The Knuth-Yao Quadrangle Inequality Speedup is a Consequence of Total Monotonicity

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Motivation

- Nothing new: material here goes back 20-30 years.
- There are two classic cookbook
  Dynamic Programming Speedups in the literature:
  Knuth-Yao technique & SMAWK algorithm.
- They "feel" similar. Are they related?
- Knuth-Yao predates online algorithms.
  Can the KY speedup be maintained online?
- Answers to the two questions turned out to be related.
- Note: major confusion arises in the analysis because certain essential terms, e.g., quadrangle-inequality, monotone and online-algorithm have been used in very different ways in the two techniques' literature.

Outline

- Background
  - Kunth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.
- Extensions
  Applying the technique to known generalizations of KY.

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- Both techniques are often used to speed up DPs.
- $\boxed{\text{How are the two techniques related?}}$

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  Computing Optimal Binary Search Trees (Optimal BST)

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  - n internal nodes corresponds to successful search

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    q_l, $(l = 0 \dots n)$ is the weight that Key$_l$ $<$ search-key $<$ Key$_{l+1}$
  - Minimize the number of comparisons

$$\sum_{1 \le l \le n} p_l \cdot (1 + \underbrace{d(p_l)}_{\text{depth}}) + \sum_{0 \le l \le n} q_l \cdot \underbrace{d(q_l)}_{\text{depth}}$$

Optimal BST

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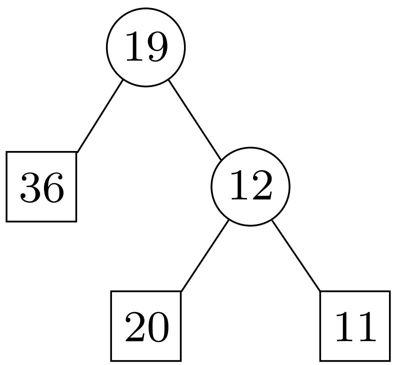

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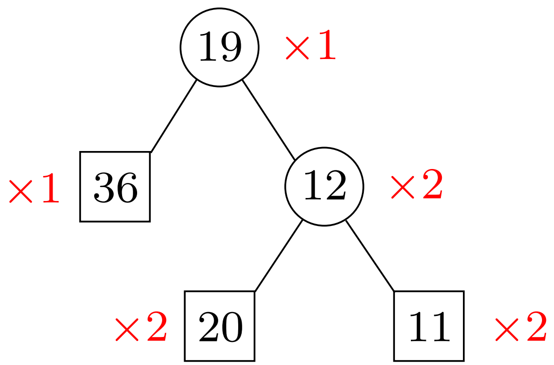

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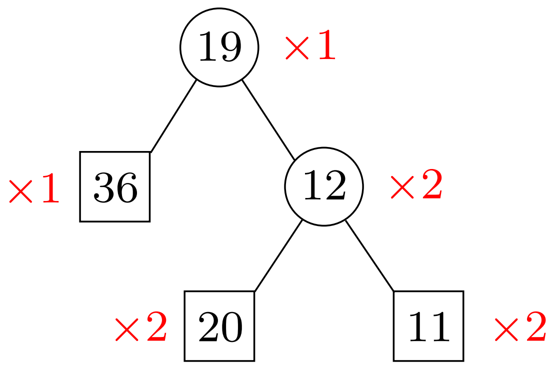

$\text{Cost} = 141$

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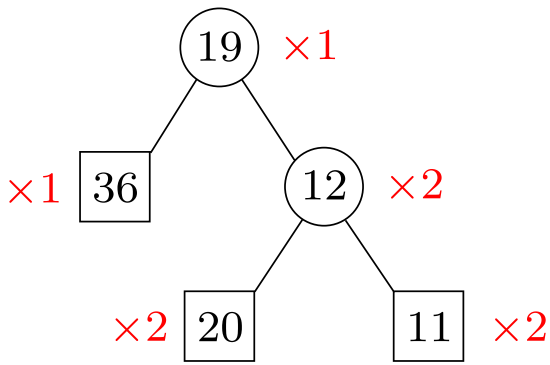

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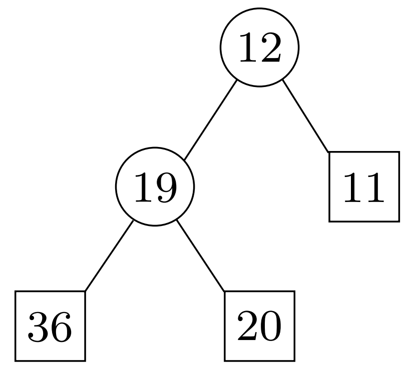

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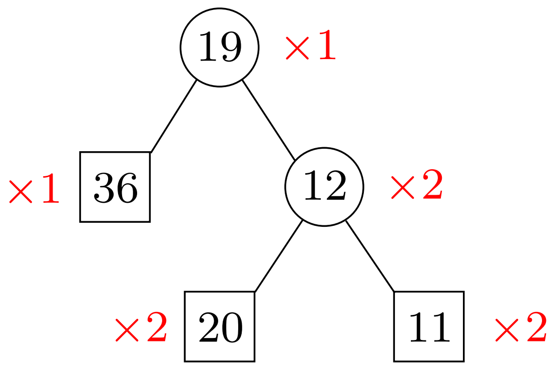
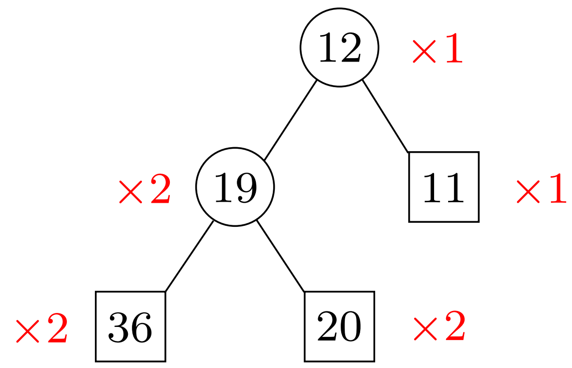


Cost $= 141$



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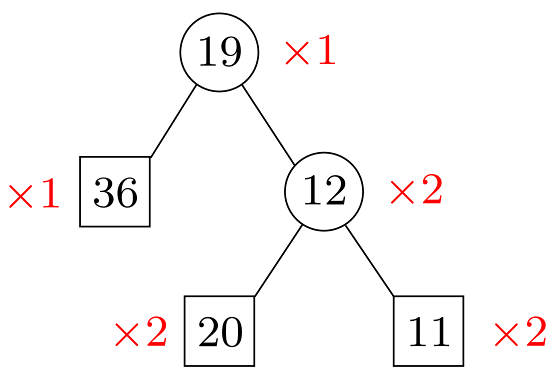

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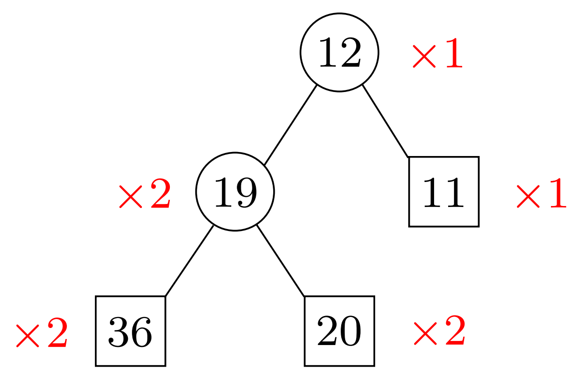

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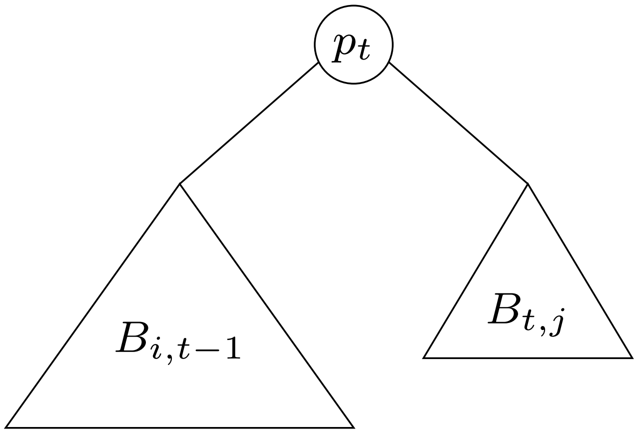
  - $B_{i,j}$ the optimal BST for the subproblem $\text{Key}_{i+1}, \ldots, \text{Key}_j$
  - DP recurrence

$$B_{i,j} = \sum_{l=i+1}^{j} p_l + \sum_{l=i}^{j} q_l + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\}$$



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	0	1	2	3	4	5	6
0	0						
1		0					
2			0				
3				0			
4					0		
5						0	
6							0

$B_{i,j}$ depends on the entries to the left and below.

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	0	1	2	3	4	5	6
0	0	230					
1		0	146				
2			0	75			
3				0	43		
4					0	44	
5						0	52
6							0

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	0	1	2	3	4	5	6
0	0	230	433				
1		0	146	260			
2			0	75	141		
3				0	43	119	
4					0	44	121
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	0	1	2	3	4	5	6
0	0	230	433	586			
1		0	146	260	349		
2			0	75	141	250	
3				0	43	119	204
4					0	44	121
5						0	52
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	0	1	2	3	4	5	6
0	0	230	433	586	698		
1		0	146	260	349	491	
2			0	75	141	250	357
3				0	43	119	204
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	0	1	2	3	4	5	6
0	0	230	433	586	698	862	
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- Naive: $O(n^3) = \sum_{i=1}^{n} \sum_{j=i}^{n} \Theta(j-i)$

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$$K_B(i,j) \leq K_B(i,j+1) \leq K_B(i+1,j+1)$$

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	i	$i+1$
j	$K_B(i,j)$	$K_B(i,j+1)$
$j+1$		$K_B(i+1,j+1)$

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- The index table

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	0	1	2	3	4	5	6
0		0					
1			1				
2				2			
3					3		
4						4	
5							5
6							

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$$K_B(i,j) \leq K_B(i,j+1) \leq K_B(i+1,j+1)$$

- The index table

	0	1	2	3	4	5	6
0		0	0				
1			1	1			
2				2	2		
3					3	4	
4						4	5
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6							

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  - $O(n^2)$ total work over all n diagonals.

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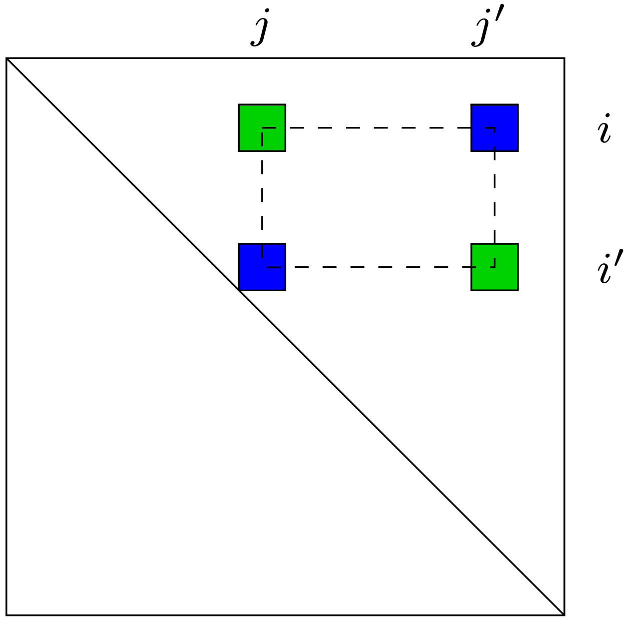

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    $$f(i,j) + f(i',j') \le f(i',j) + f(i,j')$$
  - Function $f(i,j), (0 \le i \le j \le n)$
    is Monotone over the integer lattice (MIL), if $\forall [i,j] \subseteq [i',j']$
    $$f(i,j) \le f(i',j')$$

Speedup using Quadrangle Inequality

$$B_{i,j} = w(i,j) + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\}$$

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Online Problem

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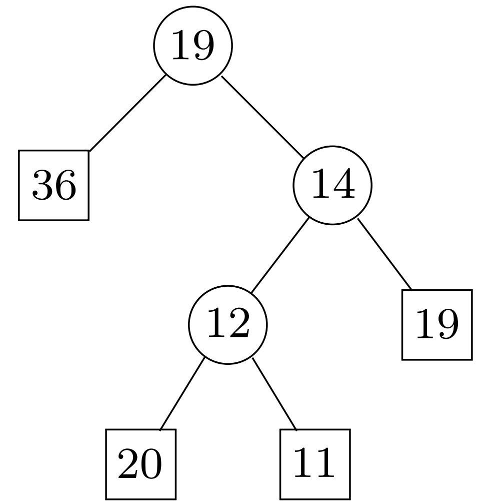
$p = (\quad 19, 12, 14 \quad) \quad q = (\quad 36, 20, 11, 19 \quad)$

	1	2	3	4	5	6
1						
2		0	75	141	250	
3			0	43	119	
4				0	44	
5					0	
6						



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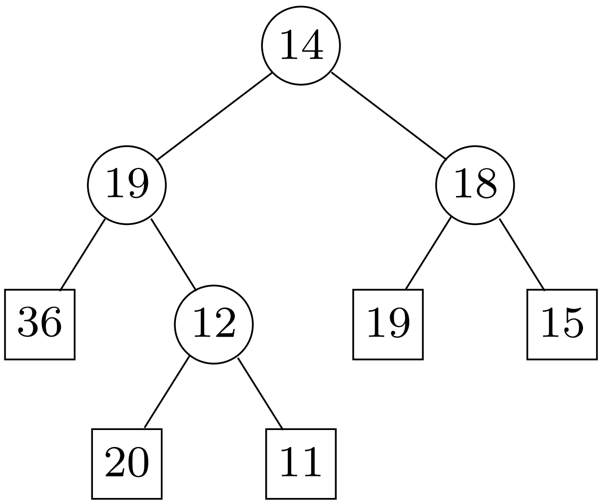
	1	2	3	4	5	6
1						
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0



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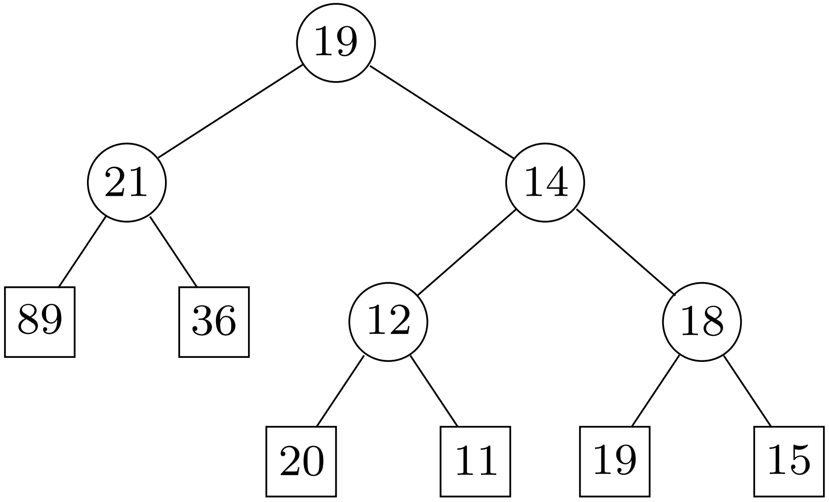
$p = (\quad 21, 19, 12, 14, 18) \quad q = (\quad 89, 36, 20, 11, 19, 15)$

	1	2	3	4	5	6
1	0	146	260	349	491	624
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0



Outline

- Background
  - Kunth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.
- Extensions
  Applying the technique to known generalizations of KY.

Totally Monotone Matrices

- Definition

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7	2	4	3	8	9
5	1	5	1	6	5
7	1	2	0	3	1
9	4	5	1	3	2
8	4	5	3	4	3
9	6	7	5	6	5

$\text{RM}_M(1) = 2$

$\text{RM}_M(2) = 4$

$\text{RM}_M(3) = 4$

$\text{RM}_M(4) = 4$

$\text{RM}_M(5) = 6$

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Totally Monotone Matrices

- Definition (Cond.)
  - A 2×2 Monotone matrix

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4	5

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- Has been used to speed up many DP problems, e.g., computational geometry, bioinformatics, k-center on a line, etc.

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  TM property is often established via Monge property.

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  An $m \times n$ matrix M is Monge if $\forall i \leq i'$ and $\forall j \leq j'$

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The Monge Property

Quadrangle Inequality

Function $f(i,j)$

$\forall i \le i' \le j \le j'$

$f(i,j) + f(i',j') \le f(i',j) + f(i,j')$

Monge

Matrix M

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- QI vs. Monge

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  - Different names for same type of inequality.
  - Used differently in literature.
    - QI: $f(i,j)$ is function to be calculated.
      Need all $f(i,j)$ entries.
    - Monge: $M_{i,j}$ implicitly given.
      Only need the row minima, but not other entries.

Monge Property

$$\forall i \le i' \quad \forall j \le j' \qquad M_{i,j} + M_{i',j'} \le M_{i',j} + M_{i,j'}$$

- Theorems

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  - M is Monge $\Rightarrow$ M is Totally Monotone
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  - If $\forall i$ and $\forall j$, $M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1}$,
    then M is Monge.

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    then M is Monge.
    - $\Rightarrow$ Only need to prove Monge property for adjacent rows and columns.

Monge Property

- General Scheme

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  1. Prove Monge Property for adjacent rows and columns

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Monge Property

- General Scheme
  1. Prove Monge Property for adjacent rows and columns
  2. (Automatically implies) Monge Property
  3. (Automatically implies) Totally Monotone Property
  4. Use SMAWK algorithm to find row minima
  5. Usually $\Theta(n)$ speedup

Relationship?

Quadrangle Inequality Totally Monotone (Monge)

Relationship?

Quadrangle Inequality

A matrix to be calculated

Totally Monotone (Monge)

A matrix given implicitly

Relationship?

Quadrangle Inequality

A matrix to be calculated

Need all $O(n^2)$ entries

Totally Monotone (Monge)

A matrix given implicitly

Need only $O(n)$ row minima

Relationship?

Quadrangle Inequality

A matrix to be calculated

Need all $O(n^2)$ entries

$O(n^3)$ to $O(n^2)$ speedup

Totally Monotone (Monge)

A matrix given implicitly

Need only $O(n)$ row minima

$O(n^2)$ to $O(n)$ speedup

Relationship?

Quadrangle Inequality	Totally Monotone (Monge)
A matrix to be calculated	A matrix given implicitly
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- This talk

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  - QI instance is decomposed into $\Theta(n)$ TM instances
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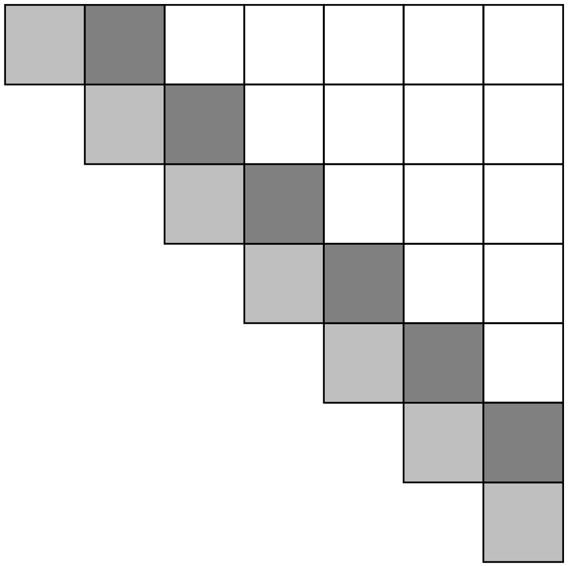

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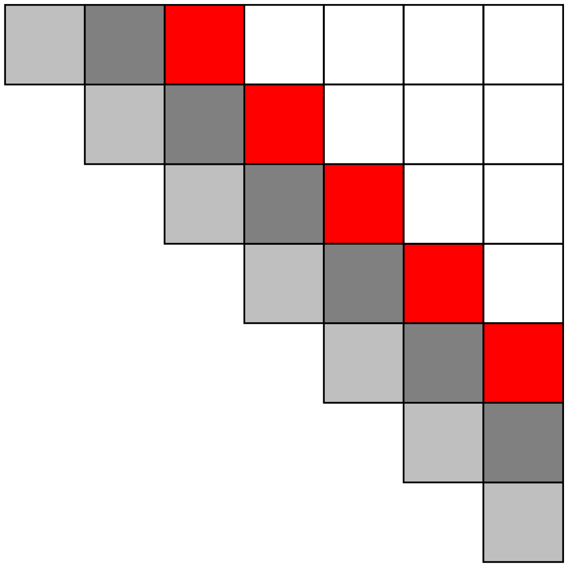

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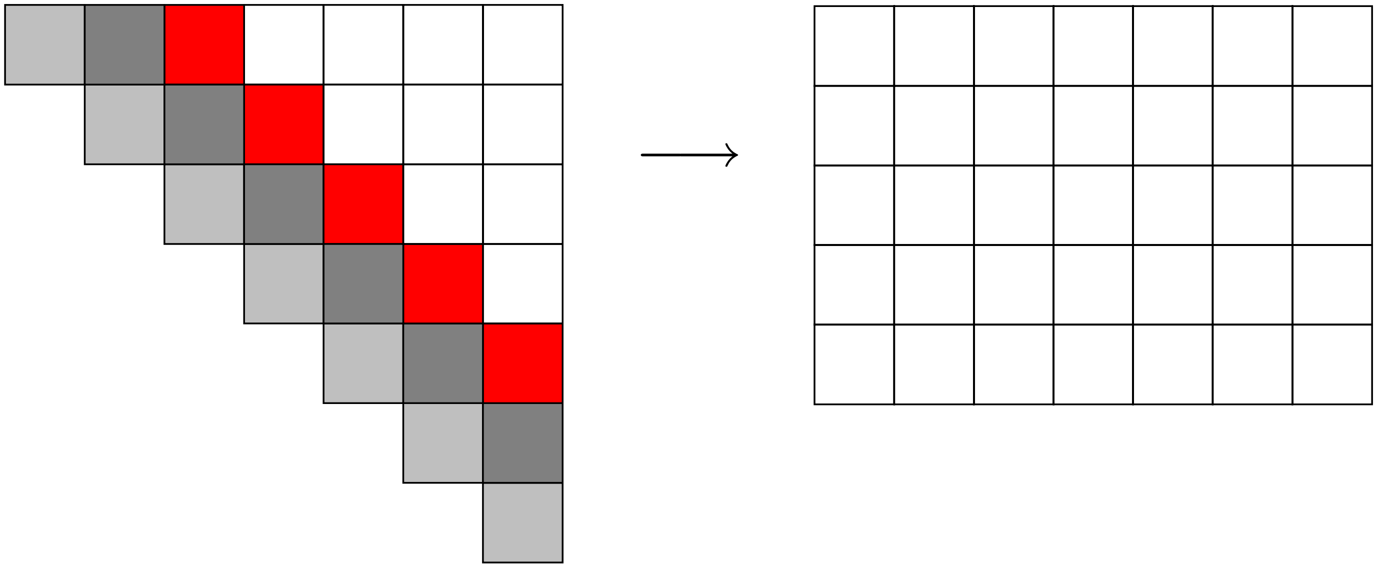

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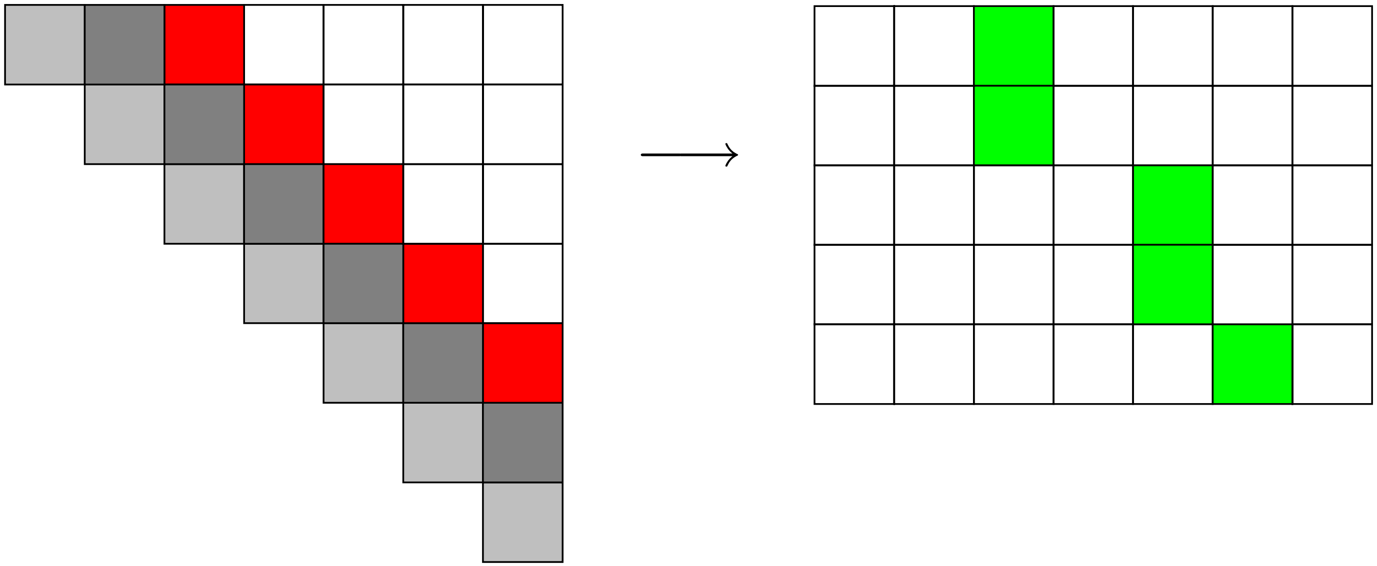

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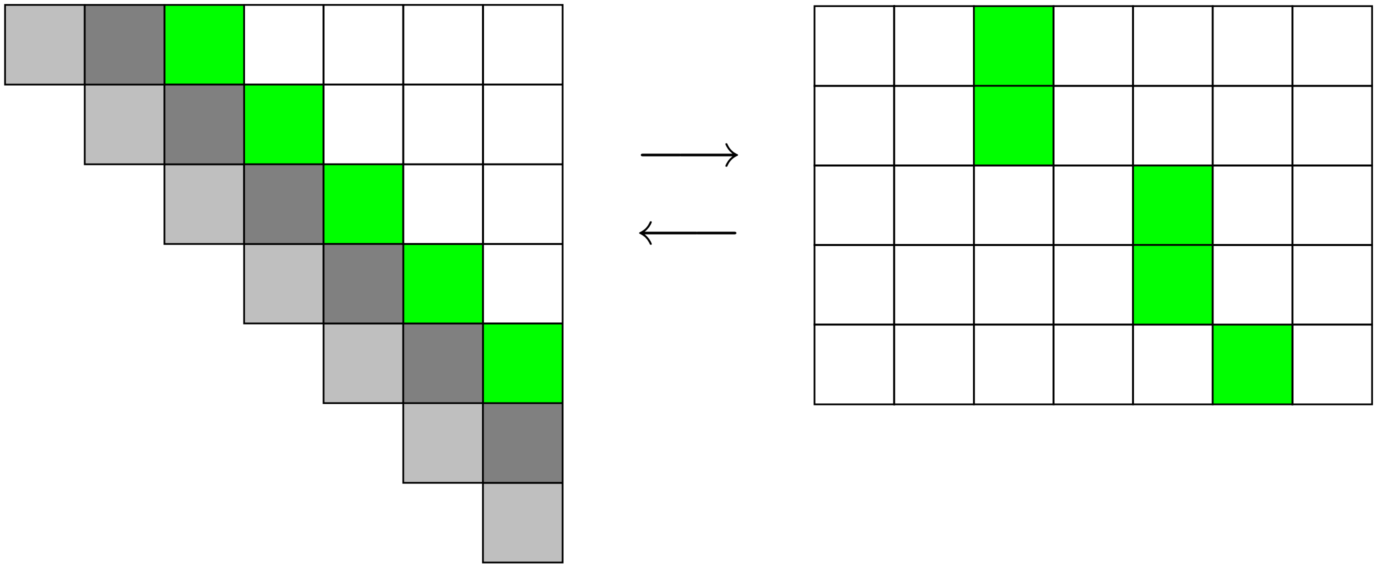

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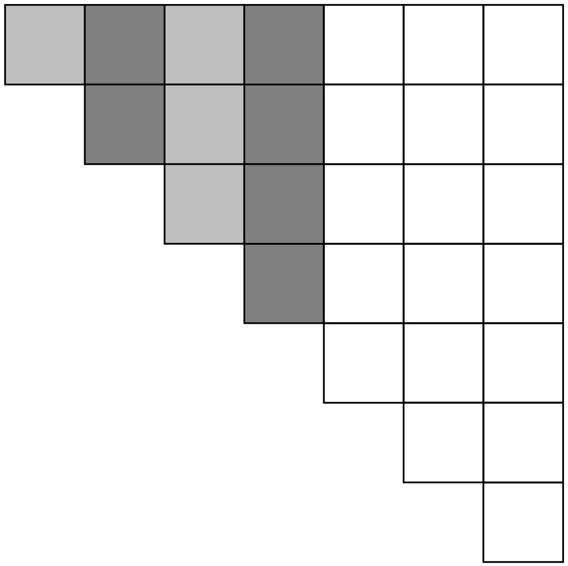

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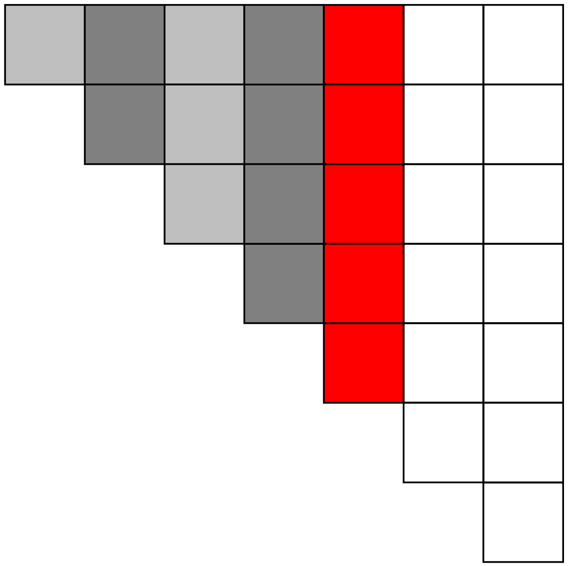

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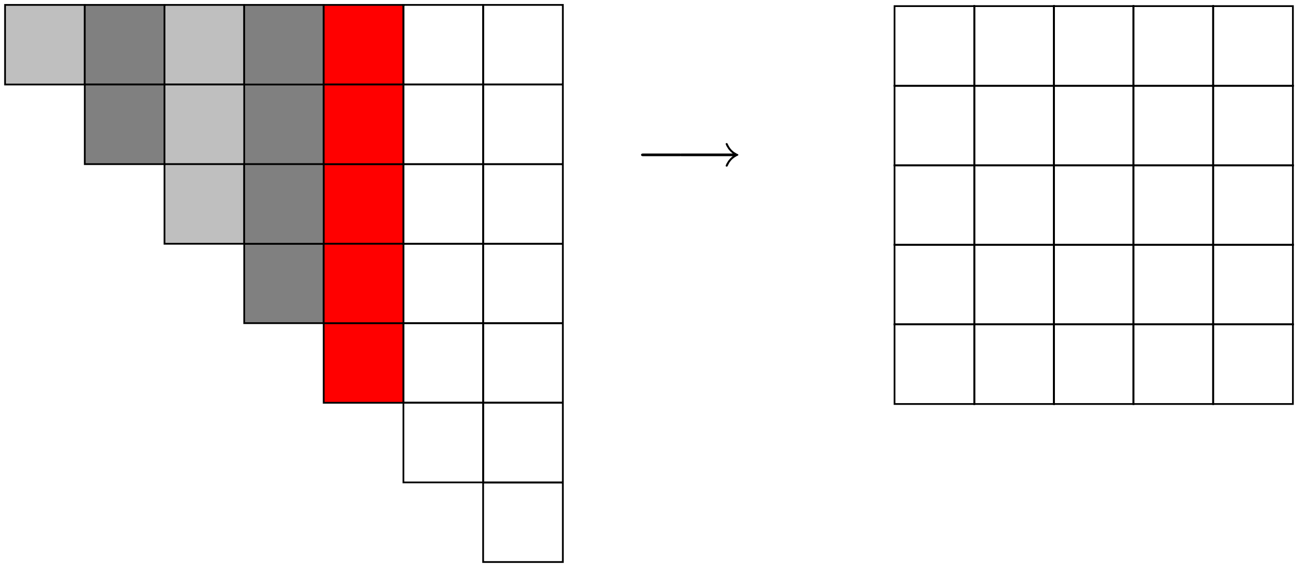

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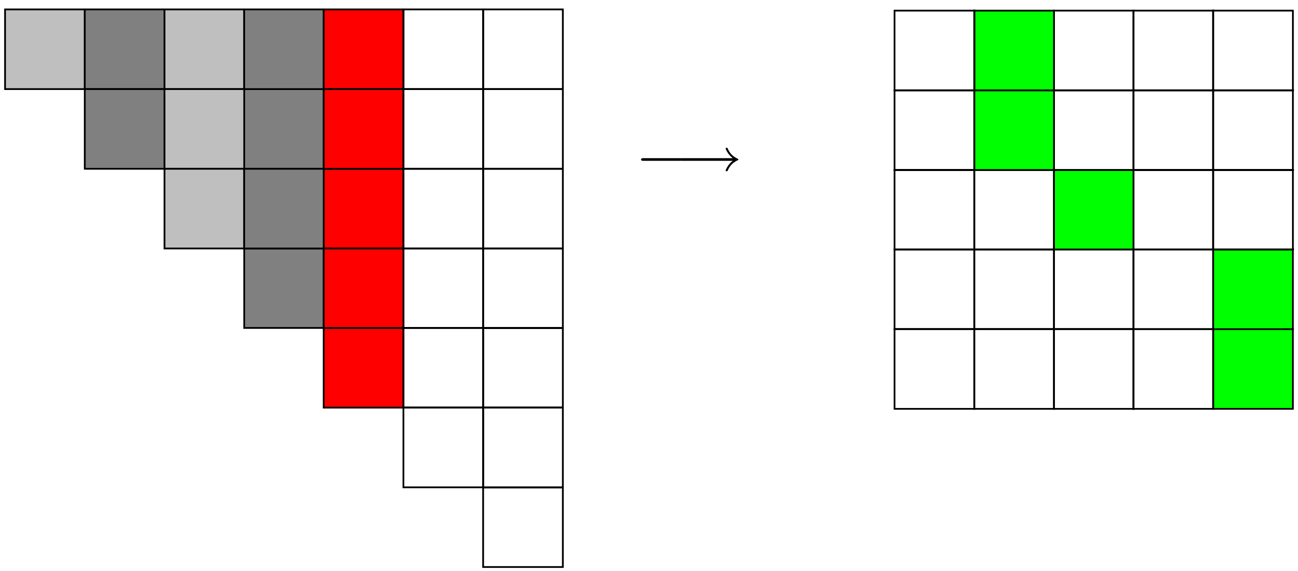

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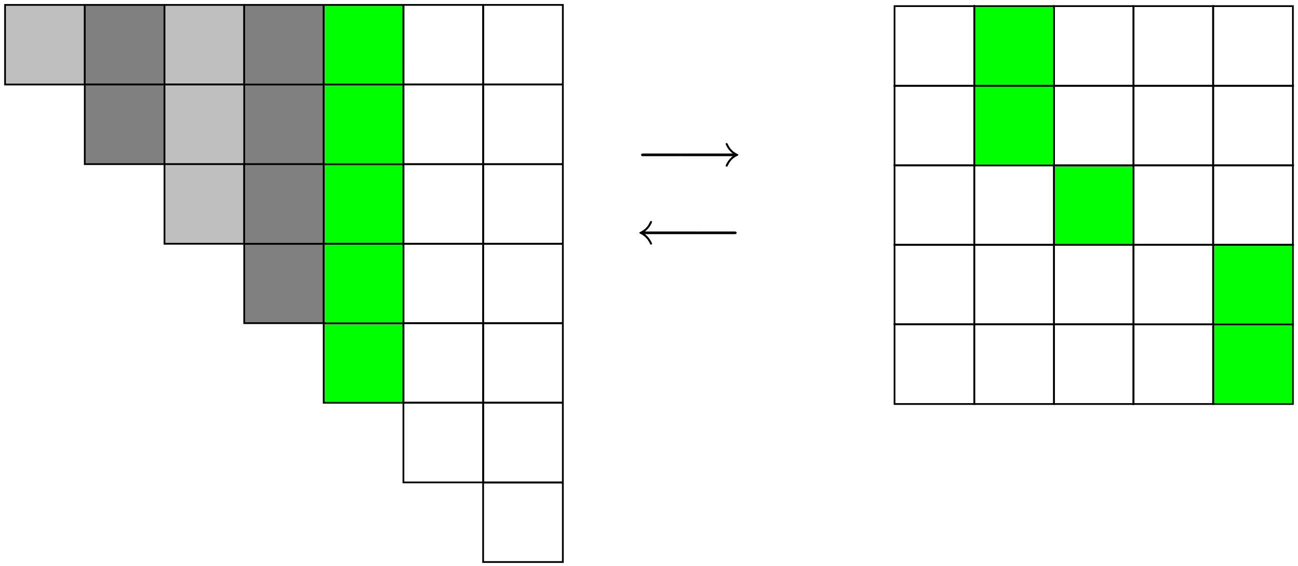

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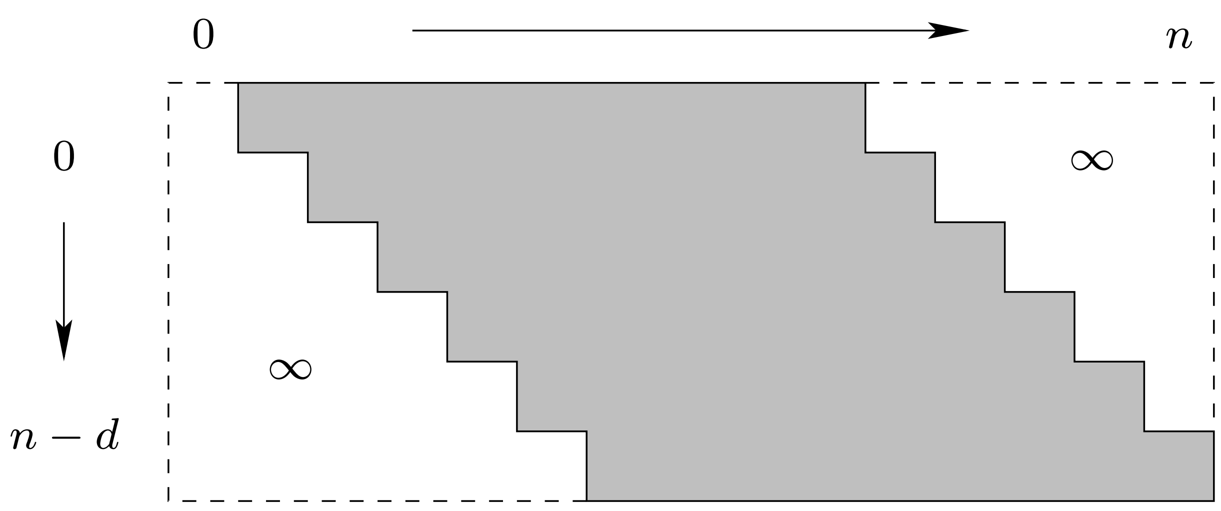

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Since B satisfies QI,

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- $\Rightarrow O(n^2)$ time for all D^d.
- Note: Must run SMAWK on D^d in the order $d = 1, 2, 3, \ldots$

  Entries in D^d depend upon row minima of $D^{d'}$ where $d' < d$.

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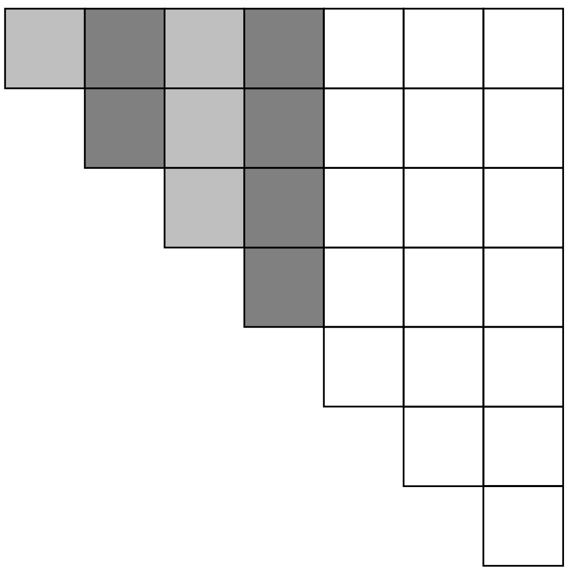

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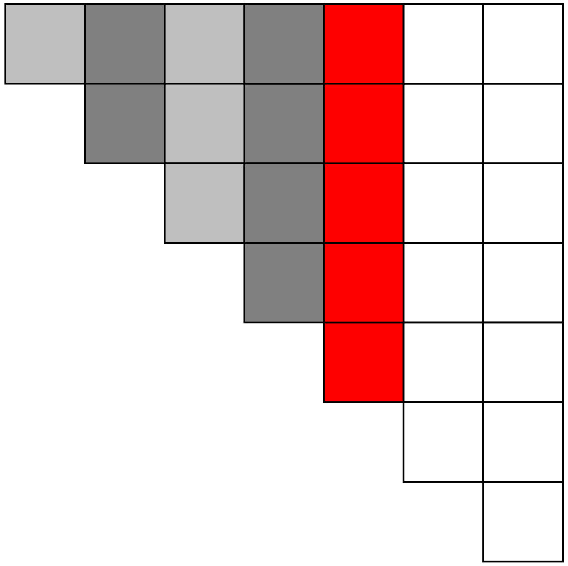

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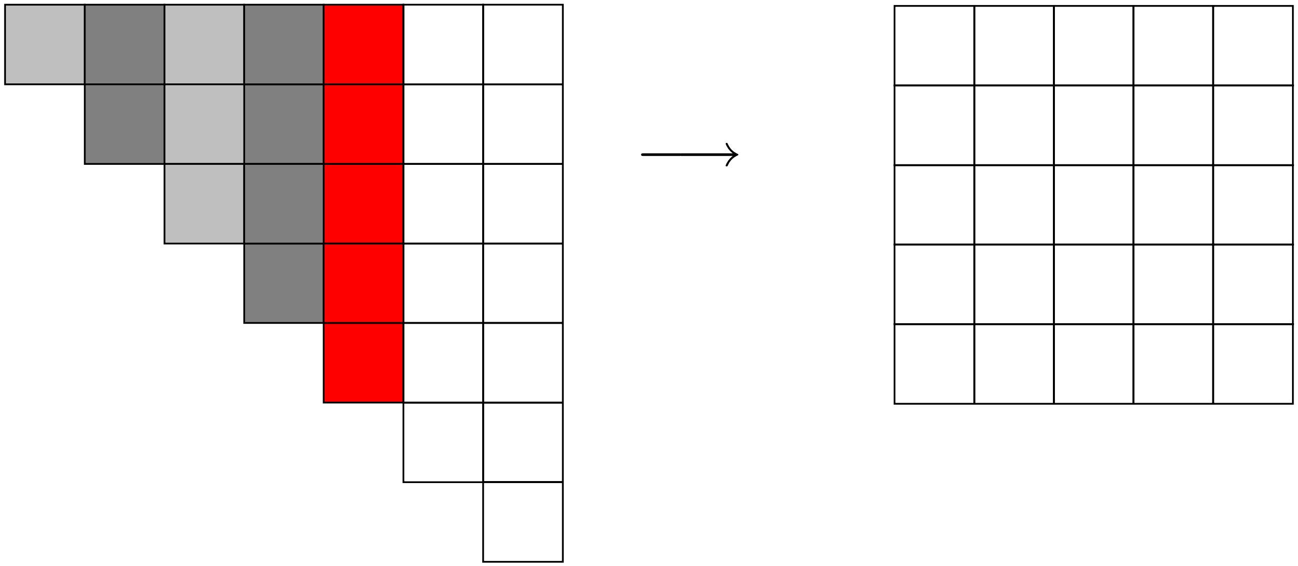

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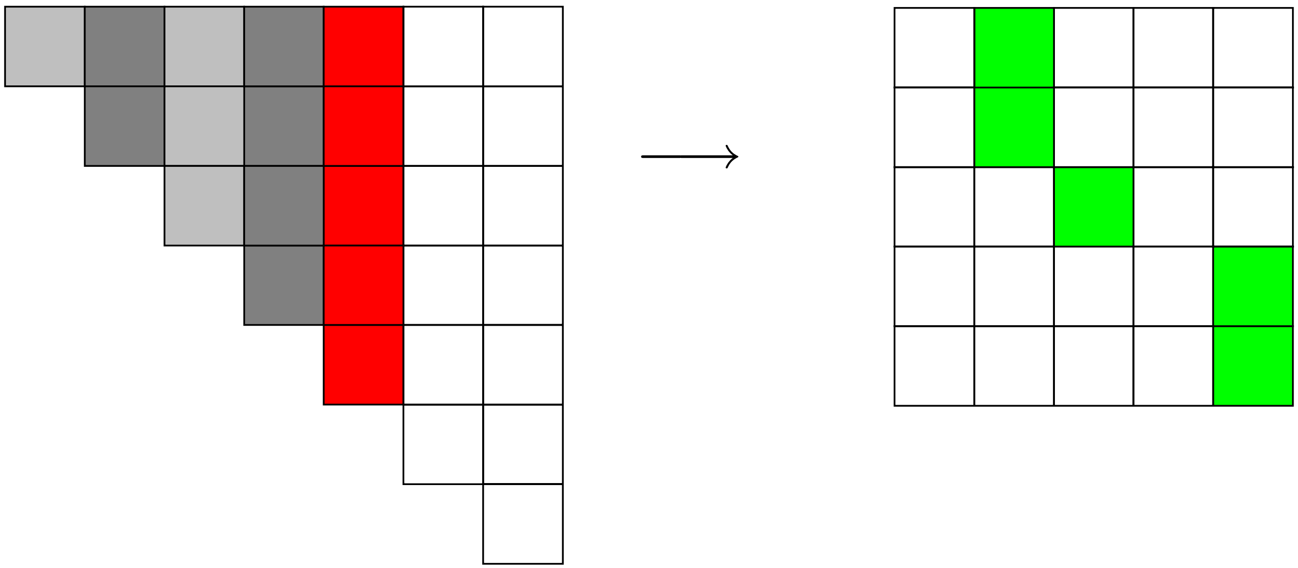

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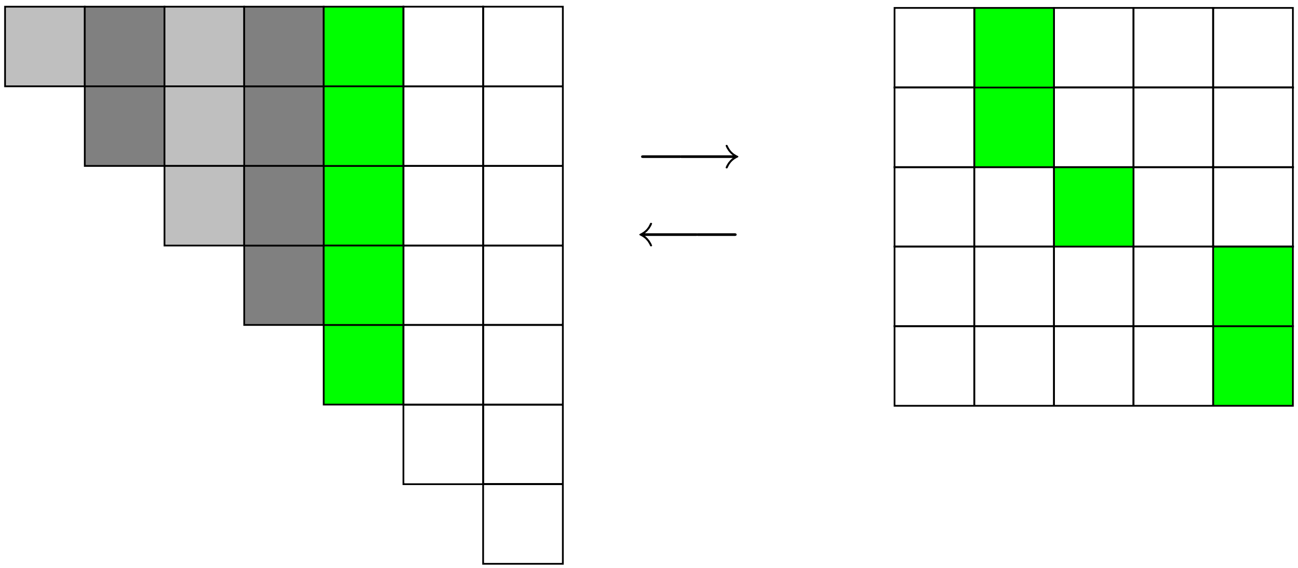

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    $$R^m_{i,j} = \begin{cases} w(i,m) + \{B_{i,j-1} + B_{j,m}\} & \text{if } 0 \le i < j \le m \\ \infty & \text{otherwise} \end{cases}$$
  - Then
    $$B_{i,m} = \min_{i<j\le m} R^m_{i,j} = \min_{0<j\le m} R^m_{i,j}$$
- Lemma
  - R^m is Monge, for each $1 \le m \le n$.

R^m Decomposition

$$R^m_{i,j} = \begin{cases} w(i,m) + \{B_{i,j-1} + B_{j,m}\} & \text{if } 0 \le i < j \le m \\ \infty & \text{otherwise} \end{cases}$$

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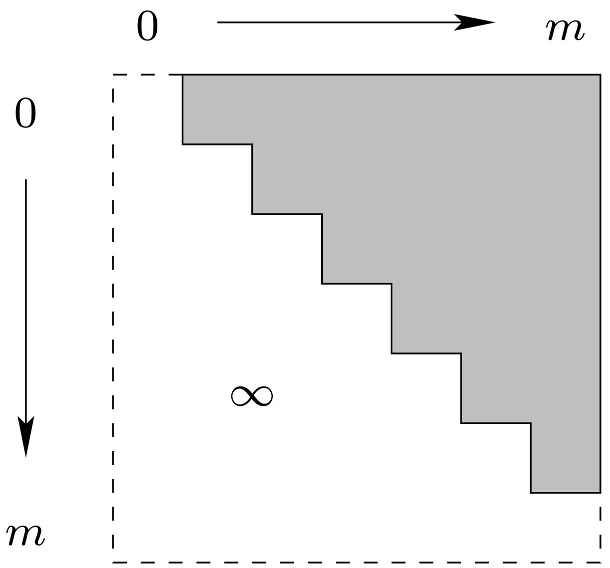

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$$R^m_{i,j} + R^m_{i+1,j+1} \;\le\; R^m_{i+1,j} + R^m_{i,j+1}$$

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So

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L^m and R^m Imply Original KY Result

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  - Recall

    $\text{RM}_{R^m}(i)$ is index of rightmost minimum of row i of R^m.

1	1	2	2	2	2	$\text{RM}_M(1) = 2$
1	1	1	1	2	2	$\text{RM}_M(2) = 4$
1	1	1	1	2	2	$\text{RM}_M(3) = 4$
1	1	1	1	2	2	$\text{RM}_M(4) = 4$
1	1	1	1	1	1	$\text{RM}_M(5) = 6$
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  - Similar

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  - $D^d_{i,j} = w(i, i+d) + \{B_{i,j-1} + B_{j,i+d}\} \qquad (0 \le i < j \le i+d \le n)$

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  - $O(n)$ time for each column $\Rightarrow O(n^2)$ in total.

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Finding row minima in totally monotone matrices with limited dependency.

This is also known as online TM problem.

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Entries of column j can depend on the row minima of rows i where $M_{i,j} = \infty$.

Green: the column j.
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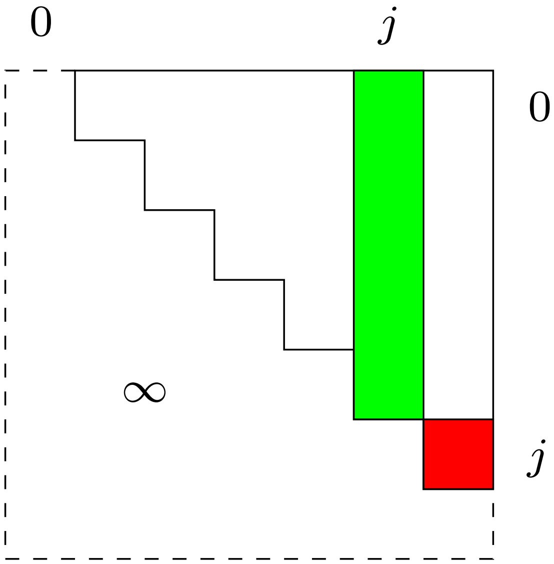

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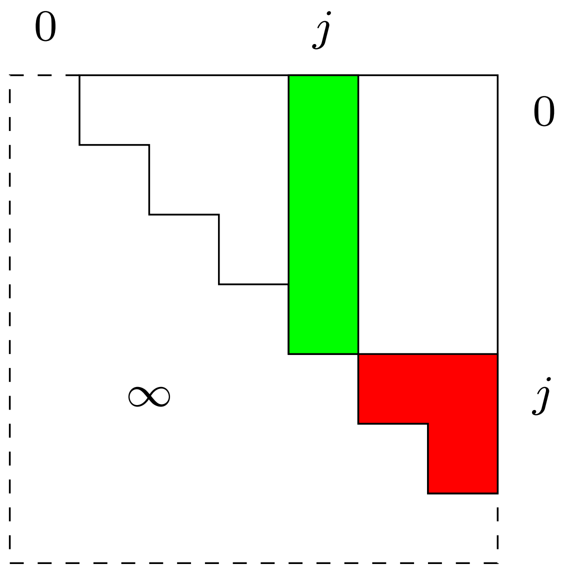

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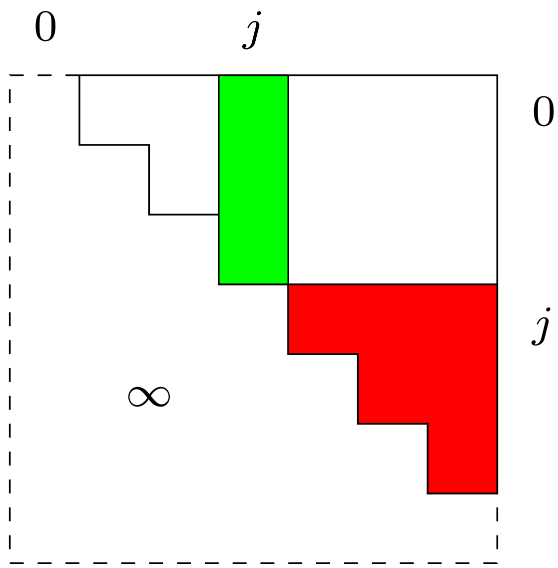

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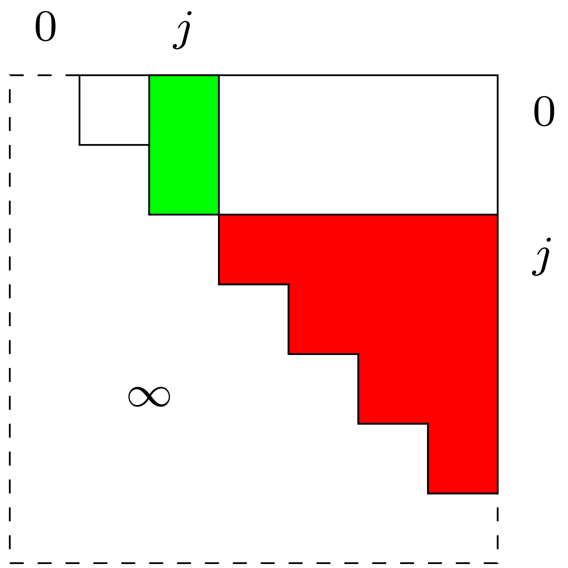

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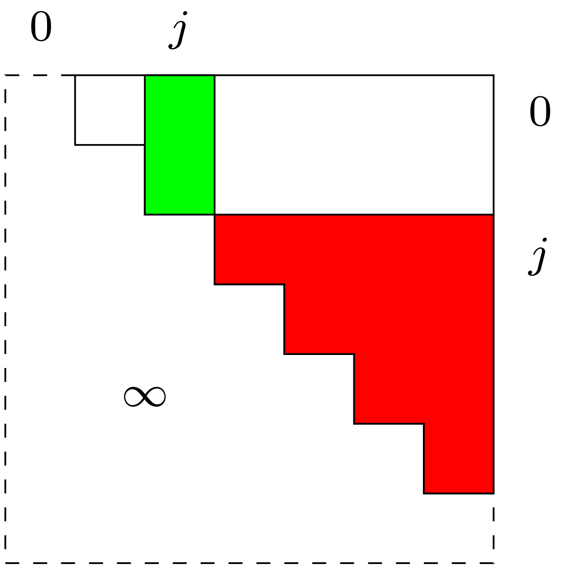

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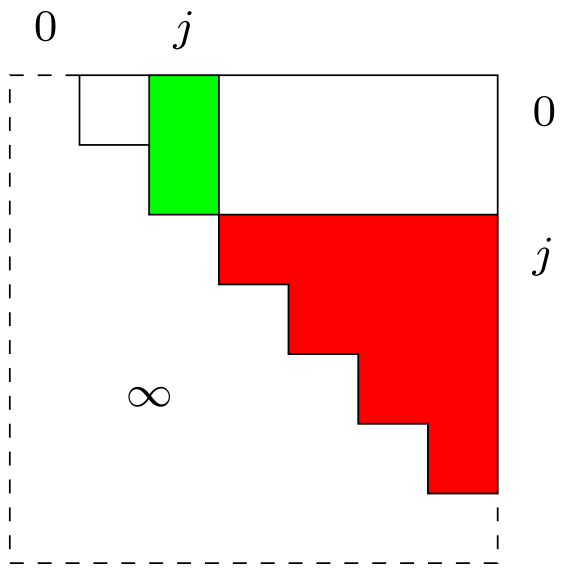

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$\boxed{R^m \text{ satisfies the condition of LARSCH.}}$

Note

- Aggarwal and Park (FOCS '88) developed a 3-D monotone matrix representation of the KY problem and then showed how to use an algorithm due to Wilber (for online computation of maxima of certain concave sequences) to calculate "tube-maxima" of their matrices.
- Careful decomposition of their work yields a decomposition similar to L^m and an $O(n)$ algorithm for calculating its row-minima. This provides an alternative derivation of the previous result (with a symmetry argument extending it to R^m).

Online Algorithm

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- Recall: Two-sided online

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  - Current step: Optimal BST for $\text{Key}_{l+1}, \ldots, \text{Key}_r$

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	1	2	3	4	5	6
1	0	146	260	349	491	624
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0

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and $\forall i < t \le t' \le j \le j'$ and $i \le i' < t'$

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- If the value of $w(i,t,j)$ is independent of t, the Borchers and Gupta definition becomes the original Knuth-Yao definition.

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  - $B_{i,j} = \min_{i<t\leq j}\{w(i,t,j) + aB_{i,t-1} + bB_{t,j}\}$
  - $w(i,t,j)$ is Monotone on the integer lattice (MIL),
    if $\forall [i,j] \subseteq [i',j']$ and $i < t \leq j$, $w(i,t,j) \leq w(i',t,j')$.

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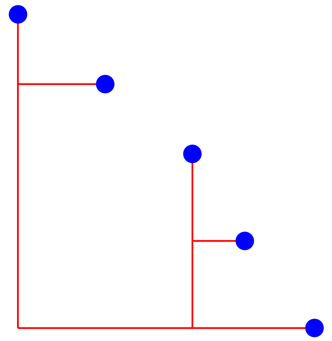

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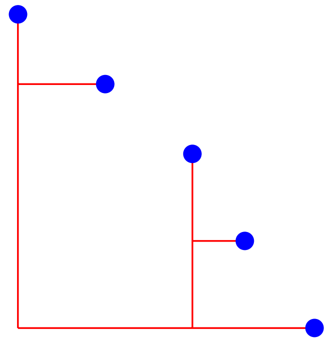

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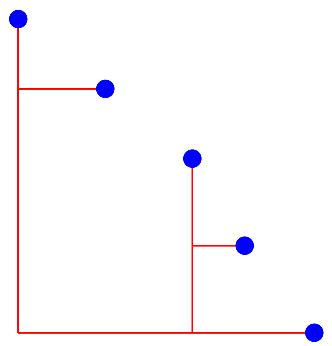

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  - $w(i,t,j)$ satisfies generalized QI and MIL.

Outline

- Background
  - Kunth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding

    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition

  A transformation from QI to TM such that

  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions

  Another transformation from QI to TM that

  (1) implies KY speedup and (2) enables online solution.
- Extensions

  Applying the technique to known generalizations of KY.

Questions?