Exercise 1 Homework Solutions

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Part I
Complex Numbers

1

1.1 Question (5 pts)

Express the following complex number in an exponential form ($z = re^{i\varphi}$):

$$z = 3 + 4i \tag{1}$$

1.2 Answer

First we'll find the radius or absolute value of the number:

$$\begin{aligned} r = |z| &= \sqrt{(a+bi)(a-bi)} = \sqrt{(3+4i)(3-4i)} = \sqrt{9 - 12i + 12i - 16i^2} \\ &= \sqrt{9+16} = \sqrt{25} = 5 \end{aligned} \tag{2}$$

The angle is given by the inverse tangent:

$$\varphi = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{4}{3} = 0.93 \text{ radians} \tag{3}$$

2

2.1 Question (10 pts)

What is the natural logarithm of $z = re^{i\varphi}$? Don't forget all possible values of φ.

2.2 Answer

$$\ln z = \ln\left(re^{i\varphi}\right) = \ln r + i\varphi \tag{4}$$

Unfortunately, this is not complete. The addition of 2π to all phases will always lead us back to the same place from which we started. Thus, a more general form of the answer looks like:

$$\ln z = \ln\left(re^{i(\varphi+2\pi n)}\right) = \ln r + i(\varphi + 2\pi n) \tag{5}$$

where n is some integer. This means that $\ln z$ is a *multivalued* function, have an infinite number of values for a single pair of r and φ. The *principal value* of $\ln z$ is the value for which $n = 0$.

Part II
Linear Algebra

3

3.1 Question (10 pts)

Write the following linear system in the canonical representation $a\vec{X} = \vec{b}$:

$$\begin{aligned} 4 - 2x + 2z &= 0 \\ y + z - 4x &= 4 \\ x + y &= z \end{aligned} \tag{6}$$

3.2 Answer

First we'll write the equation so that the parameters and variables are on one end, and the solutions on the other:

$$\begin{aligned} -x + z &= -2 \quad \text{(divided by 2)} \\ -4x + y + z &= 4 \\ x + y - z &= 0 \end{aligned} \tag{7}$$

In this stage it's easiest to first write the variables vector $\vec{X}$ and the solutions vector $\vec{b}$:

$$\vec{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \; ; \; \vec{b} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \tag{8}$$

With these two in hand we can finally write a, the coefficient matrix:

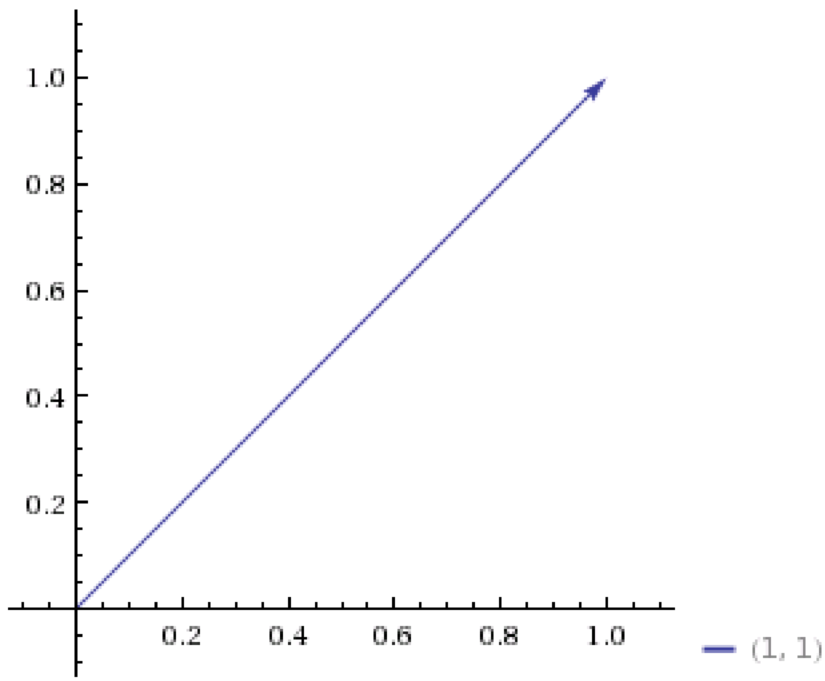

Figure 1: A vector point to $(x, y) = (1, 1)$.

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{9}$$

Try multiplying $a\vec{X}$ and check that indeed you receive equation 7.

4

4.1 Question (25 pts)

To achieve a deeper understanding of matrices, we'll see how it's used as a **transformation in space**. Assuming we have a vector on the real plane $\vec{v}_1$, we can represent its coordinates as a linear algebra vector by defining the first number of the vector to be the x-axis coordinates, and the second - y-axis coordinates (figure 1). The vector in the figure points to $(x, y) = (1, 1)$.

4.2

What is the mathematical operator that can transform this vector so that it points to $(x, y) = (2, 3)$? *Hint:* This is a question about matrices…

4.3

What is the mathematical operator that can rotate this vector to $(x, y) = (-1, 1)$? Try to find the most general form that solves all questions of this type.

4.4 Answers

4.4.1

In a 2D world we need 2D matrices to to mathematical operations, so their dimensions are 2 by 2. Assuming our vector is $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the equation we need to solve is:

$$\begin{gathered} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A\vec{v}_1 = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a + b \\ c + d \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{gathered} \tag{10}$$

While this equation has many possible solutions, the point is that the A matrix was our mathematical operator in this case. Note that it both rotated the vector by a small angle, and stretched it.

4.4.2

In this question all we want to do is rotate the vector clockwise by 90 degrees, or $\pi/2$ radians. Let's write the equation again:

$$A\vec{v}_1 = \begin{pmatrix} a + b \\ c + d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{11}$$

While we have many solutions here as well, I'd like to point out that perhaps the simplest matrix we can find that performs this rotation is:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

We can check this special matrix for a number of different vectors, for example $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and see that this matrix will rotate this vector by $\pi/2$ radians as well. A different matrix wouldn't rotate us by the same angle, indicating that we perhaps found a general representation for rotations by $\pi/2$ radians in 2D space. Generally speaking, rotations in 2D space are done with the rotation matrix:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{13}$$

which is just our previous matrix A when $\theta = \pi/2$.

Part III
Statistics and Probability

5

5.1 Question (20 pts)

In a football game, a specific player has a probability of 0.5 to not score any goals in a match. He also has a 0.25 probability to score 1 goal, 0.15 probability to score 2, and a probability of 0.1 to score 3 goals. What is the expected value of goals in the coming football season, assuming the player will play 30 games?

5.2 Answer

By looking at the expected value formulas we've seen in class, we can tell that if we find the expected value of goals for a single game, we can then just multiply that result by the number of games the player has throughout the season:

$$\begin{gathered} G = \text{Number of goals per game} \\ E[G] = 0.5 \cdot 0 + 0.25 \cdot 1 + 0.15 \cdot 2 + 0.1 \cdot 3 = 0.85 \text{ [goals per match]} \\ 0.85 \cdot 30 \text{ [games in season]]} = 25.5 \text{ [goals]} \end{gathered} \tag{14}$$

6

6.1 Question (20 pts)

Bob goes to the gym each day of the week with a probability of 40%, independently of yesterday's decision. Alice promised to go to a movie with him only if he visited the gym at least 5 times in the past week. What are the chances they'll see each other?

6.2 Answer

We have a trial experiment, with a chance of success in each trial - meaning we're dealing with a random variable that distributes binomially. The number of trials is $n = 7$, and the chance of success is $p = 0.4$. Thus we declare $X \sim B(7, 0.4)$ where X is the number of times Bob visited the gym.

Here we have to sum the probability of Bob visiting the gym 5, 6 and 7 times to reach the final answer:

$$
\begin{aligned}
P(\text{visits at least 5 times}) &= P(\text{visits 5 times}) + P(\text{visits 6 times}) + \\
&+ P(\text{visits 7 times}) = \\
&= \binom{7}{5} \cdot 0.4^5 \cdot 0.6^2 + \binom{7}{6} \cdot 0.4^6 \cdot 0.6 + \binom{7}{7} \cdot 0.4^7 \cdot 0.6^0 \\
&\simeq 0.096 = 9.6\%
\end{aligned} \tag{15}
$$