Submodular maximization

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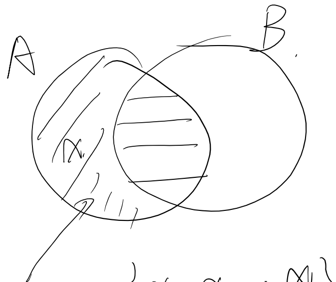
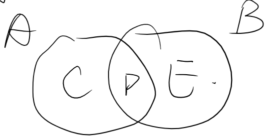
Definition.

$f: \quad f(S+\{x\}) - f(S) \geq f(T+\{x\}) - f(T) \quad \forall S \subseteq T \quad \forall x \notin T.$

$\Rightarrow. \quad f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad \forall S, T.$



$A - B = \{x_1, x_2, \cdots, x_k\}.$



$1^\circ \quad f(A) - f(A \cap B) \geq f(A \cup B) - f(B)$

$f(C+D) - f(D) \geq f(C+D+E) - f(D+E)$

$f(x_1+D) - f(D) \geq f(x_1+D+E) - f(D+E)$

$f(x_1+x_2+D) - f(x_1+D) \geq f(x_1+x_2+D+E) - f(x_1+D+E)$

$\vdots$

$f(x_1+\cdots+x_k+D) - f(D) \geq f(x_1+\cdots+x_k+D+E) - f(D+E).$

$2^\circ. \quad k=0. \quad A \subseteq B.$

$$f(A \cup B) + f(A \cap B) = f(B) + f(A)$$

Difference.

$$\Delta_y(\Delta_x f(S)) = \Delta_x f(S+\{y\}) - \Delta_x f(S).$$

$$= f(S+x+y) - f(S+y) - f(S+x) + f(S). \leq 0.$$

$S \cup T \qquad S \qquad T \qquad S \cap T$

$\Delta_x \Delta_y f(S).$ $\qquad \Delta_x \Delta_x f(S) = -\Delta_x f(S).$

$$\frac{d^2 f(x)}{dx^2}.$$

$\Delta_A f(S) =$

$A = \{x_1, x_2, \cdots, x_k\}.$

$\Delta_x f(S) \geq 0 \iff f(S+\{x\}) - f(S) \geq 0.$

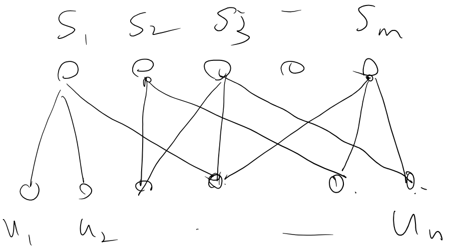
Set Cover.

$N = \{1, \cdots, n\}$ $\qquad S_1, \cdots, S_m \subseteq N. \qquad w(S_i)$

Set cover.

$N = \{u_1, \cdots, u_n\}$ $\quad S_1, \cdots, S_m \subseteq N$. $\quad w(S_i)$

$f(T) = \left| \bigcup_{S \in T} S \right|$ $\quad T$



$= \sum_{S \in T} w(S)$.

1°. submodular

2°. monotone.

$T: \{S_1, S_2\}$ $\quad f(T) = 5$.

$T + S_m$: $\quad f(T + S_m) = 6$.

$T': \{S_1, S_2, S_3\}$ $\quad f(T') = 6$

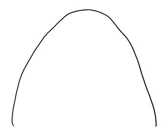
$T' + S_m$. $\quad f(T' + S_m) = 6$.

Graph Cut.

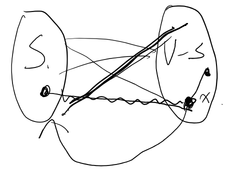
$G = (V, E)$

$f(S) = \text{cut}(S, V-S)$ $\quad S \subseteq V$.

1°. not monotone. $\quad f(\phi) = 0$. $\quad f(V) = 0$.



$\boxed{f(S + \{x\}) - f(S)}$.



$= \deg_x(V - S - \{x\}) - \deg_x(S)$.

$\boxed{f(T + \{x\}) - f(T)} = \deg_x(V - T - \{x\}) - \deg_x(T) = \deg_x(B) - (\deg_x(A) + \deg_x(S))$

$V - T - \{x\}$

S, $A = T - S$. $\quad B$.

$\deg_x(A) + \deg_x(B) - \deg_x(S)$.

max cut

$\max_S f(S)$

Greedy algorithm.

Greedy algorithm.

Algo: $\{u_1, u_2, \cdots, u_k\}$.

OPT: $\{v_1, \cdots, v_k\}$.

$$\Rightarrow \begin{cases} f(\{u_1\}) \geq f(\{v_i\}) \quad \forall i. \\ f_{\{u_1\}}(\{u_2\}) \geq f_{\{u_1\}}(\{v_i\}) \quad \forall i. \\ f_{\{u_1, \cdots, u_j\}}(\{u_{j+1}\}) \geq f_{\underbrace{\{u_1, \cdots, u_j\}}_{A_j}}(\{v_i\}) \quad \forall i. \end{cases}$$

$$\geq \frac{f_{A_j}(\{v_1\}) + f_{A_j}(\{v_2\}) + \cdots + f_{A_j}(\{v_k\})}{k}.$$

$$\geq \frac{f_{A_j}(\{v_1, \cdots, v_k\})}{k}.$$

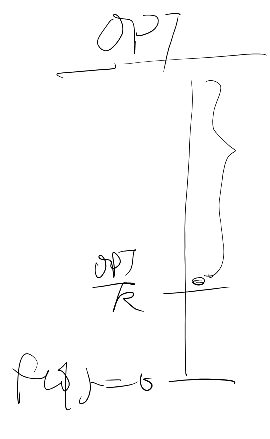
$$= \frac{f(A_j \cup \{v_1, \cdots, v_k\}) - f(A_j)}{k}.$$

$$f_{A_j}(\{u_{j+1}\}) \geq \frac{OPT - f(A_j)}{k}.$$

$$f(A_{j+1}) - f(A_j) \geq \frac{OPT - f(A_j)}{k}.$$

$f(A_0) = f(\phi) = 0.$

$f(A_k)$



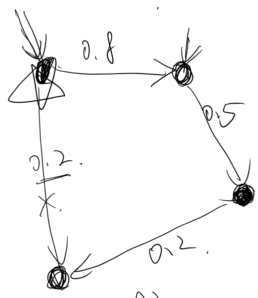
$$OPT - f(A_{j+1}) \leq \left(1 - \frac{1}{k}\right)\left(OPT - f(A_j)\right).$$

$$OPT - \underset{f(A_k)}{Algo} \leq \left(1 - \frac{1}{k}\right)^k \cdot OPT.$$

$$Algo \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot OPT. \quad \geq \left(1 - \frac{1}{e}\right) \cdot OPT$$

Social Network

Social Network

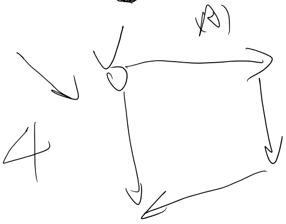


$\boxed{E(S)}$

$\max_{|S| \le 2000} \boxed{E(S)}$

(A)

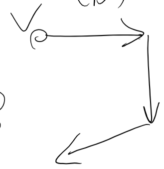
$f(S)$ 4



$0.8 \times 0.5 \times 0.2 \times 0.2$

(B)

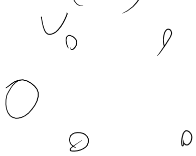
3



$0.8 \times 0.5 \times 0.2 \times 0.8$

(C)

0



$0.2 \times 0.5 \times 0.8 \times 0.8$