

# ON A QUASI-LINEAR EQUATION

RICHARD BELLMAN

**1. Introduction.** The purpose of this note is to establish some limit theorems for the non-linear recurrence relations

$$1.1 \quad x_i(n+1) = \text{Max}_q \sum_{j=1}^N a_{ij}(q) x_j(n), \quad i = 1, 2, \dots, N; n \geq 0,$$

under certain assumptions concerning the initial values  $c_i = x_i(0)$ , and the coefficient matrices  $A(q) = (a_{ij}(q))$ .

Equations of this type occur in various parts of the theory of dynamic programming, as we shall indicate below, and are, in addition, of interest in furnishing a link between the theory of linear and non-linear operations, as we have discussed elsewhere (1).

Generally speaking, these equations arise in the consideration of processes of Markoff type, see (2), in which decisions are made at various stages of the process.

Results corresponding to those obtained below hold for the more general equations of the form

$$1.2 \quad x_i(n+1) = \begin{cases} \text{Max}_q \sum_{j=1}^N a_{ij}(q) x_j(n), & i = 1, 2, \dots, K < N, \\ \sum_{j=1}^N a_{ij}(q^*) x_j(n), & i = K+1, \dots, N, \end{cases}$$

where  $q^*$  in the lower equations is determined by the upper equations.

**2. The homogeneous equation.** Let us consider the equation

$$2.1 \quad \lambda y_i = \text{Max}_q \sum_{j=1}^N a_{ij}(q) y_j \quad (i = 1, 2, \dots, N),$$

where we impose the following conditions:

2.2 (a)  $q = (q_1, q_2, \dots, q_N)$  runs over some set of values,  $S$ , with the property that the maximum is attained in (1),

(b)  $\infty > m \geq a_{ij}(q) > 0$  ( $i, j = 1, 2, \dots, N$ ) for  $q \in S$ ,

(c) for any  $q$ , let  $\phi(q)$  denote the characteristic root of  $A(q) = (a_{ij}(q))$  of largest absolute value, the Perron root, known to be positive. We assume that there exists at least one value of  $q$  for which  $\phi(q)$  assumes its maximum for  $q \in S$ .

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We shall now prove

**THEOREM 1.** *Under these conditions, there exists a unique positive  $\lambda$  with the property that 2.1 has a positive solution,  $y_i > 0$  ( $i = 1, 2, \dots, N$ ). This solution is unique up to a multiplicative constant, and*

$$2.3 \quad \lambda = \operatorname{Max}_{q \in S} \phi(q).$$

*Proof.* We begin by showing the existence of a positive  $\lambda$  and a positive set of solutions  $\{y_i\}$ . Consider the region defined by

$$y_i \geq 0, \quad \sum_{i=1}^N y_i = 1.$$

The normalized transformation

$$2.4 \quad y'_i = \left[ \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) y_j \right] / \left[ \sum_{i=1}^N \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) y_j \right],$$

is a continuous mapping of this region into itself. Hence there exists a fixed point,  $\{y_i\}$ . This fixed point is a solution of 2.1, with  $\lambda$  the denominator in 2.4. Each component  $y_i$  is positive because of the positivity of  $a_{ij}(q)$ .

To show that this solution is unique up to a multiplicative constant, let  $[\mu, z]$  be another solution of 2.1 with  $\mu > 0$  and  $z$  a positive vector. Let  $\{q\}$  be the set of values for which the maximum is attained in 2.1 and  $\{\bar{q}\}$  the similar set associated with  $z$ . Observe that we may have different sets for each  $i$ . We have then

$$2.5 \quad \begin{aligned} \lambda y_i &= \sum_j a_{ij}(q) y_j \geq \sum_j a_{ij}(\bar{q}) y_j, & i = 1, 2, \dots, N, \\ \mu z_i &= \sum_j a_{ij}(\bar{q}) z_j. \end{aligned}$$

Let us now assume, without loss of generality that  $\lambda < \mu$ . Let  $\epsilon$  be a positive constant chosen so that one, at least, of the components  $y_i - \epsilon z_i$  is zero, one at least is positive, and the others are non-negative. This can always be accomplished if  $y$  and  $z$  are not proportional. If  $i$  is an index for which  $y_i - \epsilon z_i$  is zero, we have

$$2.6 \quad 0 = \mu(y_i - \epsilon z_i) > \lambda y_i - \epsilon \mu z_i \geq \sum_{j=1}^N a_{ij}(\bar{q})(y_j - \epsilon z_j) > 0,$$

since  $a_{ij}(\bar{q}) > 0$ , a contradiction. Hence  $\lambda = \mu$ , and  $y$  and  $z$  are proportional.

To show that  $\lambda = \operatorname{Max} \phi(q)$ , we proceed as follows. It is clear that  $\lambda$ , as the characteristic root of some  $A(q)$ , satisfies the inequality  $\lambda \leq \mu$ , where  $\mu = \operatorname{Max} \phi(q)$ . Assume that actually  $\lambda < \mu$ . Let  $z = (z_1, z_2, \dots, z_n)$  be a positive characteristic vector associated with  $\mu$  and  $\bar{q}$  a set of  $q$ -values which yield  $\mu = \phi(\bar{q})$ . Then we have

$$2.7 \quad \mu z_i = \sum_{j=1}^N a_{ij}(\bar{q}) z_j \leq \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) z_j.$$

Since  $y_i$  is positive, we can find a positive constant  $m$  such that  $z_i \leq my_i$  for  $i = 1, 2, \dots, N$ . Hence 2.1 yields

$$2.8 \quad \mu z_i \leq m \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) y_j = m \lambda y_i.$$

Thus  $z_i \leq my_i \lambda / \mu$ . Iterating this, we obtain  $z_i \leq my_i (\lambda / \mu)^k$ , for arbitrary  $k$ . Since  $\lambda / \mu < 1$ , by assumption, this yields  $z_i = 0$ , a contradiction. Hence  $\lambda = \mu$ .

**3. The recurrence relation.** Let us now return to the recurrence relation of 1.1 and prove

**THEOREM 2.** *If, in addition to the conditions of 2.2, we assume that there is a unique  $q$  for which the maximum value of  $\phi(q)$  is attained and that  $c_i \geq 0$ , then*

$$3.1 \quad x_i(n) \sim a y_i \lambda^n,$$

as  $n \rightarrow \infty$ , where  $a$  is a constant dependent upon the initial values  $c_i$ .

*Proof.* Let us take  $c_i > 0$  without loss of generality. There are then two positive constants  $k$  and  $K$  such that  $ky_i \leq c_i \leq Ky_i$  ( $i = 1, 2, \dots, N$ ). Let us show inductively that

$$3.2 \quad ky_i \lambda^n \leq x_i(n) \leq Ky_i \lambda^n.$$

Assume that we have the result for  $n$ , then

$$3.3 \quad \begin{aligned} x_i(n+1) &\leq K \lambda^n \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) y_j = K \lambda^{n+1} y_i \\ &\geq k \lambda^n \operatorname{Max}_q \sum_{j=1}^N a_{ij}(q) y_j = k \lambda^{n+1} y_i. \end{aligned}$$

To establish the asymptotic behavior we show that for  $n$  sufficiently large the set of  $q$ 's which furnish the maximum in 1.1 is precisely the set which yields  $\lambda = \operatorname{Max} \phi(q)$ .

Assume the contrary. This means that infinitely often we employ a set  $\{\bar{q}\}$  which is not identical with the  $q$  which furnishes the maximum in  $\phi(q)$ .

We then have, for  $i = 1, 2, \dots, N$ ,

$$3.4 \quad x_i(n+1) = \sum_{j=1}^N a_{ij}(\bar{q}) x_j(n) \leq \left( \sum_{j=1}^N a_{ij}(\bar{q}) y_j \right) K \lambda^n.$$

For some index  $i$  we must have

$$3.5 \quad \sum_{j=1}^N a_{ij}(\bar{q}) y_j < \lambda y_i,$$

with strict inequality. For if

$$\sum_{j=1}^N a_{ij}(\bar{q}) y_j \geq \lambda y_i$$

for all  $i$ , the characteristic root of  $A(\bar{q}) = (a_{ij}(\bar{q}))$  of largest absolute value,  $\phi(\bar{q})$ , would at least equal  $\lambda = \text{Max } \phi(q)$ , which would contradict the assumption concerning the uniqueness of the maximum of  $\phi(q)$ .

Hence, for some component, say the first, we have

$$3.6 \quad x_1(n+1) \leq \theta K \lambda^{n+1} y_1, \quad 0 < \theta < 1.$$

Since  $a_{1j}(q^*) > 0$  for  $i, j$ , where  $q^*$  is the value of  $q$  for which  $\lambda = \phi(q^*)$ , we see that, for  $i = 1, 2, \dots, N$ ,

$$3.7 \quad x_i(n+2) \leq K \lambda^{n+1} \left[ \sum_{j=2}^N a_{ij}(q^*) y_j + \theta a_{1j}(q^*) y_1 \right] \leq \theta_1 K \lambda^{n+2} y_i,$$

where  $\theta < 1$ .

If therefore a set of  $q$ 's distinct from  $q^*$  are used  $R$  times, we obtain

$$3.8 \quad x_i(n) \leq \theta_1^R K \lambda^n y_i,$$

for  $n$  sufficiently large. Since  $0 < \theta_1 < 1$ , if  $R$  is too large we eventually contradict the lower bound for  $x_i(n)$ .

Hence for  $n \geq n_0 = n_0(c_i)$ , we have

$$3.9 \quad x(n+1) = A(q^*) x(n),$$

whence the asymptotic statement of 3.1 follows.

**4. A dynamic programming problem.** Suppose that we are engaged in a multi-stage decision process of the following type. At each stage we have our choice of various operations, which we number  $i = 1, 2, \dots, K$ . The  $i$ th operation has a probability distribution attached with the following properties:

4.11 There is a probability  $p_{ik}$  that we receive  $k$  units and the process continues,  $k = 1, 2, \dots, R$ ;

4.12 There is a probability  $p_{i0}$  that we receive nothing and the process terminates.

How do we proceed so as to maximize the probability that we receive at least  $n$  units before the process terminates?

Let us define the sequence

4.2  $u(n)$  = the probability of attaining at least  $n$  units before the termination of the process using an optimal procedure.

Then using the intuitive "principle of optimality" (1), we see that  $u(n)$  satisfies the recurrence relation

$$4.3 \quad u(n) = \begin{cases} \text{Max}_i \left[ \sum_{k=1}^R p_{ik} u(n-k) \right], & n > 0, \\ 1, & n \leq 0. \end{cases}$$

Using methods similar to those above, we see that for large  $n$ ,

$$4.4 \quad u(n) \sim c\rho^n,$$

where  $\rho$  is the root of largest absolute value, necessarily positive, of

$$4.5 \quad 1 = \sum_{k=1}^R p_{ik} \rho^{-k},$$

for the value of  $i$  which maximizes  $\rho$ .

**5. An analogue of a result of Markoff.** Markoff showed that if

$$5.1 \quad x_i(n+1) = \sum_{j=1}^N a_{ij} x_j(n) \quad (n = 0, 1, \dots)$$

and  $x_i(0) > 0$ , with the conditions

$$5.2 \quad a_{ij} > 0, \sum_j a_{ij} = 1, \quad (i = 1, 2, \dots, N),$$

then

$$5.3 \quad \lim_{n \rightarrow \infty} x_i(n) = c, \quad (i = 1, 2, \dots, N),$$

where  $c$  depends on the initial values.

The same proof shows that the same result holds for the sequence defined by

$$5.4 \quad x_i(n+1) = \text{Max}_q \sum_{j=1}^N a_{ij}(q) x_j(n),$$

provided that the conditions in 5.2 hold uniformly in  $q$ . The constant will, of course, in general, be different from that above.

#### REFERENCES

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*Rand Corporation,  
Santa Monica, California*