Analysis and Control of Time-Varying and Perturbed Systems

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Why Study Time-Varying Perturbed Systems?

- **Real-World Imperfections:**
  - Systems rarely time-invariant
  - Parameters drift, components age, environment shifts
  - *Ex:* Aircraft dynamics change with fuel, altitude, air density
- **Uncertainty Omnipresent:**
  - External disturbances (wind gusts, load variations)
  - Internal uncertainties (sensor noise, actuator inaccuracies, unmodeled dynamics)
  - *Ex:* Robot arm payload
- **Performance & Robustness Demands:**
  - Modern control needs high performance (precision, speed) and robust stability
  - Ignoring variations $\rightarrow$ poor performance, instability, failure

Understanding Perturbation Types

General System Form:

$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$$

Vanishing Perturbation:

- $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$
- Preserves exponential stability
- Examples: modeling errors, unmodeled dynamics

Non-Vanishing Perturbation:

- $g(\underline{x}, t) \not\to 0$ as $\underline{x} \to 0$
- Leads to ultimate boundedness
- Examples: constant disturbances, sensor noise

Lyapunov Theory for Time-Varying Systems

Assumptions:

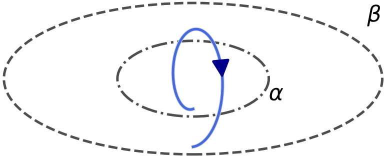
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentiable, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

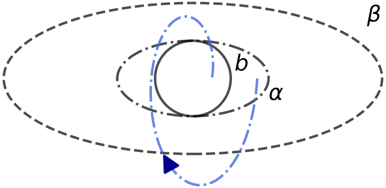
Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \, \|\underline{x}\|^{\alpha} \leq V(t, \underline{x}) \leq c_2 \, \|\underline{x}\|^{\alpha}$$

$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^{\alpha}$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Lyapunov Stability for Vanishing Perturbations

Problem: Analyze stability of $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- Nominal system ($\dot{\underline{x}} = f(\underline{x})$) exponentially stable

Assumptions for Exponential Stability:

- Perturbation $g(\underline{x}, t)$ vanishes (i.e., $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$)
- $V(t, \underline{x})$: continuously differentiable, positive definite, radially unbounded

Condition for Global Uniform Exponential Stability:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq -c_3 \|\underline{x}\|^2 \text{ and } \left\| \frac{\partial V}{\partial \underline{x}} \right\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ where } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Lemma - Example

Problem: Analyze stability of a scalar perturbed system

$$\dot{\underline{x}} = -a\,\underline{x}(t) + g(t, \underline{x}), \quad \underline{x}(0) = \underline{x}_0, \quad a > 0$$

Assumptions/Conditions:

- $\underline{x}(t) \geq 0 \quad \forall\, t \geq 0$
- $g(t, \underline{x}) \leq b\,\underline{x}(t) \quad \forall\, \underline{x} \geq 0$ and $0 \leq b < a$

Step 1: Integral form of the solution

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \dot{\underline{x}}(\tau)\, d\tau$$

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \left[-a\,\underline{x}(\tau) + b\,\underline{x}(\tau)\right] d\tau = \underline{x}_0 - (a-b) \int_{t_0}^{t} \underline{x}(\tau)\, d\tau$$

Comparison Lemma - Example

Step 2: Define comparison function $z(t)$:

$$z(t) = \underline{x}_0 - (a-b)\int_{t_0}^{t} z(\tau)\, d\tau$$

$$\dot{z}(t) = -(a-b)\, z(t), \quad z(t_0) = \underline{x}_0$$

Solution:

$$z(t) = \underline{x}_0\, e^{-(a-b)(t-t_0)}$$

Conclusion: Exponential Stability of the Perturbed System

$$\lim_{t\to\infty} \underline{x}(t) = 0$$

$$\underline{x}(t) \leq \underline{x}_0\, e^{-(a-b)(t-t_0)}$$

Non-Vanishing Perturbations: The Problem

Challenge:

- Perturbation does not vanish $\Rightarrow$ exact convergence to zero is impossible
- State $\underline{x}(t)$ will always be pushed away from origin

Goal: Ultimate Boundedness

- We seek boundedness around origin
- "Good enough" stability for real-world systems

Analogy:

- Like balancing a pencil in wind as it won't stay still
- We can bound how far it wobbles

Lyapunov Conditions for Ultimate Boundedness

System: $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- Origin is exponentially stable for nominal system ($g \equiv 0$)

Lyapunov Function Conditions:

- $V(\underline{x})$ positive definite, radially unbounded, continuously differentiable
- $\dot{V} \leq -c_3 \|\underline{x}\|^2$ for nominal dynamics
- $\left\| \frac{\partial V}{\partial \underline{x}} \right\| \leq c_4 \|\underline{x}\|$

Key Trade-off:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r, \quad \theta \in (0, 1)$$

Result: Initial Decay and Ultimate Boundedness

For $\|\underline{x}(t_0)\| \leq \sqrt{c_1/c_2}\, r$

Phase 1: Initial Exponential Decay $(t_0 \leq t \leq t_0 + T)$

$$\|\underline{x}(t)\| \leq k\, e^{-\gamma(t-t_0)} \|\underline{x}(t_0)\|$$

Phase 2: Ultimate Bound $(t \geq t_0 + T)$

$$\|\underline{x}(t)\| \leq b$$

Interpretation:

- System initially decays toward origin
- Perturbation prevents full convergence
- Final "wobble size" b depends on δ, $c_1 - c_4$

Example: Bounded Disturbance Response

Problem: Analyze Boundedness of a Perturbed Mass-Spring-Damper-System

$$\dot{\underline{x}}_1 = x_2$$
$$\dot{\underline{x}}_2 = -2x_1 - 3x_2 + d$$

with bounded disturbance $|d| \leq \delta$ and the origin of the nominal system being exponentially stable

Lyapunov Candidate Function: $V(\underline{x}) = \underline{x}^T P \underline{x}$

- $P > 0$ found by solving $A^T P + PA = -Q$
- For $Q = I$ and $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Example: Bounded Disturbance Response

Applying Lyapunov Stability Conditions:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\|d\|\|\underline{x}\| \leq -c_3\|\underline{x}\|^2 + c_4\delta\|\underline{x}\|$$

Conclusion: Ultimate Boundedness Guaranteed

- System state $\|\underline{x}(t)\|$ eventually converges to and remains within a bounded region:

$$\|\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

- The ultimate bound b and other parameters are:

$$b = \frac{c_4}{c_3} k \frac{\delta}{\theta}, \quad k = \sqrt{c_2/c_1}$$

- Size of the ultimate bound b directly **scales with the disturbance magnitude** δ

Key Insights & Practical Implications

Theoretical Insights:

- Lyapunov methods: unify time-varying and perturbed system analysis
- Perturbation type: dictates achievable stability properties
- Ultimate boundedness: models real-world robustness

Design Implications:

- Small, vanishing perturbations: maintains exponential convergence
- Persistent disturbances: design for bounded operation
- Robustness: requires accurate perturbation characterization

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