Interior Point Methods

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Outline

- Interior point method: basic idea
- Log barrier function
- Central path
- Barrier (interior point) method
- A hierarchy of convex optimization

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Inequality constrained minimization

- Convex program with inequality constraints

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b,
\end{array}
$$

  - $f_0, \ldots, f_m : \mathbf{R}^n \rightarrow \mathbf{R}$ convex, twice differentiable
  - $A \in \mathbf{R}^{p \times n}$ with rank $p < n$
  - An optimal x^* exists

Basic idea: how to solve?

- Recall 1: equality constrained QP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b, \end{array}$$

  - KKT conditions

$$Ax^\star = b, \qquad Px^\star + q + A^T \nu^\star = 0,$$

  - Directly solve

$$\left[\begin{array}{cc} P & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} x^\star \\ \nu^\star \end{array}\right] = \left[\begin{array}{c} -q \\ b \end{array}\right].$$

Basic idea: how to solve?

- Recall 2: Newton's method with equality constraints
  - A quadratic approximation at current *x*

$$\begin{array}{ll} \text{minimize} & \widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v \\ \text{subject to} & A(x+v) = b \end{array}$$

  - Solve v (i.e., Δx)

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

  - Need line search, since $\widehat{f}$ is an approximation
  - What if *starting with an infeasible x?*

Basic idea: how to solve?

- Convex program with inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

  - Decompose into a sequence of *equality* constrained problem
  - Each solved by Newton's method with equality constraints

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Logarithmic barrier function

- Reformulate inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

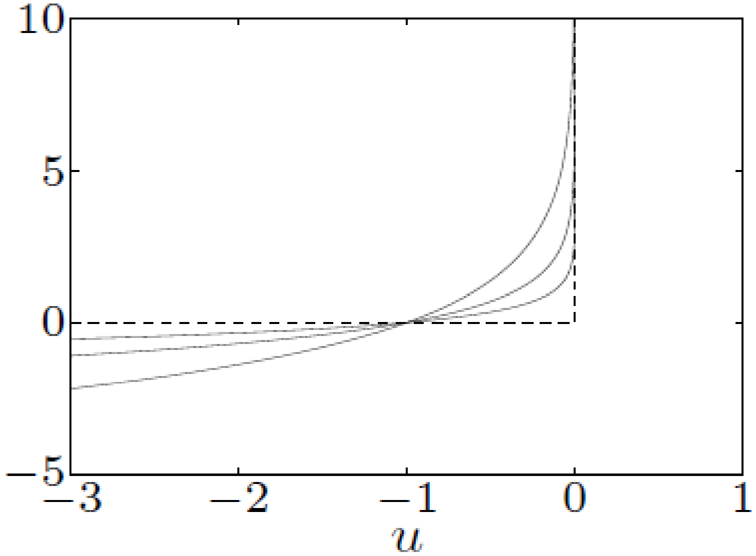
  - where

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0. \end{cases}$$

  - But indicator function is difficult to optimize

Logarithmic barrier function

- An alternative: $-(1/t)\log(-u)$
  - A smooth approximation to I_-
  - Improves as $t \to \infty$



Logarithmic barrier function

- Approximation by log barrier function

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m -(1/t)\log(-f_i(x)) \\ \text{subject to} & Ax = b. \end{array}$$

- Log barrier function $\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x))$
  - Convex
  - Twice differentiable

$$\begin{aligned} \nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\ \nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x) \end{aligned}$$

Logarithmic barrier function

- Approximation by log barrier function

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m -(1/t)\log(-f_i(x)) \\ \text{subject to} & Ax = b. \end{array}$$

- Log barrier function $\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x))$

$$\text{minimize} \quad f_0 + (1/t)\phi$$

- Rewrite (for a fixed t > 0)

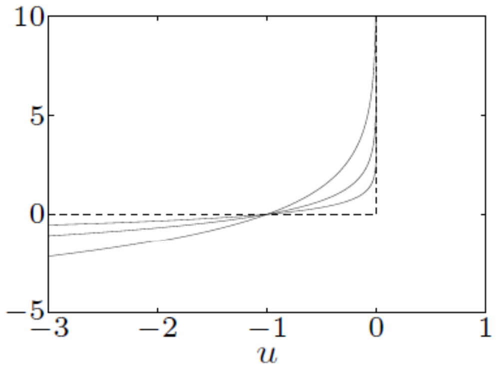
$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

Logarithmic barrier function

- Solve a sequence of problems

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

  - Increase *t* step by step
  - Why not a large *t* at the beginning?
    - Difficult to solve via Newton's method



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Central Path

- For any t > 0, define x*(t) as the solution to:

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

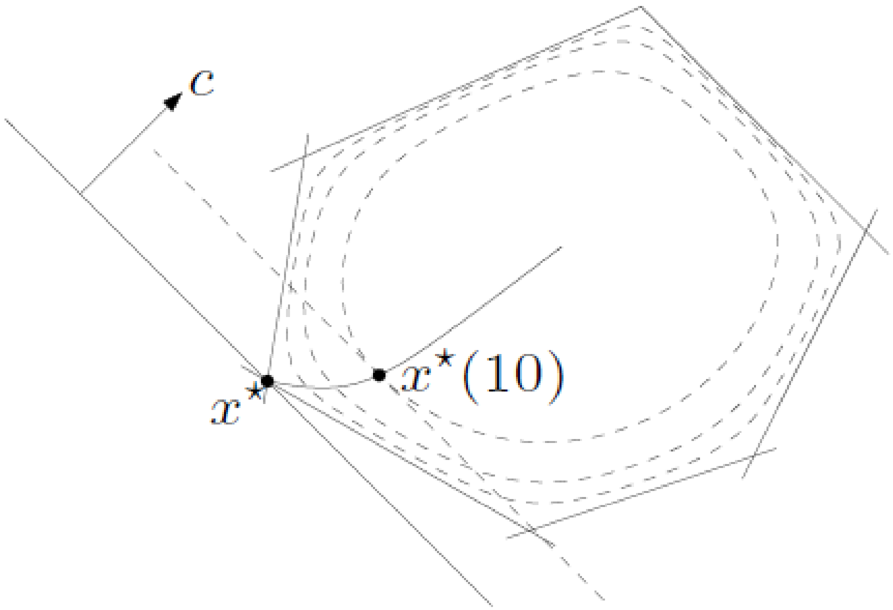
- Central path is the set:

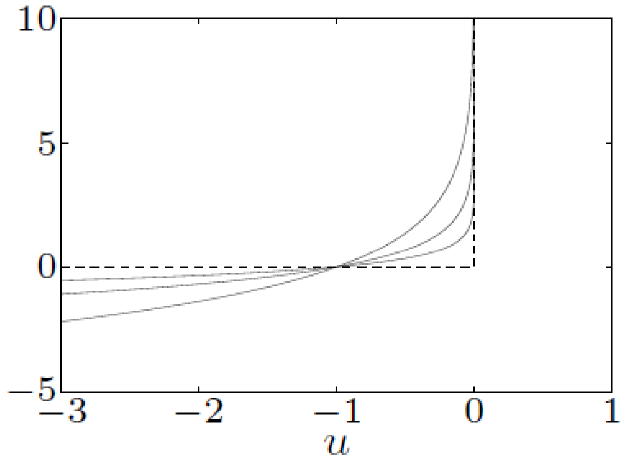
$$\{x^\star(t) \mid t > 0\}$$

Central Path for LP

- Example: an LP with inequality

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, 6 \end{array}$$





Central Path: Dual points

- For any t > 0, define x*(t) as the solution to:

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

- x*(t) is feasible to original problem
  - Not optimal
  - Leads to a dual feasible point v*(t)
  - But duality gap for x*(t) is bounded: m/t !
  - Useful as stop criteria ☺

KKT interpretations of Central Path

- Consider the problem for a fixed t:

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

- KKT conditions (see textbook)

$$\begin{array}{rcl} Ax = b, \quad f_i(x) & \leq & 0, \quad i = 1, \ldots, m \\ \lambda & \succeq & 0 \\ \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu & = & 0 \\ -\lambda_i f_i(x) & = & 1/t, \quad i = 1, \ldots, m. \end{array}$$

- Only *one* difference to KKT for original problem
  - Converge to original problem as $t \rightarrow \infty$

Force field interpretations

- Consider for a t > 0 (ignoring Ax=b)

  minimize $\quad t f_0(x) - \sum_{i=1}^m \log(-f_i(x))$

  - $t f_0(x)$: potential with the force $F_0(x) = -t\nabla f_0(x)$
  - $-\log(-f_i(x))$: potential with the force

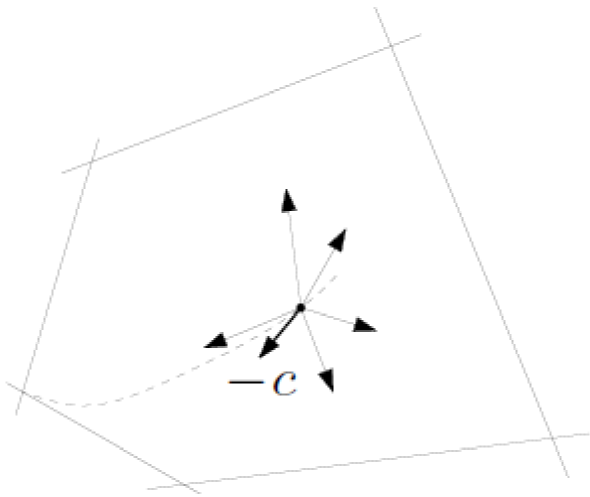
    $$F_i(x) = (1/f_i(x))\nabla f_i(x)$$

  - Force balance at optimal x*(t)

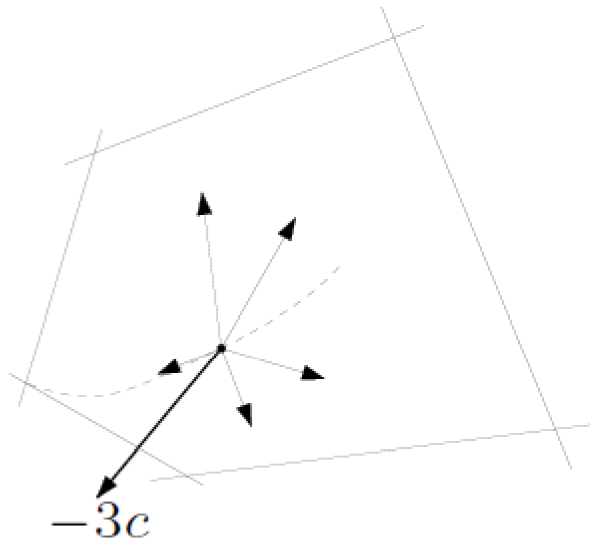
$$F_0(x^\star(t)) + \sum_{i=1}^m F_i(x^\star(t)) = 0$$

Force field for LP

- Consider the LP



$t = 1$



$t = 3$

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Barrier Method

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given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.* Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. *Update.* $x := x^\star(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

- Solve for a sequence of *t*
- Update (inner loops): Newton's, warm starting
- μ : large → fewer outer loops, more inner loops

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A hierarchy of convex optimization

- QP with equality constraints
  - Directly solve via KKT conditions
- Convex prog. with equality constraints
  - Form a quadratic approximation at *x*
  - Solve the approximation (Newton's with equality)
  - Line search along the Newton step
- Interior point method
  - Form a sequence of equality constrained problems
  - Newton's method with equality constraints

Thanks