

# Lecture 9: Support Vector Machines

## Advanced Topics in Machine Learning: COMPG113

Arthur Gretton

Gatsby Unit, CSML, UCL

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- The representer theorem
- Review of convex optimization
- Support vector classification, the  $C$ -SV and  $\nu$ -SV machines

# Representer theorem

# Learning problem: setting

Given a set of paired observations  $(x_1, y_1), \dots, (x_n, y_n)$  (regression or classification).

Find the function  $f^*$  in the RKHS  $\mathcal{H}$  which satisfies

$$f^* = \arg \min_{f \in \mathcal{H}} J(f), \quad (1)$$

where

$$J(f) = L_y(f(x_1), \dots, f(x_n)) + \Omega \left( \|f\|_{\mathcal{H}}^2 \right),$$

$\Omega$  is non-decreasing,  $y$  is the vector of  $y_i$ , **loss  $L$  depends on  $x_i$  only via  $f(x_i)$ .**

- Classification:  $L_y(f(x_1), \dots, f(x_n)) = \sum_{i=1}^n \mathbb{I}_{y_i f(x_i) \leq 0}$
- Regression:  $L_y(f(x_1), \dots, f(x_n)) = \sum_{i=1}^n (y_i - f(x_i))^2$

**The representer theorem:** a solution to

$$\min_{f \in \mathcal{H}} \left[ L_Y(f(x_1), \dots, f(x_n)) + \Omega \left( \|f\|_{\mathcal{H}}^2 \right) \right]$$

takes the form

$$f^* = \sum_{i=1}^n \alpha_i k(x_i, \cdot).$$

If  $\Omega$  is strictly increasing, the solution must have this form.

# Representer theorem: proof

**Proof:** Denote  $f_s$  projection of  $f$  onto the subspace

$$\text{span} \{k(x_i, \cdot) : 1 \leq i \leq n\}, \quad (2)$$

such that

$$f = f_s + f_{\perp},$$

where  $f_s = \sum_{i=1}^n \alpha_i k(x_i, \cdot)$ .

**Regularizer:**

$$\|f\|_{\mathcal{H}}^2 = \|f_s\|_{\mathcal{H}}^2 + \|f_{\perp}\|_{\mathcal{H}}^2 \geq \|f_s\|_{\mathcal{H}}^2,$$

then

$$\Omega \left( \|f\|_{\mathcal{H}}^2 \right) \geq \Omega \left( \|f_s\|_{\mathcal{H}}^2 \right),$$

so this term is minimized for  $f = f_s$ .

**Proof (cont.):** Individual terms  $f(x_i)$  in the loss:

$$f(x_i) = \langle f, k(x_i, \cdot) \rangle_{\mathcal{H}} = \langle f_s + f_{\perp}, k(x_i, \cdot) \rangle_{\mathcal{H}} = \langle f_s, k(x_i, \cdot) \rangle_{\mathcal{H}},$$

so

$$L_y(f(x_1), \dots, f(x_n)) = L_y(f_s(x_1), \dots, f_s(x_n)).$$

Hence

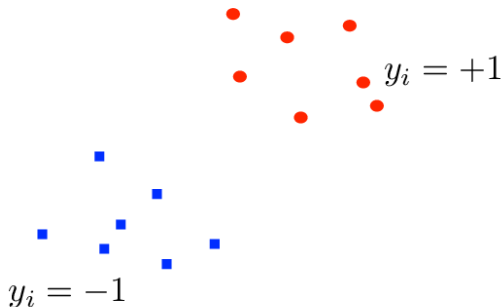
- Loss  $L(\dots)$  only depends on the component of  $f$  in the data subspace,
- Regularizer  $\Omega(\dots)$  minimized when  $f = f_s$ .
- If  $\Omega$  is non-decreasing, then  $\|f_{\perp}\|_{\mathcal{H}} = 0$  is a minimum. If  $\Omega$  strictly increasing, min. is unique.

# Short overview of convex optimization



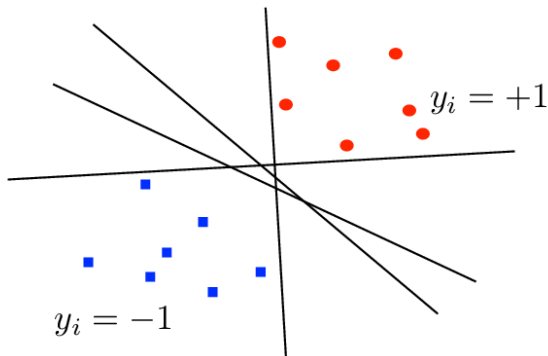
# Why we need optimization: SVM idea

Classify two clouds of points, where there exists a hyperplane which linearly separates one cloud from the other without error.



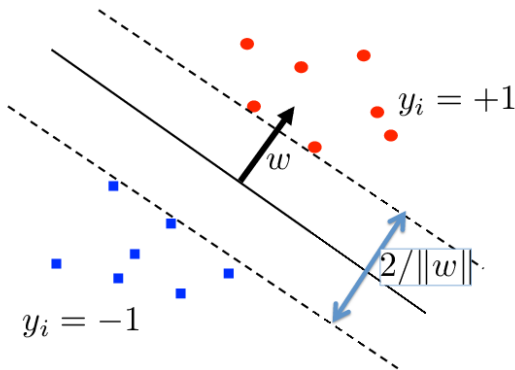
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# Why we need optimization: SVM idea

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Smallest distance from each class to the **separating hyperplane**  $w^T x + b$  is called the **margin**.

# Why we need optimization: SVM idea

This problem can be expressed as follows:

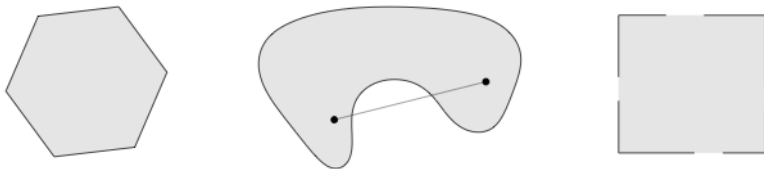
$$\max_{w,b}(\text{margin}) = \max_{w,b} \left( \frac{2}{\|w\|} \right) \quad \text{or} \quad \min_{w,b} \|w\|^2 \quad (3)$$

subject to

$$\begin{cases} (w^\top x_i + b) \geq 1 & i : y_i = +1, \\ (w^\top x_i + b) \leq -1 & i : y_i = -1. \end{cases} \quad (4)$$

This is a **convex optimization problem**.

# Convex set



(Figure from Boyd and Vandenberghe)

Leftmost set is convex, remaining two are not.

Every point in the set can be seen from any other point in the set, along a straight line that never leaves the set.

## Definition

$C$  is convex if for all  $x_1, x_2 \in C$  and any  $0 \leq \theta \leq 1$  we have  $\theta x_1 + (1 - \theta)x_2 \in C$ , i.e. every point on the line between  $x_1$  and  $x_2$  lies in  $C$ .

# Convex function: no local optima



(Figure from Boyd and Vandenberghe)

## Definition (Convex function)

A function  $f$  is **convex** if its domain  $\text{dom} f$  is a convex set and if  $\forall x, y \in \text{dom} f$ , and any  $0 \leq \theta \leq 1$ ,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$

The function is **strictly convex** if the inequality is strict for  $x \neq y$ .

# Optimization and the Lagrangian

(Generic) optimization problem on  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0 && i = 1, \dots, m \\ & && h_i(x) = 0 && i = 1, \dots, p. \end{aligned} \tag{5}$$

- $p^*$  the optimal value of (5),  $\mathcal{D}$  assumed nonempty, where...
- $\mathcal{D} := \bigcap_{i=1}^m \text{dom} f_i \cap \bigcap_{i=1}^p \text{dom} h_i$  ( $\text{dom } f_i = \text{subset of } \mathbb{R}^n \text{ where } f_i \text{ defined}$ ).

Ideally we would want an unconstrained problem

$$\text{minimize } f_0(x) + \sum_{i=1}^m l_-(f_i(x)) + \sum_{i=1}^p l_0(h_i(x)),$$

$$\text{where } l_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases} \quad \text{and } l_0(u) \text{ is the indicator of } 0.$$

Why is this hard to solve?

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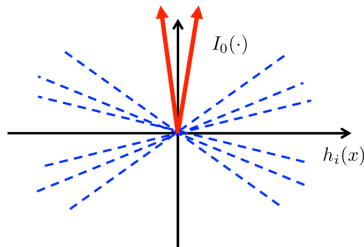
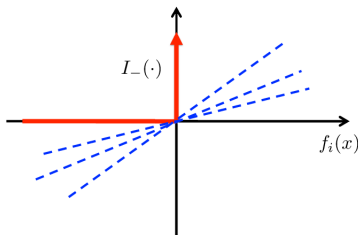
# Lower bound interpretation of Lagrangian

The **Lagrangian**  $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$  is a **lower bound** on the original problem:

$$L(x, \lambda, \nu) := f_0(x) + \sum_{i=1}^m \underbrace{\lambda_i f_i(x)}_{\leq l_-(f_i(x))} + \sum_{i=1}^p \underbrace{\nu_i h_i(x)}_{\leq l_0(h_i(x))},$$

and has domain  $\text{dom} L := \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$ . The vectors  $\lambda$  and  $\nu$  are called **lagrange multipliers** or **dual variables**.

To ensure a lower bound, we require  $\lambda \succeq 0$ .



# Lagrange dual: lower bound on optimum $p^*$

The **Lagrange dual function**: minimize Lagrangian

When  $\lambda \succeq 0$  and  $f_i(x) \leq 0$ , Lagrange dual function is

$$g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda, \nu). \quad (6)$$

A **dual feasible** pair  $(\lambda, \nu)$  is a pair for which  $\lambda \succeq 0$  and  $(\lambda, \nu) \in \text{dom}(g)$ .

We will show: (next slide) for any  $\lambda \succeq 0$  and  $\nu$ ,

$$g(\lambda, \nu) \leq f_0(x)$$

wherever

$$\begin{aligned} f_i(x) &\leq 0 \\ g_i(x) &= 0 \end{aligned}$$

(including at  $f_0(x^*) = p^*$ ).

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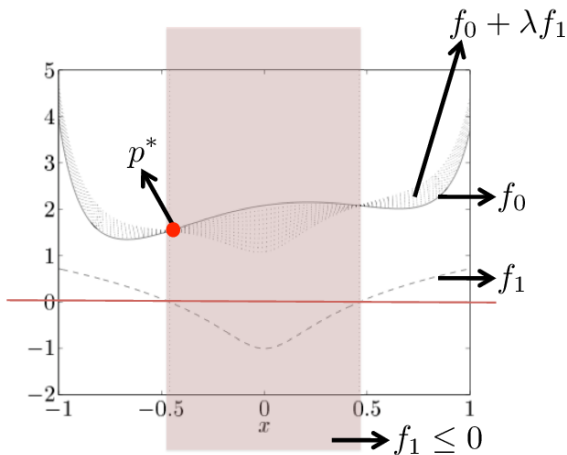
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# Lagrange dual: lower bound on optimum $p^*$

Simplest example: **minimize over  $x$**  the function

$$L(x, \lambda) = f_0(x) + \lambda f_1(x)$$

(Figure modified from Boyd and Vandenberghe)



## Reminders:

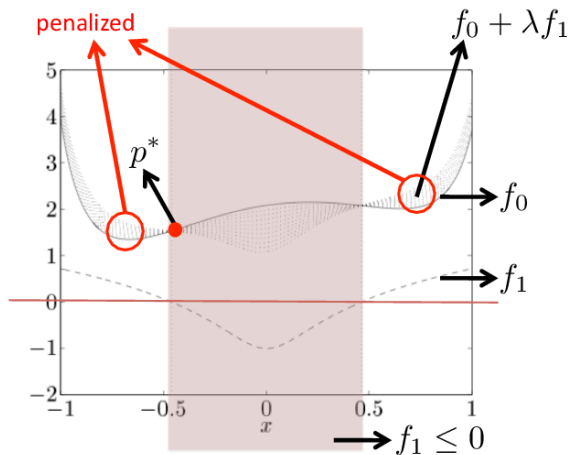
- $f_0$  is function to be minimized.
- $f_1 \leq 0$  is inequality constraint
- $\lambda \geq 0$  is Lagrange multiplier
- $p^*$  is minimum  $f_0$  in constraint set

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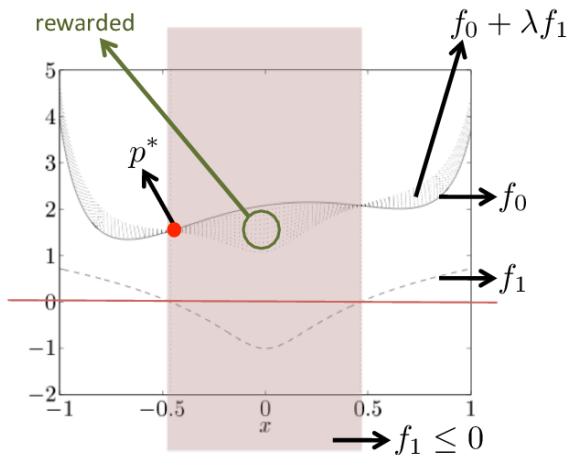
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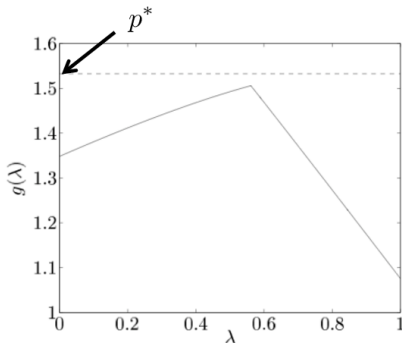
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# Lagrange dual: lower bound on optimum $p^*$

When  $\lambda \succeq 0$ , then for all  $\nu$  we have

$$g(\lambda, \nu) \leq p^* \quad (7)$$

A **dual feasible** pair  $(\lambda, \nu)$  is a pair for which  $\lambda \succeq 0$  and  $(\lambda, \nu) \in \text{dom}(g)$  (Figure from Boyd and Vandenberghe)



Reminders:

- $g(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda)$
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# Lagrange dual is lower bound on $p^*$ (proof)

We now give a formal proof that **Lagrange dual function**  $g(\lambda, \nu)$  lower bounds  $p^*$ .

**Proof:** Assume  $\tilde{x}$  is feasible, i.e.  $f_i(\tilde{x}) \leq 0$ ,  $h_i(\tilde{x}) = 0$ ,  $\tilde{x} \in \mathcal{D}$ ,  $\lambda \succeq 0$ . Then

$$\sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \leq 0$$

Thus

$$\begin{aligned} g(\lambda, \nu) &:= \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \\ &\leq f_0(\tilde{x}) + \sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \\ &\leq f_0(\tilde{x}). \end{aligned}$$

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# Best lower bound: maximize the dual

Best lower bound  $g(\lambda, \nu)$  on the optimal solution  $p^*$  of (5):

**Lagrange dual problem**

$$\begin{array}{ll}\text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0.\end{array}\tag{8}$$

**Dual feasible:**  $(\lambda, \nu)$  with  $\lambda \succeq 0$  and  $g(\lambda, \nu) > -\infty$ .

**Dual optimal:** solutions  $(\lambda^*, \nu^*)$  maximizing dual,  $d^*$  is optimal value (**dual always easy to maximize**: next slide).

**Weak duality** always holds:

$$d^* \leq p^*.$$

...but what is the point of finding a **best (largest) lower bound** on a **minimization problem**?

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**Dual optimal:** solutions  $(\lambda^*, \nu^*)$  to the dual problem,  $d^*$  is optimal value (**dual always easy to maximize**: next slide).

**Weak duality** always holds:

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**Strong duality:** (does **not** always hold, conditions given later):

$$d^* = p^*.$$

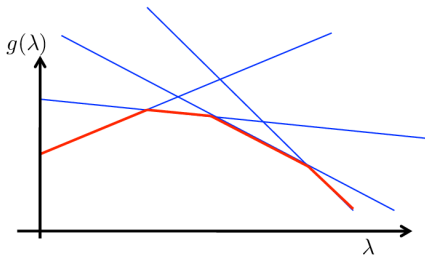
If S.D. holds: solve the **easy (concave) dual problem to find  $p^*$**

# Maximizing the dual is always easy

The **Lagrange dual function**: minimize Lagrangian (lower bound)

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu).$$

Dual function is a pointwise infimum of affine functions of  $(\lambda, \nu)$ , hence **concave** in  $(\lambda, \nu)$  with convex constraint set  $\lambda \succeq 0$ .



Example:

One inequality constraint,

$$L(x, \lambda) = f_0(x) + \lambda f_1(x),$$

and assume there are only four possible values for  $x$ . Each line represents a different  $x$ .

# How do we know if strong duality holds?

Conditions under which strong duality holds are called **constraint qualifications** (they are sufficient, but not necessary)

(Probably) best known sufficient condition: **Strong duality holds if**

- Primal problem is **convex**, i.e. of the form

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0 && i = 1, \dots, n \\ & && Ax = b \end{aligned} \quad (10)$$

for convex  $f_0, \dots, f_m$ , and

- **Slater's condition** holds: there exists some *strictly* feasible point<sup>1</sup>  $\tilde{x} \in \text{relint}(\mathcal{D})$  such that

$$f_i(\tilde{x}) < 0 \quad i = 1, \dots, m \quad A\tilde{x} = b.$$

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<sup>1</sup>We denote by  $\text{relint}(\mathcal{D})$  the relative interior of the set  $\mathcal{D}$ . This looks like the interior of the set, but is non-empty even when the set is a subspace of a larger space.

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for convex  $f_0, \dots, f_m$ , and

- **Slater's condition** for the case of **affine**  $f_i$  is trivial (inequality constraints no longer strict, reduces to original inequality constraints)

$$f_i(\tilde{x}) \leq 0 \quad i = 1, \dots, m \quad A\tilde{x} = b.$$

# A consequence of strong duality...

**Assume** primal is equal to the dual. What are the consequences?

- $x^*$  solution of **original** problem (minimum of  $f_0$  under constraints),
- $(\lambda^*, \nu^*)$  solutions to **dual**

$$\begin{aligned} f_0(x^*) &\stackrel{\text{(assumed)}}{=} g(\lambda^*, \nu^*) \\ &\stackrel{\text{(g definition)}}{=} \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) \\ &\stackrel{\text{(inf definition)}}{\leq} f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\stackrel{(4)}{\leq} f_0(x^*), \end{aligned}$$

(4):  $(x^*, \lambda^*, \nu^*)$  satisfies  $\lambda^* \succeq 0$ ,  $f_i(x^*) \leq 0$ , and  $h_i(x^*) = 0$ .

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## ...is complementary slackness

From previous slide,

$$\sum_{i=1}^m \lambda_i^* f_i(x^*) = 0, \quad (11)$$

which is the condition of **complementary slackness**. This means

$$\begin{aligned} \lambda_i^* > 0 &\implies f_i(x^*) = 0, \\ f_i(x^*) < 0 &\implies \lambda_i^* = 0. \end{aligned}$$

From  $\lambda_i$ , read off which inequality constraints are strict.

# KKT conditions for global optimum

Assume functions  $f_i, h_i$  are **differentiable** and **strong duality**. Since  $x^*$  minimizes  $L(x, \lambda^*, \nu^*)$ , derivative at  $x^*$  is zero,

$$\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) = 0.$$

**KKT conditions definition:** we are at global optimum,  $(x, \lambda, \nu) = (x^*, \lambda^*, \nu^*)$  when (a) **strong duality** holds, and (b)

$$f_i(x) \leq 0, \quad i = 1, \dots, m$$

$$h_i(x) = 0, \quad i = 1, \dots, p$$

$$\lambda_i \geq 0, \quad i = 1, \dots, m$$

$$\lambda_i f_i(x) = 0, \quad i = 1, \dots, m$$

$$\nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) + \sum_{i=1}^p \nu_i \nabla h_i(x) = 0$$

# KKT conditions for global optimum

Assume functions  $f_i, h_i$  are **differentiable** and **strong duality**. Since  $x^*$  minimizes  $L(x, \lambda^*, \nu^*)$ , derivative at  $x^*$  is zero,

$$\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) = 0.$$

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# KKT conditions for global optimum

In summary: if

- primal problem **convex** and
- constraint functions satisfy **Slater's conditions**

then strong duality holds. If in addition

- functions  $f_i, h_i$  **differentiable**

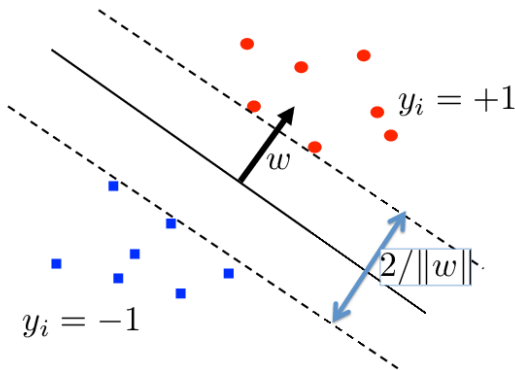
**then** KKT conditions *necessary and sufficient* for optimality.

# Support vector classification



# Linearly separable points

Classify two clouds of points, where there exists a hyperplane which linearly separates one cloud from the other without error.



Smallest distance from each class to the separating hyperplane  $w^T x + b$  is called the **margin**.

# Maximum margin classifier, linearly separable case

This problem can be expressed as follows:

$$\max_{w,b} (\text{margin}) = \max_{w,b} \left( \frac{2}{\|w\|} \right) \quad (12)$$

subject to

$$\begin{cases} \min (w^\top x_i + b) = 1 & i : y_i = +1, \\ \max (w^\top x_i + b) = -1 & i : y_i = -1. \end{cases} \quad (13)$$

The resulting classifier is

$$y = \text{sign}(w^\top x + b),$$

We can rewrite to obtain

$$\max_{w,b} \frac{1}{\|w\|} \quad \text{or} \quad \min_{w,b} \|w\|^2$$

subject to

$$y_i(w^\top x_i + b) \geq 1. \quad (14)$$

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# Maximum margin classifier: with errors allowed

Allow “errors”: points within the margin, or even on the wrong side of the decision boundary. Ideally:

$$\min_{w,b} \left( \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \mathbb{I}[y_i (w^\top x_i + b) < 0] \right),$$

where  $C$  controls the tradeoff between maximum margin and loss.

Replace with **convex upper bound**:

$$\min_{w,b} \left( \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \theta \left( y_i (w^\top x_i + b) \right) \right).$$

with hinge loss,

$$\theta(\alpha) = (1 - \alpha)_+ = \begin{cases} 1 - \alpha & 1 - \alpha > 0 \\ 0 & \text{otherwise.} \end{cases}$$

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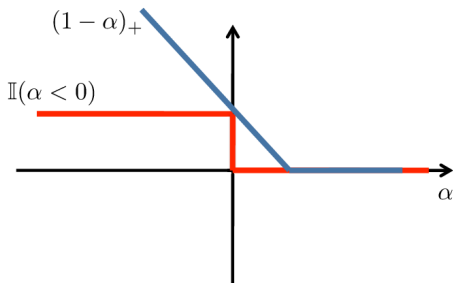
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# Hinge loss

Hinge loss:

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# Support vector classification

Substituting in the hinge loss, we get

$$\min_{w,b} \left( \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \theta \left( y_i (w^\top x_i + b) \right) \right).$$

How do you implement hinge loss with simple **inequality constraints** (for optimization)?

$$\min_{w,b,\xi} \left( \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i \right) \quad (15)$$

subject to<sup>2</sup>

$$\xi_i \geq 0 \quad y_i (w^\top x_i + b) \geq 1 - \xi_i$$

---

<sup>2</sup>Either  $y_i (w^\top x_i + b) \geq 1$  and  $\xi_i = 0$  as before, or  $y_i (w^\top x_i + b) < 1$ , and then  $\xi_i > 0$  takes the value satisfying  $y_i (w^\top x_i + b) = 1 - \xi_i$ .

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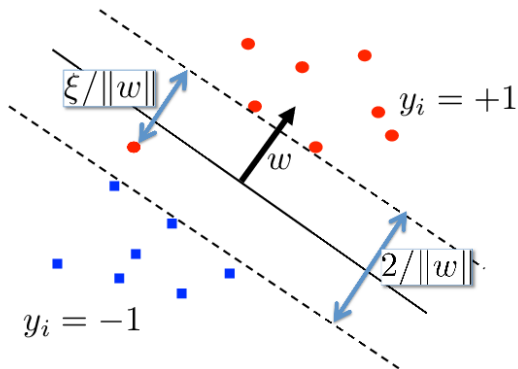
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# Support vector classification



# Does strong duality hold?

- ① Is the optimization problem **convex** wrt the variables  $w, b, \xi$ ?

$$\text{minimize} \quad f_0(w, b, \xi) := \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i$$

$$\text{subject to} \quad f_i(w, b, \xi) := 1 - \xi_i - y_i \left( w^\top x_i + b \right) \leq 0 \quad i = 1, \dots, n$$
$$Ax = b \quad (\text{absent})$$

Each of  $f_0, f_1, \dots, f_n$  are **convex**.

- ② Does **Slater's condition** hold? **Trivial** since inequality constraints **affine**, and there always exists some

$$\xi_i \geq 0$$
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# Support vector classification: Lagrangian

The Lagrangian:  $L(w, b, \xi, \alpha, \lambda)$

$$= \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i \left( 1 - y_i (w^\top x_i + b) - \xi_i \right) + \sum_{i=1}^n \lambda_i (-\xi_i)$$

with dual variable constraints

$$\alpha_i \geq 0, \quad \lambda_i \geq 0.$$

**Minimize wrt the primal variables**  $w$ ,  $b$ , and  $\xi$ .

# Support vector classification: Lagrangian

Derivative wrt  $w$ :

$$\frac{\partial L}{\partial w} = w - \sum_{i=1}^n \alpha_i y_i x_i = 0 \quad w = \sum_{i=1}^n \alpha_i y_i x_i. \quad (16)$$

Derivative wrt  $b$ :

$$\frac{\partial L}{\partial b} = \sum_i y_i \alpha_i = 0. \quad (17)$$

Derivative wrt  $\xi_i$ :

$$\frac{\partial L}{\partial \xi_i} = C - \alpha_i - \lambda_i = 0 \quad \alpha_i = C - \lambda_i. \quad (18)$$

Noting that  $\lambda_i \geq 0$ ,

$$\alpha_i \leq C.$$

# Support vector classification: Lagrangian

Now use **complementary slackness**:

**Non-margin SVs:**  $\alpha_i = C \neq 0$ :

- 1 We immediately have  $1 - \xi_i = y_i (w^\top x_i + b)$ .
- 2 Also, from condition  $\alpha_i = C - \lambda_i$ , we have  $\lambda_i = 0$  (hence can have  $\xi_i > 0$ ).

**Margin SVs:**  $0 < \alpha_i < C$ :

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# The support vectors

We observe:

- 1 The solution is sparse: points which are not on the margin, or “margin errors”, have  $\alpha_i = 0$
- 2 **The support vectors:** only those points on the decision boundary, or which are margin errors, contribute.
- 3 Influence of the non-margin SVs is bounded, since their weight cannot exceed  $C$ .

# Support vector classification: dual function

Thus, our goal is to maximize the dual,

$$\begin{aligned}g(\alpha, \lambda) &= \frac{1}{2} \|w\|^2 + C \sum_{i=1}^n \xi_i + \sum_{i=1}^n \alpha_i \left( 1 - y_i (w^\top x_i + b) - \xi_i \right) \\&\quad + \sum_{i=1}^n \lambda_i (-\xi_i) \\&= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j + C \sum_{i=1}^m \xi_i - \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j \\&\quad - \underbrace{b \sum_{i=1}^m \alpha_i y_i}_0 + \sum_{i=1}^m \alpha_i - \sum_{i=1}^m \alpha_i \xi_i - \sum_{i=1}^m (C - \alpha_i) \xi_i \\&= \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j.\end{aligned}$$

# Support vector classification: dual function

Maximize the dual,

$$g(\alpha) = \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j,$$

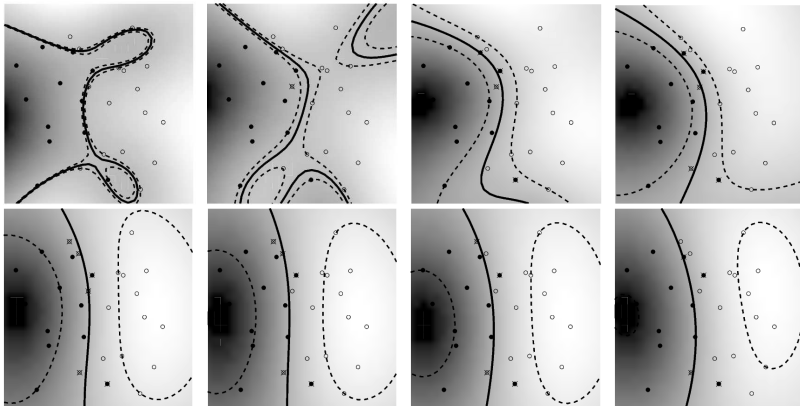
subject to the constraints

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^n y_i \alpha_i = 0$$

This is a quadratic program.

**Offset  $b$ :** for the margin SVs, we have  $1 = y_i (w^\top x_i + b)$ . Obtain  $b$  from any of these, or take an average.

# Support vector classification: kernel version



Taken from Schoelkopf and Smola (2002)

**Maximum margin classifier in RKHS:** write the hinge loss formulation

$$\min_w \left( \frac{1}{2} \|w\|_{\mathcal{H}}^2 + C \sum_{i=1}^n \theta(y_i \langle w, k(x_i, \cdot) \rangle_{\mathcal{H}}) \right)$$

for the RKHS  $\mathcal{H}$  with kernel  $k(x, \cdot)$ . Use the result of the **representer theorem**,

$$w(\cdot) = \sum_{i=1}^n \beta_i k(x_i, \cdot).$$

Maximizing the margin equivalent to minimizing  $\|w\|_{\mathcal{H}}^2$ : for many RKHSs a **smoothness constraint** (e.g. Gaussian kernel).

# Support vector classification: kernel version

Substituting and introducing the  $\xi_i$  variables, get

$$\min_{\beta, \xi} \left( \frac{1}{2} \beta^\top K \beta + C \sum_{i=1}^n \xi_i \right) \quad (19)$$

where the matrix  $K$  has  $i, j$ th entry  $K_{ij} = k(x_i, x_j)$ , subject to

$$\xi_i \geq 0 \quad y_i \sum_{j=1}^n \beta_j k(x_i, x_j) \geq 1 - \xi_i$$

Convex in  $\beta, \xi$  since  $K$  is **positive definite**: hence strong duality holds.

Dual:

$$g(\alpha) = \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j k(x_i, x_j),$$

$$\text{subject to } w(\cdot) = \sum_{i=1}^n y_i \alpha_i k(x, \cdot), \quad 0 \leq \alpha_i \leq C.$$



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$$\text{subject to } w(\cdot) = \sum_{i=1}^n y_i \alpha_i k(x, \cdot), \quad 0 \leq \alpha_i \leq C.$$

# Support vector classification: the $\nu$ -SVM

Another kind of SVM: the  $\nu$ -SVM:

Hard to interpret  $C$ . Modify the formulation to get a **more intuitive parameter  $\nu$** .

Again, we drop  $b$  for simplicity. Solve

$$\min_{w, \rho, \xi} \left( \frac{1}{2} \|w\|^2 - \nu \rho + \frac{1}{n} \sum_{i=1}^n \xi_i \right)$$

subject to

$$\begin{aligned} \rho &\geq 0 \\ \xi_i &\geq 0 \\ y_i w^\top x_i &\geq \rho - \xi_i, \end{aligned}$$

(now directly adjust **margin width  $\rho$** ).

# The $\nu$ -SVM: Lagrangian

$$\frac{1}{2}\|w\|_{\mathcal{H}}^2 + \frac{1}{n} \sum_{i=1}^n \xi_i - \nu \rho + \sum_{i=1}^n \alpha_i \left( \rho - y_i w^\top x_i - \xi_i \right) + \sum_{i=1}^n \beta_i (-\xi_i) + \gamma(-\rho)$$

for dual variables  $\alpha_i \geq 0$ ,  $\beta_i \geq 0$ , and  $\gamma \geq 0$ .

Differentiating and setting to zero for each of the primal variables  $w$ ,  $\xi$ ,  $\rho$ ,

$$w = \sum_{i=1}^n \alpha_i y_i x_i$$

$$\alpha_i + \beta_i = \frac{1}{n} \quad (20)$$

$$\nu = \sum_{i=1}^n \alpha_i - \gamma \quad (21)$$

From  $\beta_i \geq 0$ , equation (20) implies

$$0 \leq \alpha_i \leq n^{-1}.$$

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# Complementary slackness (1)

## Complementary slackness conditions:

Assume  $\rho > 0$  at the global solution, hence  $\gamma = 0$ , and

$$\sum_{i=1}^n \alpha_i = \nu. \quad (22)$$

Case of  $\xi_i > 0$ : complementary slackness states  $\beta_i = 0$ , hence from (20) we have  $\alpha_i = n^{-1}$ . Denote this set as  $N(\alpha)$ . Then

$$\sum_{i \in N(\alpha)} \frac{1}{n} = \sum_{i \in N(\alpha)} \alpha_i \leq \sum_{i=1}^n \alpha_i = \nu,$$

so

$$\frac{|N(\alpha)|}{n} \leq \nu,$$

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## Complementary slackness (2)

**Case of  $\xi_i = 0$ :**  $\beta_i > 0$  and so  $\alpha_i < n^{-1}$ . Denote by  $M(\alpha)$  the set of points  $n^{-1} > \alpha_i > 0$ . Then from (22),

$$\nu = \sum_{i=1}^n \alpha_i = \sum_{i \in N(\alpha)} \frac{1}{n} + \sum_{i \in M(\alpha)} \alpha_i \leq \sum_{i \in M(\alpha) \cup N(\alpha)} \frac{1}{n},$$

thus

$$\nu \leq \frac{|N(\alpha)| + |M(\alpha)|}{n},$$

and  $\nu$  is a lower bound on the number of support vectors with non-zero weight (both on the margin, and “margin errors”).

# Dual for $\nu$ -SVM

Substituting into the Lagrangian, we get

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j + \frac{1}{n} \sum_{i=1}^n \xi_i - \rho \nu - \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j \\ & + \sum_{i=1}^n \alpha_i \rho - \sum_{i=1}^n \alpha_i \xi_i - \sum_{i=1}^n \left( \frac{1}{n} - \alpha_i \right) \xi_i - \rho \left( \sum_{i=1}^n \alpha_i - \nu \right) \\ & = -\frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j \end{aligned}$$

Maximize:

$$g(\alpha) = -\frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^\top x_j,$$

subject to

$$\sum_{i=1}^n \alpha_i \geq \nu \quad 0 \leq \alpha_i \leq \frac{1}{n}.$$



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# Questions?

