Introduction of Approximation Algorithm

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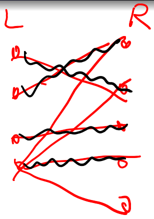
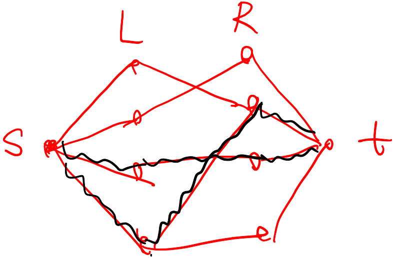
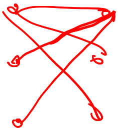


- Maximum cardinality matching of a bipartite graph





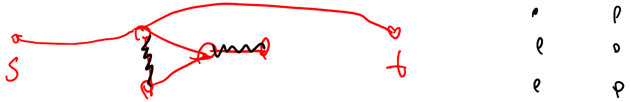
- Maximum cardinality matching of a bipartite graph
  - Based on network flow algorithm
  - Hungarian algorithm: https://en.wikipedia.org/wiki/Hungarian_algorithm
  - HopcroftKarp algorithm: https://en.wikipedia.org/wiki/Hopcroft-Karp_algorithm

Introduction of Approximation Algorithm

decision problem vs optimization problem

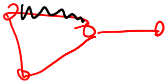
- Decision problem: YES or No answer
- Optimization problem: maximization problem or minimization problem
- Approximation ratio

Maximum matching



- General graph $G = (V, E)$, find the cardinality of maximum matching
- Polynomial time exact algorithm: very complicated
  - https://en.wikipedia.org/wiki/Blossom_algorithm

Maximum matching



- General graph $G = (V, E)$, find the cardinality of maximum matching
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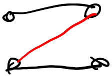
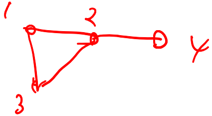
- Approximation algorithm: find a maximal matching

$$\frac{1}{2}\cdot OPT \le Algo \le OPT$$



Maximum matching

- General graph $G = (V, E)$, find the cardinality of maximum matching
- Polynomial time exact algorithm: very complicated
  - `https://en.wikipedia.org/wiki/Blossom_algorithm`
- Approximation algorithm: find a maximal matching
  - 1/2-approximation ratio
  - The analysis is tight.



- Min, Max problem
- Analysis of a particular approximation algorithm
  - Approximation ratio: $\frac{\text{Algorithm's solution}}{\text{Optimal solution}}$ in the worst case
  - Is the analysis tight?
- Hardness of Approximation

Cardinality Vertex Cover

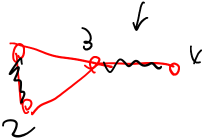
- Given graph $G = (V, E)$, find a set $S \subseteq V$ with minimum cardinality, s.t. $\forall (i,j) \in E, i \in S$ or $j \in S$.
- NP-complete

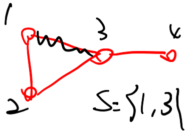
Cardinality Vertex Cover

$OPT = \{1, 3\}$

$\{2, 3\}$.

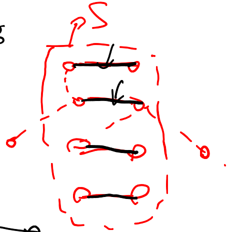
$S = \{1, 2, 3, 4\}$





$S = \{1, 3\}$

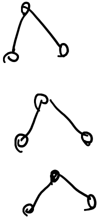
- Given graph $G = (V, E)$, find a set $S \subseteq V$ with minimum cardinality, s.t. $\forall (i,j) \in E, i \in S$ or $j \in S$.
- NP-complete
- Approximation algorithm 1: maximal matching



$OPT \le Algo \le 2 \cdot OPT$.

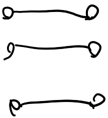
$\frac{1}{2} \cdot Algo \le OPT$

k



$|OPT| = k$

$|Algo| = 2k$.

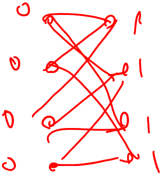


- Given graph $G = (V, E)$, find a set $S \subseteq V$ with minimum cardinality, s.t. $\forall (i, j) \in E, i \in S$ or $j \in S$.
- NP-complete
- Approximation algorithm 1: maximal matching
    - Analysis: 2-approximation ratio, tight

Cardinality Vertex Cover

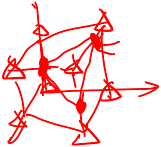
- Given graph $G = (V, E)$, find a set $S \subseteq V$ with minimum cardinality, s.t. $\forall (i, j) \in E, i \in S$ or $j \in S$.
- NP-complete
- Approximation algorithm 1: maximal matching
    - Analysis: 2-approximation ratio, tight
- Approximation algorithm 2: linear programming

weighted Vertex Cover



- (weighted) Vertex Cover

$$
\begin{aligned}
\min \quad & w(v)x_v \\
s.t \quad & x_i + x_j \geq 1 \quad \forall (i,j) \in E \\
& x_i \in \{0,1\}
\end{aligned}
$$



Algo. $\geq$ OPT_{IP} $\geq$ $\boxed{OPT_{LP} \geq \frac{1}{2} Algo.}$

- (weighted) Vertex Cover

$$x_v \geq \frac{1}{2} \cdot x_v'$$

$$\begin{aligned} & min \sum w(v)x_v \\ s.t \quad & x_i + x_j \geq 1 \quad \forall(i,j) \in E \\ & x_i \in \{0,1\} \end{aligned}$$

OPT_{IP}. $\quad$ OPT_{LP}

- integer programming $\rightarrow$ linear programming
- $x_i \in \{0,1\} \rightarrow \boxed{0 \leq x_i \leq 1}$
- Design approximation algorithm based on the optimal solution of LP

OPT of LP: if $x_i \geq \frac{1}{2} \Rightarrow i \in S \quad x_i' \leftarrow 1$

if $x_i < \frac{1}{2} \Rightarrow i \notin S \quad x_i' \leftarrow 0$.

$OPT_{LP} = \sum w(v) \cdot x_v$. $\quad$ $Algo = \sum w(v) \cdot x_v'$

- VC is APX-complete: cannot be approximated arbitrarily well unless P $=$ NP.
- VC cannot be approximated within a factor of 1.3606 unless P $=$ NP (2005, PCP).
- VC cannot be approximated within a factor of $2-\epsilon$ if unique game conjecture is true (2008).
- Difference between tight and hardness.

Set Cover

vertex cover: u_e: edge.
S_i: $\{e \mid i \text{ cover } e\}$.

- Given a universal set $U = \{u_1, \cdots, u_n\}$, m subset $S_1, \cdots, S_m \subseteq U$ and a weight function $c : \{1, \cdots, m\} \to \mathbb{N}$. Find some sets which can cover all elements with minimum cost.
- Vertex cover problem is a special case of set cover problem
- Other examples: dominating set problem, edge cover problem

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- Vertex cover problem is a special case of set cover problem
- Other examples: dominating set problem, edge cover problem
- Algorithm: greedy algorithm
  - Analysis: $O(\log n)$-approximation ratio, tight

Conclusion

- Min, Max problem
- Approximation algorithm design
  - Combinatorial algorithm
  - LP-based algorithm
- Analysis of a particular approximation algorithm
  - Approximation ratio: $\frac{\text{Algorithm's solution}}{\text{Optimal solution}}$ in the worst case
  - Is the analysis tight?
- Hardness of Approximation