U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

Notes on Linear Programming: Part II

DUALITY THEOREMS

George Dantzig
Alex Orden

RM-1265

ASTIA Document Number AD 114135

30 October 1953

Assigned to ______________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

2

SUMMARY

Since the simplex procedure itself yields as a natural by-product proofs of several important theorems concerned with "Duality" in the field of linear inequalities, we demonstrate them here.

DUALITY THEOREMS

George Dantzig
Alex Orden

We shall consider one of many forms of the duality theorem — this one due to von Neumann [1] (for references to other equivalent forms see [2], [3], [4]). Let A be an m x n matrix, b an m-component column vector, and c an n-component row vector.

Theorem 1: <u>If column vectors</u> $X = \{x_1, x_2, \cdots, x_n\}$ <u>exist which satisfy</u> (1) <u>and if the corresponding values of</u> x_o <u>in</u> (1.1) <u>have a finite upper bound, then row vectors</u> $Y = (y_1, y_2, \cdots, y_m)$ <u>exist which satisfy</u> (2) <u>and the corresponding values of</u> y_o <u>in</u> (2.1) <u>have a finite lower bound</u>:

(1) $x_i \geq 0 \quad (i = 1, \cdots, m)$ (2) $y_j \geq 0 \quad (j = 1, \cdots, n)$

$AX \leq b$ $c \leq YA$

(1.1) $cX = x_o$; (2.1) $y_o = Yb$;

<u>moreover, there exist two vectors</u> $X = X^*$ <u>and</u> $Y = Y^*$ <u>such that the corresponding values</u> $x_o = x_o^*$ <u>and</u> $y_o = y_o^*$ <u>satisfy</u>

(3) $$\text{Max } x_o = x_o^* = y_o^* = \text{Min } y_o .$$

Assuming there exist a solution Y to (2) and a solution X to (1), we may multiply $AX \leq b$ by Y on the left without affecting the inequality (because Y has nonnegative components) and similarly multiply $c \leq YA$ on the right by X to obtain

$$(4) \qquad x_0 = bX \leq YAX \leq bY = y_0 \,.$$

This shows that y_0 forms an upper bound for the values of x_0, and x_0 a lower bound for values of y_0; hence if we exhibit a pair of solutions X and Y with the property $x_0 = y_0$, this must be a maximizing solution for X and a minimizing solution for Y, and the duality theorem is established.

We shall now show that we can obtain such a pair as an immediate corollary of the properties of an optimum basic solution of the simplex method [5]. For this purpose we transform (1) into the equivalent system of linear <u>equations</u> in nonnegative variables by introducing nonnegative variables $W = \{x_{n+1}, \cdots, x_{n+m}\}$

$$(5) \qquad \begin{aligned} x_0 - cX &= 0 \qquad x_j \geq 0 \qquad (j = 1,2,\cdots,n+m) \\ AX + I_m W &= b \end{aligned}$$

where I_m is the m x m identity matrix. One of the main results of the generalized simplex method[1] is that when a system such as (5) has solutions and a finite upper bound exists for values of x_0, there exists a solution $x_j = x_j^*$ satisfying (5) and a row vector β^* (Theorem VI in [5]) with the properties

---

[1] The linear equation system in [5] (second section) contains a redundant equation to which the stated result applies more directly — however, it is a simple matter to show that it may be omitted as here.

$$(6) \qquad \beta^* P_o = 1, \quad \beta^* \begin{bmatrix} o \\ b \end{bmatrix} = x_o^*, \quad \beta^* P_j \geq 0 \qquad (j = 1,2,\cdots,n),$$

where P_j is the column vector of coefficients associated with the variable x_j in (5). It is easy to verify from $\beta^* P_o = 1$ that the first component of β^* is unity. Accordingly we define

$$(7) \qquad \beta^* = [1, y_1^*, y_2^*, \cdots, y_n^*] = [1, Y^*]$$

and set

$$(8) \qquad y_o^* = \beta^* \begin{bmatrix} o \\ b \end{bmatrix} = Y^* b \ .$$

It is also easy to verify that the other properties of β^* in (6) are precisely the same as (2) and (3), establishing the Minmax Theorem.

Theorem 2: <u>If either system has a solution but the associated linear form is unbounded, then the dual system has no solution.</u>[1]

Proof: If (on the contrary) the dual system also has a solution Y, then the linear forms denoted by x_o and y_o satisfy, by (4), $x_o \leq y_o$; whence y_o is an upper bound for x_o, contradicting our hypothesis.

Theorem 3: <u>Whenever inequality occurs in the</u> k-th <u>relation of either system for an optimizing solution, then the</u> k-th <u>variable of an optimizing solution of the dual system</u>

---

[1] Note: Both a system and its dual may have no solution; for example, $x_1 - x_2 \leq 1$, $-x_1 + x_2 \leq -2$, $2x_1 - x_2 = \max$, and $y_1 - y_2 \geq 2$, $-y_1 + y_2 \geq 1$, $y_1 - 2y_2 = \min$, where $x_i \geq 0$, $y_j \geq 0$.

vanishes. Conversely, if the k-th variable is positive of the dual system, then k-th relation of the original system is an equality.

Proof: Let $AX^* + IW^* = b$ where X^* is an optimizing solution to (1). Multiply this expression by Y^*, an optimizing solution to (2); then

$$Y^*AX^* + Y^*W^* = Y^*b \; ; \tag{9}$$

from (3) and (4) follows $Y^*AX^* = Y^*b$ or

$$Y^*W^* = \sum_1^m y_i^* x_{n+i}^* = 0 \qquad (x_{n+i}^* \geq 0, \; y_i^* \geq 0) \; . \tag{10}$$

Since $x_{n+i}^* > 0$ means an inequality in the k-th relation of the first problem, it follows $y_i^* = 0$; similarly if $y_i^* > 0$, then $x_{n+i}^* = 0$, proving the theorem.

REFERENCES

[1] J. von Neumann, "Discussion of a Maximization Problem," Institute for Advanced Study, 1947 Manuscript.

[2] D. Gale, H. Kuhn, and A. Tucker, "Linear Programming and the Theory of Games," Activity Analysis of Production and Allocation, T. C. Koopmans, Ed., John Wiley and Sons, 1951, Chapter XIX.

[3] H. Weyl, "The Elementary Theory of Convex Polyhedra" and "Elementary Proof of a Minimax Theorem due to von Neumann," Contributions to the Theory of Games, Volume I, Kuhn and Tucker, Editors, Princeton University Press, 1950.

[4] G. Dantzig and A. Orden, "A Duality Theorem Based on the Simplex Method," Symposium on Linear Inequalities, USAF Hq. SCOOP Publication No. 10, dated 1 April 1952.

[5] G. B. Dantzig, A. Orden, P. Wolfe, "The Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints," The RAND Corporation, RM-1264, 5 April 1954.