1 Lecture 5: Lovász extension and matroids

1.1 Lovász extension

Extension of an arbitrary (sub or non submodular) set function $z : 2^V \mapsto \mathbb{R}$ to $g : \mathbb{R}_+^n \mapsto \mathbb{R}$ where $n = |v|$.

Definition 1. $g\left(\mathbf{1}^A\right) = z(A)\ \forall A \subseteq V$.

For any $w \in \mathbb{R}_+^n$ let $p_1 > p_2 > \cdots > p_n$, be distinct values, where $p_n = 0$. Let $v_j = \{i | w(i) \geq p_j\}$

Definition 2. The *Lovász extension* g is given by

$$g(w) = \sum_{j=1}^{n} (p_j - p_{j+1})\, z(v_j)\,.$$

Example 1. If $w = (0.75, 0.3, 0.2, 0.3, 0)$ we get

$$\underbrace{0.75}_{p_1} > \underbrace{0.3}_{p_2} > \underbrace{0.2}_{p_3}\,.$$

$$g(w) = 0.45z(\{1\}) + 0.1z(\{1,2,4\}) + 0.2z(\{1,2,3,4\})\,.$$

Example 2. For $w \in [0,1]_1^n$ we get a positive linear combination of vertices.

Lemma 1. *Let $z : 2^V \mapsto \mathbb{R}$ be a set function with $z(\emptyset) = 0$ and $g : \mathbb{R}_+^n \mapsto \mathbb{R}$ it's Lovász extension. If z is submodular, then the following equality holds:*

$$g(w) = \max_{x \in P_z} w^T x \text{ for all } w \in [0,1]^n$$

where P_z is the submodular polyhedron for z.

Proof. Exercise 3.2. □

Theorem 1. *A set function $z : 2^V \mapsto \mathbb{R}$ is submodular (or supermodular) if and only if it's extensions g is convex (or concave).*

Proof. ($\Rightarrow$) Let $w_1, w_2 \in \mathbb{R}_+^n$, $\lambda \in (0,1)$ and $w = \lambda w_1 + (1-\lambda) w_2$. From Lemma (1) let $x^\star$ be such that

$$g(w) = w^T x^\star = \lambda w_1^T x^\star + (1-\lambda) w_2^T x^\star.$$

Then $g(\lambda w_1) \geq \lambda w_1^T x^\star$ and $g((1-\lambda)) \geq (1-\lambda) w_2^T x^\star$ and thus

$$g(w) = g(\lambda w_1 + (1-\lambda) w_2) \leq g(\lambda w_1) + g((1-\lambda) w_2) = \lambda g(w_1) + (1-\lambda) g(w_2)$$

($\Rightarrow$) Let $A, B \subseteq V$ from the definition

$$g\left(\mathbf{1}^A + \mathbf{1}^B\right) = g\left(\mathbf{1}^{A\cup B}\right) + g\left(\mathbf{1}^{A\cap B}\right).$$

By convexity we get

$$g\left(\mathbf{1}^A + \mathbf{1}^B\right) \leq g\left(\mathbf{1}^A\right) + g\left(\mathbf{1}^B\right)$$

$$\Rightarrow z(A) + z(B) \geq z(A\cup B) + z(A\cap B)\,.$$ □

1.2 Matroids and greedy algorithms

Definition 3. A *forest* is a subset of edges without any cycles.

Example 3. Let $G = (V, E)$ be a connected graph with edge weights w_e. Let $I = \{$ forests of $G\}$, find a spanning tree while maximizing $\sum_{e \in J} w_e$ where $J \subseteq E$.

Greedy algorithm (kruskal) Set $J = \emptyset$ while $\exists e \notin J$ such that $J \cup \{e\} \in I$ choose e with maximum weight w_e. Replace J with $J \cup \{e\}$.

Example 4. Let $G = (V, E)$ be a graph. $M \subseteq E$ is matching if no vertex is incident to more than one edge. Let $I = \{$matchinigs of $G\}$, and say we want to find the matching with maximum sum of the edge weights. If we try Kruskal greedy algorithm on the graph in Figure (1), The result will be to first pick the edge pq then rs for a total sum of 12. The maximum matching however is to pick ps and qr for a sum of 15. So for this example the greedy approach fails.

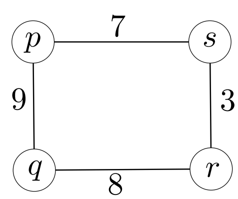

Figure 1: Graph for matching example.

Definition 4. Let V be a finite set and let I be a family of subsets of V. We call V the grounds set and I the independent sets. The pair $m = (V, I)$ is called a *matroid* if

(i) $\emptyset \in I$

(ii) If $J \in I$ and $J' \subseteq J$ then $J' \in I$. (Down-closed)

(iii) $\forall J', J \in I$ with $|J'| < |J|$ $\exists j \in J \backslash J'$ such that $J' \cup \{j\} \in I$. (Extension axiom).

Definition 5. B is a *base* of $A \subseteq V$ if $B \in I$ and $\nexists\ x \in A \backslash B$ such that $B \cup \{x\} \backslash I$.

Lemma 2. *$m = (V, I)$ is a matroid if and only if I fulfills axiom (i) and (ii) and all bases B of A where $A \subseteq I$ have the same cardinality.*

Proof. ($\Rightarrow$)Let B_1 and B_2 be basis. If $|B_1| < |B_2|$ then B_1 can be extended which is a contradiction of the definition of base.

($\Leftarrow$) For $J', J \in I$ where $|J'| < |J|$ let $A = J \cup J'$. J' is not a basis of A $\Rightarrow \exists j \in A \backslash J' = J \backslash J'$ such that $J' \cup (j) \in I$. $\square$

Example 5. Let $m = (V, I)$ be a matroid. Take a subset $A \subseteq V$ with the weights

$$w_{ij} = \begin{cases} 1 & j \in A \\ 0 & \text{otherwise.} \end{cases}$$

For $J \in I$ let $w(J) = |J \cap A|$. If $J \in I \Rightarrow J \cap A \in I$ and $w(J \cap A) = |J \cap A|$. A base of A will solve the maximum independent set.

Example 6. Continuation of the spanning tree example Let $m = (E, I)$ with $I = \{$ forests of $G\}$. The basis are spanning trees with $|B| = |V| - 1$.

Example 7. Continuation of the matching example. Let $A = \{pq, qr, rs\}$ basis of A are $\{pq, rs\}$ and $\{qr\}$.

Example 8. Let A be a matrix. A linear matroid $M = (V, I)$ is a matroid where $I = \{$linearly indepdent columns of $A\}$.