BASIC MATHEMATICS FOR MACHINE LEARNING

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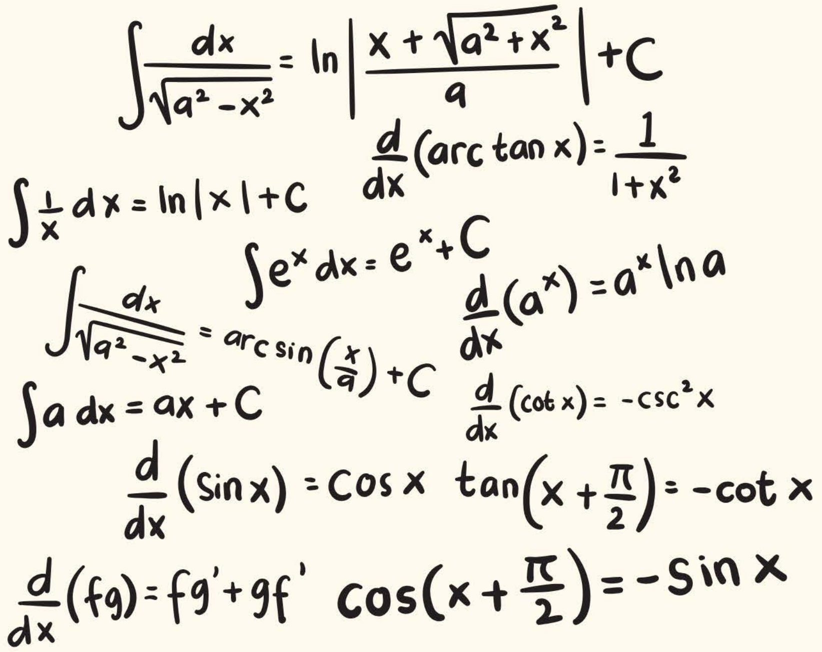
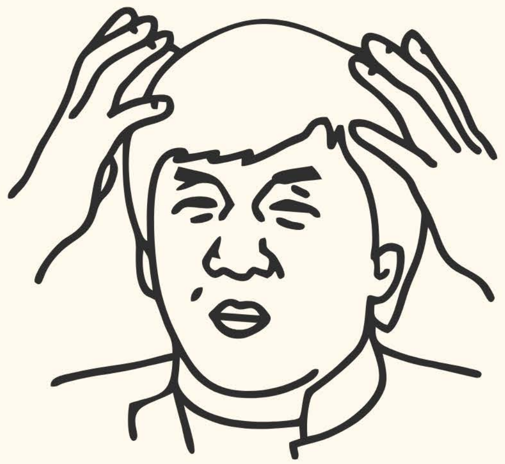
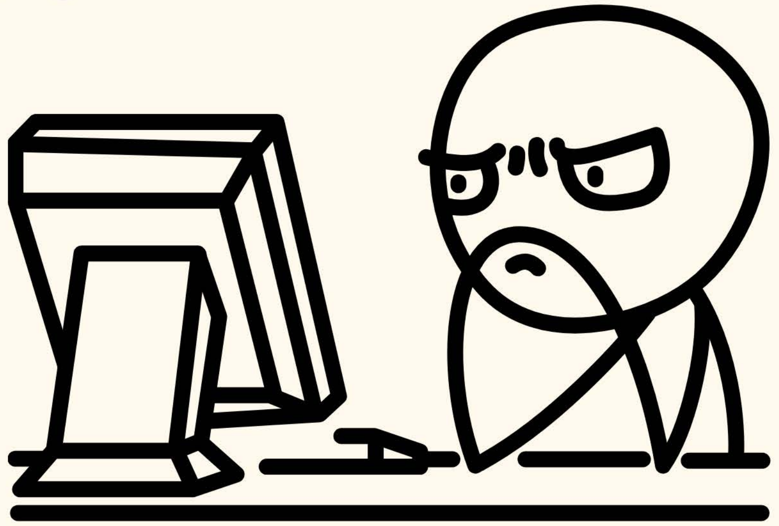
WELCOME TO MATH CLASS





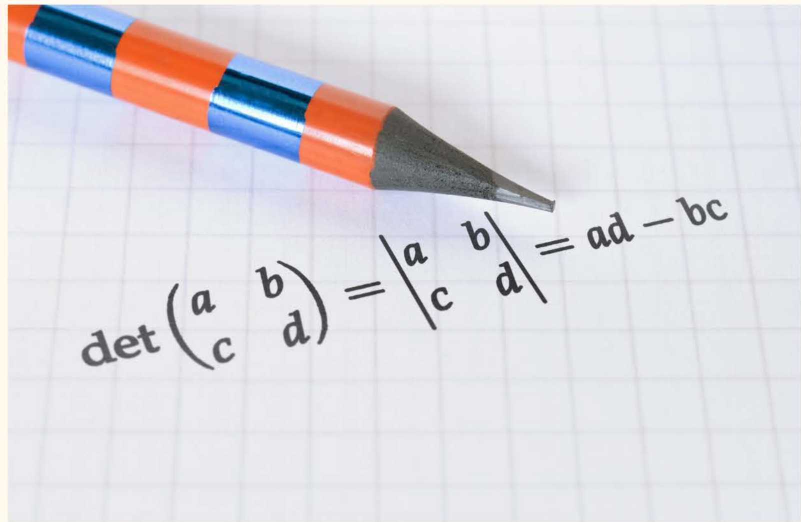


Please open your mind to math.
It’s not as daunting as you think.

LIST OUTLINE

- Matrix
- Logarithm
- Differentiation

MATRIX



Matrix

a group of numbers or other symbols arranged in a rectangle that can be used to solve particular mathematical problems

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 6 \\ 0 & 1 & 4 \end{bmatrix}$$

Cambridge Dictionary

MATRIX

Understanding Rows and Columns in a Matrix

Definition of Rows and Columns:

- Rows: A row in a matrix is a horizontal array of elements. Rows run from left to right across the matrix. Each row is indexed starting from the top of the matrix.
- Columns: A column in a matrix is a vertical array of elements. Columns run from top to bottom. Each column is indexed starting from the left side of the matrix.

$$A = \begin{bmatrix} 4 & 8 & 12 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix}$$

MATRIX

Understanding Rows and Columns in a Matrix

$$A = \begin{bmatrix} 4 & 8 & 12 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix}$$

Rows in Matrix AAA:

- First Row: [4,8,12]- contains the elements in the first horizontal array from the top.
- Second Row: [3,6,9]- contains the elements in the second horizontal array from the top.
- Third Row: [5,10,15] - contains the elements in the third horizontal array from the top.

Columns in Matrix AAA:

- First Column: [4,3,5] - contains the elements in the first vertical array from the left.
- Second Column: [8,6,10] - contains the elements in the second vertical array from the left.
- Third Column: [12,9,15] - contains the elements in the third vertical array from the left.
-

MATRIX

Understanding Matrix Dimensions

$$B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

matrix B with 3 rows and 2 columns.

3x2 Matrix

MATRIX

Identity Matrices (I)

Definition: A square matrix in which all the elements on the main diagonal are ones (1) and all other elements (off-diagonal) are zeros (0).

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any matrix multiplied by an identity matrix remains unchanged.

$$A \times I = I \times A = A$$

MATRIX

Diagonal Matrices (D)

Definition: A type of square matrix where the entries outside the main diagonal are all zero. The diagonal elements can be any number.

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

MATRIX

Zero Matrices (O)

Definition: A matrix in which all the elements are zero.

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix **Addition**

Definition: Matrix addition is the operation of adding two matrices by adding the corresponding entries together. This operation is only defined when both matrices have the same dimensions.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \longrightarrow A + B = \begin{bmatrix} 1+2 & 3+0 \\ 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix **Multiplication**

Conditions for Matrix Multiplication:

Matrix multiplication is possible when the **number of columns in the first matrix** (A) matches the **number of rows in the second matrix** (B).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{bmatrix}$$

A x B ?

B x A ?

BASIC OPERATIONS ON MATRICES

Matrix **Multiplication**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{bmatrix}$$

A x B ?

$$\begin{bmatrix} (1 \times 7 + 2 \times 10 + 3 \times 13) & (1 \times 8 + 2 \times 11 + 3 \times 14) & (1 \times 9 + 2 \times 12 + 3 \times 15) \\ (4 \times 7 + 5 \times 10 + 6 \times 13) & (4 \times 8 + 5 \times 11 + 6 \times 14) & (4 \times 9 + 5 \times 12 + 6 \times 15) \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix **Multiplication**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{bmatrix}$$

A x B ?

$$\begin{bmatrix} (1 \times 7 + 2 \times 10 + 3 \times 13) & (1 \times 8 + 2 \times 11 + 3 \times 14) & (1 \times 9 + 2 \times 12 + 3 \times 15) \\ (4 \times 7 + 5 \times 10 + 6 \times 13) & (4 \times 8 + 5 \times 11 + 6 \times 14) & (4 \times 9 + 5 \times 12 + 6 \times 15) \end{bmatrix}$$

A x B or AB $= \begin{bmatrix} 66 & 72 & 78 \\ 174 & 189 & 204 \end{bmatrix}$

BASIC OPERATIONS ON MATRICES

Matrix Transpose

Definition: The transpose of a matrix A is another matrix A^T obtained by swapping its rows with its columns. If A is an m×n matrix, then A^T will be an n×m matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix Transpose

Properties of Transpose

Symmetric Property: if $A = A^T$, then A is symmetric.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Addition and Multiplication

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

BASIC OPERATIONS ON MATRICES

Determinant of Matrix

Definition : The determinant is a scalar value that can be computed from the elements of a **square matrix** and encodes certain properties of the matrix, such as its invertibility and its effect on the volume of space.

For a 2x2 matrix, the determinant is computed as follows:

$$\text{For matrix} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{Det}(A) = ad - bc$$

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} \longrightarrow \begin{aligned} \text{Det}(A) &= 3 \times 6 - 8 \times 4 \\ &= 18 - 32 \\ &= -14 \end{aligned}$$

BASIC OPERATIONS ON MATRICES

Determinant of Matrix

For a 3x3 matrix, the determinant is computed as follows:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \text{Det}(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{aligned} \text{Det}(A) &= 1(5 \times 9 - 6 \times 8) - 2(4 \times 9 - 6 \times 7) + 3(4 \times 8 - 5 \times 7) \\ &= 1(-3) - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 \\ &= 0 \end{aligned}$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

The inverse of a matrix A, denoted as A^{-1}, is a matrix that, when multiplied with A, results in the identity matrix I.

Mathematically, this relationship is described by $A \times A^{-1} = A^{-1} \times A = I$.

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Inverses: If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.

BASIC OPERATIONS ON MATRICES

Matrix Inverse

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Calculate the Matrix of Minors

- **Minor for** $a_{11} = 3$

$$\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} = 0 \times 1 - (-2) \times 1 = 2$$

- **Minor for** $a_{12} = 0$

$$\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2 \times 1 - (-2) \times 0 = 2$$

- **Minor for** $a_{13} = 2$

$$\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \times 1 - 0 \times 0 = 2$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Calculate the Matrix of Minors

- **Minor for** $a_{21} = 2$

$$\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 0 \times 1 - 2 \times 1 = -2$$

- **Minor for** $a_{22} = 0$

$$\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3 \times 1 - 2 \times 0 = 3$$

- **Minor for** $a_{23} = -2$

$$\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 \times 1 - 0 \times 0 = 3$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Calculate the Matrix of Minors

- **Minor for** $a_{31} = 0$

$$\begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} = 0 \times -2 - 2 \times 0 = 0$$

- **Minor for** $a_{32} = 1$

$$\begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} = 3 \times -2 - 2 \times 2 = -10$$

- **Minor for** $a_{33} = 1$

$$\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} = 3 \times 0 - 2 \times 0 = 0$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

Step 1: Calculate the Matrix of Minors

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

The matrix of minors is: $\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

Step 2: Create the Cofactor Matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Apply a checkerboard of signs (+ and -) starting with a + for the top-left element. This changes some minors to cofactors.

$$\text{Minor Matrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\text{Co-factor Matrix} = \begin{bmatrix} (-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & (-1)^{1+3}M_{13} \\ (-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & (-1)^{2+3}M_{23} \\ (-1)^{3+1}M_{31} & (-1)^{3+2}M_{32} & (-1)^{3+3}M_{33} \end{bmatrix} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & -2 & +2 \\ +2 & +3 & -3 \\ 0 & +10 & 0 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 3: Adjugate Matrix

Transpose the cofactor matrix (switch rows and columns):

$$\begin{bmatrix} +2 & -2 & +2 \\ +2 & +3 & -3 \\ 0 & +10 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} +2 & +2 & 0 \\ -2 & +3 & +10 \\ +2 & -3 & 0 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Matrix Inverse

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathrm{Det}(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Step 4: Multiply by $1/\det(A)$

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(A) = 3 \cdot (0 \cdot 1 - 1 \cdot -2) - 0 \cdot (2 \cdot 1 - -2 \cdot 0) + 2 \cdot (2 \cdot 1 - 0 \cdot 0) = 10$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} +2 & +2 & 0 \\ -2 & +3 & +10 \\ +2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1.0 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

Zero Determinant: If **det(A)=0** , the matrix is called **singular or Singularity** . A singular matrix does **not** have an **inverse**.

BASIC OPERATIONS ON MATRICES

Matrix Inverse

The product of A and A^{-1} results in:

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

✓

TEST ON NUMPY

BASIC OPERATIONS ON MATRICES

Linear Equations in Matrix Form

solve systems of linear equations. By expressing linear equations in the form **Ax=b**

$$2x + 3y - z = 5$$

$$-x + 4y + 2z = 6$$

$$3x - 2y + z = -2$$

BASIC OPERATIONS ON MATRICES

Linear Equations in Matrix Form

solve systems of linear equations. By expressing linear equations in the form **Ax=b**

$$\begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 6 \\ -2 \end{bmatrix}$$

BASIC OPERATIONS ON MATRICES

Linear Equations in Matrix Form

solve systems of linear equations. By expressing linear equations in the form **Ax=b**

$$x = A^{-1}b$$

$$x = \begin{bmatrix} 0.2979 \\ 1.5106 \\ 0.1277 \end{bmatrix}$$

TEST ON NUMPY

Meaning of the exponential (e)

Which bank is better?

Bank 1

Interests
100% every year

(all your money once a year)

Bank 2

Interests
50% every 6 months

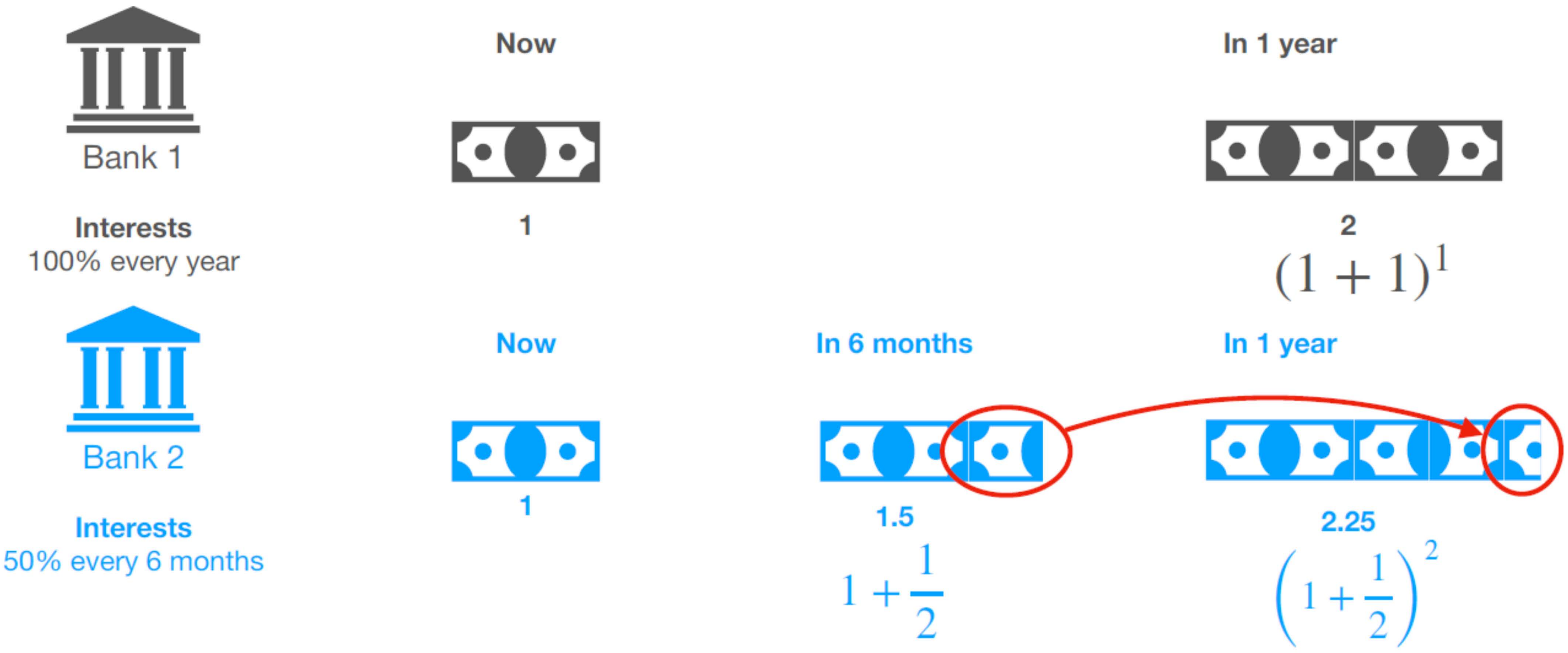
(half of your money twice a year)

Bank 3

Interests
33.3% every 4 months

(A third of your money three times a year)

Meaning of the exponential (e)

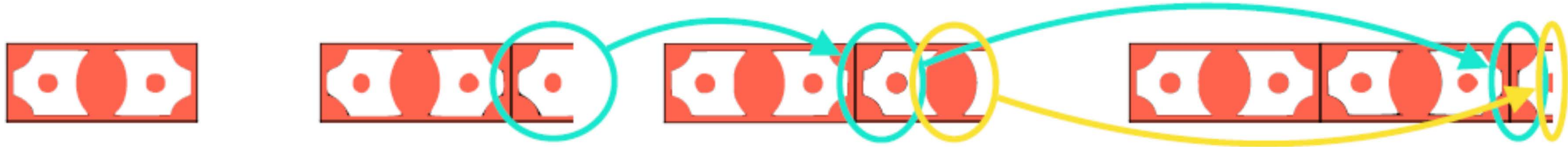


Meaning of the exponential (e)

Bank 3

Interests
33.3% every 4 months

Now	**In 4 months**	**In 8 months**	**In 1 year**



1	**1.33**	**1.77**	**2.37**
1	$1+\frac{1}{3}$	$\left(1+\frac{1}{3}\right)^2$	$\left(1+\frac{1}{3}\right)^3$
Original money	Add 1/3 of the money	Add 1/3 of the money	Add 1/3 of the money

Meaning of the exponential (e)

Bank 1

Interests
100% every year

(all your money once a year)

Now	$1
In 1 year	$2
In 2	$4
In 3	$8
In 4	$16

+100%
+100%
+100%
+100%

Bank 2

Interests
50% every 6 months

(half of your money twice a year)

Now	$1
In 1 year	$2.25
In 2	$5.06
In 3	$11.39
In 4	$25.63

+125%
+125%
+125%
+125%

Bank 3

Interests
33.3% every 4 months

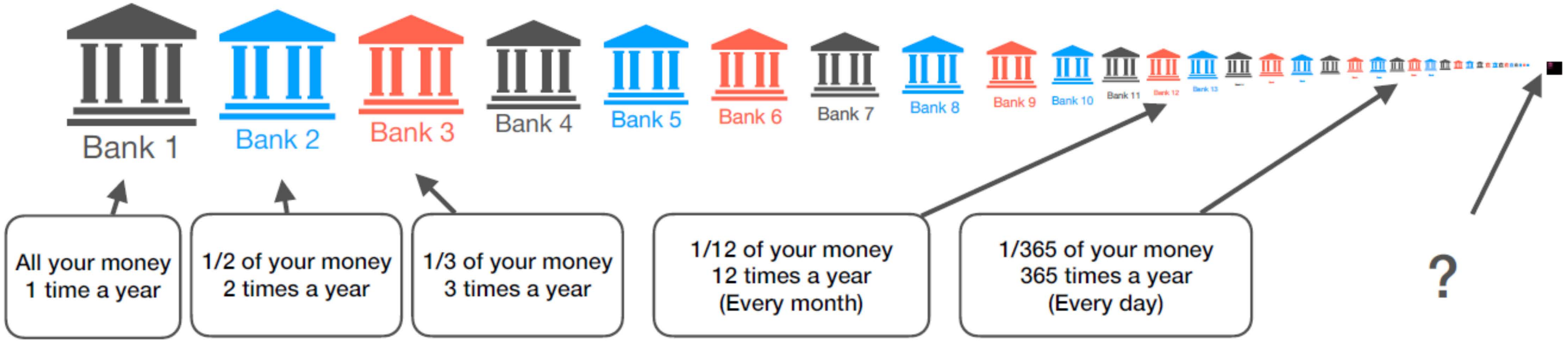
(A third of your money three times a year)

Now	$1
In 1 year	$2.37
In 2	$5.62
In 3	$13.32
In 4	$31.57

+137%
+137%
+137%
+137%

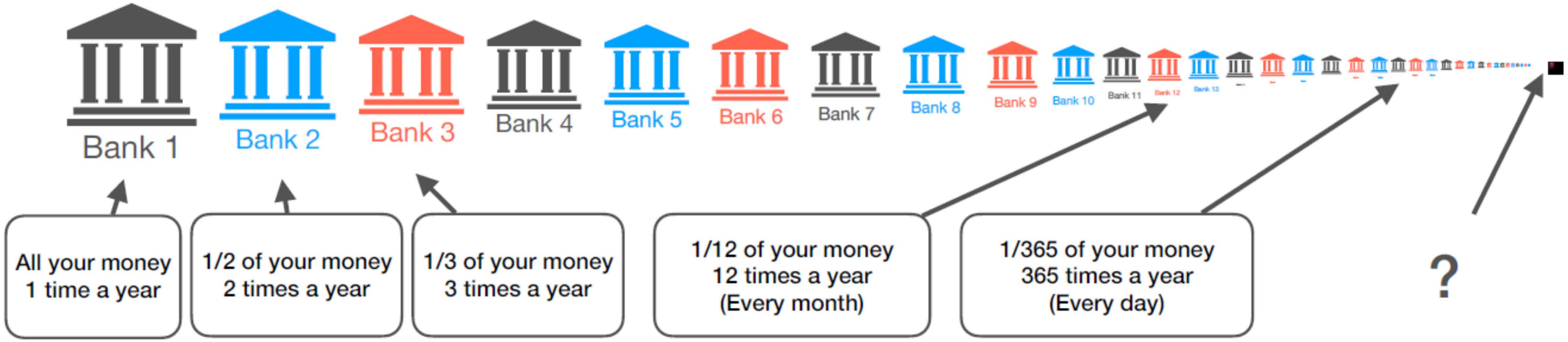
Meaning of the exponential (e)

A Lot of Banks

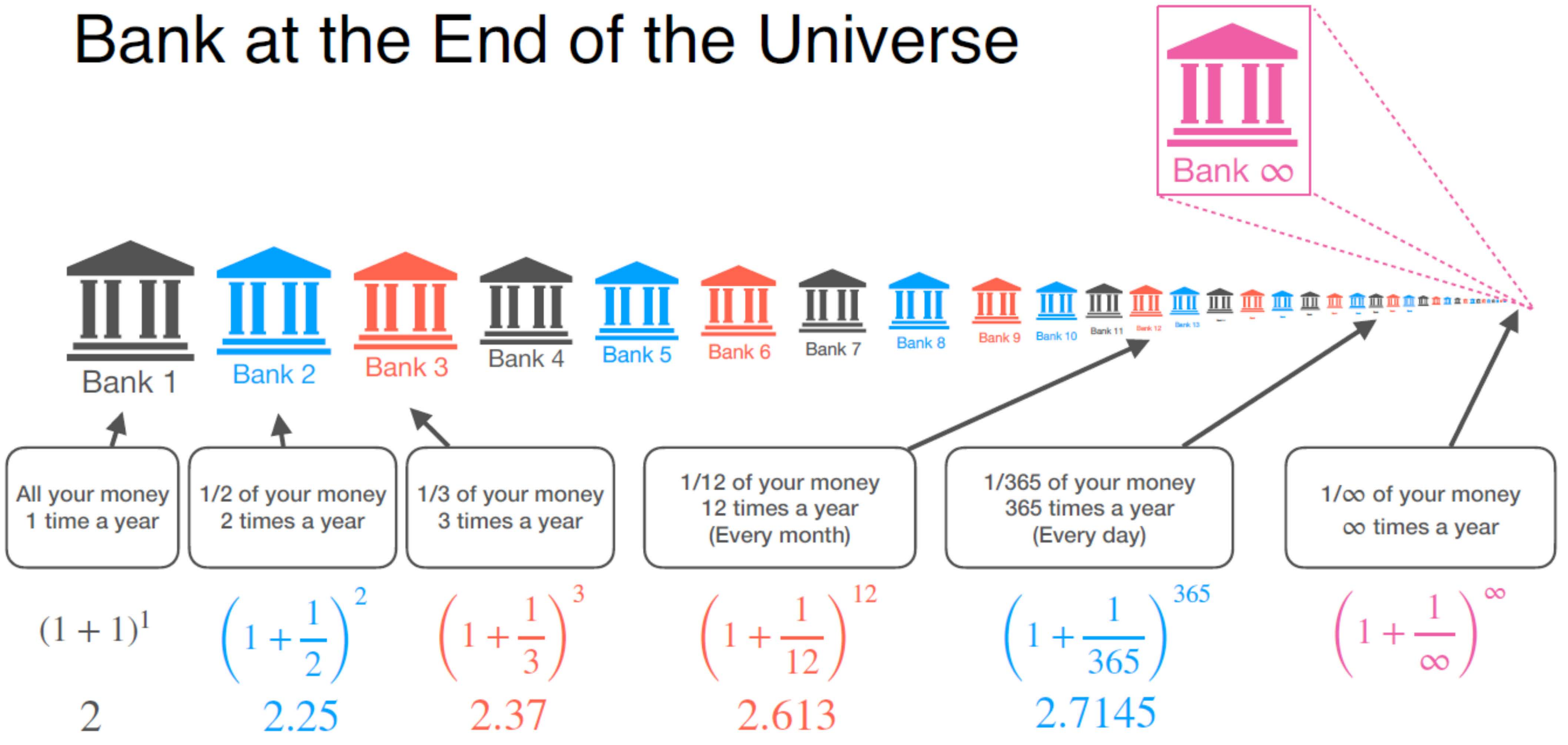


Meaning of the exponential (e)

A Lot of Banks



Meaning of the exponential (e)



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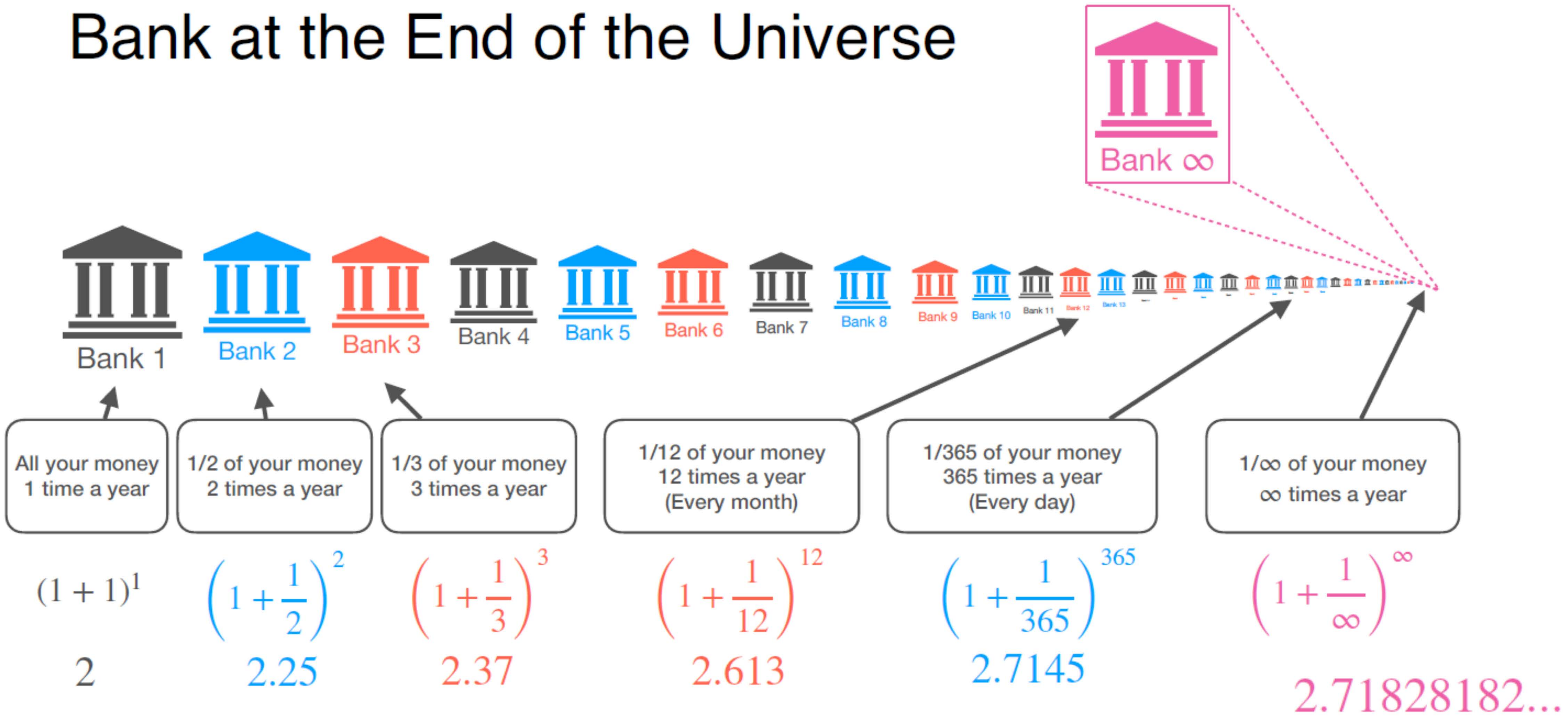
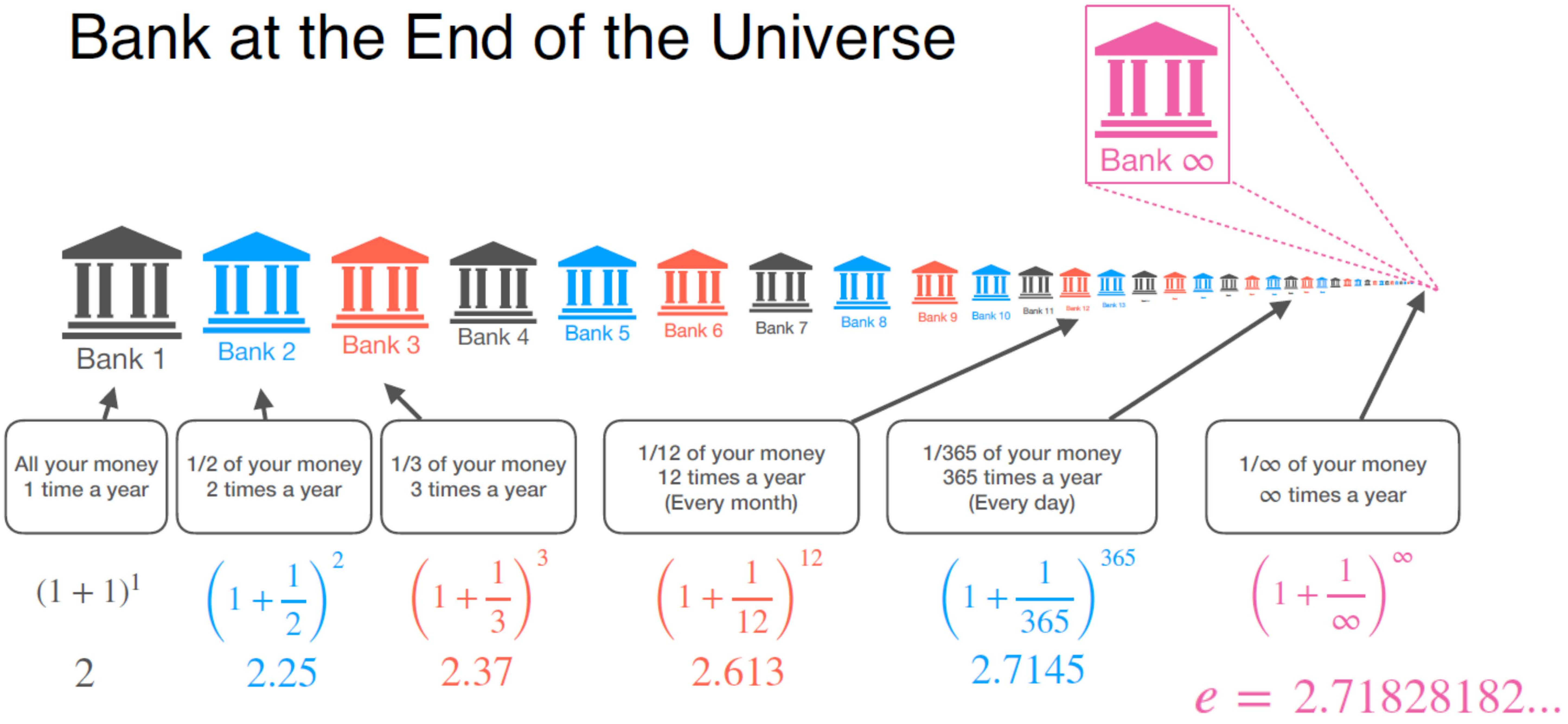


Meaning of the exponential (e)



Logarithm

$$\log(3) \qquad \log(x)$$

$$e^{?} = 3 \qquad e^{?} = x$$

Logarithm

Rules of Logarithms

Rule 1: $\log_b(M \cdot N) = \log_b M + \log_b N$

Rule 2: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

Rule 3: $\log_b\left(M^k\right) = k \cdot \log_b M$

Rule 4: $\log_b(1) = 0$

Rule 5: $\log_b(b) = 1$

Rule 6: $\log_b\left(b^k\right) = k$

Rule 7: $b^{\log_b(k)} = k$

Where:

$b > 0$ but $b \neq 1$, and M, N, and k are real numbers but M and N must be positive!

https://www.chilimath.com/lessons/advanced-algebra/logarithm-rules/

Natural logs

Properties of natural logs

1. $e^{\ln x} = x$
2. $\ln e^x = x$
3. $\ln xy = \ln x + \ln y$
4. $\ln \frac{x}{y} = \ln x - \ln y$
5. $\ln x^y = y \cdot \ln x$

Useful identities of natural logs

1. $\ln e = 1$
2. $\ln 1 = 0$

https://scientificallysound.org/2021/03/16/properties-of-exponents-and-logarithms/

The derivative

Definition of the Derivative

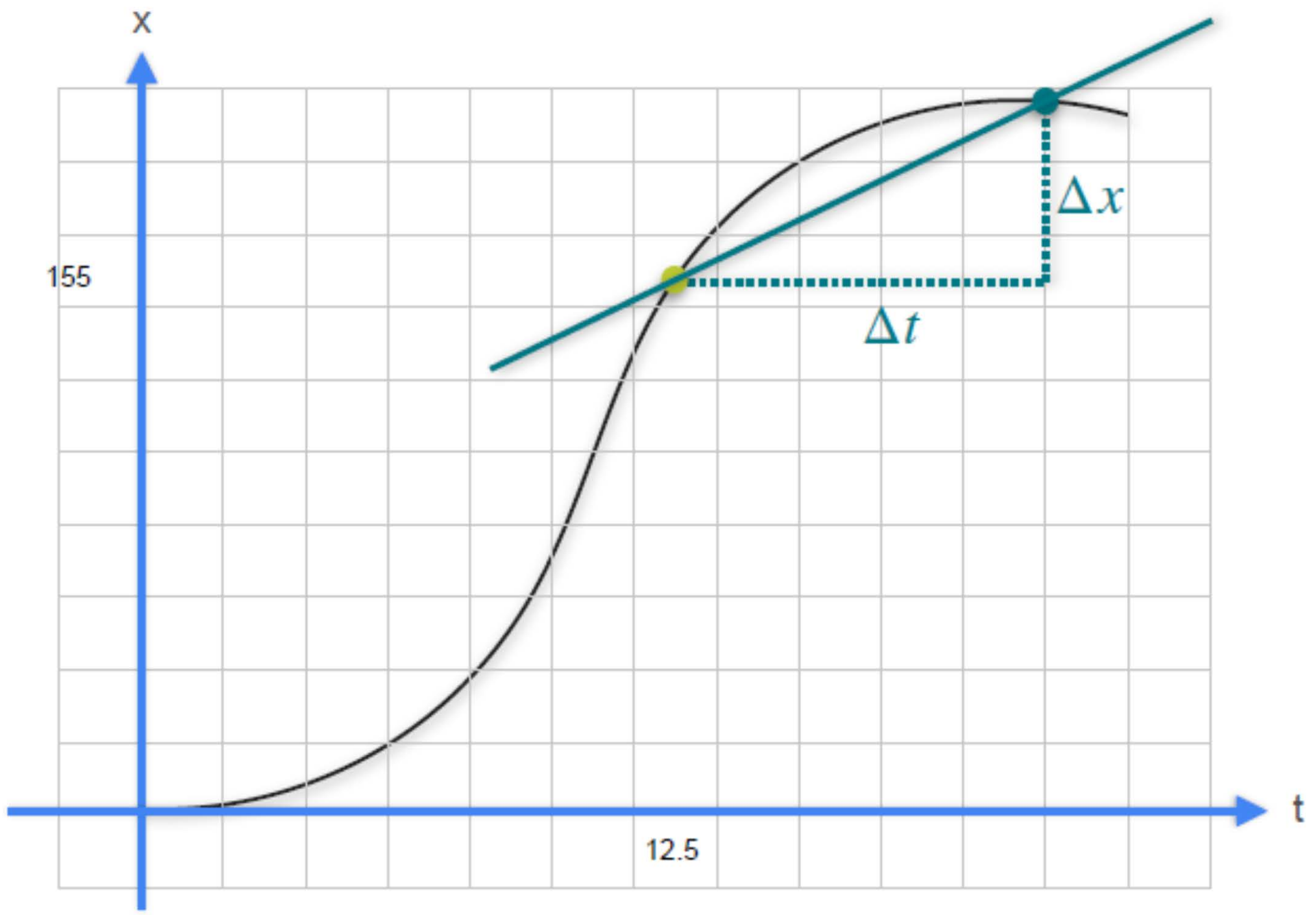
The derivative of a function $f(x)$ at a point $x = a$ is defined as the limit of the difference quotient as the increment h approaches zero:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit, if it exists, gives the slope of the tangent line to the function at the point a. The derivative $f'(a)$ is sometimes read as "the derivative of f at a", or "the rate of change" of f at a".

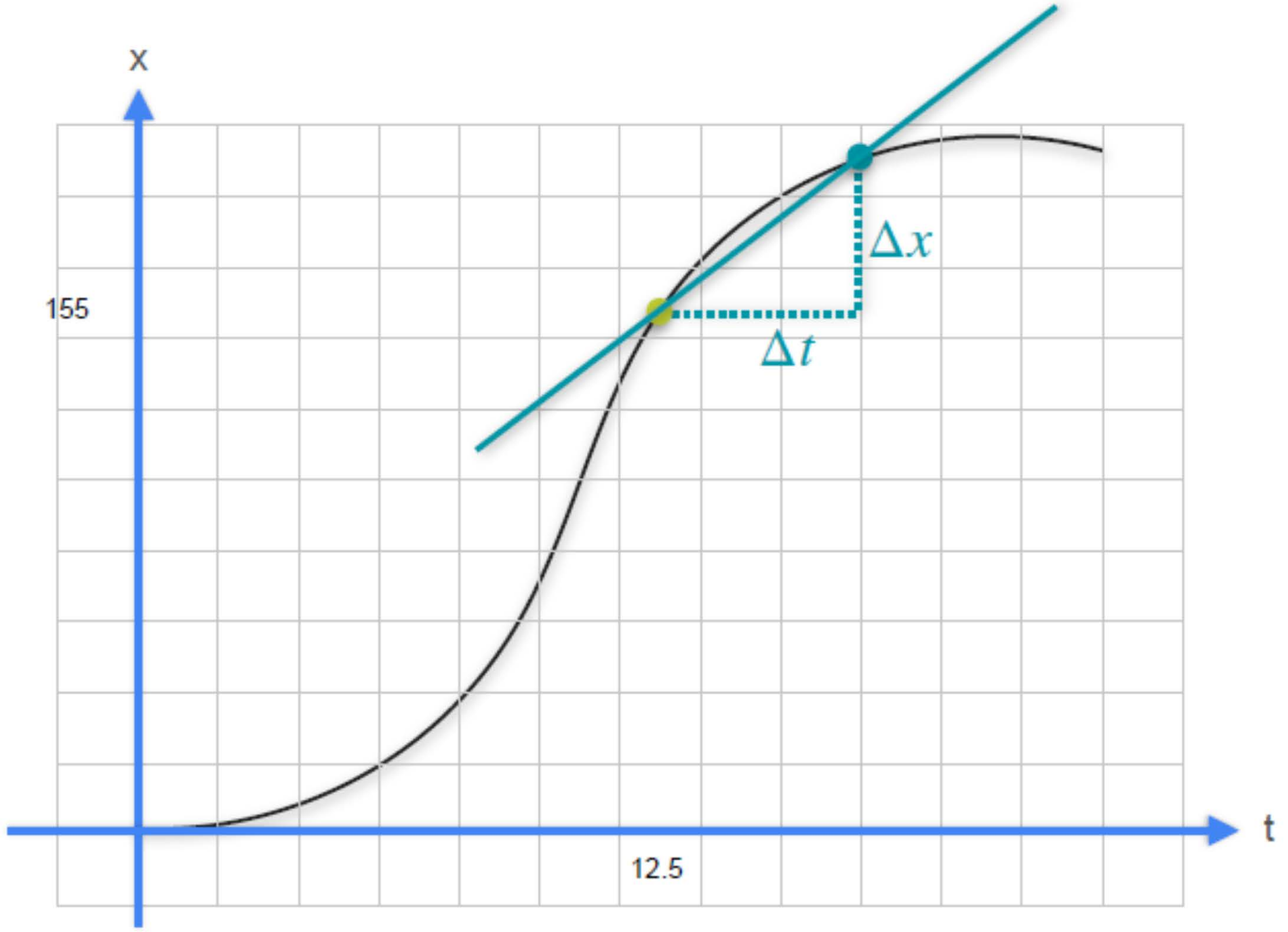
The derivative

$$\frac{\Delta x}{\Delta t}$$



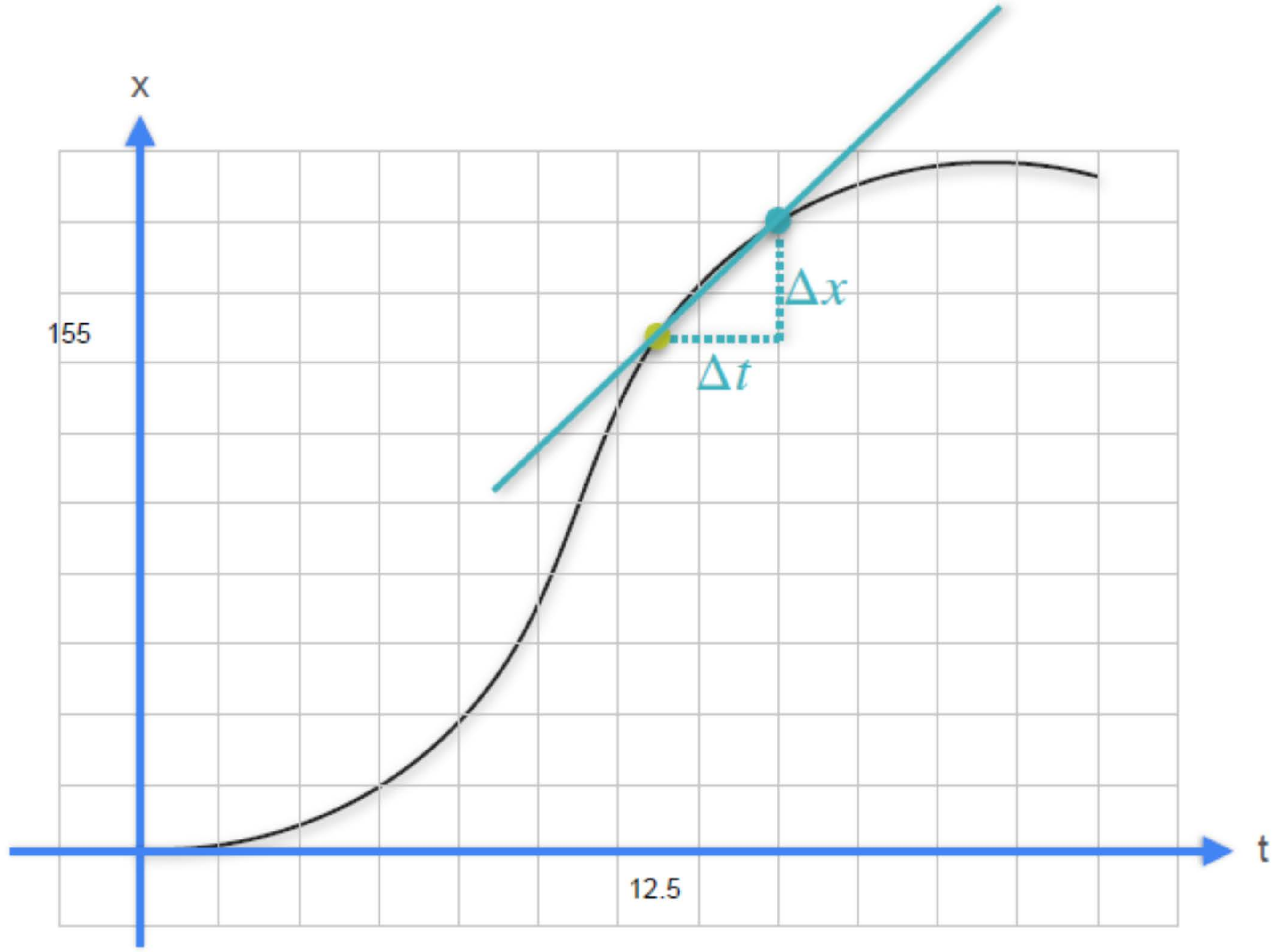
The derivative

$$\frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t}$$



The derivative

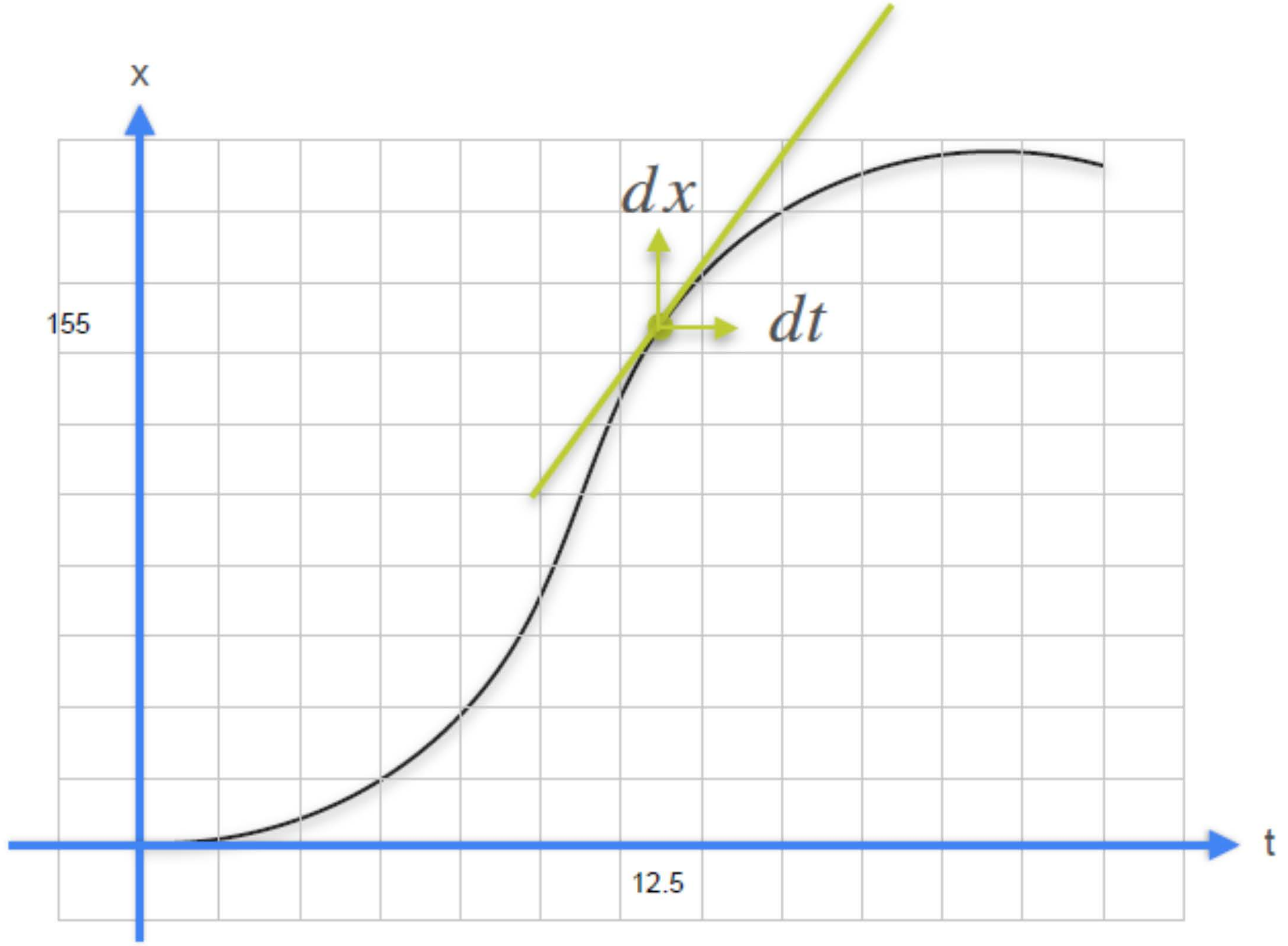
$$\frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t}$$



The derivative

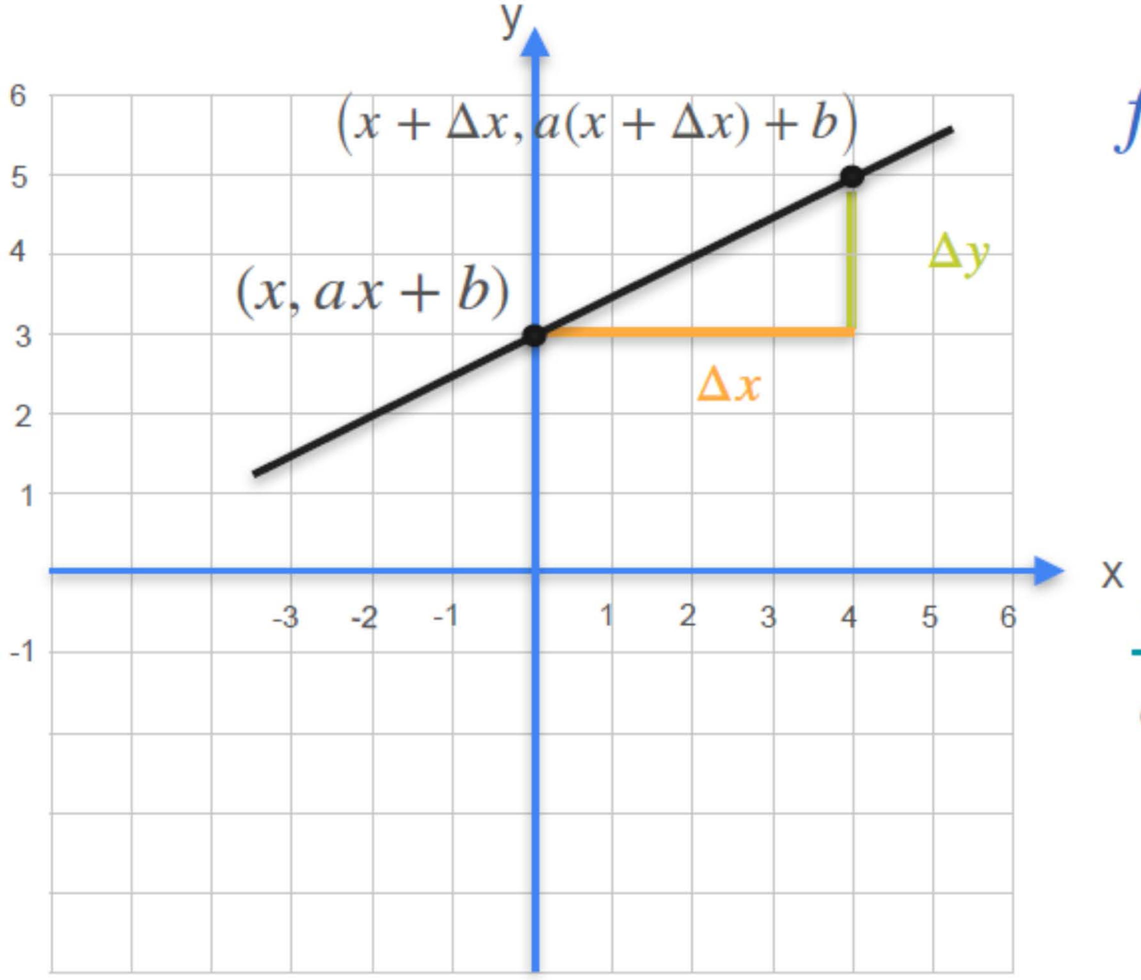
$$\frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t} \quad \frac{\Delta x}{\Delta t} \longrightarrow \frac{dx}{dt}$$

Derivative



The derivative

Derivative of a Line



$$f(x) = ax + b \Rightarrow f'(x) = a$$

$$\frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = a$$

$$\frac{df}{dx} \; \boxed{\frac{\Delta y}{\Delta x}} = \frac{a(\cancel{x} + \Delta x) + \cancel{b} - (\cancel{a}x + \cancel{b})}{\Delta x}$$

$$= a\frac{\cancel{\Delta x}}{\cancel{\Delta x}} = \boxed{a}$$

The derivative

Can you spot the pattern?

$$f(x) = x^{\boxed{2}}$$

-1

$$f'(x) = 2x^1$$

$$f(x) = x^{\boxed{3}}$$

-1

$$f'(x) = 3x^2$$

$$f(x) = x^{\boxed{-1}}$$

-1

$$f'(x) = (-1)x^{-2}$$

$$f(x) = x^n \quad \Rightarrow \quad f'(x) = \frac{d}{dx} f(x) = n x^{n-1}$$

The derivative

Can you spot the pattern?

$$f(x) = x^{\boxed{2}}$$

-1

$$f'(x) = 2x^1$$

$$f(x) = x^{\boxed{3}}$$

-1

$$f'(x) = 3x^2$$

$$f(x) = x^{\boxed{-1}}$$

-1

$$f'(x) = (-1)x^{-2}$$

$$f(x) = x^n \Rightarrow f'(x) = \frac{d}{dx} f(x) = n x^{n-1}$$

Derivative Formula Example

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Differentials

$d(\text{constant}) = 0$

$d(u^n) = nu^{n-1}\,du$

$d(e^u) = e^u\,du$

$d(\ln u) = \frac{1}{u}\,du$

$d(\sin u) = \cos u\,du$

$d(\cos u) = -\sin u\,du$

$d(\tan u) = \sec^2 u\,du$

$d(\cot u) = -\csc^2 u\,du$

$d(\sec u) = \sec u \tan u\,du$

$d(\csc u) = -\csc u \cot u\,du$

THANK YOU FOR LISTENING