AD-604705

604705

1

Functional Equations in the
Theory of Dynamic Programming—II
Nonlinear Differential Equations

by

Richard Bellman

P-671

May 5, 1955

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary

The study of dynamic programming processes of continuous type gives rise to functional equations of the form $dx/dt = \max_q f(x,t;q)$. In this paper we present a summary of some basic results concerning the existence and uniqueness of solutions of this equation. Detailed results will appear subsequently.

Functional Equations in the Theory of Dynamic Programming—II
Nonlinear Differential Equations

By
Richard Bellman
The RAND Corporation, Santa Monica, California

1. Introduction. In the first paper of this series [1], cf also [2,3], we discussed the existence and uniqueness of solutions of the functional equation

$$f(P) = \max_{Q} f(T(P,Q)), \tag{1.1}$$

the basic functional equation of the theory of dynamic programming. In this paper we shall present the first of a series of investigations of various continuous analogues of the foregoing equation. Our results here concern the nonlinear system

$$\frac{dx_i}{dt} = \max_{q} f_i(x,t;q), \quad x_i(0) = c_i, \quad i = 1,2,\ldots,n, \tag{1.2}$$

which we shall write in the vector form

$$\frac{dx}{dt} = \max_{q} f(x,t;q), \quad x(0) = c \tag{1.3}$$

The quantities x, f and c are n-dimensional vectors, while q is an m-dimensional vector whose components may range over a finite or denumerable set, or continuum of values.

In subsequent papers we shall present analogous results for partial differential equations of parabolic type, for the first order partial differential equations obtained from the calculus of variations,[4,5], and for other classes of functional equations as well. A detailed exposition of the results of this note, and a discussion of their relation to an interesting class of problems in the calculus of variations will appear elsewhere.

Equations of the above type appear in the study of continuous decision processes; some particular examples have been treated in unpublished notes by Copeland and Darling. In many ways, these equations constitute natural generalizations of linear systems, and thus play an important role in analysis, apart from their probabilistic applications.

Furthermore, as we shall see below, some important types of equations, such as the Riccati equation, can be written in this form.

2. A General Existence and Uniqueness Theorem.

Let us first state a g··neral ree t.

Theorem 1. Consider the system in (1.2). Let us assume that $f(x,q,t)$ satisfies the following conditions:

a. $\| f(x,t;q) - f(y,t;q) \| \leq k(q,t) \| x-y \|$, for x,y,t in the region R defined by the inequalities $\| x-c \| \leq a$, $\| y-c \| \leq a$, $0 \leq t \leq t_0$, for some non-zero constants a and t_0.

b. The maximum value of $f_1(x,t;q)$ is assumed for a function $q_1 = q_1(x,t)$ which is uniformly bounded in R.

c. The function $k(q,t)$ is uniformly bounded for uniformly bounded q and for t in R.

Then there is a unique solution of (1.3) for $0 \leq t \leq t_1$, where t_1 is some non-zero constant less than t_0. This solution may be obtained as the limit of the sequence defined by

$$x_0 = c, \qquad x_{n+1} = c + \int_0^t \operatorname*{Max}_q \ f(x_n, s; q)\, ds \tag{2.2}$$

There are many interesting questions concerning the structure of the solution x and the policy function q which we do not have space to discuss here.

3. Quasi-Linear Systems.

As usual, we can obtain a much more complete result for systems of linear type.

Theorem 2. Consider the system

$$\frac{dx_i}{dt} = \operatorname*{Max}_q \left[b_i(t;q) + \sum_{j=1}^{N} a_{ij}(t;q) x_j \right], \quad x_i(0) = c_i, \quad i=1,2,\ldots,n \tag{3.1}$$

Where we assume that

$$\operatorname*{Max}_q | a_{ij}(t;q) |, \quad \operatorname*{Max}_q |b_i(t,q)| \leq f(t) \tag{3.2}$$

with $f(t)$ integrable over any finite t-interval.

Then there is a unique solution for $t \geq 0$ which may be found by the same method of successive approximations described above.

In many cases, we can determine the asymptotic behavior of the solution, see [6] for a discussion of this problem for difference equations. Results of similar type may be established for differential-difference equations, and more general types of integro-differential equations.

4. Approximation in Policy Space. The approximation technique used above was the classical one. Let us now describe another type. Given $q_o(t)$, for each index i, we may compute an initial approximation, x^o, using the equation

$$\frac{dx^o}{dt} = A(t;q_o)x^o + b(q_o)x, \quad x^o(0) = c \tag{4.1}$$

To obtain a better approximation, we determine $q_1(t)$, again for each i, by the condition that $q_1(t)$ maximize the expression $\left[\sum_{j=1}^{N} a_{ij}(t;q)x_i^o + b_i(t,q)\right]$. Having determined $q_1(t)$ in this way, we compute a second approximation x^1 using (4.1) with q_o replaced by q_1. We now continue in this fashion, computing a sequence of policy functions $\{q_n\}$, and a sequence of approximations $\{x^n\}$. This type of approximation we call "approximation in policy space", see [5]. Concerning this process, we can establish the following result:

Theorem 3. If the linear system,

$$\frac{dx}{dt} = A(t,q)x + f(t), \quad x(0) = 0. \tag{4.2}$$

has, for any fixed q, the property that $f \geq 0$ for $t \geq 0$ implies that $x \geq 0$ for $t \geq 0$, then approximation in policy space yields monotone convergence to the solution.

In particular, there is always convergence when x is a scalar variable.

If $A(q)$ is a constant matrix for each allowable q, then a necessary and sufficient condition for the above condition to hold is $a_{ij} \geq 0$, for $i \neq j$; if $A(t,q)$ is variable, then a sufficient condition is $a_{ij}(t) \geq 0$, for $i \neq j$; see [7].

5. The Riccati Equation. An important tool in the qualitative and quantitative theory of the second order linear differential equation $u'' + a(t)\,u' + b(t)u = 0$ is the transformation of the equation into the form

$$v' = -v^2 - a(t)v - b(t) \tag{5.1}$$

the Riccati equation, by means of the substitution $u = \exp(\int v\,dt)$. We now observe that the Riccati equation may be written in the form

$$v' = \operatorname*{Min}_{q} \left[q^2 + 2qv(t) - a(t)v - b(t) \right] \tag{5.2}$$

an equation of the above general type. This representation furnishes a new approach to the study of the solutions of (5.1). There are several other classes of equations of both analytic and physical interest to which this quasi-linearization may be applied. This will be discussed elsewhere.

[1] R. Bellman, "Some Functional Equations in the Theory of Dynamic Programming", these PROCEEDINGS, 39, 1077–1082, 1953

[2] R. Bellman, An Introduction to the Theory of Dynamic Programming (RAND Monograph R-245, 1953)

[3] R. Bellman, "Some Functional Equations in the Theory of Dynamic Programming"—I (Functions of Points and Point Transformations), Trans. Amer. Math. Soc., to appear.

[4] R. Bellman, "Dynamic Programming and a New Formalism in the Calculus of Variations", these PROCEEDINGS, 40, 231–235, 1954

[5] R. Bellman, "Monotone Convergence in Dynamic Programming and the Calculus of Variations", these PROCEEDINGS, 40, 1073–1075, 1954

[6] R. Bellman, "Quasi-Linear Equations", to appear.

[7] R. Bellman, I. Glicksberg and O. Gross, "Some Variational Problems Occurring in the Theory of Dynamic Programming", Rendiconti del Palermo, to appear.