Analysis and Control of Time-Varying and Perturbed Systems

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Advanced Nonlinear Control

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Main Objective

Based on:

- *Nonlinear Control* (Ch. 4): Time-varying and perturbed systems
- *Nonlinear Systems* (Ch. 9, 11.5): Stability under perturbations

Objective:

- Analyze stability under **vanishing perturbations** using comparison functions
- Study ultimate boundedness for systems with **non-vanishing perturbations**
- Apply Lyapunov-based methods for robust analysis of nonlinear, time-varying systems
- Formulate practical and broadly applicable stability conditions

Lyapunov Theory for Time-Varying Systems

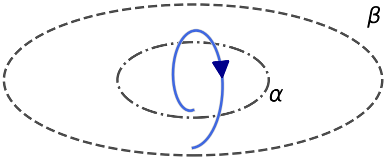
Assumptions:

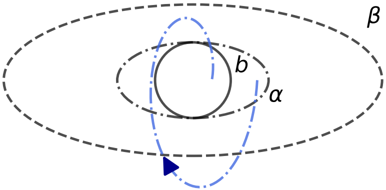
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \left\|\underline{x}\right\|^{\alpha} \leq V(t, \underline{x}) \leq c_2 \left\|\underline{x}\right\|^{\alpha}$$
$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^{\alpha}$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Perturbation Model for Vanishing Perturbation Models

- Exact Modelling rarley feasible due to modelling errors/external disturbances or parameter drift
- $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$
  - f is locally Lipschitz
  - g is piecewise continuous in t and locally Lipschitz
  - generally unknown but bounded
- $g(0, t)$ and $g(\underline{x}, t) = 0$ for $t \to \infty$

Lyapunov Stability Theorems

Assumptions:

- Origin $\underline{x} = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded

Globally Uniformly Exponentially Stable Equilibrium:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq c_3 \|\underline{x}\|^2 \text{ and } \|\frac{\partial V}{\partial \underline{x}}\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ with } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Functions

Assumptions:

- Origin $\underline{x} = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes

Exponential Stability:

$$\|g(\underline{x}, t)\| \leq \gamma(t)\|\underline{x}\| \text{ with } \int_{t_0}^{t} \gamma(\tau)d\tau < \epsilon(t - t_o) + \eta$$

$$\text{with } \epsilon < \frac{c_1 c_3}{c_2 c_4}$$

Example: Linear Time-Varying System

Perturbed Linear Time-Varying System:

$$\dot{\underline{x}} = [A(t) + B(t)]\underline{x}$$

Assumptions:

- $A(t)$ is uniformly bounded
- Origin of nominal system is uniformly exponentially stable
- $B(t) \to 0$ as $t \to \infty$
- Lyapunov function $V(t, \underline{x})$ is positive definite and derivative negative definite

$$g(t, \underline{x}) = B(t)\underline{x} \Rightarrow \|g(t, \underline{x})\| \leq \|B(t)\| \cdot \|\underline{x}\| = \gamma(t)\|\underline{x}\|$$

$\Rightarrow$ Exponential stability of nominal system is preserved under vanishing perturbations

Perturbation Model for Non-Vanishing Perturbations

- $g(t, \underline{x})$ does not vanish, persistent input or modeling error
- Origin is no longer an equilibrium
- Use ultimate boundedness: trajectories enter and remain in a compact set
- Stability becomes input-to-state-like (robustness to bounded disturbances)

Lyapunov Based Conditions for Boundednes

Assumptions:

- $\underline{x} = 0$ is exponentially stable for the nominal system
- Non-vanishing, bounded perturbation $g(\underline{x}, t)$
- Lyapunov function $V(t, \underline{x})$ is positive definite, radially unbounded

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \text{ with } \in (0,1), r > 0$$

Lyapunov Based Conditions for Boundednes

Exponetial Stability:
For all initial conditions $\|\underline{x}(t_0)| \leq \sqrt{c_1/c_2}r$

$$\|\underline{x}(t)\| \leq ke^{-\gamma(t-t_0)}\|\underline{x}(t_0)\|, t_0 \leq t \leq t_0 + T$$

$$\|\underline{x}(t)\| \leq b \forall\, t \geq t_0 + T$$

$$\text{where } k = \sqrt{\frac{c_2}{c_1}},\ \gamma = \frac{(1-\theta)c_3}{2c_2},\ b = \frac{c_4}{c_3}k\frac{\delta}{\theta}$$

Example: Non-Vanishing Perturbation in a Nonlinear System

- As a special case of non-vanishing perturbations
- Stability Theorems
- Case distinctions

Summary: Insights and Tradeoffs

Conceptual Insights

- Lyapunov methods unify time-varying and perturbed stability analysis
- Key difference: convergence to origin vs. convergence to a bound
- Ultimate boundedness reflects robustness in practical systems

Benefits & Limitations

- Systematic and broadly applicable
- But: Lyapunov construction is hard
- Often conservative; usually local

Takeaway:
Exponential stability of a nominal system can ensure robust behaviour in the presence of perturbations, but only if the perturbation is appropriately characterised.

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