected, where $W_1^- = W_2^- = 1$, and the output is a constant single value for all divisions, such as 1. Assume that only 10% values of the diesel fuel and the gasoline amounts can be regulated, that is, $a_j = x_{lj} - 10\ \%\ x_{lj}$ and $b_j = x_{lj} + 10\ \%\ x_{lj}$, for $j = 1, 2$. The δ_l-KAM, when the components of delta are introduced with Eq. 12.3, is given by Eq. 12.24 ($l = 1, 2, \ldots, 18$).

$$
\begin{aligned}
& KA^*_{\delta^*_l} = \min\left[\textstyle\sum_{j=1}^{2}\left(x_{lj} + \delta_l x_{lj} - s^-_{lj}\right)\right] / \textstyle\sum_{j=1}^{2} x_{lj}, \\
& \text{Subject to} \\
& \textstyle\sum_{i=1}^{18} \lambda_{li} x_{ij} + s^-_{lj} = x_{lj} + \delta_l x_{lj}, \ \text{for } j = 1, 2, \\
& \textstyle\sum_{i=1}^{18} \lambda_{li} = 1, \\
& -10\% x_{lj} \leq \delta_l x_{lj} - s^-_{lj}, \ \text{for } j = 1, 2, \\
& \delta_l x_{lj} - s^-_{lj} \leq 10\% x_{lj}, \ \text{for } j = 1, 2, \\
& \lambda_{li} \geq 0, \ \text{for } i = 1, 2, \ldots, 18, \\
& s^-_{lj} \geq 0, \ \text{for } j = 1, 2, \\
& \delta_l > 0.
\end{aligned}
\tag{12.24}
$$

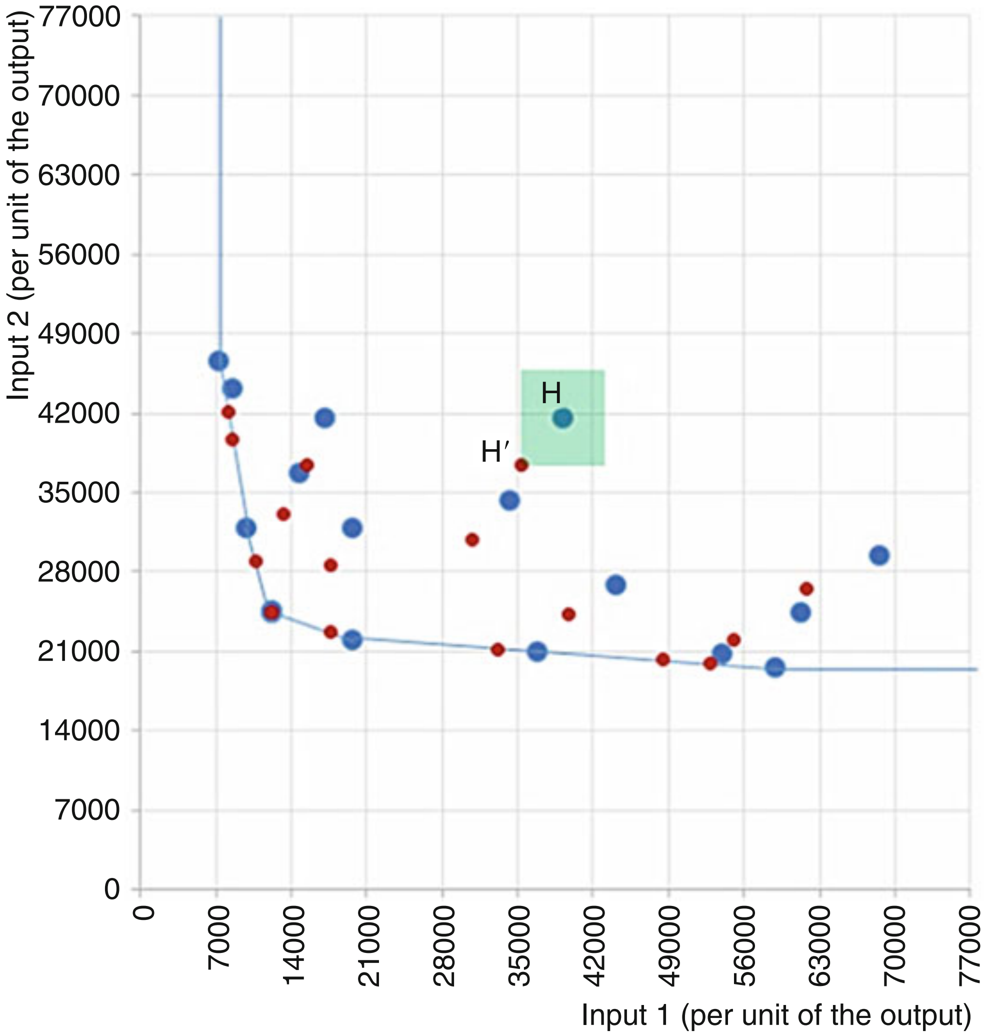

Fig. 12.2 The benchmarking by Eq. 12.24 with 10% regulation

Figure 12.2 represents the location of divisions with large blue circles, and the location of their targets with small red circles. For instance, according to the restrictions in Eq. 12.24, when H is evaluated, the feasible area is a square which is shaded with green color in Fig. 12.2. Therefore, H is compared with $H^{'}$, and the KAM's score for H represents the ratio of the efficiency of H to the efficiency of $H^{'}$. In order to decompose the inefficiency of H, the inefficiency of $H^{'}$ should also be measured.

Figure 12.3 also represents the targets for divisions when each division can regulate 30% value of the diesel fuel and the gasoline amounts. For instance, the feasible area for division H in Fig. 12.3 shows that H is compared with C, and since $H^{'}$ wholly dominates both H and C, the suggested target for H is $H^{'}$. As can be seen, H is not compared to other divisions (except C), which means the efficiency of H is not completely measured due to the extra restrictions.

As the range of regulation is increased the divisions are benchmarked toward division A, which has the best performance among the divisions where $w_1^- = w_2^- = 1$, (see Fig. 4.16).

In other words, when all divisions belong to the measured feasible area, this means that the extra restrictions do not affect the efficiency measurement by KAM.

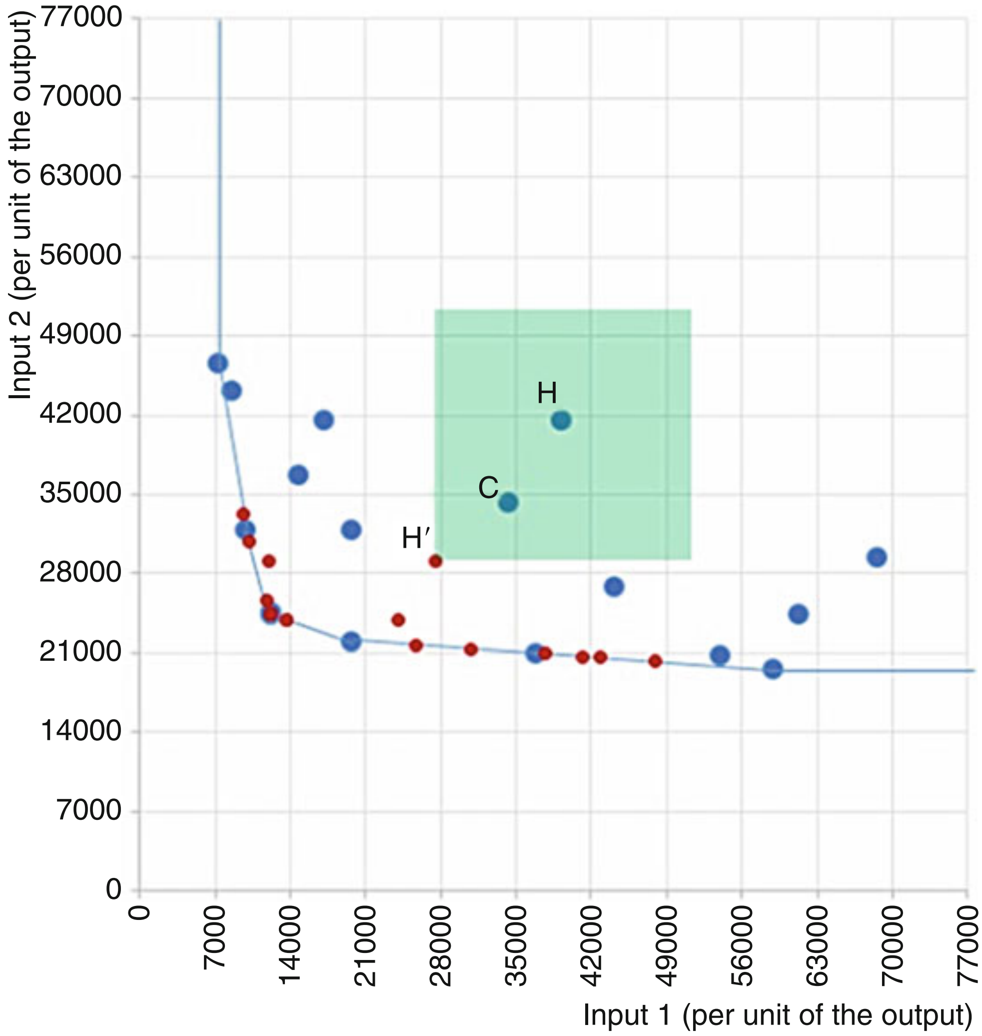

Fig. 12.3 The benchmarking by Eq. 12.24 with 30% regulation

However, if an extra restriction changes the feasible area such that some observed firms are not included in that feasible area, the efficiency is not completely measured. As a result, the effects of extra restrictions should be calculated to decompose the inefficiency as well.

Note that, as the range of regulation in Figs. 12.2 and 12.3 increases, the intersection of the shaded square with the PPS provides the feasible area for H.

From the above illustration, it is clear that an extra restriction may change the way of interpreting the outcomes of KAM and the decomposition of measured efficiencies. But, what remains the same in any situation is that, KAM robustly compares the efficiency of a firm, F_l, with the efficiency of all points in the corresponded feasible area, and the target of KAM for F_l is the point which has a better efficiency score in comparison with the efficiency score of F_l, according to the restrictions and the introduced conditions.

For more illustration, let's consider the objective of KAM, as Eq. 12.25 displays. The first fraction in the objective of KAM represents the efficiency of F_l ($l = 1, 2, \ldots, n$) and the second fraction represents the efficiency of a point in the feasible area that the corresponded components of that point can have equal, lesser or greater values than that of F_l. KAM minimizes this ratio, that is, KAM maximizes the second fraction according to Types 1–6. Note that the first fraction has a constant value, and does not influence the optimization.

$$\min \left(\frac{\sum_{k=1}^{p} W_k^+ y_{lk}}{\sum_{j=1}^{m} W_j^- x_{lj}} \right) \Big/ \left(\frac{\sum_{k=1}^{p} W_k^+ \left(y_{lk} - \delta_k^+ + s_k^+ \right)}{\sum_{j=1}^{m} W_j^- \left(x_{lj} + \delta_j^- - s_j^- \right)} \right) \tag{12.25}$$

Therefore, the optimal target has the best efficiency score in comparison with F_l in the estimated feasible area, according to the constraints and the introduced conditions.

If there are no more constraints than the constraints in the Eq. 12.23, the optimal scores of KAM are independent on the returns to scale technologies, and adding the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ (or $\sum_{i=1}^{n} \lambda_{li} \geq t$ or $\sum_{i=1}^{n} \lambda_{li} \leq t$) does not change the optimal scores, because the measured feasible area in this situation includes all the observed firms. In addition, none of the returns to scale technologies change the efficiency score of a firm which is calculated with a linear combination of output factors over a linear combination of input factors.

On the other hand, the KAM benchmarks by Eq. 12.23 depend on the introduced returns to scale technology. In other words, if the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ is added to Eq. 12.23, the efficiency scores of firms are not changed, but, KAM suggests the targets which are on the VRS-PPS.

As a result, while there is more than one input (output) factor and the relationships between the factors or units of measurement/prices/worth/weights of the factors are unknown, it is impossible to provide a fair relative score for each firm without some information from expert judgment. If the approach is introduced or prices are estimated for each factor, the δ-KAM is the most powerful tool to linearly discriminate, rank and benchmark firms as well as decomposing the efficiency scores.

12.7 Outliers and Numbers of Firms via Factors

Since the discrimination between the firms is based on the best observed performers, there was a concern in the literature of operations research that such discrimination is sensitive to the possible presence of outliers. If there is no significant error in the measurement, there is not a valid concern about the extreme data when the firms are supposed as homogenous. In addition, since KAM fairly ranks the firms based on the introduced approach and the value of delta, the firms can easily be classified according to their measured efficiency scores, so the possible outliers can easily be announced statistically and the production function can be estimated by the rest of firms.

There was also a concern in the literature of operations research about increasing the number of factors when the number of firms is a constant value, or is not large enough in comparison with the number of factors. This concern is raised due to the misinterpretation of the technical efficiency as efficiency, and misusing the technical efficiency to rank firms. As frequently explained in this book, the concept of doing the job right (technical efficiency) is not logically enough for discriminating between firms or ranking them. The concept of doing the job well (efficiency) is required to rank a set of homogenous firms with multiple input factors and multiple output factors. From this fact, there is no concern about the number of factors via the number of firms when KAM is applied.

12.8 Conclusion

The technical efficiency scores (that is, the scores to indicate whether or not a DMU does the job right) represent the optimal output factors which can be obtained from a set of input factors, but they should not be used to rank firms; they have unfortunately been inaccurately used in the literature of operations research for the last four decades. The provided scores by KAM can relatively be meaningful according to the delta parameter, and can be used to measure the efficiency scores of firms which lets us rank and benchmark the firms logically. In this chapter, the optimal value of delta is measured, and KAM is improved to measure the efficiency of firms under specific restrictions for targets. There are no concerns about the presence of outliers or the number of firms via the number of factors, when KAM is applied. The discrimination between homogenous firms requires expert judgment either to introduce a set of weights for factors or to specify a measurement approximation to estimate the efficiency scores which are relatively meaningful. From the outcomes of KAM, the inefficiency can be decomposed to non-technical inefficiency (allocative inefficiency) and technical inefficiency, and the technical inefficiency can also be decomposed to VRS-technical inefficiency and non-VRS technical inefficiency (scale inefficiency).

12.9 Exercises

12.1. Write Eq. 12.5 in VRS technology and find its dual linear programming.

12.2. Solve Eq. 12.12 when $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, and the fourth input factor is non-controllable.

12.3. Find the dual linear programming for

12.3.1. Eq. 12.12.

12.3.2. Eq. 12.21.

12.4. Prove that if the values of W_j^- and W_k^+ (prices) are multiplied to the corresponded data before applying KAM (Eqs. 12.5, 12.23), the results of KAM are not changed, (see Tone 2002 and Khezrimotlagh et al. 2012a).

12.5. Solve the example in Sect. 12.5.1 when SBM approach is applied and discuss the results.

12.6. Solve Eq. 12.22 when the VRS technology is applied, and compare the results with the data in Tables 12.8, 12.9, and 12.10.

12.7. Apply Eq. 12.23 for gemstone example, where $W^- = W^+ = 1$.

12.7.1. Compare the results with the results in Table 1.4.
12.7.2. Decompose the efficiency score of each candidate.
12.7.3. Apply Eq. 12.20, where each candidate can only regulate 10% of input and output factors, and describe the results.
12.7.4. Apply Eq. 12.20, where each candidate can only regulate 30% of the factors, and compare the results with previous results.
12.7.5. Assume that $W^- = 2$ and $W^+ = 3$ and answer the above questions.
12.7.6. Assume the SBM approach, that is, $W_l^- = 1/x_l$ and $W_l^+ = 1/\mathrm{y}_l$ and answer the above questions.
12.7.7. State the conclusion of this task.

12.8. Apply Eq. 12.23 for petroleum example in Sect. 4.4, where $W_1^+ = W_2^+ = 1$.

12.8.1. Compare the results with the results in Table 12.11.
12.8.2. Decompose the efficiency score of each division.
12.8.3. Apply Eq. 12.23, where each candidate can only regulate 5% of output factors, and describe the results.
12.8.4. Apply Eq. 12.23, where each candidate can only regulate 10% of output factors, and compare the results with previous results.
12.8.5. Assume the average approach, that is, $W_{l1}^+ = 1/\underset{l}{\mathrm{ave}}\{y_{l1}\}$ and $W_{l2}^+ = 1/\underset{l}{\mathrm{ave}}\{y_{l2}\}$, and answer the above questions.
12.8.6. Assume the maximum approach, that is, $W_{l1}^+ = 1/\max_l\{y_{l1}\}$ and $W_{l2}^+ = 1/\max_l\{y_{l2}\}$, and answer the above questions.
12.8.7. State the conclusion of this task.