Analysis and Control of Time-Varying and Perturbed Systems

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Advanced Nonlinear Control

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Main Objective

Based on:

- *Nonlinear Control* (Ch. 4): Time-varying and perturbed systems
- *Nonlinear Systems* (Ch. 9, 11.5): Stability under perturbations

Objective:

- Analyze stability under **vanishing perturbations** using comparison functions
- Study ultimate boundedness for systems with **non-vanishing perturbations**
- Apply Lyapunov-based methods for robust analysis of nonlinear, time-varying systems
- Formulate practical and broadly applicable stability conditions

Lyapunov Theory for Time-Varying Systems

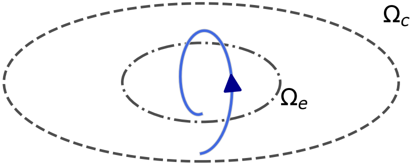
Assumptions:

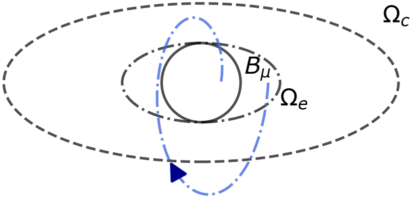
- Origin $x = 0$ is an equilibrium point
- Lyapunov function $V(t,x)$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \|x\|^\alpha \leq V(t,x) \leq c_2 \|x\|^\alpha$$
$$\dot{V}(t,x) \leq -c_3 \|x\|^\alpha$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|x(t_0)\| \leq \alpha \Rightarrow \|x(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c), \; \beta > 0, \; \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|x(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Perturbation Model for Vanishing Perturbation Models

- Exact Modelling rarley feasible due to modelling errors/external disturbances or parameter drift
- $\dot{x} = f(x) + g(x, t)$
  - f is locally Lipschitz
  - g is piecewise continuous in t and locally Lipschitz
  - generally unknown but bounded
- $g(0, t)$ and $g(x, t) = 0$ for $t \to \infty$

Lyapunov Stability Theorems

Assumptions:

- Origin $x = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes
- Lyapunov function $V(t, x)$ is continuously differentialbe, positive definite and radially unbounded

Globally Uniformly Exponentially Stable Equilibrium:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) \leq c_3 \|x\|^2 \text{ and } \|\frac{\partial V}{\partial x}\| \leq c_4 \|x\|$$

$$\|g(x, t)\| \leq \gamma \|x\| \text{ with } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Functions

Assumptions:

- Origin $x = 0$ is an exponentially stable equilibrium point
- Perturbation vanishes

Exponential Stability:

$$\|g(x,t)\| \leq \gamma(t)\|x\| \text{ with } \int_{t_0}^{t} \gamma(\tau)d\tau < \epsilon(t - t_o) + \eta$$

$$\text{with } \epsilon < \frac{c_1 c_3}{c_2 c_4}$$

Example: Linear Time-Varying System

Perturbed Linear Time-Varying System:

$$\dot{x} = [A(t) + B(t)]x$$

Assumptions:

- $A(t)$ is uniformly bounded
- Origin of nominal system is uniformly exponentially stable
- $B(t) \to 0$ as $t \to \infty$
- Lyapunov function $V(t, x)$ is positive definite and derivative negative definite

$$g(t, x) = B(t)x \Rightarrow \|g(t, x)\| \leq \|B(t)\| \cdot \|x\| = \gamma(t)\|x\|$$

$\Rightarrow$ Exponential stability of nominal system is preserved under vanishing perturbations

Perturbation Model for Non-Vanishing Perturbations

- $g(t, x)$ does not vanish, persistent input or modeling error
- Origin is no longer an equilibrium
- Use ultimate boundedness: trajectories enter and remain in a compact set
- Stability becomes input-to-state-like (robustness to bounded disturbances)

Lyapunov Based Conditions for Boundednes

Assumptions:

- Origin $x = 0$ is an exponentially stable equilibrium point
- Non vanishing perturbation
- Lyapunov function $V(t, x)$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite
- Perturbation is bounded

$$\|g(x,t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \text{ with } \in (0,1), r > 0$$

Lyapunov Based Conditions for Boundednes

Exponetial Stability:

For all initial conditions $\|x(t_0)| \leq \sqrt{c_1/c_2}r$

$$\|x(t)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0)\|, t_0 \leq t \leq t_0 + T$$

$$\|x(t)\| \leq b \forall\, t \geq t_0 + T$$

$$\text{where } k = \sqrt{\frac{c_2}{c_1}},\ \gamma = \frac{(1-\theta)c_3}{2c_2},\ b = \frac{c_4}{c_3}k\frac{\delta}{\theta}$$

Example: Non-Vanishing Perturbation in a Nonlinear System

- As a special case of non-vanishing perturbations
- Stability Theorems
- Case distinctions

Conceptual Links Between Sections

- **Lyapunov Functions:** Central in both cases, quantify energy and decay
- **Main Difference:** Convergence to origin vs. convergence to a bounded set
- **Comparison Lemma:** Powerful for estimating solution bounds in both cases
- **Ultimate Boundedness:** Reflects robustness, common in practical control
- **Design Implication:** Small perturbations can be tolerated if decay is fast; persistent ones require constraint-aware design

Takeaway: Both frameworks enhance system analysis in the presence of uncertainty. Knowing the type of perturbation guides appropriate control strategies.

Benefits and Drawbacks of Lyapunov-Based Analysis

Benefits

- General applicability:
- Rigorous and systematic:
- Insightful design tool:
- Comparison lemma:
- Handles uncertainty:

Drawbacks

- Constructing Lyapunov functions is hard:
- Results are conservative:
- Limited to local regions:
- Quantitative bounds may be loose:

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