Fourier Analysis

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Part I
Fourier Analysis

1 Preparations

1.1 Introduction

Fourier Analysis is definitely one of the wonders of “modern” Mathematics. Before we can explicitly prove and show it, we have to make sure we understand exactly what we’re doing. Our goal is to take a signal in the time domain - standard voltage recordings, for example - and *represent* it in a different domain. We’re basically translating the signal into a different language - frequency instead of time, in the Fourier case. The translation is called a transform, which is the mathematical operation we’re performing on that signal. The result is the Fourier Series - the final translation of that signal. Since both representations are equivalent, one can also invert this translation, using the Inverse Fourier Transform, back to the "original" language our sequence was written in.

These so-called translations are not necessarily between time and frequency. Another famous pair is space and wave number ($x \leftrightarrow k$), which is also “mediated” by a Fourier transform. Other transformations allow more elaborate calculations (specifically, the Z-transform is very important in signal processing), but in essence they’re all similar to the basic Fourier transform we’re dealing with now.

1.2 Orthogonality

Before diving head first into Fourier Analysis, we might benefit from the concept of orthogonality. You’ve seen the most common example of orthogonality when you were introduced with the classic $\vec{x}$ and $\vec{y}$ axes. These two vectors are orthogonal (or perpendicular) and they allow us to represent all points in a 2D space. In other words, I can use these two vectors to span a 2D space, which helps me represent each point on that space with these two vectors. For example, the point on the 2D plane $(-3, 4)$ can be represented the following equation:

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; \vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; \; (-3, 4) = -3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; + \; 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

To "prove" that $\vec{x}$ and $\vec{y}$ are orthogonal we need to multiply them and make sure the result is 0. The rules for multiplying two vectors are derived from the basic rules of linear algebra, which we visited in exercise 1:

$$\vec{x} \cdot \vec{y} = 1 \cdot 0 + 0 \cdot 1 = 0 \tag{2}$$

Just as we span the 2D space with $\vec{x}$ and $\vec{y}$, we'd like to span a different plane with different vectors. The space we'll spanning will be space of all function with a single argument x, and our base vectors will be the sine and cosine functions. In this space the sine and cosine functions are orthogonal to each other under a different multiplication rule, which we'll see in a minute. Fourier Analysis is the method that allows us to span the function space using only sines and cosines. In other words, we'll be representing all functions in a different manner, or language, using sine and cosine functions as our base vectors.

This concept is pretty hard to get, and its usability might be unclear, so let's go through it in parts, starting with multiplication of the orthogonal functions, just like the multiplication of the vectors in equations 1 and 2. The multiplication rules in our space are defined by integrating two functions:

$$f(x) \cdot g(x) \equiv \int_0^{2\pi} f(x)g(x)dx \tag{3}$$

And the orthogonality of sine and cosine follows:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \cos nx \cos mx dx = \delta_{mn} \\
&\frac{1}{\pi}\int_0^{2\pi} \sin nx \sin mx dx = \delta_{mn}, \quad m \neq 0 \\
&\frac{1}{\pi}\int_0^{2\pi} \sin nx \sin mx dx = 0, \quad m = n = 0 \\
&\frac{1}{\pi}\int_0^{2\pi} \cos nx \sin mx dx = 0
\end{aligned} \tag{4}$$

where m, n are integers and δ_{mn} is defined as

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{5}$$

First we now see how to multiply two functions in our Fourier space. To do so, you multiply them elemntwise inside an integral and integrate over their argument. The boundaries of the integral are currently limited to $[0, 2\pi)$ because of reasons we'll explain later. But how did we reach these results for sine and cosine, i.e. the values to the right side of the equals sign? We'll look at one example that will clarify the way to solve these integrals:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos nx \cos mx \, dx =$$

For the case of $m = n = 0$:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos 0 \cos 0 \, dx = 1$$

$n = m \neq 0$:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \cos nx \cos nx \, dx &= \frac{1}{2\pi}\int_0^{2\pi} \cos^2 nx dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} (1 + \cos(2nx)\, dx) \\
&= \frac{1}{2\pi}\left[x + \frac{1}{2n}\sin(2nx)\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left(2\pi + \frac{1}{2n}\sin(4n\pi)\right) = 1
\end{aligned} \tag{6}$$

$n \neq m$:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{4\pi}\int_0^{2\pi} (\cos((n-m)x) + \cos((n+m)x))\, dx \\
&= \frac{1}{4\pi}\left[\frac{1}{n-m}\sin((n-m)x) + \frac{1}{n+m}\sin((n+m)x)\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\left(\frac{1}{n-m}\sin(2\pi(n-m)) + \frac{1}{n+m}\sin(2\pi(n+m))\right)
\end{aligned}$$

and since n, m are integers their addition or subtraction also results in an integer. Thus each expression nullifies:

$$\frac{1}{2\pi}\int_0^{2\pi} \cos nx \cos mx \, dx = 0$$

With similar methods involving other trigonometric identities one can prove all of these orthogonality conditions. Don't give too much thought on the normalization factor, it's a convention that should help you get the result you want, with the right units.

To summarize this part, we've proven (kinda...) that sine and cosine are orthogonal functions under this multiplication rule, and we expect this property to help us represent every function out there with these very convenient functions.

2 Fourier Series

2.1 Definition

We now turn to derive the actual Fourier series. For starters we'll assume we have a periodic function $f(t) = f(t + nT)$, where n is an integer. The period is $T = 1/f$ with f being the frequency. The *angular frequency* is $\omega = 2\pi f$. The first step is to reduce the periodic function to a 2π period, with a simple change of variables: $t \to 2\pi\tau/T = \omega\tau$. This explains why we're always integrating over $[0, 2\pi)$ - the function is periodic and behaves the same for all other periods, or parts of the number axis.

By assuming very little on our function (Finite number of discontinuities and extrema between $[0, 2\pi)$ and convergence) we can tread the final step and represent it by means of a Fourier series. Before showing the final expression, let's make sure we're on top of variables we have:

- f - frequency [Hz] (hardly used)
- T - period [s]
- ω - angular frequency [rad/s]
- n - integer
- $f(\omega\tau)$ - periodic function

Here's the expression for the Fourier Series (swapping $\omega\tau$ for t):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{7}$$

In words - the function equals (= can be represented as) a constant $a_0/2$, which is also called the *direct current (DC)*, plus sine and cosine terms with increasing discrete frequencies and different amplitudes. Together they can all add up and recreate precisely the function $f(t)$. Notice the ω that snuck in, indicating the angular frequency of this wave. The coefficients a_n and b_n are simply numbers, which we'll discuss in a minute.

The obvious question is "how do we know it's true"? Generally there are two methods to prove it. The first one was discussed in class - take the expression in equation 7 and try to minimize the gap between it and the original function $f(t)$ we wish to transform. The second method requires us to discuss the converges of the Fourier series.

Happily, we'll do none of that. Instead, we'll just show the method to find a_n and b_n, basically assuming that the series truly represents the original function $f(t)$. To this end, we'll multiply both sides of equation 7 by $\cos n\omega t$ and integrate in respect to t over a period of the function, $0 \to 2\pi$ for example.

The left side of the equation is simply

$$\int_0^{2\pi} f(t)\cos(n\omega t)dx \tag{8}$$

On the right side of the equation, we can imagine writing explicitly all of the summed terms, and integrating each one seperately. Every term of the following shape:

$$\frac{1}{\pi}\int_0^{2\pi} \cos m\omega t \cos n\omega t dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{9}$$

while every term

$$\int_0^{2\pi} \cos m\omega t \sin n\omega t dt = 0 \tag{10}$$

Thus, for $m = n$,

$$a_n \int_0^{2\pi} \cos^2 n\omega t dt = \pi a_n \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(n\omega t)dx \tag{11}$$

And in the same manner, but by integrating both sides of equation 7 with $\sin(n\omega t)dx$ we receive an expression for b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(n\omega t)dx \tag{12}$$

The most important thing we can see immediately from equation 11 is that the DC term $a_0/2$ is just the average of the function over its period:

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt \tag{13}$$

The sine and cosine waves are superimposed on this value. If the function is symmetric about the x-axis the DC is 0, indicating that its average is 0.

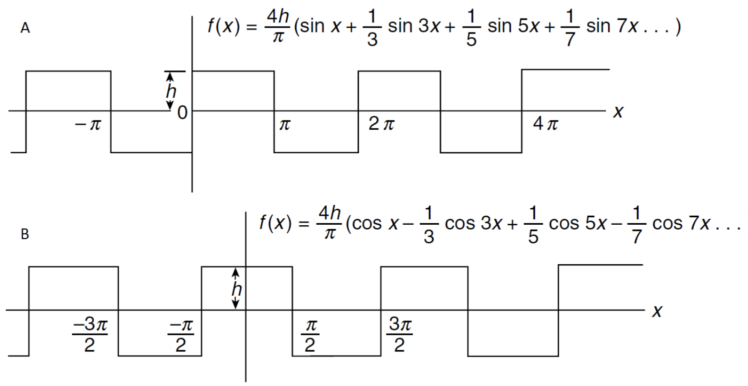

Figure 1: A 0 DC ("zero-offset") square wave. **A)** function represented as sine waves. **B)** After translating the function by a quarter of a period, it's now even. From Pain, *The Physics of Vibrations and Waves*.

2.2 Even and odd functions

One more important property of functions, that should help us understand what are we looking forward to receive when calculating their Fourier series, is the concept of odd and even functions. An *even* function is a function that is symmetric about the y-axis, meaning that it looks the same when going from $x = 0$ to $x \to \infty$ or from $x = 0$ to $x \to -\infty$. An *odd* function is a function that is reflected across the x axis when looking at its negative part - in other words it's symmetric about the origin.

There are mathematical expressions to check whether some function $f(x)$ is even, odd, or niether. An even functions holds this true: $f(-t) = f(t)$. In words - going n steps to the left on the x-axis will result in the same value as going n step to the right on the x-axis. An odd function holds: $-f(-t) = f(t)$, and this is why we say that it's symmetric over the origin. The most known example of an even function is the cosine, and the sine function is the most known odd one. Every function can be decomposed to its even and odd parts:

$$f(t) = f_{\text{odd}}(t) + f_{\text{even}}(t) \tag{14}$$

If $f(t)$ is an even function ($f(t) = f(-t)$) then it will only be represented by the even cosine functions, and vice-versa regarding the uneven sine waves ($\sin \omega t = -\sin(-\omega t)$, figure 1). This means that only by looking at a function we can determine if it could only be represented with either cosine or sine terms only. An odd function, for example, as seen in the top part of figure 1, will only be constructed of sine terms, meaning that all of the a_n terms in its Fourier series are equal to 0. In the next example we won't assume this behavior - we'll show it.

2.3 Example

We'll now find the Fourier series of a square wave of height h:

$$f(x) = \begin{cases} h & 0 < x < \pi \\ -h & \pi < x < 2\pi \end{cases} \tag{15}$$

Make sure you take note that this is still the regular x, with units of "space". The coefficients are given by:

$$a_n = \frac{1}{\pi}\left[h\int_0^{\pi} \cos n\omega t dt - h\int_{\pi}^{2\pi} \cos n\omega t dt\right] = 0 \tag{16}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[h\int_0^{\pi} \sin n\omega t dt - h\int_{\pi}^{2\pi} \sin n\omega t dt\right] \\ &= \frac{h}{n\pi}\left[(\cos n\omega t)_{\pi}^{0} + (\cos n\omega t)_{\pi}^{2\pi}\right] \\ &= \frac{h}{n\pi}\left[(1-\cos n\omega\pi) + (1-\cos n\omega\pi)\right] \end{aligned} \tag{17}$$

which leaves us with $b_n = 0$ for even n and $b_n = {}^{4h}/_{n\pi}$ for odd n. The final "translation" of the original square wave function after eliminating all cosines (due to a_n) is:

$$\hat{f}(t) = \frac{4h}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \dots\right) \tag{18}$$

A representation of a few of the first terms in equation 18 is seen in figure 2. Each term is sometimes also referred to as an harmonic component of the signal. As we can tell, these first few frequencies (or hamonics) can only give us the general shape of the square wave, while missing out on the fine details, like the edges. The higher frequencies (high n) will be responsible for that part. Keep that in mind for the latter part of the course on image processing.

Furthermore, You can almost imagine how the function becomes more square-like the more frequencies we insert into that long expression at the top of the figure. There are many very nice animations of this sort in the web.[1]

2.4 Complex Fourier series

It's very helpful sometimes to think of a Fourier series as composed of complex elements, and not as we presented it above, with a_0, a_n and b_n. The mathematical definitions is as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \;\; ; \;\; c_n = \frac{1}{T}\int_T f(t)e^{-in\omega t}dt \tag{19}$$

[1] Demonstration of the creation of a square wave.

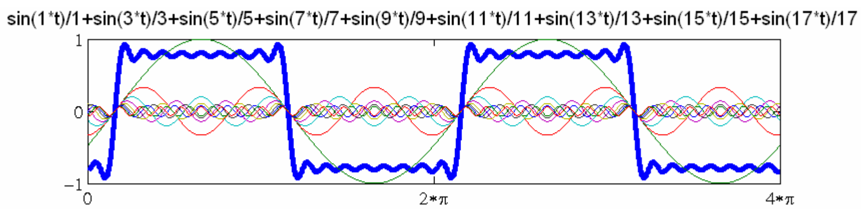

Figure 2: Building a square wave (in blue) from sinusoidal waves with increasing frequencies.

where the lower limit T on the integral states that we're integrating over a single period of the periodical function $f(t)$.

The two important difference are the change in the sum's limits to $(-\infty, \infty)$ and the replacement of the sine and cosine terms with the complex exponential. As we discussed in our first recitation, this can be done due to Euler's formula:

$$e^{ix} = \cos x + i \sin x \tag{20}$$

A full explanation of the equality between equations 19 and 7 can be found in the course's book, but it seems plausible even without proving it, just by looking at equation 20 above.

As seen in equation 19, the periodicity of our series is located in the argument of the exponential. As we might remember, a few key properties of exponents allow us for some very easy calculations with them. For example:

$$\begin{aligned} e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)} \Leftrightarrow \cos\theta \cdot \cos\phi &= \frac{1}{2}(\cos(\theta+\phi) + \cos(\theta-\phi)) \\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} &= \mathrm{Re}(e^{i\theta}) \\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} &= \mathrm{Im}(e^{i\theta}) \end{aligned} \tag{21}$$

3 The Fourier transform

The Fourier transform is the actual complete mathematical tool that will allow us to turn any signal into a function of its frequencies. It's the operation that decomposes any function, not necessarily a periodic one, into a different "plane", one that is spanned with frequencies $2\pi f \leftrightarrow \omega$, rather than regular "time" t.

If we take a regular function in time, or a signal as we sometimes like calling them, and Transform them, we'll receive a new function, its argument being the frequency ω. The function will have an amplitude which tells how strong this frequency was in the original time signal, and a phase offset, which is represented exponentially, depicting the offset from a zero-offset sine wave. This new transformed function can be transformed back, using the Inverse Fourier Transform, to receive our original signal back.

Fourier Transforms is a very important tool all forms of Science and Engineering. One doesn't have to transform time-based signals only. A Fourier transform on a signal representing locations (in space) will result in a function which argument is the k wave number. Many other Fourier-like transforms are used in many fields of research.

3.1 Continuous Fourier transform

The basic form of the Fourier transform is the continuous-time Fourier transform. This is the analog of the Fourier series in the following limits:

- Infinitely-long period. This means we'll define the Fourier series for any function, even if it's a-periodic, since it can still be considered periodic with a period of $T \to \infty$.
- Decrease the step size of the sum to infinitesimal steps $d\omega$. This effectively create the limit $n/T \to n\omega \Rightarrow \omega$.
- Replacing the discrete coefficients c_n from equation 19 with a function $F(\omega)$.
- Change the sum to an integral, since our summing steps are minimal.

Thus the left side of equation 19 becomes:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \tag{22}$$

This is actually the *Inverse* Fourier transform - it's the operation that takes us from the frequency domain (the function inside the integral = integrand) back to the time domain. The transform itself is defined, based on the exact same considerations, as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{23}$$

The clear difference is that in equation 22 the integration is done over ω, thus leading to its disappearance by the end of the calculation, leaving us only with t, i.e. back in the time domain. On the other hand, equation 23 leaves us only with ω, after the time was "integrated out". The $1/2\pi$ normalization is just there so that if you transform, and then inverse-transform a function, the result will be identical, and specifically on the same scale. If you convert the ω into $2\pi f$ everywhere then it's not needed.

3.2 Examples

3.2.1 Question

What is the Fourier transform of the sine $sin(t)$?

3.2.2 Answer

we know that the sine function can be represented by one sigle frequency. Lets look what we get out of the tranform.
By substituting all needed functions into equation 23, we'll get:

$$F(\omega) = \int_{-\infty}^{\infty} sin(at)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{e^{iat} - e^{-iat}}{2} e^{i\omega t}dt = \frac{1}{2}\int_{-\infty}^{\infty} 1e^{i(\omega+a)t} - \int_{-\infty}^{\infty} 1e^{i(\omega-a)t}dt = \frac{\sqrt{2\pi}}{2}\left[\delta(\omega - a) + \delta(\omega + a)\right] \tag{24}$$

In the frequency domain the sin function is a to deltas at the sin positive and negative frequencies. It takes all available frequencies.