Course 67562 Spring 2025

**Dynamics, Networks and Computation**

**Homework Exercise #4**

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# 1 Submission Guidelines

Submission deadline is Sunday, the 5th of June 2025, at 23:59

## 1.1 Submission

Submit a single .ipynb file (Jupyter notebook) with your answers to the questions. (Python code cells for code questions, MarkDown cells for text/math) If there are any other files required to run the code in the notebook, supply them as well.

If you wish to submit in any other way (e.g. want to use another language), Okay it with me (Hagai) first.

## 1.2 Working in Groups

Your'e allowed to discuss questions and code implementations with other students, but the final answers should be yours alone. In particular, you should write all code yourself.

# 2 Cellular Automata and Traffic Flow

In this question, we'll use a simple cellular automaton (Rule 184) to model traffic flow (see also recitation 5 notes).

We'll work with a 1D grid of length $L = 1000$ with periodic boundary conditions. All averaging will be with respect to a random ensemble of initial conditions, each of which we'll run for 500 steps to equilibrate before analyzing the resulting system. For a given simulation, all realizations in the ensemble will have the same initial number of cars $n \leq L$. Note that $n$ is invariant under rule 184, and should also be under the other rules you'll work out in this question. We'll define the *density* of such a run as $\rho = \frac{n}{L}$.

## 2.1 Average Speed in Rule 184

Simulate rule 184 for different values of $\rho$ between 0 and 1. Plot the average speed $\overline{v}$ (total distance traveled per time unit, so in this case a number between 0 and 1) as a function of $\rho$.

## 2.2 Different Speeds

Generalize rule 184 to include speed and a speed limit $m$. Namely, a car in location $i$ will move to $\min\{i + m, i + g - 1\}$ where $g$ is the distance to the next car. What did you need to change in the CA definition?

Plot the *flow* $\phi = \rho\overline{v}$ as a function of $\rho$ for $m \in \{1, \cdots .5\}$. What is the effect of the speed limit?

### 2.3 Anticipatory Driving

So far, our cars were maximally cautious. If there was a car ahead, they assumed the most conservative scenario, i.e. that said car will stay put, and thus did not advance past its location.

Let's now allow each car to observe $k$ steps ahead, and incorporate the knowledge that other cars can also look $k$ steps ahead, and operates via the same rules. Let's start with the case of $m = 1$.

The car at site $i$ checks whether in a block of $k$ sites ahead at least one site is empty. If it is, it moves one site to the right, even if that site is occupied, as it now knows that the car at site $i + 1$ will move (as all drivers follow the same rule).

What did you need to change in the CA definition?

Plot the flow $\phi$ as a function of $\rho$ for $k \in \{1, \cdots 5\}$. What is the effect of $k$?

### 2.4 Car Accidents

Generalize one of the models you developed in the previous questions to include a mechanism of car accidents (collisions). The modelling and implementation details are up to you. Analyze the number of accidents as a function of some parameter of interest and discuss your results.

## 3 Reaction Diffusion and the Gray-Scott System

**Note:** In this question you'll be asked to simulate PDEs. The size of the grid to simulate is not specified, as the feasible size will depend heavily on the efficiency of your implementation, and on whether you have access to a GPU, which will allow you to accelerate these simulations dramatically.

The larger the grid you use, the better (you'll be able to see more interesting patterns), so use the largest grid size that you can within reasonable runtime constraints.

### 3.1 Gray-Scott

The general form of the reaction-diffusion equations which we discussed in recitation is

$$\frac{\partial u}{\partial t} = D_u \cdot \nabla^2 u - k_u \cdot u + F\left(u, v\right)$$

$$\frac{\partial v}{\partial t} = D_v \cdot \nabla^2 v - k_v \cdot v + G\left(u, v\right)$$

where $u$ and $v$ are concentrations of two morphogens which diffuse with diffusion coefficients $D_u$ and $D_v$ and are influenced by some interaction between them captured by $F$ and $G$.

The *Gray-Scott model* is a reaction diffusion system which is often used to exhibit many of the interesting properties and behaviors such systems possess. In this system we consider two morphogens, $U$ and $V$ with concentrations $u$ and $v$ respectively, a parameter $f$ which represents a process which feeds $U$ and drains both $U$ and $V$, and a parameter $k$ which represents draining only of $V$.

The two morphogens react according to

$$U + 2V \to 3V$$

and the resulting equations are

$$\frac{\partial u}{\partial t} = D_u \cdot \nabla^2 u - uv^2 + f\left(1 - u\right)$$

$$\frac{\partial v}{\partial t} = D_v \cdot \nabla^2 v + uv^2 - \left(f + k\right) v$$

### 3.2 Gray-Scott Simulation

Simulate the Gray-Scott equations on a square grid with periodic boundary conditions and the following parameter values: $D_u = 1, D_v = 0.5, f = 0.0367, k = 0.0649$.

As an initial condition, start with $u = 1$ and $v = 0$ everywhere except for a small patch in the middle with $v = 1$ (you may want to try different shapes for this initial patch).

You can experiment with visualizing either $u$ or $v$ concentrations or both.

Describe the resulting dynamic behavior. Can you give an intuitive explanation of how this dynamic comes about?

### 3.3 Extended Gray-Scott

Suggest some modification to the system you simulated in the previous question. How do you expect the behavior to change? Simulate the modified system. Were you correct?

Some ideas for modifications:

- Orientation: Diffusion can occur faster in one direction than another. You can also rotate the Laplacian.

- Local effects: The diffusion rates, as well as the feed and kill rates, can vary across the grid to give different patterns in different areas.

- Flow: The chemicals can flow across the grid (add external forcing, or bias, to the equations) to give various dynamic effects.

- Noise: You can add stochasticity to the dynamics in various ways.

# Good Luck!