Closest String and Closest Substring Problems

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Problem Statement I

Closest String

Given a set $\mathcal{S} = \{s_1, s_2, \cdots, s_n\}$ of strings each length m, find a center string s of length m minimizing d such that for every string $s_i \in \mathcal{S}$, $d(s, s_i) \leq d$. Here $d(s, s_i)$ is the Hamming distance between s and s_i.

Closest Substring

Given a set $\mathcal{S} = \{s_1, s_2, \cdots, s_n\}$ of strings each length m and an integer L, find a center string s of length L minimizing d such that for every string $s_i \in \mathcal{S}$ there is a length L substring t_i of s_i with $d(s, t_i) \leq d$.

Example of Closest String

Given 4 strings s_1, s_2, s_3, s_4,

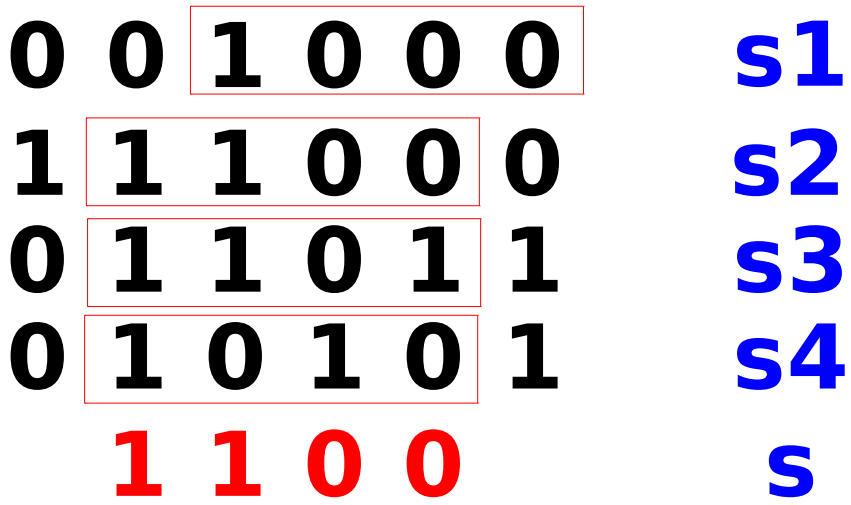
0	0	1	0	0	0	s1
1	1	1	0	0	0	s2
0	1	1	0	1	1	s3
0	1	0	1	0	1	s4
0	1	1	0	0	1	s

Optimal center string $s = 011000$

$$d = \max_{i=1}^{4} d(s_i, s) = \max\{2, 2, 1, 2\} = 2$$

Example of Closest Substring

Given 4 strings s_1, s_2, s_3, s_4, $L = 4$,



Optimal center string $s = 1100$,

$$d = \max_{i=1}^{4} d(t_i, s) = \max\{1, 0, 1, 2\} = 2$$

Basic Idea

In polynomial time, we can enumerate all $\binom{n}{r}$ strings for any fixed r. We can prove that, on positions that r strings all agree, denote as Q, it is a good approximate solution. We can also prove that, $|Q| \geq m - r \cdot d_{opt}$.
On other positions, we use LP + Random rounding technique to obtain an approximate solution.

Algorithm I

Input : $s_1, s_2, \cdots, s_n \in \Sigma^m$, an integer $r \geq 2$ and a small number $\epsilon > 0$.

Output : a center string $s \in \Sigma^m$

Algorithm :

1. **for** each r-element subset $\{s_{i_1}, s_{i_2}, \cdots, s_{i_r}\}$ of the n input strings **do**
   1. let $Q = \{j | s_{i_1}[j] = s_{i_2}[j] = \cdots = s_{i_r}[j]\}$, $P = \{1, 2, \cdots, m\} - Q$

Algorithm II

2. solve the following optimization problem

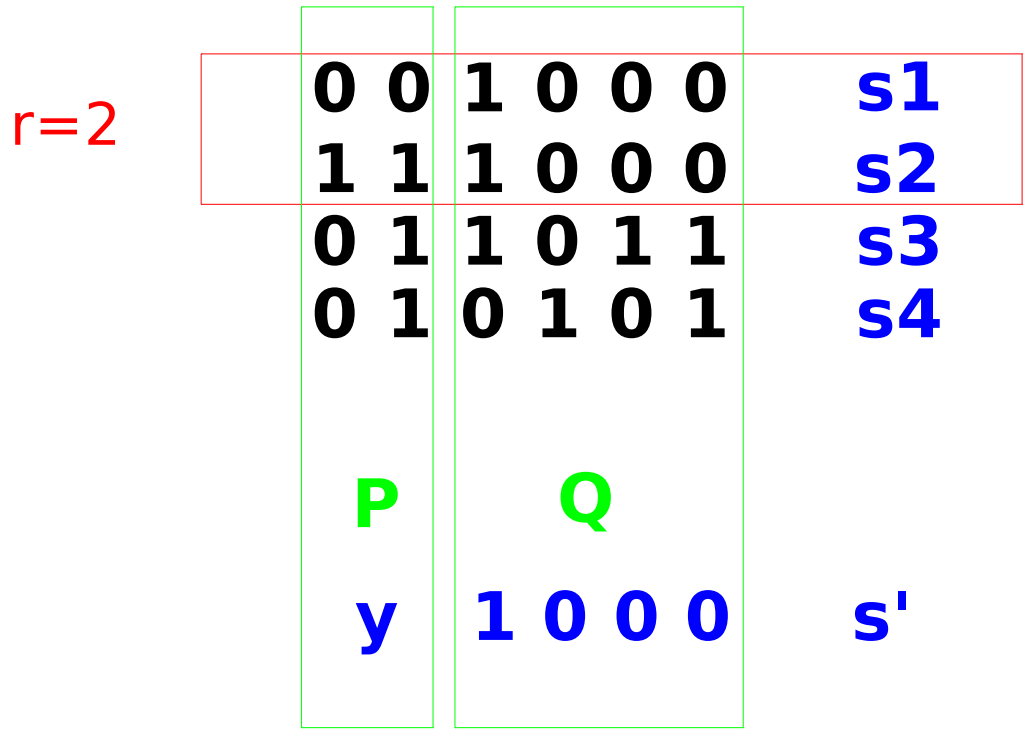
$$\begin{aligned} & \min d \\ s.t. \quad & d(s_i|_P, y) + d(s_i|_Q, s_{i_1}|_Q) \leq d, i = 1, 2, \cdots, n \end{aligned}$$

   Random rounding the fractional solution $\bar{y}$ to get a approximation solution y. Use derandomization technique to make this step deterministic rather than random.
3. let $s'|_Q = s_{i_1}|_Q, s'|_P = y$. Calculate the radius of the solution with s' as the center string.

2. **for** $i = 1, 2, \cdots, n$ **do**
   1. calculate the radius of the solution with s_i as the center string.
3. output the best solution of the above two steps.

Example of Closest String I

Suppose $r = 2$. In step 1, we should enumerate all $\binom{4}{2} = 6$ cases. In each cases, we select 2 lines, calculate P and Q.



Example of Closest String II

In step 2, we fix $s'|_Q = 1000$, solve the following optimization problem :

$$
\begin{aligned}
& \min d \\
s.t. \quad & y_{10} + y_{11} = 1 \\
& y_{20} + y_{21} = 1 \\
& y_{11} + y_{21} + 0 \leq d \\
& y_{10} + y_{20} + 0 \leq d \\
& y_{11} + y_{20} + 2 \leq d \\
& y_{11} + y_{20} + 2 \leq d
\end{aligned}
$$

Example of Closest String III

Solve this linear programming, and random rounding the fractional solution to integer solution :

$$y_{10} = y_{21} = 1, y_{11} = y_{20} = 0$$

So,

$$s'|_P = 01, s' = 011000$$

$$d = \max_{i=1}^{4} d(s_i, s') = \max\{1, 1, 2, 3\} = 3$$

Try all $\binom{4}{2} = 6$ cases, obtain the minimum, denote as d_0. Then we finish step 1.

Example of Closest String IV

In step 2, we calculate the radius when s_i is the center string.

$$\begin{aligned} d_1 &= \max_{i=1}^{4} d(s_1, s_i) \\ d_2 &= \max_{i=1}^{4} d(s_2, s_i) \\ d_3 &= \max_{i=1}^{4} d(s_3, s_i) \\ d_4 &= \max_{i=1}^{4} d(s_4, s_i) \end{aligned}$$

Calculate the minimal radius in both step 1 and step 2.

$$d = \min\{d_0, d_1, d_2, d_3, d_4\}$$

Analysis I

Theorem

The above algorithm is a PTAS for CLOSEST STRING problem.

Analysis II

Proof.

Obviously, the time complexity of the algorithm is $O\left((nm)^r n^{O(\log|\Sigma|\cdot r^2/\epsilon^2)}\right)$, which is polynomial in terms of n,m.

The proof of approximation guarantee is organized as 3 lemmas as follows :

Lemma 1 proves $s'|_Q$ is a good approximation to s with approximation rate $1+\frac{1}{2r-1}$.

Lemma 2 proves $|P| < r \cdot d_{opt}$.

Based on the above 2 lemmas, Lemma 3 proves step 1.2 obtains a approximate solution with rate $(1+\frac{1}{2r-1}+\epsilon)$.

Analysis III

Lemma 1

If $\max_{i\leq i,j\leq n} d(s_i, s_j) > (1 + \frac{1}{2r-1})d_{opt}$, then there **exists** r indices $1 \leq i_1, i_2, \cdots, i_r \leq n$ such that for any $1 \leq l \leq n$,

$$d(s_l|_Q, s_{i_1}|_Q) - d(s_l|_Q, s|_Q) \leq \frac{1}{2r-1} d_{opt}$$

where Q is the set of positions that $s_{i_1}, s_{i_2}, \cdots, s_{i_r}$ all agree.

- if $\max_{i\leq i,j\leq n} d(s_i, s_j) \leq (1 + \frac{1}{2r-1})d_{opt}$, then any s_i will be a good center string. (Recall the step 2 of the algorithm)

Analysis IV

Lemma 2

Let $P = \{1, 2, \cdots, m\} - Q$, then $|P| \leq r \cdot d_{opt}$ and $|Q| \geq m - r \cdot d_{opt}$.

Proof.

Let $q \in P$, then there exists some s_{i_j} such that $s_{i_j}[q] \neq s[q]$. Since $d(s_{i_j}, s) \leq d_{opt}$, each s_{i_j} contributes at most d_{opt} positions for P. Thus $|P| \leq r \cdot d_{opt}$. □

- this lemma gives the lower bound of d_{opt}, $d_{opt} \geq \frac{|P|}{r}$, which is essential for the analysis of LP + random rounding!

Analysis V

Lemma 3

Given a string s' and a position set Q and P, $|P| < r \cdot d_{opt}$, such that for any $i = 1, 2, \cdots, n$,

$$d(s_i|_Q, s'|_Q) - d(s_i|_Q, s|_Q) \leq \rho \cdot d_{opt},$$

then step 1.2 gives a solution with approximate rate $(1 + \rho + \epsilon)$ in polynomial time for any fixed $\epsilon \geq 0$.

Before the proof, we can see that the conditions of this lemma are all satisfied by lemma 2, where $\rho = \frac{1}{2r-1}$.

Proof

Analysis VI

Recall the optimization problem in step 1.2

$$\begin{aligned} & \min d \\ s.t. \quad & d(s_i|_P, y) + d(s_i|_Q, s'|_Q) \leq d, i = 1, 2, \cdots, n \end{aligned}$$

First, we show that $y = s|_P$ is a solution with cost $d \leq (1+\rho)d_{opt}$. In fact, for $i = 1, 2, \cdots, n$

$$\begin{aligned} & d(s_i|_P, s|_P) + d(s_i|_Q, s'|_Q) \\ \leq \quad & d(s_i|_P, s|_P) + d(s_i|_Q, s|_Q) + \rho \cdot d_{opt} \\ \leq \quad & (1+\rho)d_{opt} \end{aligned}$$

Second, rewirte the optimization problem to an ILP problem. In order for this, define 0-1 variables $y_{j,a}$, $1 \leq j \leq |P|$, $a \in \Sigma$,

Analysis VII

$y_{j,a} = 1$ means $y[j] = a$; define $\chi(s_i[j], a) = 0$ if $s_i[j] = a$ and 1 if $s_i[j] \neq a$. Then the above optimization problem can be formulated as follows

$$\begin{aligned} & \min d \\ s.t. \quad & \begin{cases} \sum_{a \in \Sigma} y_{j,a} = 1, 1 \leq j \leq |P| \\ \sum_{1 \leq j \leq |P|} \sum_{a \in \Sigma} \chi(s_i[j], a) y_{j,a} + d(s_i|_Q, s'|_Q) \leq d, 1 \leq i \leq n \end{cases} \end{aligned}$$

Solve it by LP to get a fractional solution $\bar{y}$ with cost $\bar{d}$.
Random rounding $\bar{y}$ to y' by independently set $y'_{j,a} = 1$ and $y'_{j,b} = 0, b \in \Sigma - \{a\}$ with probability $y_{j,a}^{-}$.
So $d(s_i|_P, y') = \sum_{i=1}^{|P|} \sum_{a \in \Sigma} \chi(s_i[j], a) y_{j,a}$, which is the sum of $|P|$ independent random variables.

Analysis VIII

$$\begin{aligned} E(d(s_i|_P)) &= \sum_{1\le j\le |P|}\sum_{a\in\Sigma}\chi(s_i[j],a)y_{j,a}^{-} \\ &\le \bar{d} - d(s_i|_Q, s'|_Q) \\ &\le (1+\rho)\cdot d_{opt} - d(s_i|_Q, s'|_Q) \end{aligned}$$

Employ Chernoff Bound

$$\Pr(X > \mu + \epsilon n) \le \exp(-\frac{1}{3}n\epsilon^2)$$

we have

$$\Pr(d(s_i|_P, y') > (1+\rho)d_{opt} - d(s_i|_Q, s'|_Q) + \epsilon'|P|) \le \exp(-\frac{1}{3}\epsilon'^2|P|)$$

Analysis IX

Let $s'|_P = y'$ and consider all n strings, we claim

$$\Pr(d(s_i, s') < (1+\rho)d_{opt} + \epsilon'|P| \text{ for all i}) \geq 1 - n\exp(-\frac{1}{3}\epsilon'^2|P|)$$

Use standard derandomization methods, we can obtain a deterministic s' that satisfies

$$d(s_i, s') < (1+\rho)d_{opt} + \epsilon'|P|, \quad 1 \leq i \leq n$$

Recall that $|P| < r \cdot d_{opt}$, let $\epsilon' = \frac{\epsilon}{r}$, we have

$$d(s_i, s') < (1+\rho+\epsilon)d_{opt}, \quad 1 \leq i \leq n$$

Then we finish the proof.

Basic Idea

We want to follow the algorithm of Closest String algorithm. However, we do not know how to construct an optimization problem, for the reason that we do not konw the optimal substring in each string. Thus, the choice of a "good" substring from every string s_i is the only obstacle on the way to the solution.
As we do in the algorithm of Closest String, first we try all the choices of r substrings from $\mathcal{S}$, we can assume that $t_{i_1}, t_{i_2}, \cdots, t_{i_r}$ is a good partial solution. After that, we randomly pick $O(\log(mn))$ positions from P. By trying all length $|R|$ strings, we can assume we know $s|_R$. Then for each $i \leq i \leq n$, we find the substring t'_i from s_i such that $f(t'_i) = d(s|_R, t'_i|_R) \cdot \frac{|P|}{|R|} + d(t_{i_1}|_Q, t'_i|_Q)$ is minimized. We use $t'_i, 1 \leq i \leq n$ to construct the optimization problem.

Algorithm for Closest Substring problem I

Input : $s_1, s_2, \cdots, s_n \in \Sigma^m$, an integer $1 \leq L \leq m$, an integer $r \geq 2$ and a small number $\epsilon > 0$.

Ouput : center string s

Algorithm :

1. **for** each r-element subset $\{t_{i_1}, t_{i_2}, \cdots, t_{i_r}\}$, where t_{i_j} is a substring of length L from s_{i_j} **do**
   1. let $Q = \{j | t_{i_1}[j] = t_{i_2}[j] = \cdots = t_{i_r}[j]\}$, $P = \{1, 2, \cdots, m\} - Q$
   2. randomly select $\frac{4}{\epsilon^2}\log(mn)$ positions from P, denote as R
   3. **for** every string x of length $|R|$ **do**
      1. for $1 \leq i \leq n$, let t'_i be a length L substring of s_i minimizing $f(t'_i) = d(x, t'_i|_R) \cdot \frac{|P|}{|R|} + d(t_{i_1}|_Q, t'_i|_Q)$.

Algorithm for Closest Substring problem II

2. solve the optimization problem

$$\begin{aligned} & \min d \\ s.t. \quad & d(t_i'|_P, y) + d(t_i'|_Q, t_{i-1}|_Q) \leq d, 1 \leq i \leq n \end{aligned}$$

   Random rounding the fractional solution $\bar{y}$ to get a approximation solution y. Use derandomization technique to make this step deterministic rather than random.
3. Let $s'|_Q = t_{i_1}|_Q$ and $s'|_P = y$. Let $c = \max_{i=1}^{n} d(s', t_i')$.

2. **for** every length L substring s' of s_1 **do**
   1. Let $c = \max_{i=1}^{n} \min_{t_i} d(s', ti)$.
3. output the s' with minimum c in step 1.3.3 and step 2.1.

Example of Closest Subtring I

Suppose $r = 2$. In step 1, we should enumerate all $\binom{4}{2} \times 3 \times 3 = 54$ cases. In each cases, we select 2 substring of length 4 , calculate P and Q.
We fix $s'|_Q = 00$.
Randomly select $|R| = O(\log(mn))$ positions, say, $|R| = 1$. Then enumerate all length $|R|$ strings, say, $s'|_R = 0$.

Example of Closest Substring II

r=2

0	0	1	0	0	0	s1
1	1	1	0	0	0	s2
0	1	1	0	1	1	s3
0	1	0	1	0	1	s4

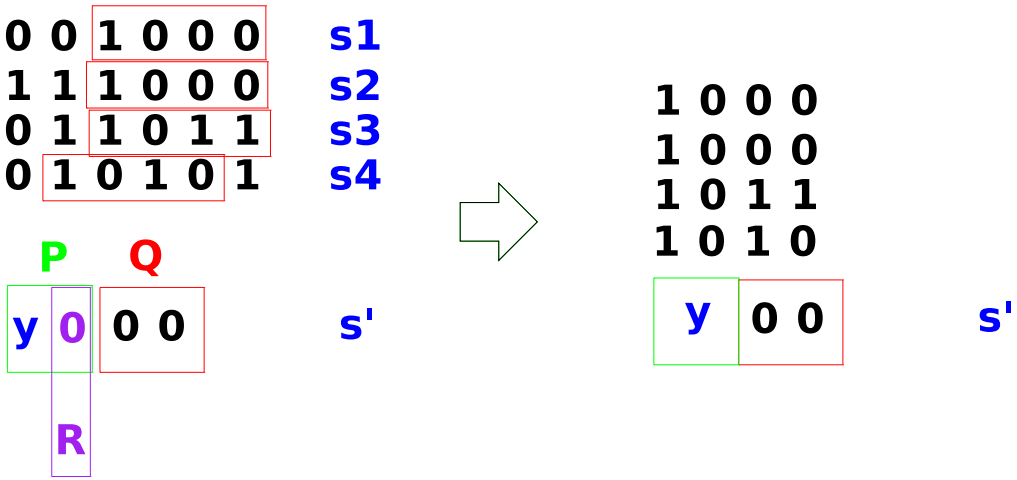
P Q

0	1	0	0	t1
1	0	0	0	t2
y	0	0	0	s'

R

Example of Closest Substring III

Now, for each s_i, find out the t'_i minimizing $f(u) = d(u|_R, s'|_R) \times 2 + d(u|_Q, s'|_Q)$.



Example of Closest Substring IV

Solve the following optimization problem :

$$
\begin{aligned}
& \min d \\
s.t. \quad & y_{10} + y_{11} = 1 \\
& y_{20} + y_{21} = 1 \\
& y_{10} + y_{21} + 0 \leq d \\
& y_{10} + y_{21} + 0 \leq d \\
& y_{10} + y_{21} + 2 \leq d \\
& y_{10} + y_{21} + 1 \leq d
\end{aligned}
$$

Example of Closest Subtring V

Solve this linear programming, and random rounding the fractional solution to integer solution :

$$y_{10} = y_{21} = 0, y_{11} = y_{20} = 1$$

So,

$$s' = 1000$$

$$d = \max_{i=1}^{4} d(t_i, s') = \max\{0, 0, 2, 1\} = 2$$

Try all 54 cases, obtain the minimum, denote as d_0. Then we finish step 1.

Example of Closest Subtring VI

In step 2, we calculate the radius when t_i is the center string where t_i is a length L substring of s_i. denote the minimun d_1. Calculate the minimal radius in both step 1 and step 2.

$$d = \min\{d_0, d_1\}$$

Analysis I

Theorem

The above algorithm is a PTAS for Closest Substring problem.

Analysis II

Proof.

The time complexity of the algorithm is $O\left((n^m)^{O(\log|\Sigma|/\delta^4)}\right)$, which is polynomial in terms of n,m.

The proof of approximation guarantee is organized as 3 lemmas as follows :

Lemma 1 proves $s'|_Q$ is a good approximation to s with approximation rate $1+\frac{1}{2r-1}$.

Define $s^*|_P = s|_P$ and $s^*|_Q = t_{i_1}|_Q$, lemma 2 proves $d(s^*, t_i') \leq d(s^*, t_i) + 2\epsilon|P|$ for all $1 \leq i \leq n$.

Based on the above 2 lemmas, Lemma 3 proves the algorithm obtains a approximate solution with rate $(1+\frac{1}{2r-1}+3\epsilon r)$. □

Analysis III

Lemma 1

There exists $t_{i_1}, t_{i_2}, \cdots, t_{i_r}$ chosen in step 1, such that for any $1 \leq l \leq n$

$$d(t_l|_Q, t_{i_1}|_Q) - d(s_l|_Q, s|_Q) \leq \frac{1}{2r-1} \cdot d_{opt}$$

Proof.

The fact follow from Lemma 1 of Closest String directly. □

Analysis IV

Lemma 2

Define $s^*|_P = s|_P$ and $s^*|_Q = t_{i_1}|_Q$. Then we have, with high probability

$$d(s^*, t_i') \leq d(s^*, t_i) + 2\epsilon|P|$$

for all $1 \leq i \leq n$.

Proof.

□

- The randomness comes from the randomly selected $|R|$. Use standard method, we can derandomize it to make this deterministic.

Analysis V

Lemma 3

step 1.3.3 gives an approximation solution s' with approximation rate $(1 + \frac{1}{2r-1} + 3\epsilon r)$.

Proof.

Recall the optimization problem in step 1.2

$$\begin{aligned} & \min d \\ s.t. \quad & d(t_i'|_P, y) + d(t_i'|_Q, s'|_Q) \leq d, i = 1, 2, \cdots, n \end{aligned}$$

$y = s|_P$ is a feasible solution. Now we calculate its radius when s^* is the center string. Recall that $s^*|_P = s|_P$ and $s^*|_Q = t_{i_1}|_Q$.

Analysis VI

According to Lemma 2 and Lemma 1

$$\begin{aligned} d(s^*, t_i') &\leq d(s^*, t_i) + 2\epsilon|P| \\ &\leq d(s|_P, t_i|_P) + d(t_{i_1}|_Q, t_i|_Q) + 2\epsilon|P| \\ &\leq d(s|_P, t_i|_P) + d(s|_Q, t_i|Q) + 2\epsilon|P| + \frac{1}{2r-1} d_{opt} \\ &\leq (1 + \frac{1}{2r-1}) d_{opt} + 2\epsilon|P|) \end{aligned}$$

In a short word, $y = s|_P$ is a solution of the optimization problem with cost at most $(1 + \frac{1}{2r-1}) d_{opt} + 2\epsilon|P|)$.
Next, as we do in the Closest String problem, we rewirte the optimization problem to an ILP and solve it and random rounding the fractional solution $\bar{y}$ to y. Define $s'|_Q = t_{i_1}|_Q$ and $s'|_P = y$.

Analysis VII

$$E(d(s', t_i')) \leq \bar{d} \leq (1 + \frac{1}{2r-1} + 2\epsilon|P|)d_{opt}$$

So, chernoff bound ensures that, with high probability,

$$\begin{aligned} & d(s', t_i') \\ \leq & (1 + \frac{1}{2r-1})d_{opt} + 2\epsilon|P| + \epsilon|P| \\ \leq & (1 + \frac{1}{2r-1})d_{opt} + 3\epsilon|P| \\ \leq & (1 + \frac{1}{2r-1} + 3\epsilon r)d_{opt} \end{aligned}$$

After derandomization, we can obtain an approximation solution s' with rate $(1 + \frac{1}{2r-1} + 3\epsilon r)d_{opt}$.