University of Chinese Academy of Sciences

模式识别与机器学习

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Chap 4 课程作业解答

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Problem 1

设有如下三类模式样本集 ω_1, ω_2 和 ω_3, 其先验概率相等, 求 S_w 和 S_b.

$$\begin{aligned}
&\omega_1 : \left\{(1,0)^{\mathrm{T}}, (2,0)^{\mathrm{T}}, (1,1)^{\mathrm{T}}\right\}; \\
&\omega_2 : \left\{(-1,0)^{\mathrm{T}}, (0,1)^{\mathrm{T}}, (-1,1)^{\mathrm{T}}\right\}; \\
&\omega_3 : \left\{(-1,-1)^{\mathrm{T}}, (0,-1)^{\mathrm{T}}, (0,-2)^{\mathrm{T}}\right\}
\end{aligned}$$

Solution: 易知 S_w, S_b 的计算公式为

$$S_b = \sum_{i=1}^{M} P(\omega_i)(\boldsymbol{m}_i - \boldsymbol{m}_0)(\boldsymbol{m}_i - \boldsymbol{m}_0)^{\mathrm{T}}, \tag{1}$$

$$S_w = \sum_{i=1}^{M} P(\omega_i)\, \mathbf{E}\left\{(\boldsymbol{x} - \boldsymbol{m}_i)(\boldsymbol{x} - \boldsymbol{m}_i)^{\mathrm{T}} \,|\, x \in \omega_i\right\} = \sum_{i=1}^{M} P(\omega_i)\, \boldsymbol{C}_i \tag{2}$$

根据题意有: $P(\omega_1) = P(\omega_2) = P(\omega_3) = \frac{1}{3}$, 而且计算可知:

$$\boldsymbol{m}_0 = \begin{pmatrix} 1/9 \\ -1/9 \end{pmatrix}, \boldsymbol{m}_1 = \begin{pmatrix} 4/3 \\ 1/3 \end{pmatrix}, \boldsymbol{m}_2 = \begin{pmatrix} -2/3 \\ 2/3 \end{pmatrix}, \boldsymbol{m}_3 = \begin{pmatrix} -1/3 \\ -4/3 \end{pmatrix}$$

于是根据 (1) 式可知:

$$\begin{aligned}
S_b &= \frac{1}{3}\left[\begin{pmatrix} 11/9 \\ 4/9 \end{pmatrix}\begin{pmatrix} 11/9 & 4/9 \end{pmatrix} + \begin{pmatrix} -7/9 \\ 7/9 \end{pmatrix}\begin{pmatrix} -7/9 & 7/9 \end{pmatrix} + \begin{pmatrix} -4/9 \\ -11/9 \end{pmatrix}\begin{pmatrix} -4/9 & -11/9 \end{pmatrix}\right] \\
&= \frac{1}{81}\begin{pmatrix} 62 & 13 \\ 13 & 62 \end{pmatrix}
\end{aligned}$$

根据协方差矩阵的估计式:

$$\boldsymbol{C}_i = \frac{1}{N_i}\sum_{j=1}^{N_i}\left(\boldsymbol{x}^{(j)} - \boldsymbol{m}_i\right)\left(\boldsymbol{x}^{(j)} - \boldsymbol{m}_i\right)^{\mathrm{T}} (i = 1, 2, 3) \tag{3}$$

于是根据上述 (3) 式有:

$$\begin{aligned}
\boldsymbol{C}_1 &= \frac{1}{3}\left\{\begin{pmatrix} -1/3 & -1/3 \end{pmatrix}\begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} + \begin{pmatrix} 2/3 & -1/3 \end{pmatrix}\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix} + \begin{pmatrix} -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}\right\} = \frac{1}{9}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\
\boldsymbol{C}_2 &= \frac{1}{3}\left\{\begin{pmatrix} -1/3 & -2/3 \end{pmatrix}\begin{pmatrix} -1/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}\right\} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\
\boldsymbol{C}_3 &= \frac{1}{3}\left\{\begin{pmatrix} -2/3 & 1/3 \end{pmatrix}\begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}\right\} = \frac{1}{9}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\end{aligned}$$

因而求得

$$S_w = \sum_{i=1}^{M} P(\omega_i)\, \boldsymbol{C}_i = \frac{1}{3}\sum_{i=1}^{3}\boldsymbol{C}_i = \frac{1}{27}\begin{pmatrix} 6 & -1 \\ -1 & 6 \end{pmatrix}$$

Problem 2

设有如下两类样本集, 其出现的概率相等:

$$\omega_1 : \left\{ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^{\mathrm{T}} \right\}$$
$$\omega_2 : \left\{ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}} \right\}$$

运用 *Karhunen-Loeve* (*K-L*) 变换, 分别把特征空间维数降到 2 维和 1 维, 并画出样本在该空间中的位置.

Solution: 先计算样本均值

$$\boldsymbol{m} = \sum_{i=1}^{2} P(\omega_i)\, \boldsymbol{m}_i = \frac{1}{2} \begin{pmatrix} 3/4 \\ 1/4 \\ 1/4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1/4 \\ 3/4 \\ 3/4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

再平移样本 ($\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{m}$) 以符合最佳条件:

$$\omega_1' : \left\{ \begin{pmatrix} -1/2 & -1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & -1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & -1/2 & 1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & 1/2 & -1/2 \end{pmatrix}^{\mathrm{T}} \right\}$$
$$\omega_2' : \left\{ \begin{pmatrix} -1/2 & -1/2 & 1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} -1/2 & 1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} -1/2 & 1/2 & 1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & 1/2 & 1/2 \end{pmatrix}^{\mathrm{T}} \right\}$$

再计算 $\boldsymbol{z}$ 的自相关矩阵:

$$\boldsymbol{R} = \sum_{i=1}^{2} P(\omega_i)\, \mathbf{E}\left(\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}}\right) = \frac{1}{2} \left[\frac{1}{4} \sum_{\boldsymbol{z}^j \in \omega_1'} \boldsymbol{z}^j \left(\boldsymbol{z}^j\right)^{\mathrm{T}} \right] + \frac{1}{2} \left[\frac{1}{4} \sum_{\boldsymbol{z}^j \in \omega_2'} \boldsymbol{z}^j \left(\boldsymbol{z}^j\right)^{\mathrm{T}} \right] = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

显然 $\boldsymbol{R}$ 的特征值为 1/4, 1/4, 1/4. 对应的特征向量分别为

$$\boldsymbol{\varphi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{\varphi}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{\varphi}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

若要将特征空间维数 (即 3) 降到 2, 则做如下变换即可:

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{m}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 - 1/2 \\ x_2 - 1/2 \\ x_3 - 1/2 \end{pmatrix} = \begin{pmatrix} x_1 - 1/2 \\ x_2 - 1/2 \end{pmatrix}$$

故得到降维后的样本为:

$$\omega_1 : \left\{ \begin{pmatrix} -1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^{\mathrm{T}} \right\}$$
$$\omega_2 : \left\{ \begin{pmatrix} -1/2 & -1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} -1/2 & 1/2 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^{\mathrm{T}} \right\}$$

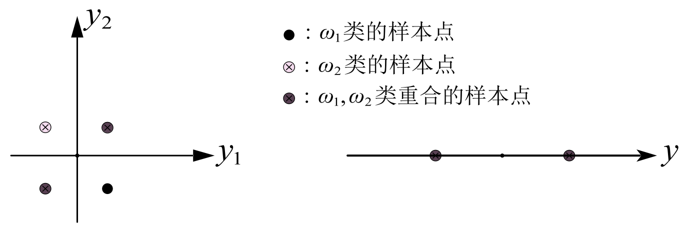
若要将特征空间维数 (即 3) 降到 1, 则做如下变换即可:

$$y = \boldsymbol{\varphi}_1{}^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{m}) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 - 1/2 \\ x_2 - 1/2 \\ x_3 - 1/2 \end{pmatrix} = x_1 - 1/2$$

故得到降维后的样本为:

$$\omega_1 : \{-1/2, 1/2\}; \quad \omega_2 : \{-1/2, 1/2\}$$

降维后的样本在空间中的分布位置如下图1中所示.



(**a**). 降到2维后的样本分布 (**b**). 降到1维后的样本分布

图 1: 样本在降维空间中的位置

至此, Chap 4 的作业解答完毕.

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