Analysis and Control of Time-Varying and Perturbed Systems

Keno Bürger
Advanced Nonlinear Control
16 June 2025

Chair of Automatic Control Engineering
Technical University of Munich

Main Objective

Based on:

- *Nonlinear Control* (Ch. 4): Time-varying and perturbed systems
- *Nonlinear Systems* (Ch. 9, 11.5): Stability under perturbations

Objective:

- Analyze stability under **vanishing perturbations** using comparison functions
- Study ultimate boundedness for systems with **non-vanishing perturbations**
- Apply Lyapunov-based methods for robust analysis of nonlinear, time-varying systems
- Formulate practical and broadly applicable stability conditions

Lyapunov Theory for Time-Varying Systems

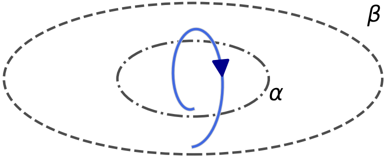
Assumptions:

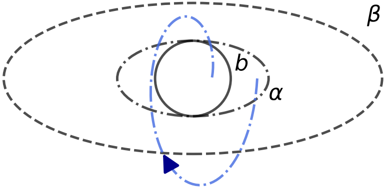
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \left\|\underline{x}\right\|^{\alpha} \leq V(t, \underline{x}) \leq c_2 \left\|\underline{x}\right\|^{\alpha}$$
$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^{\alpha}$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Perturbation Models: Vanishing vs. Non-Vanishing

Motivation: Exact modeling is rarely possible due to disturbances, unmodeled dynamics, or parameter drift.

System: $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- f locally Lipschitz
- g piecewise continuous in t, locally Lipschitz in $\underline{x}$
- g generally unknown, but bounded

Vanishing Perturbation: $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$
$\Rightarrow$ Exponential stability of the origin is preserved (robustness)

Non-Vanishing Perturbation: $g(\underline{x}, t) \not\to 0$ as $\underline{x} \to 0$
$\Rightarrow$ Solutions remain bounded (ultimate boundedness)

Lyapunov Stability Theorems

Assumptions:

- Origin $\underline{x} = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded

Globally Uniformly Exponentially Stable Equilibrium:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq c_3 \|\underline{x}\|^2 \text{ and } \|\frac{\partial V}{\partial \underline{x}}\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ with } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Lemma – Example

System:

$$\dot{\underline{x}} = -a\,\underline{x}(t) + g(t,\underline{x}), \quad \underline{x}(0) = 0, \quad a > 0$$

Assumptions:

- $\underline{x}(t) \geq 0 \quad \forall\, t \geq 0$
- $g(t,\underline{x}) \leq b\,\underline{x}(t) \quad \forall\, x \geq \underline{x} \geq 0$

Integral Condition:

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \gamma(\tau)\, d\tau = \underline{x}_0 + \int_{t_0}^{t} [-a\,\underline{x}(\tau) + b\,\underline{x}(\tau)]\, d\tau$$

Bound for Derivative:

$$\dot{\underline{x}}(t) \leq -a\,\underline{x}(t) + b\,\underline{x}(t) = -(a-b)\,\underline{x}(t)$$

Comparison Lemma – Example

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \dot{\underline{x}}(\tau)\, d\tau \leq \underline{x}_0 - (a-b) \int_{t_0}^{t} \underline{x}(\tau)\, d\tau$$

If $(a-b)\,\underline{x}$ is continuous, positive definite, and non-decreasing, then:

$$\lim_{t \to \infty} \underline{x}(t) = 0$$

which ensures that the system loses more than it gains.

Furthermore, exponential decay is guaranteed:

$$\underline{x}(t) \leq \underline{x}_0\, e^{-(a-b)t}$$

Lyapunov-Based Conditions for Boundedness

Assumptions:

- $\underline{x} = 0$ is exponentially stable for the nominal system
- Non-vanishing, bounded perturbation $g(\underline{x}, t)$
- Lyapunov function $V(t, \underline{x})$ is positive definite and radially unbounded
- Perturbation bound:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad \theta \in (0, 1),\ r > 0$$

Lyapunov-Based Conditions for Boundedness

Exponential Stability:
For all initial conditions satisfying $\|\underline{x}(t_0)\| \leq \sqrt{c_1/c_2}\, r$:

$$\|\underline{x}(t)\| \leq k\, e^{-\gamma(t-t_0)} \|\underline{x}(t_0)\|, \quad t_0 \leq t \leq t_0 + T$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

Parameters:

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{c_4}{c_3} k \frac{\delta}{\theta}$$

Example: Bounded Disturbance Response

System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + d, \quad |d| \leq \delta$$

Interpretation:

- Mass-spring-damper system with constant external force
- Nominal system $(d = 0)$: exponentially stable

Lyapunov Candidate: $V(\underline{x}) = \underline{x}^T P \underline{x}$

- $P > 0$ solves $A^T P + PA = -Q$, with $Q = I$
- $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Bounds: $c_1 \|\underline{x}\|^2 \leq V(\underline{x}) \leq c_2 \|\underline{x}\|^2$

Example: Bounded Disturbance Response

Lyapunov Derivative:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\delta\|\underline{x}\|$$

Compare to:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\|g(\underline{x}, t)\|\|\underline{x}\|$$

Boundedness Condition:

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad \theta \in (0, 1)$$

Then:

- $\|\underline{x}(t)\| \leq b$ for $t \geq t_0 + T$
- $b = \dfrac{c_4}{c_3}k\dfrac{\delta}{\theta}, \quad k = \sqrt{c_2/c_1}, \quad \gamma = \dfrac{(1-\theta)c_3}{2c_2}$

Conclusion: State converges to a ball around the origin; size scales with δ

Summary: Insights and Tradeoffs

Conceptual Insights

- Lyapunov methods unify time-varying and perturbed stability analysis
- Key difference: convergence to origin vs. convergence to a bound
- Ultimate boundedness reflects robustness in practical systems

Benefits & Limitations

- Systematic and broadly applicable
- But: Lyapunov construction is hard
- Often conservative; usually local

Takeaway:
Exponential stability of a nominal system can ensure robust behaviour in the presence of perturbations, but only if the perturbation is appropriately characterised.

References I

PDF Chaoqun He, Renjie Luo, Yuzhuo Bai, Shengding Hu, Zhen Leng Thai, Junhao Shen, Jinyi Hu, Xu Han, Yujie Huang, Yuxiang Zhang, Jie Liu, Lei Qi, Zhiyuan Liu and Maosong Sun.
OlympiadBench: A Challenging Benchmark for Promoting AGI with Olympiad-Level Bilingual Multimodal Scientific Problems. 2024. eprint: 2402.14008.

PDF Andreas Hochlehnert, Hardik Bhatnagar, Vishaal Udandarao, Samuel Albanie, Ameya Prabhu and Matthias Bethge.
A Sober Look at Progress in Language Model Reasoning: Pitfalls and Paths to Reproducibility. Apr. 2025. DOI: 10.48550/arXiv.2504.07086.

PDF Jie Huang and Kevin Chen-Chuan Chang. **Towards Reasoning in Large Language Models: A Survey**. May 2023. DOI: 10.48550/arXiv.2212.10403.

PDF Naman Jain, King Han, Alex Gu, Wen-Ding Li, Fanjia Yan, Tianjun Zhang, Sida Wang, Armando Solar-Lezama, Koushik Sen and Ion Stoica.
LiveCodeBench: Holistic and Contamination Free Evaluation of Large Language Models for Code. June 2024. DOI: 10.48550/arXiv.2403.07974.

PDF Niklas Muennighoff, Zitong Yang, Weijia Shi, Xiang Lisa Li, Li Fei-Fei, Hannaneh Hajishirzi, Luke Zettlemoyer, Percy Liang, Emmanuel Candès and Tatsunori Hashimoto. **s1: Simple test-time scaling**. Mar. 2025. DOI: 10.48550/arXiv.2501.19393.

PDF Simon Ott, Konstantin Hebenstreit, Valentin Liévin, Christoffer Egeberg Hother, Milad Moradi, Maximilian Mayrhauser, Robert Praas, Ole Winther and Matthias Samwald. **ThoughtSource: A central hub for large language model reasoning data**. In: *Scientific Data* 10.1 (Aug. 2023). ISSN: 2052-4463. DOI: 10.1038/s41597-023-02433-3.

References II

Fengli Xu, Qianyue Hao, Zefang Zong, Jingwei Wang, Yunke Zhang, Jingyi Wang, Xiaochong Lan, Jiahui Gong, Tianjian Ouyang, Fanjin Meng, Chenyang Shao, Yuwei Yan, Qinglong Yang, Yiwen Song, Sijian Ren, Xinyuan Hu, Yu Li, Jie Feng, Chen Gao and Yong Li. **Towards Large Reasoning Models: A Survey of Reinforced Reasoning with Large Language Models**. Jan. 2025. DOI: 10.48550/arXiv.2501.09686.

Kunhao Zheng, Jesse Michael Han and Stanislas Polu. **MiniF2F: a cross-system benchmark for formal Olympiad-level mathematics**. Feb. 2022. DOI: 10.48550/arXiv.2109.00110.