Introduction to Matroid Theory

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Outline

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1. Fundamentals of Matroid
   - General Definition of Matroids
   - Equivalent Definitions
   - Operations
2. Some Classes of Representable Matroids
   - Representable Matroids
   - Excluded Minors Summary
   - Relationships between Various Classes of Matroids
3. Summary

General Definition of Matroids

Vector Independence

Properties of Vector Independence

1. ϕ is independent to any vector;
2. Hereditary: subset of some independent vectors are also independent;
3. Augmentation: we can add a new vector to a smaller independent set to keep independency.

Example

Let A be the following matrix over field $\mathbb{R}$ of real numbers.

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Hereditary: $\{1,2,3\} \to \{1,2\}$

Augmentation:

$\{1,2\}, \{2,3,5\} \to \{1,2,3\}$

General Definition of Matroids

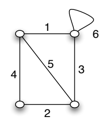
Cycles in Graphs

Properties of Graphs

1. ϕ contains no cycle;
2. Hereditary: subgraph of a cycle-free graph is cycle-free;
3. Augmentation: Add a new edge to a smaller cycle-free subgraph.

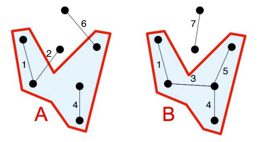
Example

Graph associated with A



Example

Let A, B be the following subgraphs.



Hereditary:
$\{1,3,4,5\} \to \{1,3,4\}$
Augmentation:
$\{1,4\},\{1,3,4,5\} \to \{1,3,4\}$

General Definition of Matroids

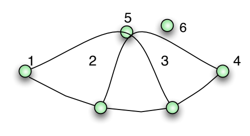
Collinear & Coplanar in Geometry

Geometry Diagram

1. Collinear/planar points are dependent (at least 3);
2. ϕ is not collinear/coplanar to any point;
3. Hereditary: subset of some non-collinear/coplanar points are still non-collinear/coplanar;
4. Augmentation: we can add a new point to a smaller set of some non-collinear/coplanar points.

Example

See the following geometry diagram.



Associated with

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$A = \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right]$$

Generalize$\Longrightarrow$Matroid

General Definition of Matroids

Independent sets

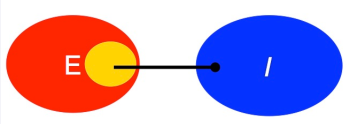
Definition

A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following three properties:

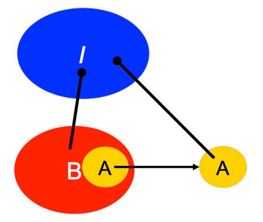
1. $\phi \in \mathcal{I}$;
2. Hereditary:
   If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;
3. Augmentation:
   If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Demo

$\mathcal{I}$ is a collection of subset of E



Hereditary



General Definition of Matroids

Independent sets

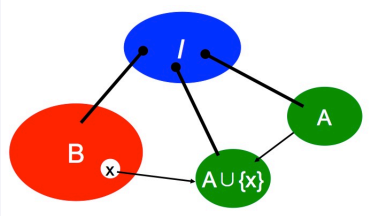
Definition

A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following three properties:

1. $\phi \in \mathcal{I}$;
2. Hereditary:
   If $I \in \mathcal{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathcal{I}$;
3. Augmentation:
   If I_1 *and* I_2 *are in* $\mathcal{I}$ *and* $|I_1| < |I_2|$, *then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

Demo

Augmentation



Outline

Equivalent Definations

Base/Basis

Definition

Bases $\mathcal{B}(M)$ of a matroid are the maximal independent sets.

Properties

- Same cardinality for all bases;
- The matroid is a span of any base;

Example

Consider

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ A = & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5}

Equivalent Definations

Base/Basis

Definition

A collection of subsets $\mathcal{B} \subseteq 2^E$ of a ground set E are the bases of a matroid if and only if:

- $\mathcal{B}$ is non-empty;
- base exchange: If $B_1, B_2 \in \mathcal{B}$, for any $x \in (B_1 - B_2)$, there is an element $y \in (B_2 - B_1)$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

Example

Consider

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5}

Equivalent Definitions
Circuit

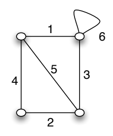
Definition

The circuits $\mathcal{C}(M)$ of a matroid are the minimal dependent sets.

Property

- If any one element is deleted from a circuit, it will become an independent set.
- Independent sets $\mathcal{I}(M)$ does not contain any element in $\mathcal{C}(M)$.

Consider the graph



Circuits: $\{1,2,3,4\}$, $\{1,3,5\}$, $\{2,4,5\}$, $\{6\}$

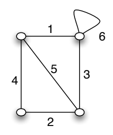
Equivalent Definitions
Circuit

Definition

A collection of sets $\mathcal{C}$ is the collection of circuits of a matroid if and only if:

- $\phi \notin \mathcal{C}$
- if $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$ then $C_1 = C_2$
- if $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there is some $C_3 \in C$ with $C_3 \subseteq (C_1 \cup C_2) - e$.

Consider the graph



Circuits: {1,2,3,4}, {1,3,5}, {2,4,5}, {6}

Equivalent Definitions
Rank Function

Definitions

Rank: maximun number of linearly independent vectors.

For any base of a matroid, $r(B) = r(M)$.

Formally, a rank function is $r : 2^E \to \mathbb{N} \cup \{0\}$, for which

- For $X \subseteq E, 0 \leq r(X) \leq |X|$
- If $X \subseteq Y \subseteq E, r(X) \leq r(Y)$
- $\forall X, Y \subseteq E, r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$

Example

Consider

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5};

Rank: $r(M) = r(B) = 3$

Equivalent Definitions

Closure

Definitions

Given a rank function of a matroid, the closure operation $\mathrm{cl} : 2^{\mathrm{E}} \to 2^{\mathrm{E}}$ is defined as

$$\mathrm{cl}\ X = \{x \in E | r(X \cup x) = r(X)\}.$$

Closure is a span of a subspace.
$\mathrm{cl}\ B = E(M)$;
Independent Set:
$\mathcal{I} = \{X \subseteq E | x \notin \mathrm{cl}(X - x) \forall x \in X\}$

Definition

A subset X of $E(M)$ is said to be a flat if $\mathrm{cl}\ \mathrm{X} = \mathrm{X}$.

Example

Consider

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

cl $\{1,2\}$=$\{1,2,\}$ $\to$ flat
cl $\{1,3\}$=$\{1,3,5\}$

Equivalent Definitions

Closure

Definitions

A function $\mathrm{cl} : 2^{\mathrm{E}} \to 2^{\mathrm{E}}$ is closure for a matroid $M = (E, \mathcal{I})$ iff

- If $X \subseteq E$ then $X \subseteq \mathrm{cl}X$
- If $X \subseteq Y \subseteq E$, then $\mathrm{cl}X \subseteq \mathrm{cl}Y$
- If $X \subseteq E$ then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}X$
- If $X \subseteq E$ and $x \in E$ and $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$ then $x \in \mathrm{cl}(X \cup y)$

Example

Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

cl $\{1,2\}=\{1,2,\} \to$ flat
cl $\{1,3\}=\{1,3,5\}$

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Operations
Dual

Definition

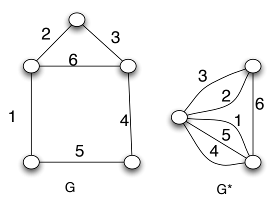
The dual matroid M^* to a matroid M is the matroid with bases that are complements of the bases of M.

$$\mathcal{B}(M^*) = \{E - B | B \in \mathcal{B}(M)\}.$$

- $(M^*)^* = M$ and X is a base if and only if $E - X$ is a cobase.
- Rank function of the dual matroid M^*:

$$r^*(X) = |X| + r(E - X) - r(M).$$

Example



Operations
Contraction, Restriction, Minor

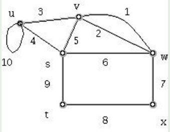
Definitions

Restriction of M to X ($X \subset E$) is the matroid M' with ground set X and independent sets $\mathcal{I}' = \{A \in \mathcal{I} | A \subset X\}$. We denote it as $M|X$.

Definitions

Deletion: $M \backslash T$ is the restriction of M on $E - T$; $M \backslash T = M|(E - T)$

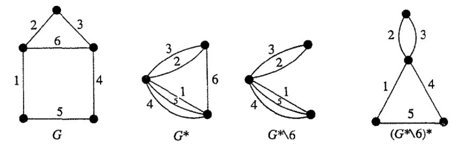
Example



Operations
Contraction, Restriction, Minor

Definitions

Contraction is the dual operation of deletion. $M/T = (M^* \backslash T)^*$



Definition

Minor: If a matroid M' can be obtained by operating combination of restrictions and contractions on a matroid M, then M' is called a minor of M.

Operations

Excluded Minor

Definition

Minor-closed matroids are a class of matroids of which every minor of a member is also in the class.

- Minor-closed classes of matroids have excluded minor characterization, that is, matroids that are not in the class but have all their proper minors in the class.

Example

If a matroid contains $U_{2,4}$ as a minor, it cannot be represented by binary vector space. $U_{2,4}^* = U_{2,4}$.

Outline

Representable Matroids

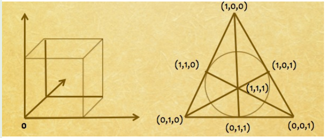
Binary

Definition

A matroid that is isomorphic to a vector matroid, where all elements take values from field $\mathbb{F}$, is called $\mathbb{F}$-representable.

Example

Fano matroid is only representable over field of characteristic 2 (Binary).



Binary matroids have excluded minor: $U_{2,4}$

Representable Matroids

Ternary

Definition

Ternary matroids are those who can be represented over field with characteristic 3.

Regular Matroids

- Representable over every field
- Both Binary and Ternary

Example

$U_{2,k}$ is $\mathbb{F}$-representable if and only if $|\mathbb{F}| \geq k-1$. $|U_{2,4}|$ is 3-representable (Ternary).

$$B = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Ternary matroids have excluded minor: $U_{2,5}, U_{3,5}, F_7$ *and* F_7^*.

Representable Matroids
Graphic Matroid

Definition

In a graph $G = (V, E)$, let $\mathcal{C} \subseteq 2^E$ be the sets of the graph cycles. Then $\mathcal{C}$ forms a set of circuits for a matroid with ground set E. The matroid $M(G)$ derived in this manner is called a cycle matroid (Graphic Matroid).

Excluded minors for graphic matroids:
$U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})$

Example

K_5 matroid and $K_{3,3}$ matroid:

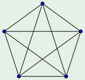


Figure: K_5 matroid

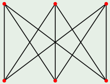
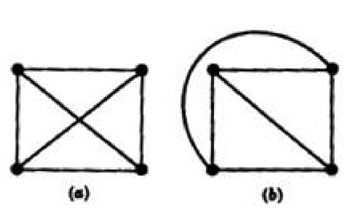


Figure: $K_{3,3}$ matroid

Representable Matroids
Cographic and Planar

Definition
A matroid is called cographic if its dual is graphic.
Ex: $M^*(K_5)$ and $M^*(K_{3,3})$

Definitions
A both graphic and cographic matroid is called planar, isomorphic to a cycle matroid derived from a planar. In a planar, all edges can be drawn on a plane without intersections.

Example


Excluded minors for cographic: $U_{2,4}, F_7, F_7^*, M(K_5), M(K_{3,3})$
Excluded minors for planar: those excluded minors of graphic and cographic

Outline

Excluded Minors

Various Classes of Representable Matroids

Excluded Minors:

Matroid Class	Excluded Minor(s)
Binary	$U_{2,4}$
Ternary	$U_{2,5}, U_{3,5}, F_7, F_7^*$
Regular	$U_{2,4}, F_7, F_7^*$
Graphic	$U_{2,4}, F_7, F_7^*, K_5^*, K_{3,3}^*$
Cographic	$U_{2,4}, F_7, F_7^*, K_5, K_{3,3}$
Planar	$U_{2,4}, F_7, F_7^*, K_5, K_{3,3}, K_5^*, K_{3,3}^*$

Outline

Relationships

Relationships between various classes of matroids:

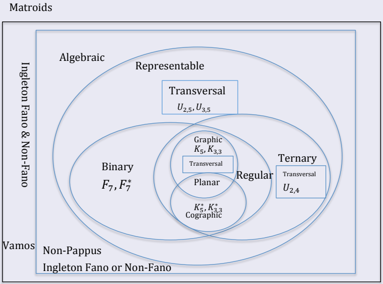


Figure: Relationships between various classes matroids: $U_{2,4}$, F_7, etc. in the graph are examples of the matroid class in which they lie. If an example is not contained in one class, it is called an excluded minor of this class.

Summary

Today's talk

1. Some Equivalent Definitions of Matroids: Sets, Base, Circuit, Rank, Closure
2. Operations of Matroids: Dual, Restriction, Contraction, Minor
3. Representable Matroids: Binary, Ternary, Regular, Graphic, Cographic, Planar
4. Excluded Minors and relationships between various classes of matroids

Q&A

Thank you!

Next: Matroid, Network and Coding.

James Oxley, Matroid Theory, Oxford University Press, 2011