Analysis and Control of Time-Varying and Perturbed Systems

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Advanced Nonlinear Control

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Main Objective

Based on:

- *Nonlinear Control* (Ch. 4): Time-varying and perturbed systems
- *Nonlinear Systems* (Ch. 9, 11.5): Stability under perturbations

Objective:

- Formulate practical and broadly applicable stability conditions
- Analyze stability under **vanishing perturbations** using comparison functions
- Study ultimate boundedness for systems with **non-vanishing perturbations**

Lyapunov Theory for Time-Varying Systems

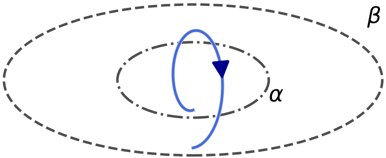
Assumptions:

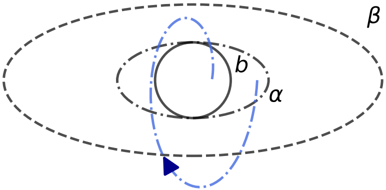
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \left\|\underline{x}\right\|^{\alpha} \leq V(t, \underline{x}) \leq c_2 \left\|\underline{x}\right\|^{\alpha}$$
$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^{\alpha}$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Perturbation Models: Vanishing vs. Non-Vanishing

Motivation: Real-world systems are subject to time dependence, modelling errors and external disturbances

System: $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- f locally Lipschitz
- g piecewise continuous in t, locally Lipschitz in $\underline{x}$
- g generally unknown, but bounded

Vanishing Perturbation: $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$
$\Rightarrow$ Exponential stability of the origin is preserved (robustness)

Non-Vanishing Perturbation: $g(\underline{x}, t) \not\to 0$ as $\underline{x} \to 0$
$\Rightarrow$ Solutions remain bounded (ultimate boundedness)

Lyapunov Stability Theorems

Assumptions:

- Origin $\underline{x} = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded

Globally Uniformly Exponentially Stable Equilibrium:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq c_3 \|\underline{x}\|^2 \text{ and } \|\frac{\partial V}{\partial \underline{x}}\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ with } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Lemma – Example

System:

$$\dot{\underline{x}} = -a\,\underline{x}(t) + g(t,\underline{x}), \quad \underline{x}(0) = 0, \quad a > 0$$

Assumptions:

- $\underline{x}(t) \geq 0 \quad \forall\, t \geq 0$
- $g(t,\underline{x}) \leq b\,\underline{x}(t) \quad \forall\, x \geq \underline{x} \geq 0$

Integral Condition:

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \gamma(\tau)\, d\tau = \underline{x}_0 + \int_{t_0}^{t} [-a\,\underline{x}(\tau) + b\,\underline{x}(\tau)]\, d\tau$$

Bound for Derivative:

$$\dot{\underline{x}}(t) \leq -a\,\underline{x}(t) + b\,\underline{x}(t) = -(a-b)\,\underline{x}(t)$$

Comparison Lemma – Example

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \dot{\underline{x}}(\tau)\, d\tau \leq \underline{x}_0 - (a-b) \int_{t_0}^{t} \underline{x}(\tau)\, d\tau$$

If $(a-b)\,\underline{x}$ is continuous, positive definite, and non-decreasing, then:

$$\lim_{t\to\infty} \underline{x}(t) = 0$$

which ensures that the system loses more than it gains.

Furthermore, exponential decay is guaranteed:

$$\underline{x}(t) \leq \underline{x}_0\, e^{-(a-b)t}$$

Lyapunov-Based Conditions for Boundedness

Assumptions:

- $\underline{x} = 0$ is exponentially stable for the nominal system
- Non-vanishing, bounded perturbation $g(\underline{x}, t)$
- Lyapunov function $V(t, \underline{x})$ is positive definite and radially unbounded
- Perturbation bound:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\,\theta r, \quad \theta \in (0, 1),\ r > 0$$

Lyapunov-Based Conditions for Boundedness

Exponential Stability:
For all initial conditions satisfying $\|\underline{x}(t_0)\| \leq \sqrt{c_1/c_2}\, r$:

$$\|\underline{x}(t)\| \leq k\, e^{-\gamma(t-t_0)} \|\underline{x}(t_0)\|, \quad t_0 \leq t \leq t_0 + T$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

Parameters:

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{c_4}{c_3} k \frac{\delta}{\theta}$$

Example: Bounded Disturbance Response

System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + d, \quad |d| \leq \delta$$

Interpretation:

- Mass-spring-damper system with constant external force
- Nominal system $(d = 0)$: exponentially stable

Lyapunov Candidate: $V(\underline{x}) = \underline{x}^T P \underline{x}$

- $P > 0$ solves $A^T P + PA = -Q$, with $Q = I$
- $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Bounds: $c_1 \|\underline{x}\|^2 \leq V(\underline{x}) \leq c_2 \|\underline{x}\|^2$

Example: Bounded Disturbance Response

Lyapunov Derivative:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\delta\|\underline{x}\|$$

Compare to:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\|g(\underline{x}, t)\|\|\underline{x}\|$$

Boundedness Condition:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad \theta \in (0, 1)$$

Then:

- $\|\underline{x}(t)\| \leq b$ for $t \geq t_0 + T$
- $b = \frac{c_4}{c_3}k\frac{\delta}{\theta}, \quad k = \sqrt{c_2/c_1}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}$

Conclusion: State converges to a ball around the origin; size scales with δ

Summary: Insights and Tradeoffs

Conceptual Insights

- Lyapunov methods unify time-varying and perturbed stability analysis
- Key difference: convergence to origin vs. convergence to a bound
- Ultimate boundedness reflects robustness in practical systems

Benefits & Limitations

- Systematic and broadly applicable
- But: Lyapunov construction is hard
- Often conservative; usually local

Takeaway:

Exponential stability of a nominal system can ensure robust behaviour in the presence of perturbations, but only if the perturbation is appropriately characterised.

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