Submodular maximization

Jialin Zhang
zhangjialin@ict.ac.cn

Institute of Computing Technology, Chinese Academy of Sciences

Jan 3, 2019

Submodular function

Definition

Function $f : 2^N \to \mathbb{R}$ is a submodular function if $\forall S, T \subseteq N$, we have $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.

$f(S) + f(T) = \underline{f(S \cup T)} + \cancel{f(S \cap T)}$

$f(\phi) = 0$

S T

$f(S) = \sum_{x \in S} f(\{x\})$.

Submodular function

Definition

Function $f : 2^N \rightarrow \mathbb{R}$ is a submodular function if $\forall S, T \subseteq N$, we have $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.

Definition (equivalent definition)

Function $f : 2^N \rightarrow \mathbb{R}$ is a submodular function if $\forall S \subset T \subseteq N$, $\forall x \notin T$, we have $f(S + \{x\}) - f(S) \geq f(T + \{x\}) - f(T)$.

- Always assume $f(\emptyset) = 0$
- Usually denote $f_S(x) \triangleq f(S + \{x\}) - f(S)$, and $f_S(T) \triangleq f(S \cup T) - f(S)$

饭	菜	饭菜
15	3	19
17	5	21

Submodular function

Definition

Function $f : 2^N \to \mathbb{R}$ is a submodular function if $\forall S, T \subseteq N$, we have $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.

Definition (equivalent definition)

Function $f : 2^N \to \mathbb{R}$ is a submodular function if $\forall S \subset T \subseteq N$, $\forall x \notin T$, we have $f(S + \{x\}) - f(S) \geq f(T + \{x\}) - f(T)$.

$\Delta_x f(S) \geq \Delta_x f(T)$.

- Always assume $f(\emptyset) = 0$
- Usually denote $f_S(x) \triangleq f(S + \{x\}) - f(S)$, and $f_S(T) \triangleq f(S \cup T) - f(S)$
- Difference: $\Delta_x f(S) \triangleq f(S + \{x\}) - f(S)$
- Relation between difference and differential calculus

$\frac{df(x)}{dx}$. $\lim \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$f(n+1)-f(n)$.

Examples

- Set cover
- Graph cut
- Independent cascade model in social network

Properties

$f_S(T) \triangleq f(S+T) - f(S)$.

- If f, g are submodular functions, so is $f + g$ and cf $(c \geq 0)$;
- If f is submodular functions, so is f_S $(S \subseteq N)$;
- If f is submodular functions, so is $g : g(S) \triangleq f(V - S)$;
- If f is submodular functions, then $\forall S \subseteq T, f(T) \leq f(S) + \sum_{e \in T \setminus S} f_S(e)$.

Submodular maximization problem under cardinality constraint

$N = \{u_1, \cdots, u_n\}$

poly(n)

$2^N \to \mathbb{R}$ $\quad$ S, $f(S)$

- Given a set function f which is monotone and submodular where $f(\emptyset) = 0$, and given integer k, solve the problem: $\max_{|S| \le k} f(S)$.

Submodular maximization problem under cardinality constraint

- Given a set function f which is monotone and submodular where $f(\emptyset) = 0$, and given integer k, solve the problem: $\max_{|S| \leq k} f(S)$.
- How is f given?
- Complexity: there is no polynomial time algorithm for submodular maximization problem.

Submodular maximization problem under cardinality constraint

- Given a set function f which is monotone and submodular where $f(\emptyset) = 0$, and given integer k, solve the problem: $\max_{|S| \leq k} f(S)$.
- How is f given?
- Complexity: there is no polynomial time algorithm for submodular maximization problem.
- Algorithm: greedy algorithm

Greedy algorithm (cardinality constraint)

- Basic idea: at each step, the item with **the largest marginal gain** will be selected.

GreedyCardinalityConstraint(k, N)

$S = \phi$;
while $|S| < k$ **do**
  $\hat{x} = argmax_{x \in N} f_S(x)$;
  $S = S \cup \{\hat{x}\}$;
  $N = N - \{\hat{x}\}$;
end while
return S;

max $f(\{x\})$

$f_{\{x_1\}}(\{x_2\})$