Analysis and Control of Time-Varying and Perturbed Systems

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Why Study Time-Varying Perturbed Systems?

- **Real-World Imperfections:**
  - Systems rarely time-invariant
  - Parameters drift, components age, environment shifts
  - *Ex:* Aircraft dynamics change with fuel, altitude, air density
- **Uncertainty Omnipresent:**
  - External disturbances (wind gusts, load variations)
  - Internal uncertainties (sensor noise, actuator inaccuracies, unmodeled dynamics)
  - *Ex:* Robot arm payload
- **Performance & Robustness Demands:**
  - Modern control needs high performance (precision, speed) and robust stability
  - Ignoring variations $\rightarrow$ poor performance, instability, failure

Understanding Perturbation Types

General System Form:

$$\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$$

Vanishing Perturbation:

- $g(\underline{x}, t) \to 0$ as $\underline{x} \to 0$
- Preserves exponential stability
- Examples: modeling errors, unmodeled dynamics

Non-Vanishing Perturbation:

- $g(\underline{x}, t) \not\to 0$ as $\underline{x} \to 0$
- Leads to ultimate boundedness
- Examples: constant disturbances, sensor noise

Lyapunov Theory for Time-Varying Systems

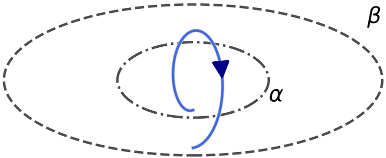
Assumptions:

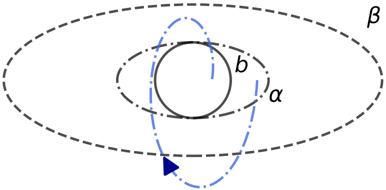
- Origin $\underline{x} = 0$ is an equilibrium point
- Lyapunov function $V(t, \underline{x})$ is continuously differentialbe, positive definite and radially unbounded
- Derivative of Lyapunov function is negative definite

Globally uniformly exponentially stable:

$$\exists\, c_i, \alpha > 0 : c_1 \left\|\underline{x}\right\|^\alpha \leq V(t, \underline{x}) \leq c_2 \left\|\underline{x}\right\|^\alpha$$
$$\dot{V}(t, \underline{x}) \leq -c_3 \|\underline{x}\|^\alpha$$

Boundedness and Ultimate Boundedness





Boundedness:

$$\|\underline{x}(t_0)\| \leq \alpha \Rightarrow \|\underline{x}(t)\| \leq \beta,$$
$$c > 0, \alpha \in (0, c),\ \beta > 0,\ \forall\, t \geq t_0$$

Ultimate Boundedness:

$$\|\underline{x}(t)\| \leq b$$
$$\forall\, t \geq t_0 + T$$

Lyapunov Stability for Vanishing Perturbations

Problem: Analyze stability of $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$

- Nominal system $(\dot{\underline{x}} = f(\underline{x}))$ exponentially stable

Assumptions for Exponential Stability:

- Perturbation $g(\underline{x}, t)$ vanishes (i.e., $g(\underline{x}, t) \rightarrow 0$ as $\underline{x} \rightarrow 0$)
- $V(t, \underline{x})$: continuously differentialbe, positive definite, radially unbounded

Condition for Global Uniform Exponential Stability:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \underline{x}} f(\underline{x}) \leq -c_3 \|\underline{x}\|^2 \text{ and } \left\| \frac{\partial V}{\partial \underline{x}} \right\| \leq c_4 \|\underline{x}\|$$

$$\|g(\underline{x}, t)\| \leq \gamma \|\underline{x}\| \text{ where } 0 \leq \gamma(t) < \frac{c_3}{c_4}$$

Comparison Lemma - Example

Problem: Analyze stability of a scalar perturbed system

$$\dot{\underline{x}} = -a\,\underline{x}(t) + g(t,\underline{x}), \quad \underline{x}(0) = \underline{x}_0, \quad a > 0$$

Assumptions/Conditions:

- $\underline{x}(t) \geq 0 \quad \forall\, t \geq 0$
- $g(t,\underline{x}) \leq b\,\underline{x}(t) \quad \forall\, \underline{x} \geq 0$ and $0 \leq b < a$

Step 1: Integral form of the solution

$$\underline{x}(t) = \underline{x}_0 + \int_{t_0}^{t} \dot{\underline{x}}(\tau)\, d\tau$$

$$\underline{x}(t) \leq \underline{x}_0 + \int_{t_0}^{t} \left[-a\,\underline{x}(\tau) + b\,\underline{x}(\tau)\right] d\tau = \underline{x}_0 - (a-b)\int_{t_0}^{t} \underline{x}(\tau)\, d\tau$$

Comparison Lemma - Example

Step 2: Define comparison function $z(t)$:

$$z(t) = \underline{x}_0 - (a-b)\int_{t_0}^{t} z(\tau)\,d\tau$$

$$\dot{z}(t) = -(a-b)\,z(t), \quad z(t_0) = \underline{x}_0$$

Solution:

$$z(t) = \underline{x}_0\, e^{-(a-b)(t-t_0)}$$

Conclusion: Exponential Stability of the Perturbed System

$$\lim_{t\to\infty} \underline{x}(t) = 0$$

$$\underline{x}(t) \le \underline{x}_0\, e^{-(a-b)(t-t_0)}$$

Non-Vanishing Perturbations: The Problem

The Challenge:

- Perturbation doesn't disappear, perfect asymptotic stability to zero is not possible
- State $\underline{x}$ will perpetually move away from the origin

Conclusion/Goal: Ultimate Boundedness

- We aim for Ultimate Boundedness
- Not exactly zero, but "close enough" for practical purposes

Intuition:

- Imagine balencing a pencil on its tip in a sufficiently strong wind
- It won't stay perfectly still
- Find out how big that wobble area is $\Rightarrow$ guarantee the pencil always stays inside

Lyapunov Conditions for Ultimate Boundedness

Given:

- System $\dot{\underline{x}} = f(\underline{x}) + g(\underline{x}, t)$
- Nominal system origin $\underline{x} = 0$ exponentially stable

Criteria (Assumptions & Conditions):

- **Lyapunov Function:** Cont. diff., pdf, radially unbounded
- **Nominal Decay Condition:** Derivative of V along nominal system is ndf
- **Lyapunov Function Gradient Bound:** Gradient of V is bounded
- **Perturbation Bound:** Non-vanishing $g(\underline{x}, t)$ is bounded by a constant δ

The Key Trade-off: δ directly impacts size of ultimate bound

$$\|g(\underline{x}, t)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad \theta \in (0,1),\ r > 0$$

Result: Initial Decay and Ultimate Boundedness

Outcome for any initial condition $\|\underline{x}(t_0)\| \leq \sqrt{c_1/c_2}\, r$

Phase 1: Initial Exponential Decay

$$\underline{x}(t)\| \leq k\, e^{-\gamma(t-t_0)} \|\underline{x}(t_0)\|, \quad t_0 \leq t \leq t_0 + T$$

Phase 2: Ultimate Boundedness

$$\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

What it Means:

- $\underline{x}(t_0)$ sufficiently close to origin $\Rightarrow$ system initially heads towards origin rapidly
- Due to non-vanishing perturbation it will settle into ball around origin
- b depends directly on δ and how "stable" the system is (c_1, c_2, c_3, c_4)

Example: Bounded Disturbance Response

Problem: Analyze Boundedness of a Perturbed Mass-Spring-Damper-System

$$\begin{aligned}\dot{\underline{x}}_1 &= x_2 \\ \dot{\underline{x}}_2 &= -2x_1 - 3x_2 + d\end{aligned}$$

with bounded disturbance $|d| \leq \delta$ and the origin of the nominal system being exponentially stable

Lyapunov Candidate Function: $V(\underline{x}) = \underline{x}^T P \underline{x}$

- $P > 0$ found by solving $A^T P + PA = -Q$
- For $Q = I$ and $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, P can be calculated
- $c_1 \|\underline{x}\|^2 \leq V(\underline{x}) \leq c_2 \|\underline{x}\|^2$

Example: Bounded Disturbance Response

Applying Lyapunov Stability Conditions:

$$\dot{V} \leq -c_3\|\underline{x}\|^2 + c_4\|d\|\|\underline{x}\| \leq -c_3\|\underline{x}\|^2 + c_4\delta\|\underline{x}\|$$

Conclusion: Ultimate Boundedness Guaranteed

- System state $\|\underline{x}(t)\|$ eventually converges to and remains within a bounded region:

$$\|\underline{x}(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

- The ultimate bound b and other parameters are:

$$b = \frac{c_4}{c_3} k \frac{\delta}{\theta}, \quad k = \sqrt{c_2/c_1}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}$$

- Size of the ultimate bound b directly **scales with the disturbance magnitude** δ

Key Insights & Practical Implications

Theoretical Insights:

- Lyapunov methods: unify time-varying and perturbed system analysis
- Perturbation type: dictates achievable stability properties
- Ultimate boundedness: models real-world robustness

Design Implications:

- Small, vanishing perturbations: maintains exponential convergence
- Persistent disturbances: design for bounded operation
- Robustness: requires accurate perturbation characterization

References I

[1] Giovanni Gallavotti. **Perturbation Theory**. In: *Perturbation Theory*. Ed. by Giuseppe Gaeta. Series Title: Encyclopedia of Complexity and Systems Science Series. New York, NY: Springer US, 2009, pp. 1–14. ISBN: 978-1-0716-2620-7 978-1-0716-2621-4. DOI: 10.1007/978-1-0716-2621-4_396.

[2] Hassan K. Khalil. **Nonlinear control**. Global edition. Boston Munich: Pearson, 2015. ISBN: 978-1-292-06069-9.

[3] Hassan K. Khalil. **Nonlinear systems**. Pearson new internat. ed., 3. ed. Always learning. Harlow: Pearson Education, 2014. ISBN: 978-1-292-05385-1.

[4] Shenyu Liu. **Unified stability criteria for perturbed LTV systems with unstable instantaneous dynamics**. Feb. 2022. DOI: 10.48550/arXiv.2111.07443.

[5] James Murdok. **1. Root Finding**. en. In: *Perturbations: Theory and Methods*. Society for Industrial and Applied Mathematics, Jan. 1999, pp. 3–81. ISBN: 978-0-89871-443-2 978-1-61197-109-5. DOI: 10.1137/1.9781611971095.ch1.