AN ALTERNATIVE METHOD FOR LINEAR PROGRAMMING

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ABSTRACT. The method of leading variables is presented as a new method of solving Linear Programming, and is compared with the Simplex and the Dual Simplex methods.

1. *Introduction.* Linear Programming is the name given to the problem of finding non-negative x_j to minimize

$$C = \sum_{j=1}^{n} c_j x_j, \tag{1}$$

subject to the linear constraints

$$\Sigma a_{ij} x_j = b_i \quad (i = 1, \ldots, m). \tag{2}$$

Dantzig (2) has devised a method of solving this problem, known as the Simplex Method. He first finds a 'feasible solution', that is, a set of non-negative x_j satisfying (2), and shows how to find another feasible solution giving a smaller value of C if there is one. This leads to an iterative solution of the problem on the following lines.

It can be shown that the smallest value of C is normally obtainable at some basic feasible solution, that is, one where only m of the x_j are non-zero. We therefore start from a basic feasible solution and use the constraints to solve for the x_j that are positive in the basic solution in terms of the others. Thus (2) becomes

$$x_k = z_{k0} + \sum_l z_{kl} x_l, \tag{3}$$

where k ranges over all values such that $x_k > 0$ in the basic solution, and l is summed over all the other x_j.

Using (3) we express C in the form

$$C = C_0 + \sum_l s_l x_l.$$

Now if some s_l, say s_p, is negative, C can be decreased by increasing x_p from zero, leaving the other x_l as zero. Then from (3)

$$x_k = z_{k0} + z_{kp} x_p,$$

so that we must stop increasing x_p as soon as $z_{k0} + z_{kp} x_p$ becomes zero for some k, and we therefore put $x_p = \dfrac{z_{v0}}{-z_{vp}}$, where $\dfrac{z_{v0}}{-z_{vp}} = \min \dfrac{z_{k0}}{-z_{kp}}$ for all k such that $z_{kp} < 0$. (If no $z_{kp} < 0$, x_p can be increased indefinitely and the solution is infinite.)

Since the coefficient of x_p in the equation

$$x_v = z_{v0} + \Sigma z_{vl} x_l$$

is $z_{vp} < 0$ (and hence does not vanish), we can solve for x_p in terms of the other x_l and x_v; if the latter are set equal to zero a new basic feasible solution is obtained.

This process continues until either an infinite solution is obtained, or else the s_l are all non-negative, in which case the required solution has been obtained. For a rigorous account of the method see for example (1) and (3).

2. *Outline of the alternative method.* This paper describes an alternative iterative method, which we call the Method of Leading Variables. It is formally very similar to the Simplex Method, but it does not require that one first finds a feasible solution. It is based on the fact that if there is only one constraint the solution is obvious. For suppose that we wish to minimize

$$C = C_0 + \sum_{l=1}^{t} s_l x_l,$$

where $x_1, \ldots, x_t$ are non-negative and satisfy

$$\sum_{l=1}^{t} q_l x_l = 1. \tag{4}$$

Later we refer to (4) as the leading equation, and the variables occurring in it as the leading variables.

Now if all the q_l are non-positive there can be no solution with non-negative x_l. Otherwise we find all variables x_p such that $q_p > 0$ and

$$\frac{s_p}{q_p} = \min_{q_l > 0} \frac{s_l}{q_l}.$$

We call the x_p principal variables. For the present we assume that there is only one principal variable and that none of the q_l vanish. Then from (4)

$$x_p = \frac{1}{q_p} - \sum_{l \neq p} \frac{q_l}{q_p} x_l.$$

Hence

$$\begin{aligned} C &= C_0 + \frac{s_p}{q_p} - \sum_{l \neq p} s_p \frac{q_l}{q_p} x_l + \sum_{l \neq p} s_l x_l \\ &= C_0 + \frac{s_p}{q_p} + \sum_{l \neq p} q_l \left(\frac{s_l}{q_l} - \frac{s_p}{q_p}\right) x_l. \end{aligned} \tag{5}$$

Now $\dfrac{s_p}{q_p} = \min\limits_{q_l > 0} \dfrac{s_l}{q_l}$, so $q_l \left(\dfrac{s_l}{q_l} - \dfrac{s_p}{q_p}\right) > 0$ for all l such that $q_l > 0$. There are now two possibilities to consider:

(*a*) If $\left(\dfrac{s_l}{q_l} - \dfrac{s_p}{q_p}\right) > 0$ for some l such that $q_l < 0$, say $l = r$, then we can put

$$x_r = \frac{-A}{q_r}, \quad x_p = \frac{1+A}{q_p} \quad (x_l = 0, l \neq r \text{ or } p),$$

so that $C = C_0 + \dfrac{s_p}{q_p} - A\left(\dfrac{s_r}{q_r} - \dfrac{s_p}{q_p}\right)$, which can be given arbitrarily large negative values by choosing A large enough. In this case we say that the problem has an infinite solution. Let us assume that this is not so. Then

(*b*) $\left(\dfrac{s_l}{q_l} - \dfrac{s_p}{q_p}\right) \leqslant 0$ for all l such that $q_l < 0$, so that $q_l\left(\dfrac{s_l}{q_l} - \dfrac{s_p}{q_p}\right) \geqslant 0$ for all l, and it is clear from (5) that the smallest possible value of C is $C_0 + \dfrac{s_p}{q_p}$, given by putting $x_l = 0$ for $l \neq p$ and hence putting $x_p = \dfrac{1}{q_p}$. We call this the principal solution.

If any q_l, say q_r, vanishes, x_r can be given any non-negative value without affecting the values of the other leading variables. If $s_r < 0$, C can be given arbitrarily large negative values by making x_r large enough, and we have assumed that this does not happen. If $s_r > 0$, C is minimized when $x_r = 0$. The case $s_r = 0$ will be formally excluded by the device introduced later to ensure having only one principal solution.

Now suppose that there are further constraints expressed in the form

$$x_k = \sum_{l=1}^{t} z_{kl} x_l \geqslant 0 \quad (k = t+1, \ldots, n).$$

At the principal solution $x_p = \dfrac{z_{kp}}{q_p}$, so if $z_{kp} \geqslant 0$ for all k all the constraints are satisfied, and we have the required solution. However, if $z_{kp} < 0$ for $k = v$, say, the principal solution is not the required solution and we must find one where x_v is non-negative. Therefore we use the equation

$$x_v = \sum_{l=1}^{t} z_{vl} x_l$$

to substitute for x_p in terms of x_v and the other leading variables in the other constraints and in the expression for C.

The leading equation (4) is thus expressed in terms of new leading variables. We therefore minimize C for non-negative values of these variables in a similar way. Now (5) expresses C in terms of all the x_l other than x_p, and the coefficients of all the x_l are positive. The last principal solution is the only one for which all these x_l vanish, and it is no longer attainable. Therefore if these variables are still restricted to non-negative values the minimum value of C must have increased. It follows that if the process is continued we can neither reach a set of leading variables previously encountered, nor can we reach an infinite solution.

In the next section a complete method of solving linear programming on these lines is described, and in §4 we show that it must lead to the required solution if one exists. Meanwhile the significance of the terminology may become clearer if the procedure is summarized in words.

We find at each stage the minimum value of C for non-negative values of the leading variables, having used the constraints to express the other, non-leading, variables in terms of the leading variables. This minimum is attained at the principal solution, where all leading variables except the principal variable vanish. The non-leading variables have, in general, non-zero values. In Dantzig's terminology this is a basic solution, all but one of the variables in the basis being non-leading variables. It is also an optimal solution in that it minimizes C while allowing the variables to take any values permitted in the complete problem as well as some that are not permitted. However, it is not feasible if some, non-leading, variable is negative. If so we ensure that this variable is non-negative in the next principal solution by making it a leading variable, replacing the last principal variable. Thus we have a sequence of basic optimal solutions with larger and larger values of C culminating in a solution that is also feasible and therefore the required solution. Dantzig has a sequence of basic feasible solutions with smaller and smaller values of C culminating in a solution that is also optimal and therefore the required solution.

3. *The method of leading variables.* The problem is to find non-negative x_j to minimize C given by (1) subject to the linear constraints (2).

From the first constraint we express one of the x_j in terms of the others, and we substitute for this x_j in the other constraints.

From the second constraint we express another x_j in terms of the $n-2$ remaining x_j, and we substitute for this x_j in the other constraints (including the first one).

Proceeding in this way, if we obtain an identity the original system of equations was redundant and this one is omitted. If we obtain a contradiction the equations are inconsistent. Otherwise the constraints can all be reduced to the form

$$x_k = z_{k0} + \sum_l z_{kl} x_l, \tag{3}$$

where only $n-m$ of the variables occur in the right-hand sides of (3), and where m now denotes the number of non-redundant equations in (2). Using (3) we express C in terms of these $n-m$ variables only.

Later work is shortened if as far as possible variables expected to be non-zero in the required solution are chosen for the x_k.

The z_{k0} and z_{kl} of (3) must also be computed when the Simplex Method is used, but the x_k must there be chosen so that the z_{k0} are all non-negative. This is discussed in §7.

It may be worth noting that if for some k the z_{kl} (including z_{k0}) are all non-negative the equation may be dropped since x_k cannot then be negative for non-negative x_l. Similarly, if $z_{k0} < 0$ and all the z_{kl} are non-positive for some k, the problem can have no solution. If $z_{k0} = 0$ and all the z_{kl} are non-positive we must have $x_k = 0$ and $x_l = 0$ for all l such that $z_{kl} < 0$, so that these variables may be dropped from later work.

For convenience of exposition the variables are renumbered so that the $n-m$ variables occurring on the right-hand sides of (3) are $x_1, \ldots, x_{n-m}$.

We next introduce a further artificial constraint*

$$1 = x_0 + \sum_{l=1}^{n-m} \omega x_l, \tag{6}$$

where x_0 is restricted to non-negative values and ω is an arbitrarily small positive constant. No infinite solution can then arise if (6) is taken as the leading equation. This constraint is equivalent to

$$x_1 + x_2 + \ldots + x_{n-m} \leqslant \frac{1}{\omega},$$

and it will not affect any finite solution of the original problem if ω is small enough. An infinite solution of the original problem will be recognized by the fact that some of the x_j will be of order $1/\omega$ in the modified problem.

We now replace (3) by

$$x_k = z_{k0} x_0 + \sum_l z_{kl} x_l. \tag{7}$$

The consequences of this are discussed in §4.

The iteration described in §2 is now started, with (6) as the leading equation and (7) as the further constraints. We always treat ω as negligibly small compared with unity.

* Charnes ((1), p. 61) considers a similar equation in connexion with the Simplex Method.

If at any stage x_0 becomes the principal variable without yielding the required solution at once we obtain an expression of the form

$$x_0 = \Sigma z_{0l} x_l \tag{8}$$

to substitute in the other equations, and the leading equation will become

$$1 = \Sigma z_{0l} x_l$$

when terms in ω are neglected. Equation (8) can now be dropped and ω plays no further part in the computation.

Note that the z_{kl}, q_l and s_l all change when the leading variables change. In fact, using the notation of § 2, their new values, distinguished by primes, are given by

$$z'_{kv} = \frac{z_{kp}}{z_{vp}}, \quad q'_v = \frac{q_p}{z_{vp}}, \quad s'_v = \frac{s_p}{z_{vp}},$$

$$z'_{kl} = z_{kl} - z_{kp}\frac{z_{vl}}{z_{vp}}, \quad q'_l = q_l - q_p\frac{z_{vl}}{z_{vp}}, \quad s'_l = s_l - s_p\frac{z_{vl}}{z_{vp}}, \quad \text{for} \quad l \neq v.$$

Here the definition of the z_{kl} has been extended by putting $z_{l_1 l_1} = 1$ and $z_{l_1 l_2} = 0$ if $l_1 \neq l_2$, where x_{l_1} and x_{l_2} are leading variables. Using these definitions the principal solution can be expressed concisely as

$$x_j = \frac{z_{jp}}{q_p}, \quad x_0 = 1 \quad (j = 1, \ldots, n)$$

unless x_0 is a leading variable other than the principal variable when it is zero.

Next we consider what to do if there is more than one principal variable, or if the principal solution makes more than one x_k negative.

In § 2 we assumed that there was only one principal variable, and that the value of C at the principal solution certainly increased when the leading variables changed. We ensure this by using a device similar to that described by Charnes (1) for avoiding degeneracy in the Simplex Method. We suppose that the original expression for C has been modified by subtracting from it the polynomial

$$\sum_{j=0}^{n} \epsilon^{j+1} x_j,$$

where ϵ is some arbitrarily small positive quantity. If we write these extra terms at the end of the expression for C, without ever substituting for the non-leading variables in them, we see that we must choose from the principal solutions for the original C the one giving the largest value of x_0, or if there is more than one we choose from them the one giving the largest value of x_1, etc. This means in fact that if x_0 is a principal variable we choose x_0. Otherwise we choose x_p so as to maximize z_{1p}/q_p, or if several principal variables maximize this we choose from them the one that maximizes z_{2p}/q_p, etc.

Of course no ϵ need be written down now that we have the above rule of procedure.

Finally, having selected the principal solution, if more than one z_{kp} is negative we may be able to shorten later work by choosing the new leading variable x_v (from those with negative z_{vp}) so as to maximize $z_{vp}^2/\sum z_{vl}^2$ where l is summed over all leading variables. The reason is that in the t-dimensional Cartesian space with the leading variables as coordinates this represents the square of the cosine of the angle between the normal to the hyperplane $x_v = 0$ and the line joining the origin to the present solution. By making this as large as possible we improve our chances of moving the principal solution appreciably nearer the required solution.

Alternatively, one could take x_v as the variable corresponding to the smallest z_{kp}. This quantity depends partly on the 'distance' of the hyperplane $x_k = 0$ from the principal solution and partly on the scale in which x_k is measured. Since the latter factor is irrelevant the more complicated non-dimensional criterion may be preferable.

Another possibility is to use the simpler criterion having previously rescaled the variables. For example, when the constraints have been first reduced to the form (3) one could define new x_k proportional to the old ones such that $\sum_l z_{kl}^2 \doteqdot 1$ for all k.

4. *Justification of the method.* At each stage of the method we have a new selection of t leading variables not previously encountered, for if not we would have the same principal solution as before, which is impossible because the value of C at the principal solution steadily increases. There is only a finite number of possible ways of selecting t leading variables out of $n+1$. Therefore the process must terminate, and it can only do so if the q_l all become non-positive, in which case the problem has no solution, or if the principal solution leaves no negative x_k, in which case it is the required solution.

Having proved this result, we can show that it is indeed plausible. We use the fact that, if the problem has a finite solution, C can be written in the form

$$C = C_0 + \sum_j s_j x_j, \tag{9}$$

where the summation extends over $n-m$ of the x_j that are zero in the required solution and where the s_j are all non-negative. This follows from the fact that the Simplex Method always converges. From it we deduce that the same solution would be obtained if only the $n-m$ of the x_j occurring in (9) were restricted to non-negative values, the others being allowed to take any values. Therefore as soon as these $n-m$ of the x_j are all among the $t\,(=n-m+1)$ leading variables the required solution will be obtained.

The replacement of (3) by (7) must now be considered. Since (3) is equivalent to (7) with $x_0 = 1$ we need only consider the possibility of reaching a solution with $x_0 = 0$. This can only happen if x_0 is still a leading variable and x_p is of order $1/\omega$. In practice we are rarely interested in equations yielding such infinite solutions but for the sake of completeness this possibility must be discussed. The solution to the original problem is that x_p is arbitrarily large and the other leading variables vanish (except for x_0 which must be unity). The values of the non-leading variables are given by

$$x_k = z_{k0} + z_{kp} x_p$$

(and not by $x_k = z_{kp} x_p$ as with a finite solution). If the method has terminated none of the z_{kp} can be negative, but if some z_{kp} vanish and the corresponding z_{k0} are negative it remains to see whether the equations can be satisfied by non-negative x_j. We therefore proceed as follows:

Denote the present x_p by x_q.

Record all the z_{kq} and the expressions for the x_k for which z_{kq} is positive.

Replace the expression for C by zero.

Replace the leading equation by

$$1 = x_0 + \sum_l \omega x_l,$$

where l is summed over all leading variables other than x_q.

Omit all constraints except those for which z_{kq} vanished, and continue with the method. By our rules for breaking ties the first principal variable will be x_0. Therefore we will eventually find either that no solution is possible, or else a finite solution. We can then substitute these values in the expressions for the x_k omitted, and regard x_q as arbitrarily large. We then have a valid solution of the original problem with an arbitrarily large negative value of C.

Theoretically the above may seem to be a tiresome complication, but it must be observed that it will rarely be required even for problems not having a finite solution.

5. *Numerical example.* The method can be illustrated by a simple numerical example.

Suppose that we wish to maximize $x_1 + x_2$ where

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 1, \\ 3x_1 + 2x_2 + x_4 &= 1, \\ x_1 + 5x_2 + x_5 &= 1, \end{aligned}$$

and x_1, x_2, x_3, x_4 and x_5 are all non-negative.

We write

$$\begin{aligned} x_3 &= x_0 - 3x_1 - x_2, \\ x_4 &= x_0 - 3x_1 - 2x_2, \\ x_5 &= x_0 - x_1 - 5x_2, \\ 1 &= x_0 + \omega x_1 + \omega x_2, \\ C &= -x_1 - x_2. \end{aligned}$$

The work can be checked by considering the solution

$$\omega = 0, \quad x_0 = x_1 = x_2 = 1, \quad x_3 = -3, \quad x_4 = -4, \quad x_5 = -5, \quad C = -2.$$

We have $\dfrac{s_1}{q_1} = \dfrac{s_2}{q_2} = -\dfrac{1}{\omega}, \dfrac{s_0}{q_0} = 0$, so using the rules for breaking ties x_1 is the principal variable. The principal solution makes x_3, x_4 and x_5 negative, but

$$\frac{z_{31}^2}{\Sigma z_{3l}^2} = \frac{9}{11}, \quad \frac{z_{41}^2}{\Sigma z_{4l}^2} = \frac{9}{14}, \quad \frac{z_{51}^2}{\Sigma z_{5l}^2} = \frac{1}{27},$$

so x_3 is taken as the new leading variable.

Hence

$$\begin{array}{rllll} & 1 & -3 & 1 & \text{check solution} \\ x_1 = & \tfrac{1}{3}x_0 & -\tfrac{1}{3}x_3 & -\ \tfrac{1}{3}x_2 & 1 \\ x_4 = & & \quad x_3 & -\quad x_2 & -4 \\ x_5 = & \tfrac{2}{3}x_0 & +\tfrac{1}{3}x_3 & -\tfrac{14}{3}x_2 & -5 \\ 1 = & \quad x_0 & -\tfrac{1}{3}\omega x_3 & +\ \tfrac{2}{3}\omega x_2 & 1 \\ C = & -\tfrac{1}{3}x_0 & +\tfrac{1}{3}x_3 & -\ \tfrac{2}{3}x_2 & -2 \end{array}$$

Now x_2 is the principal variable and x_5 the best new leading variable, so

$$\begin{array}{rlll} & 1 & -3 & -5 & \text{check solution} \\ x_2 = & \frac{1}{7}x_0 & +\frac{1}{14}x_3 & -\frac{3}{14}x_5 & 1 \\ x_1 = & \frac{2}{7}x_0 & -\frac{5}{14}x_3 & +\frac{1}{14}x_5 & 1 \\ x_4 = & -\frac{1}{7}x_0 & +\frac{13}{14}x_3 & +\frac{3}{14}x_5 & -4 \\ 1 = & x_0 & -\frac{2}{7}\omega x_3 & -\frac{1}{7}\omega x_5 & 1 \\ C = & -\frac{3}{7}x_0 & +\frac{2}{7}x_3 & +\frac{1}{7}x_5 & -2 \end{array}$$

Now x_0 is the principal variable, and x_4 the new leading variable:

$$\begin{array}{rllll} & -4 & -3 & -5 & \text{check solution} \\ x_0 = & -7x_4 & +\frac{13}{2}x_3 & +\frac{3}{2}x_5 & 1 \\ x_2 = & -x_4 & +x_3 & & 1 \\ x_1 = & -2x_4 & +\frac{3}{2}x_3 & +\frac{1}{2}x_5 & 1 \\ 1 = & -7x_4 & +\frac{13}{2}x_3 & +\frac{3}{2}x_5 & 1 \\ C = & 3x_4 & -\frac{5}{2}x_3 & -\frac{1}{2}x_5 & -2 \end{array}$$

The principal solution is now the required solution. We have

$$x_3 = \tfrac{2}{13}, \quad x_4 = x_5 = 0, \quad x_2 = \tfrac{2}{13}, \quad x_1 = \tfrac{3}{13}, \quad C = -\tfrac{5}{13}.$$

The method can be illustrated geometrically in a space of $n-m$ dimensions, by taking Cartesian coordinate axes for the original leading variables (Fig. 1). Thus in this example $n-m = 2$ and, taking axes for x_1 and x_2, we plot the lines

$$\begin{aligned} x_3 &\equiv 1 - 3x_1 - x_2 = 0, \\ x_4 &\equiv 1 - 3x_1 - 2x_2 = 0, \\ x_5 &\equiv 1 - x_1 - 5x_2 = 0. \end{aligned}$$

The sides of the lines $x_j = 0$ corresponding to negative and therefore inadmissible values of x_j are shaded.

The line
$$x_0 \equiv 1 - \omega x_1 - \omega x_2 = 0$$
is represented with $\omega = \frac{1}{2}$.

We note that C decreases as $x_1 = x_2$ increases, as indicated by the arrow.

We first consider the triangle formed by the lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, and find that the best point not outside it is the vertex where $x_0 = x_2 = 0$, marked 1 in the diagram. However, this makes x_3 negative and the best point for which x_0, x_2 and x_3 are all non-negative is the vertex $x_0 = x_3 = 0$, marked 2 in the diagram. This makes x_5 negative and the best point for which x_0, x_3 and x_5 are all non-negative is the vertex $x_3 = x_5 = 0$, marked 3 in the diagram. Finally we reach the required solution marked 4.

If the Simplex Method were used we might start at the point

$$x_1 = x_2 = 0 \quad (x_3 = x_4 = x_5 = 1),$$

marked A in the diagram, and proceed via B to the final solution at C.

6. *Finding all optimal solutions.* Having found one feasible solution giving the smallest possible value of C, we may wish to find all such solutions. We here consider only the case where the minimum value of C is finite, in which case our first solution will also be finite, even if the problem can be satisfied with arbitrarily large x_j, since the method of breaking ties ensures that x_0 is always made principal variable if possible.

It is clear that the principal solution found to be feasible is the unique optimal solution unless

$$\frac{s_p}{q_p} = \frac{s_l}{q_l} \tag{10}$$

for some other leading variable x_l. In this section we call all leading variables satisfying (10) principal variables even if the corresponding q_l are non-positive. If now $z_{kp} \geqslant 0$ for all k and all principal variables x_p, the optimal solutions can all be written down. They are, using the extended definitions of the z_{kl} introduced in § 3,

$$x_j = \sum_p \lambda_p \frac{z_{jp}}{q_p} \quad (j = 1, \ldots, n),$$

where the λ_p are arbitrary numbers of the same signs as the q_p satisfying

$$\Sigma \lambda_p = 1.$$

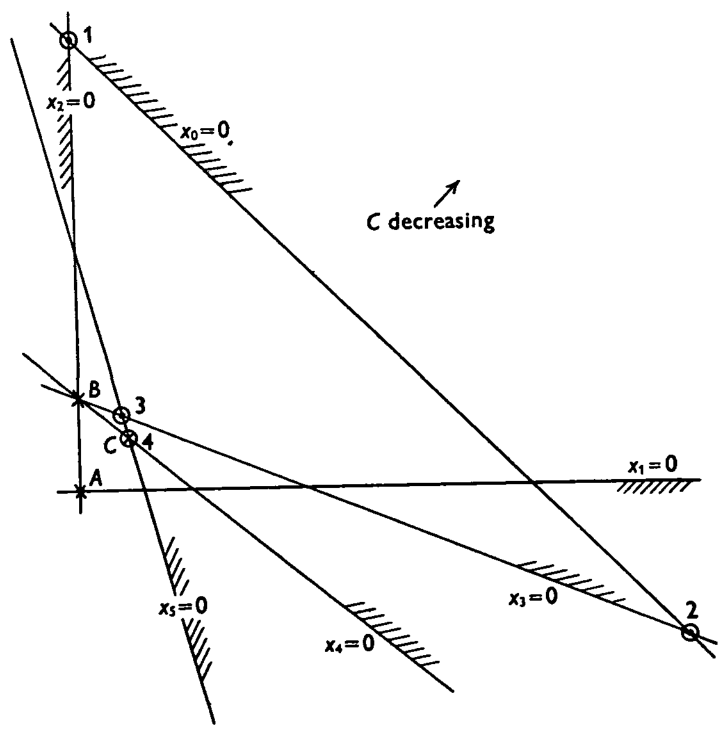

Fig. 1

However, if some of the z_{kp} are negative these solutions may include some for which the corresponding x_k are negative, and if there are several principal variables the situation may become rather complicated. Charnes (1) has described an elegant procedure for finding all optimal solutions using the Simplex Method. However, it may sometimes be easier to proceed as follows:

Consider the equations

$$\Sigma q_p x_p = 1,$$
$$x_k = \Sigma z_{kp} x_p,$$

where the summation is now only over principal variables and where the equation for x_k is included only if some z_{kp} is negative (for a principal variable x_p).

Suppose either that all the q_p are positive, or else that we know that there are no optimal solutions with arbitrarily large x_j. Then if P is the number of principal variables we have a set of linear equations with $P-1$ more variables than equations. In general they can be solved with any $P-1$ variables put equal to zero, and the solution can be recorded if no variable turns out to be negative. The x_k for which the z_{kp} were all non-negative can then be computed from

$$x_k = \Sigma z_{kp} x_p.$$

Omitting all combinations of $P-1$ variables in turn we obtain all optimal basic feasible solutions which may be denoted by $x_j^{(1)}, x_j^{(2)}, \ldots$. All optimal solutions are then given by $\sum_\alpha \lambda_\alpha x_j^{(\alpha)}$ for all non-negative λ_α satisfying $\Sigma\lambda_\alpha = 1$.

However, if some q_p is non-positive we may have to consider the possibility of a solution with arbitrarily large x_j. We then use the equation $\Sigma q_p x_p = 1$ to express some x_p, say x_h, in terms of the others, and we substitute for x_h in the other constraints. After introducing the artificial constraint

$$1 = x_0 + \sum_{p \neq h} \omega x_p,$$

we proceed as above, regarding ω as an arbitrarily small positive quantity.

7. *Comparison with other methods*. This method is formally very similar to the Simplex Method. Having suggested that the chief disadvantage of the latter is that it has to start from a feasible solution, I should perhaps indicate how this is done. The standard procedure is to introduce an extra variable in each equation, associating with it an arbitrarily large cost coefficient. So (2) becomes

$$x_{n+i} = |b_i| \pm \sum_{j=1}^{n} a_{ij} x_j,$$

while (1) becomes

$$C = \sum_{j=1}^{n} c_j x_j + \sum_{j=n+1}^{n+m} \Omega x_j,$$

where Ω is arbitrarily large.

We now have the feasible solution $x_{n+i} = |b_i|$ for $i = 1, \ldots, m$, $x_i = 0$ for $i = 1, \ldots, n$, and the constraints are reduced to the form (3). If Ω is large enough, a solution with all the extra variables put equal to zero will always be preferred to one where some are positive; so we eventually return to the original problem.

The whole analysis usually involves about the same amount of work as first reducing (2) to (3) by conventional methods and then starting an iteration on (3). However, it may sometimes be inconvenient to have to start with so many variables and hence such a large matrix of coefficients.

I am grateful to the referee for drawing my attention to Dr C. E. Lemke's Dual Simplex Method, described in (4). This can be regarded as an application of the Simplex Method to the dual linear programming problem. Alternatively, it can be explained in our terminology as follows.

As with the other methods the constraints are expressed in the form (3) and C is written as

$$C = C_0 + \Sigma s_l x_l.$$

In this method the s_l must all be positive, which can be arranged using the artificial constraint

$$x_0 + \Sigma x_l = \Omega, \tag{11}$$

where Ω is arbitrarily large and x_0 is an additional variable (closely related to the x_0 in the Leading Variables method). For if $s_r = \min s_l < 0$, we rewrite C as

$$C = (C_0 + s_r \Omega) - s_r x_0 + \sum_{l \neq r} (s_l - s_r) x_l,$$

and use (11) to substitute for x_r in terms of x_0 and the remaining x_l.

Now if the s_l are all positive and the x_l are restricted to non-negative values, C is minimized if all the x_l vanish. If all the $z_{k0} \geqslant 0$ the remaining variables are then all non-negative and we have the required solution. However, if $z_{v0} < 0$ we define x_p by

$$\frac{z_{vp}}{s_p} = \max \frac{z_{vl}}{s_l},$$

and use the equation

$$x_v = z_{v0} + \Sigma z_{vl} x_l$$

to substitute for x_p in terms of x_v and the remaining x_l.

This leaves the problem in the same form as before with a larger value of C_0 and thus defines an iterative solution. If similar rules are used for choosing x_v from the variables that are negative at the trial solution, this method leads to the same sequence of trial solutions as the Leading Variables Method. There seems to be little to choose between the two methods. Our method has a possible advantage for hand computation in that the equations whose coefficients must be compared to determine the principal variable are written next to each other. The Dual Simplex Method may be more convenient when solving dual problems simultaneously, as, for example, in the theory of games. It also seems to lend itself more easily to special methods when the variables must be both non-negative and not greater than some given constant.

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