Analysis and Control of Time-Varying and Perturbed Systems

Keno Bürger

Advanced Nonlinear Control

16 June 2025

Chair of Automatic Control Engineering
Technical University of Munich

Why Study Time-Varying Perturbed Systems?

- **Real-World Imperfections:**
  - Systems rarely time-invariant
  - Parameters drift, components age, environment shifts
  - *Ex:* Aircraft dynamics change with fuel, altitude, air density
- **Uncertainty Omnipresent:**
  - External disturbances (wind gusts, load variations)
  - Internal uncertainties (sensor noise, actuator inaccuracies, unmodeled dynamics)
  - *Ex:* Robot arm payload
- **Performance & Robustness Demands:**
  - Modern control needs high performance (precision, speed) and robust stability
  - Ignoring variations $\rightarrow$ poor performance, instability, failure

Understanding Perturbation Types

General System Form:

$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$$

Vanishing Perturbation:

- $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$
- Preserves exponential stability
- Examples: modeling errors, unmodeled dynamics

Non-Vanishing Perturbation:

- $g(\underline{x}, t) \not\to 0$ as $\underline{x} \to 0$
- Leads to ultimate boundedness
- Examples: constant disturbances, sensor noise

Lyapunov Theory for Time-Varying Systems

Assumptions:

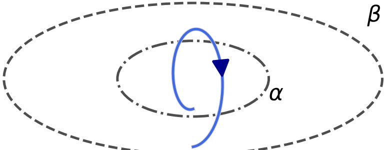
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentiable, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

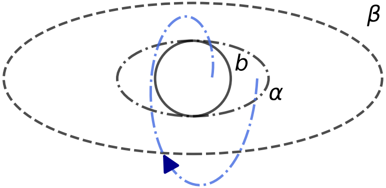
Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \left\|\underline{x}\right\|^{\alpha} \leq V(t, \underline{x}) \leq c_2 \left\|\underline{x}\right\|^{\alpha}$$

$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^{\alpha}$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Lyapunov Stability for Vanishing Perturbations

Problem: Analyze stability of $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- Nominal system ($\dot{\underline{x}} = f(\underline{x})$) exponentially stable

Assumptions for Exponential Stability:

- Perturbation $g(\underline{x}, t)$ vanishes (i.e., $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$)
- $V(t, \underline{x})$: continuously differentiable, positive definite, radially unbounded

Condition for Global Uniform Exponential Stability:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq -c_3 \|\underline{x}\|^2 \text{ and } \left\| \frac{\partial V}{\partial \underline{x}} \right\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ where } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Lemma - Example

Problem: Analyze stability of a scalar perturbed system

$$\dot{\underline{x}}(t) = -a\,\underline{x}(t) + g(t,\underline{x}), \quad \underline{x}(t_0) = \underline{x}_0, \quad a > 0$$

Assumptions/Conditions:

- $\underline{x}(t) \geq 0$ for all $t \geq t_0$
- $g(t,\underline{x})$ is bounded: $g(t,\underline{x}) \leq b\,\underline{x}(t) \quad \forall\,\underline{x} \geq 0$ and $0 \leq b < a$.

Step 1: Formulate the comparison inequality

$$\dot{\underline{x}}(t) \leq -a\,\underline{x}(t) + b\,\underline{x}(t)$$

$$\dot{\underline{x}}(t) \leq -(a-b)\,\underline{x}(t)$$

Step 2: Define a comparison system

$$\dot{\underline{z}}(t) = -(a-b)\,\underline{z}(t)$$

$$\underline{z}(t_0) = \underline{x}(t_0)$$

Comparison Lemma - Example

Step 3: Solve the comparison system

$$\underline{z}(t) = \underline{x}(t_0)\, e^{-(a-b)(t-t_0)}$$

Step 4: Apply the Comparison Lemma (Grönwall Inequality)

$$\underline{x}(t) \leq \underline{z}(t) \quad \forall\, t \geq t_0$$

Conclusion: Exponential Stability of the Perturbed System

$$\lim_{t\to\infty} e^{-(a-b)(t-t_0)} = \lim_{t\to\infty} \underline{x}(t) = 0$$

$$\underline{x}(t) \leq \underline{x}(t_0)\, e^{-(a-b)(t-t_0)}$$

Non-Vanishing Perturbations: The Problem

Challenge:

- Perturbation does not vanish $\Rightarrow$ exact convergence to zero is impossible
- State $\underline{x}(t)$ will always be pushed away from origin

Goal: Ultimate Boundedness

- We seek boundedness around origin
- “Good enough” stability for real-world systems

Analogy:

- Like balancing a pencil in wind as it won’t stay still
- We can bound how far it wobbles

Lyapunov Conditions for Ultimate Boundedness

System: $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- Origin is exponentially stable for nominal system $(g \equiv 0)$

Lyapunov Function Conditions:

- $V(\underline{x})$ positive definite, radially unbounded, continuously differentiable
- $\dot{V} \leq -c_3 \|\underline{x}\|^2$ for nominal dynamics
- $\left\| \frac{\partial V}{\partial \underline{x}} \right\| \leq c_4 \|\underline{x}\|$

Key Trade-off:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r, \quad \theta \in (0, 1)$$

Result: Initial Decay and Ultimate Boundedness

For $\|\underline{x}(t_0)\| \leq \sqrt{c_1/c_2}\, r$

Phase 1: Initial Exponential Decay $(t_0 \leq t \leq t_0 + T)$

$$\|\underline{x}(t)\| \leq k\, e^{-\gamma(t-t_0)} \|\underline{x}(t_0)\|$$

Phase 2: Ultimate Bound $(t \geq t_0 + T)$

$$\|\underline{x}(t)\| \leq b$$

Interpretation:

- System initially decays toward origin
- Perturbation prevents full convergence
- Final "wobble size" b depends on δ, $c_1 - c_4$

Example: Bounded Disturbance Response

Problem: Analyze Boundedness of a Perturbed Mass-Spring-Damper-System

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1 - 3x_2 + d\end{aligned}$$

with bounded disturbance $|d| \leq \delta$ and the origin of the nominal system being exponentially stable

Lyapunov Candidate Function: $V(\underline{x}) = \underline{x}^T P \underline{x}$

- $P > 0$ found by solving $A^T P + PA = -Q$
- For $Q > 0$ and $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Example: Bounded Disturbance Response

Applying Lyapunov Stability Conditions:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\|d\|\,\|\underline{x}\| \leq -c_3\|\underline{x}\|^2 + c_4\delta\|\underline{x}\|$$

Conclusion: Ultimate Boundedness Guaranteed

- System state $\|\underline{x}(t)\|$ eventually converges to and remains within a bounded region:

$$\|\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

- The ultimate bound b and other parameters are:

$$b = \frac{c_4}{c_3} k \frac{\delta}{\theta}, \quad k = \sqrt{c_2/c_1}$$

- Size of the ultimate bound b directly **scales with the disturbance magnitude** δ

Key Insights & Practical Implications

Theoretical Insights:

- Lyapunov methods: unify time-varying and perturbed system analysis
- Perturbation type: dictates achievable stability properties
- Ultimate boundedness: models real-world robustness

Design Implications:

- Small, vanishing perturbations: maintains exponential convergence
- Persistent disturbances: design for bounded operation
- Robustness: requires accurate perturbation characterization

References

[1] Giovanni Gallavotti. **Perturbation Theory**. In: *Perturbation Theory*. Ed. by Giuseppe Gaeta. Series Title: Encyclopedia of Complexity and Systems Science Series. New York, NY: Springer US, 2009, pp. 1–14. ISBN: 978-1-0716-2620-7 978-1-0716-2621-4. DOI: 10.1007/978-1-0716-2621-4_396.

[2] Hassan K. Khalil. **Nonlinear control**. Global edition. Boston Munich: Pearson, 2015. ISBN: 978-1-292-06069-9.

[3] Hassan K. Khalil. **Nonlinear systems**. Pearson new internat. ed., 3. ed. Always learning. Harlow: Pearson Education, 2014. ISBN: 978-1-292-05385-1.

[4] Shenyu Liu. **Unified stability criteria for perturbed LTV systems with unstable instantaneous dynamics**. Feb. 2022. DOI: 10.48550/arXiv.2111.07443.

[5] James Murdok. **1. Root Finding**. en. In: *Perturbations: Theory and Methods*. Society for Industrial and Applied Mathematics, Jan. 1999, pp. 3–81. ISBN: 978-0-89871-443-2 978-1-61197-109-5. DOI: 10.1137/1.9781611971095.ch1.

Grönwall's Inequality

$\underline{w}(t) = \underline{x}(t) - \underline{z}(t)$ with $\underline{w}(t_0) = 0$

$\dot{\underline{w}}(t) = \dot{\underline{x}}(t) - \dot{\underline{z}}(t) \leq -(a-b)(\underline{x}(t) - \underline{z}(t)) = -(a-b)\underline{w}(t)$

$\underline{w}(t) = 0 \implies \underline{w}(t) \leq 0 \implies \underline{w}(t) \leq \underline{z}(t)$

Lyapunov Candidate Function and Derivative

Candidate Function:

$$V(x) = x^T P x$$

Time Derivative:

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

Substitute $\dot{x} = Ax + Bd$:

$$\dot{V} = (Ax + Bd)^T P x + x^T P (Ax + Bd)$$

$$\dot{V} = x^T A^T P x + d^T B^T P x + x^T P A x + x^T P B d$$

$$\dot{V} = x^T (A^T P + P A) x + d^T B^T P x + x^T P B d$$

Simplifying the Expression

Use the Lyapunov Equation:

$$A^T P + PA = -Q \Rightarrow \dot{V} = -x^T Qx + d^T B^T Px + x^T PBd$$

Group the Cross Terms:

$$\dot{V} = -x^T Qx + 2x^T PBd$$

Bounding the Quadratic Term: Since $Q > 0$, $-x^T Qx \leq -\lambda_{\min}(Q)\|x\|^2$
Let $c_3 = \lambda_{\min}(Q)$, then:

$$\dot{V} \leq -c_3\|x\|^2 + 2x^T PBd$$

Bounding the Cross Term

Use Cauchy-Schwarz Inequality:

$$|2x^T PBd| \leq 2\|x\|\|PB\|\|d\|$$

Define $c_4 = 2\|PB\|$, then:

$$\dot{V} \leq -c_3\|x\|^2 + c_4\|x\|\|d\|$$

If $\|d\| \leq \delta$:

$$\dot{V} \leq -c_3\|x\|^2 + c_4\delta\|x\|$$

This inequality bounds the Lyapunov function derivative under bounded disturbances.