Zero-Sum Two Person Games

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Article Outline

Introduction

Conflicts are an inevitable part of human existence. This is a consequence of the competitive stances of greed and the scarcity of resources, which are rarely balanced without open conflict. Epic poems of the Greek, Roman, and Indian civilizations which document wars between nation-states or clans reinforce the historical legitimacy of this statement. It can be deduced that domination is the recurring theme in human conflicts. In a primitive sense this is historically observed in the domination of men over women across cultures while on a more refined level it can be observed in the imperialistic ambitions of nation-state actors. In modern times, a new source of conflict has emerged on an international scale in the form of economic competition between multinational corporations.

While conflicts will continue to be a perennial part of human existence, the real question at hand is how to formalize mathematically such conflicts in order to have a grip on potential solutions. We can use mock conflicts in the form of parlor games to understand and evaluate solutions for real conflicts. Conflicts are unresolvable when the participants have no say in the course of action. For example one can lose interest in a parlor game whose entire course of action is dictated by chance. Examples of such games are Chutes and Ladders, Trade, Trouble etc. Quite a few parlor games combine tactical decisions with chance moves. The game Le Her and the game of Parcheesi are typical examples. An outstanding example in this category is the game of backgammon, a remarkably deep game. In chess, the player who moves first is usually determined by a coin toss, but the rest of the game is determined entirely by the decisions of the two players. In such games, players make strategic decisions and attempt to gain an advantage over their opponents.

A game played by two rational players is called zero-sum if one player's gain is the other player's loss. Chess, Checkers, Gin Rummy, Two-finger Morra, and Tic-Tac-Toe are all examples of zero-sum two-person games. Business competition between two major airlines, two major publishers, or two major automobile manufacturers can be modeled as a zero-sum two-person games (even if the outcome is not precisely zero-sum). Zero-sum games can be used to construct Nash equilibria in many dynamic non-zero-sum games [64].

Games with Perfect Information

Emptying a Box

Example 1 A box contains 15 pebbles. Players I and II remove between one and four pebbles from the box in alternating turns. Player I goes first, and the game ends when all pebbles have been removed. The player who empties the box on his turn is the winner, and he receives \$1 from his opponent.

The players can decide in advance how many pebbles to remove in each of their turn. Suppose a player finds x pebbles in the box when it is his turn. He can decide to remove 1, 2, 3 or at most 4 pebbles. Thus a *strategy* for a player is any function f whose domain is $X = \{1, 2 \ldots, 15\}$ and range is $R \subseteq \{1, 2, 3, 4\}$ such that $f(x) \leq \min(x, 4)$. Given strategies f, g for players I and II respectively, the game evolves by executing the strategies decided in advance. For example if, say

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd}, \end{cases}$$

$$g(x) = \begin{cases} 3 & \text{if } x \geq 3 \\ x & \text{otherwise}. \end{cases}$$

The alternate depletions lead to the following scenario

move by	*I*	*II*	*I*	*II*	*I*	*II*	*I*	*II*
removes	1	3	1	3	1	3	1	2
leaving	14	11	10	7	6	3	2	0 .

In this case the winner is Player II. Actually in his first move Player II made a bad move by removing 3 out of 14. Player I could have exploited this. But he did not! Though he made a good second move, he reverted back to his naive strategy and made a bad third move. The question is: Can

Player II ensure victory for himself by intelligently choosing a suitable strategy? Indeed Player II can win the game with any strategy satisfying the conditions of g^* where

$$g^*(x) = \begin{cases} 1 & \text{if } x-1 \quad \text{is a multiple of 5} \\ 2 & \text{if } x-2 \quad \text{is a multiple of 5} \\ 3 & \text{if } x-3 \quad \text{is a multiple of 5} \\ 4 & \text{if } x-4 \quad \text{is a multiple of 5}\,. \end{cases}$$

Since the game starts with 15 pebbles, Player I must leave either 14 or 13 or 12, or 11 pebbles. Then Player II can in his turn remove 1 or 2 or 3 or 4 pebbles so that the number of pebbles Player I finds is a multiple of 5 at the beginning of his turn. Thus Player II can leave the box empty in the last round and win the game.

Many other combinatorial games could be studied for optimal strategic behavior. We give one more example of a combinatorial game, called the *game of Nim* [13].

Nim Game

Example 2 Three baskets contain 10, 11, and 16 oranges respectively. In alternating turns, Players I and II choose a non-empty basket and remove at least one orange from it. The player may remove as many oranges as he wishes from the chosen basket, up to the number the basket contains. The game ends when the last orange is removed from the last non-empty basket. The player who takes the last orange is the winner.

In this game as in the previous example at any stage the players are fully aware of what has happened so far and what moves have been made. The full history and the state of the game at any instance are known to both players. Such a game is called a game with *perfect information*. How to plan for future moves to one's advantage is not at all clear in this case. Bouton [13] proposed an ingenious solution to this problem which predates the development of formal game theory.

His solution hinges on the binary representation of any number and the inequality that $1+2+4+\ldots+2^n < 2^{n+1}$. The numbers 10, 11, 16 have the binary representation

$$\begin{array}{rcl} \text{Number} & & \text{Binary representation} \\ 10 & = & 1010 \\ 11 & = & 1011 \\ 16 & = & 10000 \\ \text{Column totals} & & \\ \text{(in base 10 digits)} & = & 12021\,. \end{array}$$

Bouton made the following key observations:

1. If at least one column total is an odd number, then the player who is about to make a move can choose one basket and by removing *a suitable number* of oranges leave all column totals even.
2. If at least one basket is nonempty and if all column totals are even, then the player who has to make a move will end up leaving an odd column total.

By looking for the first odd column total from the left, we notice that the basket with 16 oranges is the right choice for Player I. He can remove the left most 1 in the binary expansion of 16 and change all the other binary digits to the right by 0 or 1. The key observation is that the new number is strictly less than the original number. In Player I's move, at least one orange will be removed from a basket. Furthermore, the new column totals can all be made even. If an original column total is even we leave it as it is. If an original column total is odd, we make it even by making any 1 a 0 and any 0 a 1 in those cases which correspond to the basket with 16 oranges. For example the new binary expansion corresponds to removing all but 1 orange from basket 3. We have

$$\begin{array}{rcl} 10 & = & 1010 \\ 11 & = & 1011 \\ 1 & = & 0001 \\ \text{Column totals in base 10} & = & 2022\,. \end{array}$$

In the next move, no matter what a player does, he has to leave one of the 1's a 0 and the column total in that column will be odd and the move is for Player I. Thus Player I will be the first to empty the baskets and win the game.

For the game of Nim we found a constructive and explicit strategy for the winner regardless of any action by the opponent. Sometimes one may be able to assert who should be the winner without knowing any winning strategy for the player!

Definition 3 A zero-sum two person game has perfect information if, at each move, both players know the complete history so far.

There are many variations of nim games and other combinatorial games like Chess and Go that exploit the combinatorial structure of the game or the end games to develop winning strategies. The classic monographs on combinatorial game theory is by Berlekamp, Conway, and Guy [8] on Winning Ways for your Mathematical Plays, whose mathematical foundations were provided by Conway's earlier book On Numbers and Games. These are often characterized by sequential moves by two players and the outcome is either a win or lose kind. Since the entire history of past moves is common knowledge, the main thrust is in developing winning strategies for such games.

Definition 4 A zero-sum two person game is called a win-lose game if there are no chance moves and the final outcome is either Player I wins or loses (Player II wins) the game. (In other words, there is no way for the game to end in a tie.)

The following is a fundamental theorem of Zermelo [70].

Theorem 5 *Any zero-sum two person perfect information win-lose game Γ with finitely many moves and finitely many choices in each move has a winner with an optimal winning strategy.*

Proof Let Player I make the first move. Then depending on the choices available, the game evolves to a new set of subgames which on their own right are also win-lose games of perfect information. Among these subgames the one with the longest play will have fewer moves than the original game. By an induction on the length of the longest play, we can find a winner with a winning strategy, one for each subgame. Each player can develop good strategies for the original game as follows. Suppose the subgames are $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. Now among these subgames, let Γ_s be a game where Player I can ensure a victory for himself, no matter what Player II does in the subgame. In this case, Player I can determine at the very beginning, the right choice of action which leads to the subgame Γ_s. A good strategy for Player I is simply the choice s in the first move followed by his good strategy in the subgame Γ_s. Player II's strategy for the original game is simply a k-tuple of strategies, one for each subgame. Player II must be ready to use an optimal strategy for the subgame Γ_r in case the first move of Player I leads to playing Γ_r, which is favorable to Player II. Suppose no subgame Γ_s has a winning strategy for Player I. Then Player II will be the winner in each subgame. To achieve this, Player II must use his winning strategy in each subgame they are lead to. Such a k-tuple of winning strategies, one for each subgame, is a winning strategy for Player II for the original game Γ. □

The Game of Hex

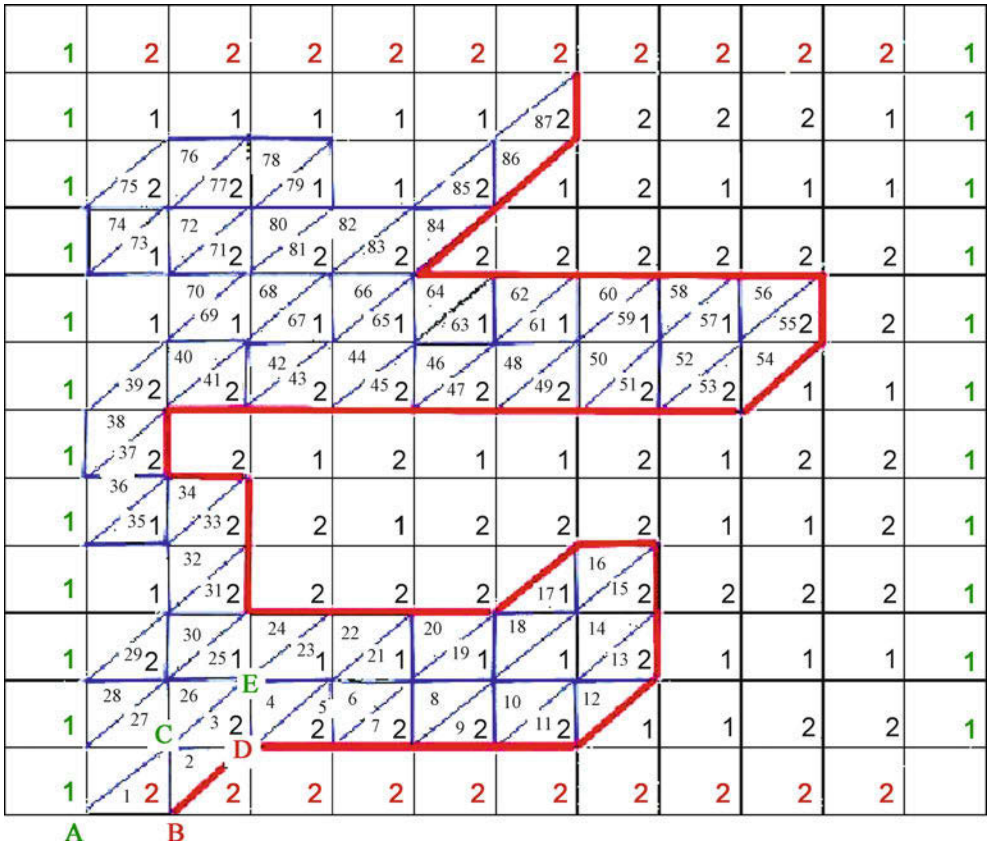
An interesting win-lose game was made popular by John Nash in late forties among Princeton graduate students. While the original game is aesthetically pleasing with its hexagonal tiles forming a 10×10 rhombus, it is more convenient to use the following equivalent formulation for its mathematical simplicity. The version below is extendable to multi person games and is useful for developing important algorithms [28].

Let B_n be a square board consisting of lattice points $\{(i, j)\colon 1 \le i \le n, 1 \le j \le n\}$. The game involves occupation of unoccupied vertices by players I and II. The board is enlarged with a frame on all sides. The frame $\mathbf{F}$ consists of lattice points $\mathbf{F} = \{(i, j)\colon 0 \le i \le n+1, 0 \le j \le n+1$ where either $i = 0$ or $n+1$, or $j = 0$ or $n+1\}$. The frame on the west side $W = \{(i, j)\colon i = 0\} \cap \mathbf{F}$ and the frame on the east side $E = \{(i, j)\colon i = n+1\} \cap \mathbf{F}$ are reserved for Player I. Similarly the frame on the south side $S = \{(i, j)\colon 0 < i < n+1, j = 0\} \cap \mathbf{F}$ and frame on the north side $N = \{(i, j)\colon (0 < i < n+1, j = n+1)\} \cap \mathbf{F}$ are reserved for Player II. Two lattice points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ are called *adjacent vertices* iff either $x_1 \le x_2, y_1 \le y_2$ or $x_1 \ge x_2, y_1 \ge y_2$ and $\max(|x_1 - x_2|, |y_1 - y_2|) = 1$. For example the lattice points $(4, 10)$ and $(5, 11)$ are adjacent while $(4, 10)$ and $(5, 9)$ are not adjacent. Six vertices are adjacent to any interior lattice point of the Hex board B_n while lattice points on the frame will have fewer than six adjacent vertices.

The game is played as follows: Players I and II, in alternate turns, choose a vertex from the available set of unoccupied vertices. The aim of Player I is to occupy a bridge of adjacent vertices that links a vertex on the west boundary with a vertex on the east boundary. Player II has a similar objective to connect the north and south boundary with a bridge.

Theorem 6 *The game of Hex can never end in a draw. For any $T \subseteq B_n$ occupied by Player I and the complement T^c occupied by Player II, either T contains a winning bridge for Player I or T^c contains a winning bridge for Player II. Further only one can have a winning bridge.*

Proof We label any vertex with 1 or 2 depending on who (Player I or Player II) occupies the vertex. Consider triangles Δ formed by vertices that are mutually adjacent to each other. Two such triangles are called *mates* if they share a common side. Either all the 3 vertices of the triangle are occupied by one player or two vertices by one player and the third by the other player. For example if $P = (x_1, y_1)$, $Q = (x_2, y_2)$, $R = (x_3, y_3)$ are adjacent to each other, and if P, Q, R are occupied by say, I, II, and I, they get the labels 1, 2 and 1 respectively. The triangle has exactly 2 sides (PQ and QR) with vertices labeled 1 and 2. The algorithm described below involves entering such a triangle via one side with vertex labels 1 and 2 and exiting via the other side with vertex labels 1 and 2. Suppose we start at the south west corner triangle Δ_0 (in the above figure) with vertex $A = (0, 0)$ occupied by player I (labeled 1), $B = (1, 0)$ occupied by player II (labeled 2), and suppose $C = (1, 1)$ is occupied by, player I (labeled 1). Since we want to stay inside the framed Hex board, the only way to exit Δ_0 via a side with vertices labeled 1 and 2 is to exit via BC. We move to the unique mate triangle Δ_1 which shares the common side BC which



Zero-Sum Two Person Games, Figure 1
Hex path via mate triangles

has vertex labels 1 and 2. The mate triangle Δ_1 has vertices $(1,0)$, $(1,1)$, and $(1,0)-(0,0)+(1,1)=(2,1)$. Suppose $D=(2,1)$ is labeled 2, then we exit via the side CD to the mate triangle with vertices C, D, and $E=(1,1)-(1,0)+(2,1)=(2,2)$. Each time we find the player of the new vertex with his label, we drop out the other vertex of the same player from the current triangle and move into the new mate triangle. In each iteration there is exactly one new mate triangle to move into. Since in the initial step we had a unique mate triangle to move into from Δ_0, there is no way for the algorithm to reenter a mate triangle visited earlier. This process must terminate at a vertex on the North or East boundary. One side of these triangles will all have the same label forming a bridge which joins the appropriate boundaries and forms a winning path. The winning player's bridge will obstruct the bridge the losing player attempted to complete. The game of Hex and its winning strategy is a powerful tool in developing algorithms for computing approximate fixed points. Hex is an example of a game where we do know that the first player can win, but we don't know how (for a sufficiently large board). □

Approximate Fixed Points

Let I^2 be the unit square $0 \le x, y \le 1$. Given any continuous function: $\mathbf{f}=(f_1,f_2)\colon I^2 \to I^2$, Brouwer's fixed point theorem asserts the existence of a point (x^*, y^*) such that $\mathbf{f}(x^*, y^*)=(x^*, y^*)$. Our Hex path building algorithm due to Gale [28] gives a constructive approach to locating an approximate fixed point.

Given $\epsilon > 0$, by uniform continuity we can find a $\delta > \frac{1}{n} > 0$ such that if (i,j) and (i',j') are adjacent vertices of a Hex board B_n, then

$$\begin{aligned} \left| f_1\left(\frac{i}{n},\frac{j}{n}\right) - f_1\left(\frac{i'}{n},\frac{j'}{n}\right)\right| &\le \epsilon\,, \\ \left| f_2\left(\frac{i}{n},\frac{j}{n}\right) - f_2\left(\frac{i'}{n},\frac{j'}{n}\right)\right| &\le \epsilon\,. \end{aligned} \tag{1}$$

Consider the the 4 sets:

$$H^+ = \left\{(i,j)\in B_n\colon f_1\left(\frac{i}{n},\frac{j}{n}\right) - \frac{i}{n} > \epsilon\right\}, \tag{2}$$

$$H^- = \left\{(i,j)\in B_n\colon f_1\left(\frac{i}{n},\frac{j}{n}\right) - \frac{i}{n} < -\epsilon\right\}, \tag{3}$$

$$V^{+} = \left\{ (i, j) \in B_n \colon f_2\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{j}{n} > \epsilon \right\}, \tag{4}$$

$$V^{-} = \left\{ (i, j) \in B_n \colon f_2\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{j}{n} < -\epsilon \right\}. \tag{5}$$

Intuitively the points in H^{+} under $\mathbf{f}$ are moved further to the right (with increased x coordinate) by more than ϵ. Points in V^{-} under $\mathbf{f}$ are moved further down (with decreased y coordinate) by more than ϵ. We claim that these sets cannot cover all the vertices of the Hex board. If it were so, then we will have a winner, say Player I with a winning path, linking the East and West boundary frames. Since points of the East boundary have the highest x coordinate, they cannot be moved further to the right. Thus vertices in H^{+} are disjoint with the East boundary and similarly vertices in H^{-} are disjoint with the West boundary. The path must therefore contain vertices from both H^{+} and H^{-}. However for any $(i, j) \in H^{+}, (i', j') \in H^{-}$ we have

$$f_1\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n} > \epsilon\,,$$

$$-f_1\left(\frac{i'}{n}, \frac{j'}{n}\right) + \frac{i'}{n} > \epsilon\,.$$

Summing the above two inequalities and using (1) we get

$$\frac{i'}{n} - \frac{i}{n} > 2\epsilon\,.$$

Thus the points (i, j) and (i', j') cannot be adjacent and this contradicts that they are part of a connected path. We have a contradiction.

Remark 7 The algorithm attempts to build a winning path and advances by entering mate triangles. Since the algorithm will not be able to cover the Hex board, partial bridge building should fail at some point, giving a vertex that is outside the union of sets $H^{+}, H^{-}, V^{+}, V^{-}$. Hence we reach an approximate fixed point while building the bridge.

An Application of the Algorithm

Consider the continuous map of the unit square into itself given by:

$$f_1(x, y) = \frac{x + \max(-2 + 2x + 6y - 6xy, 0)}{1 + \max(-2 + 2x + 6y - 6xy, 0) + \max(2x - 6xy, 0)}$$

$$f_2(x, y) = \frac{y + \max(2 - 6x - 2y + 6xy, 0)}{1 + \max(2 - 6x - 2y + 6xy, 0) + \max(2y - 6xy, 0)}.$$

Zero-Sum Two Person Games, Table 1
Table giving the Hex building path

(x, y)	$\lvert f_1 - x\rvert$	$\lvert f_2 - y\rvert$	L
(.0, .0)	0	.6667	1
(.1, 0)	.0167	.5833	2
(.1, .1)	.01228	.4667	2
(0, .1)	.0	.53333	1
(.1, .2)	.007	.35	2
(0, .2)	0	.4	1
(.1, .3)	.002	.233	2
(0, .3)	0	.26667	1
(.1, .4)	.238	.116	1
(.2, .4)	.194	.088	1
(.2, .3)	.007	.177	2
(.3, .4)	.153	.033	1
(.3, .3)	.017	.067	2
(.4, .4)	.116	0	1
(.4, .3)	.0296	0	*

With $\epsilon = .05$, we can start with a grid of $\delta = .1$ (hopefully adequate) and find an approximate fixed point. In fact for a spacing of .1 units we have the following iterations. The iterations according to Hex rule passed through the following points with $\lvert f_1(x, y) - x\rvert$ and $\lvert f_2(x, y) - y\rvert$ given by Table 1. Thus the approximate fixed point is $x^{*} = .4$, $y^{*} = .3$.

Extensive Games and Normal Form Reduction

Any game as it evolves can be represented by a rooted tree Γ where the root vertex corresponds to the initial move. Each vertex of the tree represents a particular move of a particular player. The alternatives available in any given move are identified with the edges emanating from the vertex that represents the move. If a vertex is assigned to chance, then the game associates a probability distribution with the the descending edges. The terminal vertices are called plays and they are labeled with the payoff to Player I. In zero-sum games Player II's payoff is simply the negative of the payoff to Player I. The vertices for a player are further partitioned into *information sets*. Information sets must satisfy the following requirements:

- The number of edges descending from any two moves within an information set are same.
- No information set intersects the unique unicursal path from the root to any end vertex of the tree in more than one move.
- Any information set which contains a chance move is a singleton.

We will use the following example to illustrate the extensive form representation:

Example 8 Player I has 3 dice in his pocket. Die 1 is a fake die with all sides numbered one. Die 2 is a fake die with all sides numbered two. Die 3 is a genuine unbiased die. He chooses one of the 3 dice secretly, tosses the die once, and announces the outcome to Player II. Knowing the outcome but not knowing the chosen die, Player II tries to guess the die that was tossed. He pays $1 to Player I if his guess is wrong. If he guesses correctly, he pays nothing to Player I.

The game is represented by the above tree with the root vertex assigned to Player I. The 3 alternatives at this move are to choose the die with all sides 1 or to choose the die with all sides 2 or to choose the unbiased die. The end vertices of these edges descending from the root vertex are moves for chance. The certain outcomes are 1 and 2 if the die is fake. The outcome is one of the numbers 1, …, 6 if the die chosen is genuine. The other ends of these edges are moves for Player II. These moves are partitioned into information sets V_1 (corresponding to outcome 1), V_2 (corresponding to outcome 2), and singleton information sets V_3, V_4, V_5, V_6 corresponding to outcomes 3, 4, 5 and 6 respectively. Player II must guess the die based on the information given. If he is told that the game has reached a move in information set V_1, it simply means that the outcome of the toss is 1. He has two alternatives for each move of this information set. One corresponds to guessing the die is fake and the other corresponds to guessing it is genuine. The same applies to the information set V_2. If the outcome is in $V_3, \dots, V_6$, the clear choice is to guess the die as genuine. Thus a *pure strategy* (master plan) for Player II is to choose a 2-tuple with coordinates taking the values F or G. Here there are 4 pure strategies for Player II. They are: $(F_1, F_2), (F_1, G), (G, F_2), (G, G)$. For example, the first coordinate of the strategy indicates what to guess when the outcome is 1 and the second coordinate indicates what to guess for the outcome 2. For all other outcomes II guesses the die is genuine (unbiased). The payoff to Player I when Player I uses a pure strategy i and Player II uses a pure strategy j is simply the expected income to Player I when the two players choose i and j *simultaneously*. This can as well be represented by a matrix $A = (a_{ij})$ whose rows and columns are pure strategies and the corresponding entries are the expected payoffs. The payoff matrix A given by

$$A = \begin{array}{c} \\ F_1 \\ F_2 \\ G \end{array} \begin{array}{c} \begin{array}{cccc} (F_1,F_2) & (F_1,G) & (G,F_2) & (G,G) \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix} \end{array}$$

is called the normal form reduction of the original extensive game.

Saddle Point

The normal form of a zero sum two person game has a saddle point when there is a row r and column c such that the entry a_{rc} is the smallest in row r and the largest in column c. By choosing the pure strategy corresponding to row r Player I guarantees a payoff $a_{rc} = \min_j a_{rj}$. By choosing column c, Player II guarantees a loss no more than $\max_i a_{ic} = a_{rc}$. Thus row r and column c are good pure strategies for the two players. In a payoff matrix $A = (a_{ij})$ row r is said to *strictly dominate* row t if $a_{rj} > a_{tj}$ for all j. Player I, the maximizer, will avoid row t when it is dominated. If rows r and t are not identical and if $a_{rj} \geq a_{tj}$, then we say that row r *weakly dominates* row t.

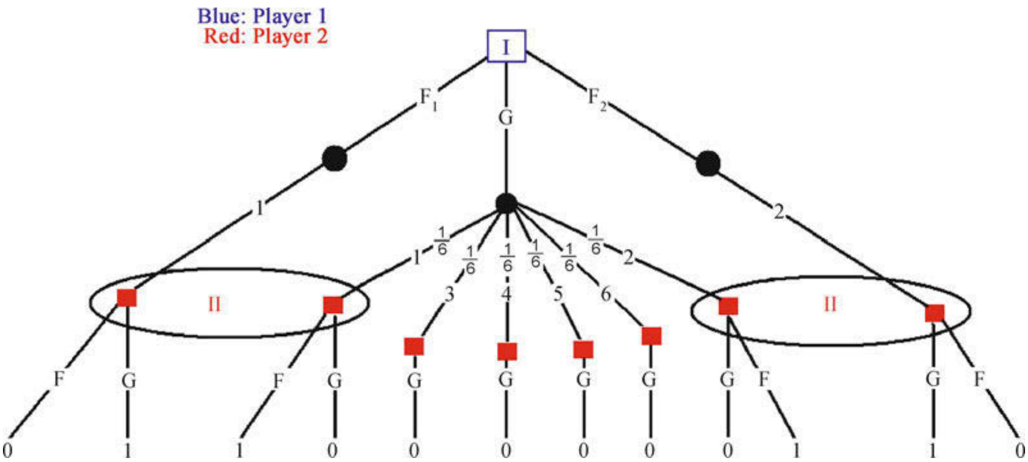
Example 9 Player I chooses either 1 or 2. Knowing player I's choice Player II chooses either 3 or 4. If the total T is odd, Player I wins $\$T$ from Player II. Otherwise Player I pays $\$T$ to Player II.

The pure strategies for Player I are simply σ_1 = choose 1, σ_2 = choose 2. For Player II there are four pure strategies given by: τ_1: choose 3 no matter what I chooses. τ_2: choose 4 no matter what I chooses. τ_3: choose 3 if I chooses 1 and choose 4 if I chooses 2. τ_4: choose 4 if I chooses 3 1 and choose 3 if I chooses 2. This results in a normal form with payoff matrix A for Player I given by:

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} (3,3) & (4,4) & (3,4) & (4,3) \end{array} \\ \begin{bmatrix} -4 & 5 & -4 & 5 \\ 5 & -6 & -6 & 5 \end{bmatrix} \end{array} .$$

Here we can delete column 4 which dominates column 3. We don't have row domination yet. We can delete column 2 as it weakly dominates column 3. Still we have no row domination after these deletions. We can delete column 1 as it weakly dominates column 3. Now we have strict row domination of row 2 by row 1 and we are left with the row 1, column 3 entry = −4. This is a saddle point for this game. In fact we have the following:

Theorem 10 *The normal form of any zero sum two person game with perfect information admits a saddle point. A saddle point can be arrived at by a sequence of row or column deletions. A row that is weakly dominated by another row can be deleted. A column that weakly dominates another column can be deleted. In each iteration we can always find a weakly or strictly dominated row or a weakly or strictly dominating column to be deleted from the current submatrix.*



Zero-Sum Two Person Games, Figure 2
Game tree for a single throw with fake or genuine dice

Mixed Strategy and Minimax Theorem

Zero sum two person games do not always have saddle points in pure strategies. For example, in the game of guessing the die (Example 8) the normal form has no saddle point. Therefore it makes sense for players to choose the pure strategies via a random mechanism. Any probability distribution on the set of all pure strategies for a player is called a *mixed strategy*. In Example 8 a mixed strategy for Player I is a 3-tuple $x = (x_1, x_2, x_3)$ and a mixed strategy for Player II is a 4-tuple $y = (y_1, y_2, y_3, y_4)$. Here x_i is the probability that player I chooses pure strategy i and y_j is the probability that player II chooses pure strategy j. Since the players play independently and make their choices simultaneously, the expected payoff to Player I from Player II is $K(x, y) = \sum_i \sum_j a_{ij} x_i y_j$ where a_{ij} are elements in the payoff matrix A.

We call $K(x, y)$ the mixed payoff where players choose mixed strategies x and y instead of pure strategies i and j. Suppose $x^* = (\frac{1}{8}, \frac{1}{8}, \frac{3}{4})$ and $y^* = (\frac{3}{4}, 0, 0, \frac{1}{4})$. Here x^* guarantees Player I an expected payoff of $\frac{1}{4}$ against any pure strategy j of Player II. By the affine linearity of $K(x^*, y)$ in y it follows that Player I has a guaranteed expectation $= \frac{1}{4}$ against any mixed strategy choice of II. A similar argument shows that Player II can choose the mixed strategy $(\frac{3}{4}, 0, 0, \frac{1}{4})$ which limits his maximum expected loss to $\frac{1}{4}$ against any mixed strategy choice of Player I. Thus

$$\max_x K(x, y^*) = \min_y K(x^*, y) = \frac{1}{4}\,.$$

By replacing the rows and columns with mixed strategy payoffs we have a saddle point in mixed strategies.

Historical Remarks

The existence of a saddle point in mixed strategies for Example 8 is no accident. All finite games have a saddle point in mixed strategies. This important theorem, called the *minimax theorem*, is the very starting point of game theory. While Borel (see under Ville [65]) considered the notions of pure and mixed strategies for zero sum two person games that have symmetric roles for the players, he was able to prove the theorem only for some special cases. It was von Neumann [66] who first proved the minimax theorem using some intricate fixed point arguments. While several proofs are available for the same theorem [29,34,44,48,49,65,69], the proofs by Ville and Weyl are notable from an algorithmic point of view. The proofs by Nash and Kakutani allow immediate extension to Nash equilibrium strategies in many person non zero sum games. For zero-sum two person games, optimal strategies and Nash equilibrium strategies coincide. The following is the seminal minimax theorem for matrix games.

Theorem 11 (von Neumann) *Let $A = (a_{ij})$ be any $m \times n$ real matrix. Then there exists a pair of probability vectors $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ such that for a unique constant v*

$$\sum_{i=1}^{m} a_{ij} x_i \geq v \qquad j = 1, 2, \ldots, n\,,$$
$$\sum_{j=1}^{n} a_{ij} y_j \leq v \qquad i = 1, 2, \ldots, m\,.$$

The probability vectors x, y are called optimal mixed strategies for the players and the constant v is called the value of the game.

H. Weyl [69] gave a complete algebraic proof and proved that the value and some pair of optimal strategies for the two players have all of their coordinates lie in the same ordered subfield as the smallest ordered field containing the payoff entries. Unfortunately his proof was non-constructive. It turns out that the minimax theorem can be proved via linear programming in a constructive way which leads to an efficient computational algorithm *a la* the simplex method [18]. The key idea is to convert the problem to dual linear programming problems.

Solving for Value and Optimal Strategies via Linear Programming

Without loss of generality we can assume that the payoff matrix $A = (a_{ij})_{m \times n} > 0$, that is $a_{ij} > 0$ for all (i, j). Thus we are looking for some v such that:

$$v = \min v_1 \tag{6}$$

such that

$$\sum_{j=1}^{n} a_{ij} y_j \leq v_1 \,, \tag{7}$$

$$y_1, \ldots, y_n \geq 0 \,, \tag{8}$$

$$\sum_{j=1}^{n} y_j = 1 \,. \tag{9}$$

Since the payoff matrix is positive, any v_1 satisfying the constraints above will be positive, so the problem can be reformulated as

$$\max \frac{1}{v_1} = \max \sum_{j=1}^{n} \eta_j \tag{10}$$

such that

$$\sum_{j=1}^{n} a_{ij} \eta_j \leq 1 \quad \text{for all } j \,, \tag{11}$$

$$\eta_j \geq 0 \quad \text{for all } j \,. \tag{12}$$

With $A > 0$, the η_j's are bounded. The maximum of the linear function $\sum_j \eta_j$ is attained at some extreme point of the convex set of constraints (11) and (12). By introducing nonnegative slack variables $s_1, s_2, \ldots, s_m$ we can replace the inequalities (11) by equalities (13). The problem reduces to

$$\max \sum_{j=1}^{n} \eta_j \tag{13}$$

subject to

$$\sum_{j=1}^{n} a_{ij} \eta_j + s_i = 1 \,, \quad i = 1, 2, \ldots, m \,, \tag{14}$$

$$y_j \geq 0 \,, \quad j = 1, 2, \ldots, n \,, \tag{15}$$

$$s_i \geq 0 \,, \quad i = 1, 2, \ldots, m \,. \tag{16}$$

Of the various algorithms to solve a linear programming problem, the simplex algorithm is among the most efficient. It was first investigated by Fourier (1830). But no other work was done for more than a century. The need for its industrial application motivated active research and lead to the pioneering contributions of Kantarowich [1939] (see a translation in Management Science [35]) and Dantzig [18]. It was Dantzig who brought out the earlier investigations of Fourier to the forefront of modern applied mathematics.

Simplex Algorithm Consider our linear programming problem above. Any solution $\eta = (y_1, \ldots, y_n), s = (s_1, \ldots, s_m)$ to the above system of equations is called a feasible solution. We could also rewrite the system as

$$\eta_1 C^1 + \eta_2 C^2 + \cdots + \eta_n C^n + s_1 e^1 + s_2 e^2 + s_m e^m = \mathbf{1}$$
$$\eta_1, \eta_2, \ldots, \eta_n, s_1, s_2, \ldots, s_m \geq 0 \,.$$

Here $C^j, j = 1 \ldots, n$ are the columns of the matrix A and e^i are the columns of the $m \times m$ identity matrix. The vector $\mathbf{1}$ is the vector with all coordinates unity. With any extreme point $(\eta, s) = (\eta_1, \eta_2, \ldots, \eta_n, s_1, \ldots, s_m)$ of the convex polyhedron of feasible solutions one can associate with it a set of m linearly independent columns, which form a basis for the column span of the matrix (A, I). Here the coefficients η_j and s_i are equal to zero for coordinates *other* than for the specific m linearly independent columns. By slightly perturbing the entries we can assume that any extreme point of feasible solutions has *exactly m* positive coordinates. Two extreme feasible solutions are called *adjacent* if the associated bases differ in exactly one column.

The key idea behind the simplex algorithm is that an extreme point $P = (\eta^*, s^*)$ is an optimal solution if and only if there is no adjacent extreme point Q for which the objective function has a higher value. Thus when the algorithm is initiated at an extreme point which is not optimal, there must be an adjacent extreme point that strictly improves the objective function. In each iteration, a column from outside the basis replaces a column in the current basis corresponding to an adjacent extreme point. Since there are $m + n$ columns in all for the matrix (A, I), and in each iteration we have strict improvement by our nondegeneracy assumption on the extreme points, the procedure must terminate in a finite number of steps resulting in an optimal solution.

Example 12 Players I and II simultaneously show either 1 or 2 fingers. If T is the total number of fingers shown then Player I receives from Player II \$$T$ when T, is odd and loses \$$T$ to Player II when T is even.

The payoff matrix is given by

$$A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}.$$

Add 5 to each entry to get a new payoff matrix with all entries strictly positive. The new game is strategically same as A.

$$\begin{bmatrix} 3 & 8 \\ 8 & 1 \end{bmatrix}.$$

The linear programming problem is given by

$$\max 1.y_1 + 1.y_2 + 0.s_1 + 0.s_2$$

such that

$$\begin{bmatrix} 3 & 8 & 1 & 0 \\ 8 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can start with the trivial solution $(0, 0, 1, 1)^{\mathrm{T}}$. This corresponds to the basis e^1, e^2 with $s_1 = s_2 = 1$. The value of the objective function is 0. If we make $y_2 > 0$, then the value of the objective function can be increased. Thus we look for a solution to

$$s_1 = 0\,, \quad s_2 > 0\,, \quad y_2 > 0$$

satisfying the constraints

$$\begin{aligned} 8y_2 + 0s_2 &= 1 \\ y_2 + s_2 &= 1 \end{aligned}$$

or to

$$s_2 = 0\,, \quad s_1 > 0\,, \quad y_2 > 0$$

satisfying the constraints

$$\begin{aligned} 8y_2 + s_1 &= 1 \\ y_2 + 0s_1 &= 1 \end{aligned}$$

Notice that $y_2 = \frac{1}{8}, s_2 = \frac{7}{8}$ is a solution to the first system and that the second system has no nonnegative solution. The value of the objective function at this extreme solution is $\frac{1}{8}$. Now we look for an adjacent extreme point. We find that $y_1 = \frac{7}{61}, y_2 = \frac{5}{61}$ is such a solution. The procedure terminates because no adjacent solution with $y_1 > 0, s_1 > 0$ or $y_2 > 0, s_1 > 0$ or $y_1 > 0, s_2 > 0$ or $y_2 > 0, s_2 > 0$ if any has higher objective function value. The algorithm terminates with the optimal value of $\frac{1}{v_1} = \frac{12}{61}$. Thus the value of the modified game is $\frac{61}{12}$, and the value of the original game is $\frac{61}{12} - 5 = \frac{1}{12}$. A good strategy for Player II is obtained by normalizing the optimal solution of the linear program, it is $\eta_1 = \frac{7}{12}, \eta_2 = \frac{5}{12}$. Similarly, from the dual linear program we can see that the strategy $\xi_1 = \frac{7}{12}, \xi_2 = \frac{5}{12}$ is optimal for Player I.

Fictitious Play

Though optimal strategies are not easily found, even naive players can learn to steer their average payoff towards the value of the game from past plays by certain iterative procedures. This learning procedure is known as *fictitious play*. The two players make their next choice under the assumption that the opponent will continue to choose pure strategies at the same frequencies as what he/she did in the past. If $x^{(n)}$, $y^{(n)}$ are the empirical mixed strategies used by the two players in the first n rounds, then in round $n+1$ Player I pretends that Player II will continue to use $y^{(n)}$ in the future and selects any row i^* such that

$$\sum_j a_{i^* j} y_j^{(n)} = \max_i \sum_j a_{ij} y_j^{(n)}\,.$$

The new empirical mixed strategy is given by

$$x^{(n+1)} = \frac{1}{n+1} I_{i^*} + \frac{n}{n+1} x^{(n)}\,.$$

(Here I_{i^*} is the degenerate choice of pure strategy i^*.) This intuitive learning procedure was proposed by Brown [14] and the following convergence theorem was proved by Robinson [56].

Theorem 13

$$\lim_n \min_j \sum_i a_{ij} x_i^{(n)} = \lim_n \max_i \sum_j a_{ij} y_j^{(n)} = v\,.$$

We will apply the fictitious play algorithm to the following example and get a bound on the value.

Example 14 Player I picks secretly a card of his choice from a deck of three cards numbered 1, 2, and 3. Player II proceeds to guess player I's choice. After each guess player I announces player II's guess as *"High"*, *"Low"* or *"Correct"* as the case may be. The game continues till player II guesses player I's choice correctly. Player II pays to Player I \$$N$ where N is the number of guesses he made.

Zero-Sum Two Person Games, Table 2

Row choices	Total so far					Column choices	Total so far		
R_1	**1**	1	2	2	3	C_1	1	2	**3**
R_3	4	**3**	4	3	4	C_2	2	**5**	5
R_2	6	6	**5**	6	6	C_3	4	6	**7**
R_3	9	8	**7**	7	7	C_3	6	7	**9**
R_3	12	10	9	**8**	8	C_4	8	**10**	10
R_2	15	12	11	**9**	9	C_4	10	**13**	11
R_2	17	15	12	12	**11**	C_5	13	**15**	12
R_2	19	18	**13**	15	13	C_3	15	**16**	14
R_2	21	21	**14**	18	15	C_3	**17**	17	16
R_1	22	22	**16**	20	18	C_3	**19**	18	18

The payoff matrix is given by

$$\begin{array}{cc} & \begin{array}{ccccc} (1,2) & (1,3) & (2) & (3,1) & (3,2) \end{array} \\ A = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left(\begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 1 & 3 & 2 \\ 3 & 2 & 2 & 1 & 1 \end{array} \right) . \end{array}$$

Here the row labels are possible cards chosen by Player I, and the column labels are pure strategies for Player II. For example, the pure strategy $(1, 3)$ for Player II, means that 1 is the first guess and if 1 is incorrect then 3 is the second guess. The elements of the matrix are payoffs to Player I. We can use cumulative total, instead average for the players to make their next choice of row or column based on the totals. We choose the row or column with the least index in case more than one row or one column meets the criterion. The total for the first 10 rounds using fictitious play is given in Table 2.

The bold entries give the approximate lower and upper bounds for the total payoff in 10 rounds giving $1.6 \leq v \leq 1.9$.

Remark 15 Fictitious play is known to have a very poor rate of convergence to the value. While it works for all zero sum two person games, it fails to extend to Nash equilibrium payoffs in bimatrix games even when the game has a unique Nash equilibrium. It extends only to some very special classes like 2×2 bimatrix games and to the so called potential games.

(See Miyasawa [1961], Shapley [1964], Monderer and Shapley [46], Krishna and Sjoestrom [41], and Berger [7]).

Search Games

Search games are often motivated by military applications. An object is hidden in space. While the space where the object is hidden is known, the exact location is unknown. The search strategy consists of either targeting a single point of the space and paying a penalty when the search fails or continue the search till the searcher gets closer to the hidden location. If the search consists of many attempts, then a pure strategy is simply a function of the search history so far. Example 14 is a typical search game. The following are examples of some simple search games that have unexpected turns with respect to the value and optimal strategies.

Example 16 A pet shop cobra of known length $t < 1$ escapes out of the shop and has settled in a nearby tree somewhere along a particular linear branch of unit length. Due to the camouflage, the exact location $[x, x + t]$ where it has settled on the branch is unknown. The shop keeper chooses a point y of his choice and aims a bullet at the point y. In spite of his 100% accuracy the cobra will escape permanently if his targeted point y is outside the settled location of the cobra.

We treat this as a game between the cobra (Player I) and the shop keeper (Player II). Let the probability of survival be the payoff to the cobra. Thus

$$K(x, y) = \begin{cases} 1 & \text{if } y < x\,, \text{ or } y > x + t \\ 0 & \text{otherwise}\,. \end{cases}$$

The pure strategy spaces are $0 \leq x \leq 1 - t$ for the snake and $0 \leq y \leq 1$ for the shop keeper. It can be shown that the game has no saddle point and has optimal mixed strategies. The value function $v(t)$ is a discontinuous function of t. In case $\frac{1}{t}$ is an integer n then, a good strategy for the snake is to hide along $[0, t]$, or $[t, 2t]$ or $[(n-1)t, 1]$ chosen with equal chance. In this case the optimal strategy for the shop keeper is to choose a random point in $[0, 1]$. The value is $1 - \frac{1}{n}$. In case $\frac{1}{t}$ is a fraction, let $n = [\frac{1}{t}]$ then the optimal strategy for the snake is to hide along $[0, t]$, or $[t, 2t], \dots$ or $[(n-1)t, nt]$. An optimal strategy for the shop keeper is to shoot at one of the points $\frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1}$ chosen at random.

Example 17 While mowing the lawn a lady suddenly realizes that she has lost her diamond engagement ring some where in her lawn. She has maximum speed s and will be able to locate the diamond ring from its glitter if she is sufficiently close to, say within a distance ϵ from the ring. What is an optimal search strategy that minimizes her search time.

If we treat Nature as a player against her, she is playing a zero sum two person game where Nature would find pleasure in her delayed success in finding the ring.

Search Games on Trees

The following is an elegant search game on a tree. For many other search games the readers can refer to the monographs by Gal [27] and Alpern and Gal [1]. Also see [55].

Example 18 A bird has to look for a suitable location to build its nest for hatching eggs and protecting them against predator snakes. Having identified a large tree with a single predator snake in the neighborhood, the bird has to further decide where to build its nest on the chosen tree. The chance for the survival of the eggs is directly proportional to the distance the snake travels to locate the nest.

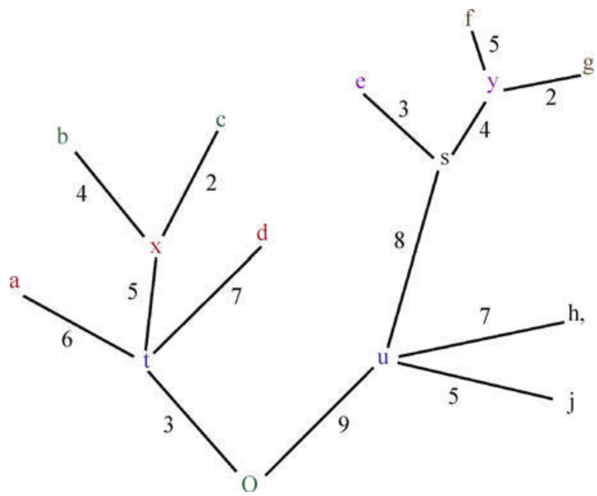
While birds and snakes work out their strategies based on instinct and evolutionary behavior, we can surely approximate the problem by the following zero sum two person search game. Let $T = (X, \mathcal{E})$ be a finite tree with vertex set X and edge set $\mathcal{E}$. Let $O \in X$ be the root vertex. A hider hides an object at a vertex x of the tree. A searcher starts at the root and travels along the edges of the tree such that the path traced covers all the terminal vertices. The search ends as soon as the searcher crosses the hidden location and the payoff to the hider is the distance traveled so far.

By a simple domination argument we can as well assume that the optimal hiding locations are simply the terminal vertices.

Theorem 19 *The search game has value and optimal strategies. The value coincides with the sum of all edge lengths. Any optimal strategy for the hider will necessarily restrict to hide at one of the terminal vertices. Let the least distance traveled to exhaust all end vertices one by one correspond to a permutation σ of the end vertices in the order $w_1, w_2, \ldots, w_k$. Let σ^{-1} be its reverse permutation. Then an optimal strategy for the searcher is to choose one of these two permutations by the toss of a coin. The hider has a unique optimal mixed strategy that chooses each end vertex with positive probability.*

Suppose the tree is a path with root O and a single terminal vertex x. Since the search begins at O, the longest trip is possible only when hider hides at x and the theorem holds trivially. In case the tree has just two terminal vertices besides the root vertex, the possible hiding locations are say, O, x_1, x_2 with edge lengths a_1, a_2. The possible searches are via paths: $O \to x_1 \to O \to x_2$ abbreviated Ox_1Ox_2 or $O \to x_2 \to O \to x_1$, abbreviated Ox_2Ox_1. The payoff matrix can be written as

$$\begin{array}{cc} & \begin{array}{cc} Ox_1Ox_2 & Ox_2Ox_1 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} a_1 & 2a_2 + a_1 \\ 2a_1 + a_2 & a_2 \end{bmatrix} \end{array} .$$



Zero-Sum Two Person Games, Figure 3
Bird trying to hide at a leaf and snake chasing to reach the appropriate leaf via optimal Chinese postman route starting at root *O* and ending at *O*

The value of this game is $a_1 + a_2 =$ sum of the edge lengths. We can use an induction on the number of subtrees to establish the value as the sum of edge lengths. We will use an example to just provide the intuition behind the formal proof.

Given any permutation τ of the end vertices (leaves), of the above tree let P be the shortest path from the root vertex that travels along the leaves in that order and returns to the root. Let the reverse path be P^{-1}. Observe that it will cover all edges twice. Thus if the two paths P and P^{-1} are chosen with equal chance by the snake, the average distance traveled by the snake when it locates the bird's nest at an end vertex will be independent of the particular end vertex. For example along the closed path $O \to t \to d \to t \to a \to t \to x \to b \to x \to c \to x \to t \to O \to u \to s \to e \to s \to y \to f \to y \to g \to y \to s \to u \to h \to u \to j \to u \to O$ the distance traveled by the snake to reach leaf e is $(3 + 7 + 7 + \cdots + 8 + 3) = 74$. If the snake travels along the reverse path to reach e the distance traveled is $(9 + 5 + 5 + \cdots + 5 + 4 + 3) = 66$. For example if it is to reach the vertex d then via path P it is $(3 + 7)$. Via P^{-1} it is to make travel to e and travel from e to d by the reverse path. This is $(66 + 3 + \cdots + 6 + 6 + 7) = 130$. Thus in both cases the average distance traveled is 70. The average distance is the same for every other leaf when P and P^{-1} are used. The optimal Chinese postman route can allow all permutations subject to permuting any leaf of any subtree only among themselves. Thus the subtree rooted at t has leaves a, b, c, d and the subtree rooted at u has leaves e, f, g, h, j. For example while permuting a, b, c, d only among themselves we have the further restriction that between b, c we cannot allow insertion of a or d. For example a, b, c, d and a, d, c, b are acceptable permutations, but not a, b, d, c. It can never be the optimal permuting choice. The same way it applies to the tree rooted at u. For example h, j, e, g, f is part of the optimal Chinese postman route, but h, g, j, e, f is not. We can think of the snake and bird playing the game as follows: The bird chooses to hide in a leaf of the subgame G_t rooted at t or at a leaf of the subgame G_u rooted at u. These leaves exhaust all leaves of the original game. The snake can restrict to only the optimal route of each subgame. This can be thought of as a 2×2 game where the strategies for the two players (bird) and snake are:

Bird:
Strategy 1: Hide optimally in a leaf of G_t,
Strategy 2: Hide optimally in a leaf of G_u.

Snake:
Strategy 1: Search first the leaves of G_t along the optimal Chinese postman route of G_t and then search along the leaves of G_u.
Strategy 2: Search first the leaves of G_u along the optimal Chinese postman route and then search the leaves of G_t along the optimal postman route. The expected outcome can be written as the following 2×2 game. (Here $v(G_t)$, $v(G_u)$ are the values of the subgames rooted at t, u respectively.)

$$\begin{array}{cc} & \begin{array}{cc} G_t G_u & \qquad G_u G_t \end{array} \\ \begin{array}{c} G_t \\ G_u \end{array} & \begin{bmatrix} [3+v(G_t)] & 2[9+v(G_u)]+[3+v(G_t)] \\ 2[3+v(G_t]+[9+v(G_u)] & [9+v(G_u)] \end{bmatrix} \end{array}$$

Observe that the 2×2 game has no saddle point and hence has value $3 + 9 + v(G_t) + v(G_u)$. By induction we can assume $v(G_t) = 24, v(G_u) = 34$. Thus the value of this game is 70. This is also the sum of the edge lengths of the game tree. An optimal strategy for the bird can be recursively determined as follows.

Umbrella Folding Algorithm Ladies, when storing umbrellas inside their handbag shrink the central stem of the umbrella and then the stems around all in one stroke. We can mimic a somewhat similar procedure also for our above game tree. We simultaneously shrink the edges $[xc]$ and $[xb]$ to x. In the next round $\{a, x, d\}$ edges $[a, t], [x, t], [d, t]$ can be simultaneously shrunk to t and so on till the entire tree is shrunk to the root vertex O. We do know that the optimal strategy for the bird when the tree is simply the subtree with root x and with leaves $b,\ c$ is given by $p(b) = \frac{4}{(4+2)},\ p(c) = \frac{2}{(4+2)}$. Now for the subtree with vertex t and leaves $\{a, b, c, d\}$, we can treat this as collapsing the previous subtree to x and treat stem length of the new subtree with vertices $\{t, a, x, d\}$ as though the three stems $[ta], [tx], [td]$ have lengths $6, 5 + (4 + 2), 7$. We can check that for this subtree game the leaves a, x, d are chosen with probabilities $p(a) = \frac{6}{(6+9+7)},\ p(x) = \frac{9}{(6+9+7)},\ p(d) = \frac{7}{(6+9+7)}$. Thus the optimal mixed strategy for the bird for choosing leaf b for our original tree game is to pass through vertices t, x, b and is given by the product $p(t)p(x)p(b)$. We can inductively calculate these probabilities.

Completely Mixed Games and Perron's Theorem on Positive Matrices

A mixed strategy x for player I is called *completely mixed* if it is strictly positive ($x > \mathbf{0}$). A matrix game A is *completely mixed* if and only all optimal mixed strategies for Player I and Player II are completely mixed. The following elegant theorem was proved by Kaplanski [36].

Theorem 20 *A matrix game A with value v is completely mixed if and only if*

1. *The matrix is square.*
2. *The optimal strategies are unique for the two players.*
3. *If $v \neq 0$, then the matrix is nonsingular.*
4. *If $v = 0$, then the matrix has rank $n - 1$ where n is the order of the matrix.*

The theory of completely mixed games is a useful tool in linear algebra and numerical analysis [4]. The following is a sample application of this theorem.

Theorem 21 (Perron 1909) *Let A be any $n \times n$ matrix with positive entries. Then A has a positive eigenvalue with a positive eigenvector which is also a simple root of the characteristic equation.*

Proof Let I be the identity matrix. For any $\lambda > 0$, the maximizing player prefers to play the game A rather than the game $A - \lambda I$. The payoff gets worse when the diagonal entries are reached. The value function $v(\lambda)$ of $A - \lambda I$ is a non-increasing continuous function. Since $v(0) > 0$ and $v(\lambda) < 0$ for large λ we have for some $\lambda_0 > 0$ the value of $A - \lambda_0 I$ is 0. Let y be optimal for player II, then $(A - \lambda_0 I)y \leq \mathbf{0}$ implies $\mathbf{0} < Ay \leq \lambda_0 y$. That is $\mathbf{0}$. Since the optimal y is completely mixed, for any optimal x of player I, we have $(A - \lambda_0 I)x = \mathbf{0}$. Thus $x > \mathbf{0}$ and the game is completely mixed. By (2) and (4) if $(A - \lambda_0 I)u = 0$ then u is a scalar multiple of y and so the eigenvector y is geometrically simple. If $B = A - \lambda_0 I$, then B is singular and of rank $n - 1$. If (B_{ij}) is the cofactor matrix of the singular matrix B then

$\sum_j b_{ij}B_{kj} = 0\,, \quad i = 1, \dots, n$. Thus row k of the cofactor matrix is a scalar multiple of y. Similarly each column of B is a scalar multiple of x. Thus all cofactors are of the same sign and are different from 0. That is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\det(A-\lambda I)\Big|_{\lambda_0} = \sum_i B_{ii} \neq 0\,.$$

Thus λ_0 is also algebraically simple. See [4] for the most general extensions of this theorem to the theorems of Perron and Frobenius and to the theory of M-matrices and power positive and polynomially matrices). $\square$

Behavior Strategies in Games with Perfect Recall

Consider any extensive game Γ where the unique unicursal path from an end vertex w to the root x_0 intersects two moves x and y of say, Player I. We say $x \prec y$ if the the unique path from y to x_0 is via move x. Let $U \ni x$ and $V \ni y$ be the respective information sets. If the game has reached a move $y \in V$, Player I will know that it is his turn and the game has progressed to *some* move in V. The game is said to have *perfect recall* if each player can remember all his past moves and the choices made in those moves. For example if the game has progressed to a move of Player I in the information set V he will remember the specific alternative chosen in any earlier move. A move x is *possible* for Player I with his pure strategy π_1, if for some suitable pure strategy π_2 of Player II, the move x can be reached with positive probability using π_1, π_2. An information set U is *relevant* for a pure strategy π_1, for Player I, if some move $x \in U$ is possible with π_1. Let Π_1, Π_2 be pure strategy spaces for players I and II.

Let $\mu_1 = \{q_{\pi_1,\ \pi_1 \in \Pi_1}\}$ be any mixed strategy for Player I. The information set U for Player I is *relevant for the mixed strategy* μ_1 if for some $q_{\pi_1} > 0$, U is relevant for π_1. We say that the information set U for Player I is not relevant for the mixed strategy μ_1 if for all $q_{\pi_1} > 0,\ U$ is not relevant for π_1. Let

$$S_\nu = \{\pi_1 \colon U \text{ is relevant for } \pi_1 \text{ and } \pi_1(U) = \nu\}\,,$$

$$S = \{\pi_1 \colon U \text{ is relevant for } \pi_1\}\,,$$

$$T = \{\pi_1 \colon U \text{ is not relevant for } \pi_1 \ \text{ and } \pi_1(U) = \nu\}\,.$$

The behavior strategy induced by a mixed strategy pair (μ_1, μ_2) at an information set U for Player I is simply the conditional probability of choosing alternative ν in the information set U, given that the game has progressed to a move in U, namely

$$\beta_1(U,\nu) = \begin{cases} \dfrac{\sum_{\pi_1 \in S_\nu} q_{\pi_1}}{\sum_{\pi_1 \in S} q_{\pi_1}} & \text{if } U \text{ is relevant for } \mu_1\,, \\ \sum_{\pi_1 \in T} q_{\pi_1} & \text{if } U \text{ is not relevant for } \mu_1\,. \end{cases}$$

The following theorem of Kuhn [42] is a consequence of the assumption of perfect recall.

Theorem 22 *Let μ_1, μ_2 be mixed strategies for players I and II respectively in a zero sum two person finite game Γ of perfect recall. Let β_1, β_2 be the induced behavior strategies for the two players. Then the probability of reaching any end vertex w using μ_1, μ_2 coincides with the probability of reaching w using the induced behavior strategy β_1, β_2. Thus in zero-sum two person games with perfect recall, players can play optimally by restricting their strategy choices just to behavior strategies.*

The following analogy may help us understand the advantages of behavior strategies over mixed strategies. A book has 10 pages with 3 lines per page. Someone wants to glance through the book reading just 1 line from each page. A master plan (pure strategy) for scanning the book consists of choosing one line number for each page. Since each page has 3 lines, the number of possible plans is 3^{10}. Thus the set of mixed strategies is a set of dimension $3^{10} - 1$. There is another randomized approach for scanning the book. When page i is about to be scanned choose line 1 with probability x_{i1}, line 2 with probability x_{i2} and line 3 with probability x_{i3}. Since for each i we have $x_{i1} + x_{i2} + x_{i3} = 1$ the dimension of such a strategy space is just 20. Behavior strategies are easier to work with. Further Kuhn's theorem guarantees that we can restrict to behavior strategies in games with perfect recall.

In general if there are k alternatives at each information set for a player and if there are n information sets for the player, the dimension of the mixed strategy space is $k^n - 1$. On the other hand the dimension of the behavior strategy space is simply $n(k-1)$. Thus while the dimension of mixed strategy space grows exponentially the dimension of behavior strategy space grows linearly. The following example will illustrate the advantages of using behavior strategies.

Example 23 Player I has 7 dice. All but one are fake. Fake die F_i has the same number i on all faces $i = 1, \dots, 6$. Die G is the ordinary unbiased die. Player I selects one of them secretly and announces the outcome of a single toss of the die to player II. It is Player II's turn to guess which die was selected for the toss. He gets no reward for correct guess but pays \$1 to Player I for any wrong guess.

Player I has 7 pure strategies while Player II has 2^6 pure strategies. As an example the pure strategy $(F_1, G, G, F_4,$

G, F_6) for Player II is one which guesses the die as fake when the outcome revealed is 1 or 4 or 6, and guesses the die as genuine when the outcome is 2 or 3 or 5. The normal form game is a payoff matrix of size 7×64. For example if G is chosen by Player I, and (F_1, G, G, F_4, G, F_6) is chosen by Player II, the expected payoff to Player I is $\frac{1}{6}[1+0+0+1+0+1] = \frac{1}{2}$. If F_2 is chosen by Player I, the expected payoff is 1 against the above pure strategy of Player II. Now Player II can use the following behavior strategy. If the outcome is i, then with probability q_i he can guess that the die is genuine and with probability $(1-q_i)$ he can guess that it is from the fake die F_i. The expected behavioral payoff to Player I when he chooses the genuine die with probability p_0 and chooses the fake die F_i with probability p_i, $i = 1, \ldots, i = 6$ is given by

$$K(p,q) = p_0 \frac{1}{6} \sum_{i=1}^{6} (1-q_i) + \sum_{i=1}^{6} p_i q_i \,.$$

Collecting the coefficients of q_i's, we get

$$K(p,q) = \sum_{i=1}^{6} q_i \left[p_i - \frac{1}{6} p_0 \right] + p_0 \,.$$

By choosing $p_i - \frac{1}{6}p_0 = 0$, we get $p_1 = p_2 = \ldots, p_6 = \frac{1}{6}p_0$. Thus $p = (\frac{1}{2}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$. For this mixed strategy for Player I, the payoff to Player I is independent of Player II's actions. Similarly, we can rewrite $K(p,q)$ as a function of p_i's for $i = 1, \ldots, 6$ where

$$K(p,q) = \sum_{k \neq 0} p_k \left[q_k - \frac{1}{6} \sum_{r=1}^{6} (1-q_r) \right] + \frac{1}{6} \left(\sum_{k=1}^{6} (1-q_k) \right) .$$

This expression can be made independent of p_i's by choosing $q_i = \frac{1}{2}$, $\quad i = 1, \ldots, 6$. Since the behavioral payoff for these behavioral strategies is $\frac{1}{2}$, the value of the game is $\frac{1}{2}$, which means that Player I cannot do any better than $\frac{1}{2}$ while Player II is able to limit his losses to $\frac{1}{2}$.

Efficient Computation of Behavior Strategies

Introduction

In our above example with one genuine and six fake dice, we used Kuhn's theorem to narrow our search among optimal behavior strategies. Our success depended on exploiting the inherent symmetries in the problem. We were also lucky in our search when we were looking for optimals among equalizers.

From an algorithmic point of view, this is not possible with any arbitrary extensive game with perfect recall. While normal form is appropriate for finding optimal mixed strategies, its complexity grows exponential with the size of the vertex set. The payoff matrix in normal form is in general not a sparse matrix (a sparse matrix is one which has very few nonzero entries) a key issue for data storage and computational accuracies. By sticking to the normal form of a game with perfect recall we cannot take full advantage of Kuhn's theorem in its drastically narrowed down search for optimals among behavior strategies. A more appropriate form for these games is the *sequence form* [63] and *realization probabilities* to be described below. The behavioral strategies that induce the *realization probabilities* grow only linearly in the size of the terminal vertex set. Another major advantage is that the sequence form induces a sparse matrix. It has at most as many non-zero entries as the number of terminal vertices or plays.

Sequence Form

When the game moves to an information set U_1 of say, player I, the perfect recall condition implies that wherever the true move lies in U_1, the player knows the actual alternative chosen in any of the past moves. Let σ_{u_1} denote the sequence of alternatives chosen by Player I in his *past moves*. If no past moves of player I occurs we take $\sigma_{u_1} = \emptyset$. Suppose in U_1 player I selects an action "c" with behavioral probability $\beta_1(c)$ and if the outcome is c the new sequence is $\sigma_{u_1} \cup c$. Thus any *sequence* s_1 for player I is the string of choices in his moves along the partial path from the initial vertex to any other vertex of the tree. Let S_0, S_1, S_2 be the set of all sequences for Nature (via chance moves), Player I and Player II respectively. Given behavior strategies $\beta_0, \beta_1, \beta_2$ Let

$$r_i(s_i) = \prod_{c \in s_i} \beta_i(c) \,, \quad i = 0,1,2 \,.$$

The functions: $r_i \colon S_i :\longrightarrow R \colon i = 0,1,2$ satisfy the following conditions

$$r_i(\emptyset) = 1 \tag{17}$$

$$r_i(\sigma_{u_i}) = \sum_{c \in A(U_i)} r_i(\sigma_{u_i}, c) \,, \quad i = 0,1,2 \tag{18}$$

$$r_i(s_i) \geq 0 \quad \text{for all } s_i \,, \quad i = 0,1,2 \,. \tag{19}$$

Conversely given any such realization functions r_1, r_2 we can define behavior strategies, β_1 say for player I, by

$$\beta_1(U_1, c) = \frac{r_1(\sigma_{u_1} \cup c)}{r_1(\sigma_{u_1})} \qquad \text{for } c \in A(U_1)\,, \text{ and } r_1(\sigma_{u_1}) > 0\,.$$

When $r_1(\sigma_{u_1}) = 0$ we define $\beta_1(U_1, c)$ arbitrarily so that $\sum_{c \in A(U_1)} \beta_1(U_1, c) = 1$. If the terminal payoff to player I at terminal vertex ω is $h(\omega)$, by defining $h(a) = 0$ for all nodes a that are not terminal vertices, we can easily check that the behavioral payoff

$$H(\beta_1, \beta_2) = \sum_{s \in S} h(s) \prod_{i=0}^{2} r_i(s_i)\,.$$

When we work with realization functions $r_i,\ i = 1, 2$ we can associate with these functions the sequence form of payoff matrix whose rows correspond to sequence $s_1 \in S_1$ for Player I and columns correspond to sequence $s_2 \in S_2$ for Player II and with payoff matrix

$$K(s_1, s_2) = \sum_{s_0 \in S_0} h(s_0, s_1, s_2)\,.$$

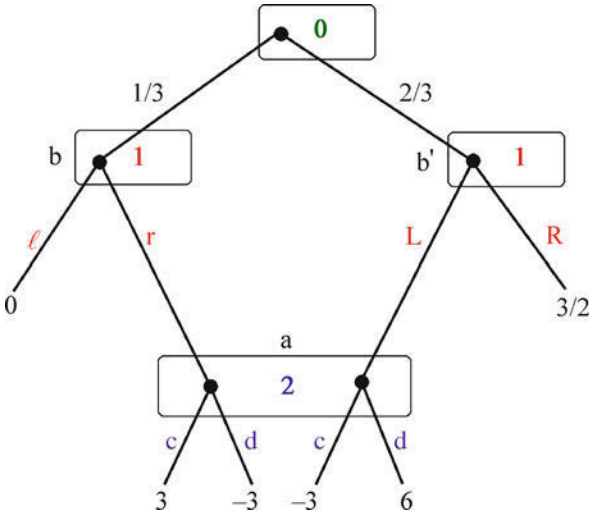
Unlike the mixed strategies we have more constraints on the sequences r_1, r_2 for each player given by the linear constraints above. It may be convenient to denote the sequence functions r_1, r_2 by vectors x, y respectively. The vector x has $|S_1|$ coordinates and vector y has $|S_2|$ coordinates. The constraints on x and y are linear given by $Ex = e, Fy = f$ where the first row is the unit vector $(1, 0, \ldots, 0)$ of appropriate size in both E and F. If $\mathcal{U}_1$ is the collection of information sets for player I then the number of rows in E is $1 + |\mathcal{U}_1|$. Similarly the number of rows in F is $1 + |\mathcal{U}_2|$. Except for the first row, each row has the starting entry as -1 and some 1's and 0's. Consider the following extensive game with perfect recall.

The set of sequences for player I is given by $S_1 = \{\emptyset, l, r, L, R\}$. The set of sequences for player II is given by $S_2 = \{\emptyset, c, d\}$. The sequence form payoff matrix is given by

$$K(s_1, s_2) = A = \begin{bmatrix} * & * & * \\ \mathbf{0} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

The constraint matrices E and F are given by

$$E = \begin{bmatrix} 1 & & & & \\ -1 & 1 & 1 & & \\ -1 & & & 1 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & & \\ -1 & 1 & 1 \end{bmatrix}.$$



Zero-Sum Two Person Games, Figure 4

Since no end vertex corresponds $s_1 = \emptyset$, for Player I, the first row of A is identically 0 and so it is represented by $*$ entries.

We are looking for a pair of vectors x^*, y^* such that y^* is the best reply vector y which minimizes $(x^*\ Ay)$ among all vectors satisfying $Fy = f, y \geq 0$. Similarly the best reply vector is x^* where it maximizes (x, Ay^*) subject to $E^{\mathrm{T}} x = e, x \geq 0$. The duals to these two linear programming problems are

$$\begin{array}{ll} \max\,(f \quad q) & \min\,(e \quad p) \\ \text{such that} & \text{such that} \\ F^{\mathrm{T}} q \leq A^{\mathrm{T}} x^*\,, & Ep \geq Ay^*\,, \\ q \text{ unrestricted.} & p \text{ unrestricted.} \end{array}$$

Since $E^{\mathrm{T}} x^* = e$, and $x^* \geq 0$, $Fy^* = f$, and $y^* \geq 0$ these two problems can as well be viewed as the dual linear programs

Primal: $\max(f \quad q)$
such that

$$\begin{bmatrix} F^{\mathrm{T}} & -A^{\mathrm{T}} \\ 0 & -E^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} q \\ x \end{bmatrix} \begin{bmatrix} \leq & 0 \\ = & e \end{bmatrix}, \qquad \begin{array}{l} x \geq 0\,, \\ q \text{ unrestricted.} \end{array}$$

Dual: $\min(e \quad p)$
such that

$$\begin{bmatrix} F & 0 \\ -A & E \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix} \begin{bmatrix} = & f \\ \geq & 0 \end{bmatrix}, \qquad \begin{array}{l} y \geq 0\,, \\ p \text{ unrestricted.} \end{array}$$

We can essentially prove that:

Theorem 24 *The optimal behavior strategies of a zero sum two person game with perfect recall can be reduced to solv-*

ing for optimal solutions of dual linear programs induced by its sequence form. The linear program has a size which in its sparse representation is linear in the size of the game tree.

General Minimax Theorems

The minimax theorem of von Neumann can be generalized if we notice the possible limitations for extensions.

Example 25 Players I and II choose secretly positive integers i, j respectively. The payoff matrix is given by

$$a_{ij} = \begin{cases} 1 & \text{if } i > j \\ -1 & \text{if } i < j \end{cases}$$

the value of the game does not exist.

The boundedness of the payoff is essential and the following extension holds.

S-games

Given a closed bounded set $S \subseteq \mathbf{R}^m$, let Players I and II play the following game. Player II secretly selects a point $s = (s_1, \ldots, s_m) \in S$. Knowing the set S but not knowing the point chosen by Player II, Player I selects a coordinate $i \in \{1, 2, \ldots, m\}$. Player I receives from Player II an amount s_i.

Theorem 26 *Given that S is a compact subset of $\mathbf{R}^m$, every S-game has a value and the two players have optimal mixed strategies which use at most m pure strategies. If the set S is also convex, Player II has an optimal pure strategy.*

Proof Let T be the convex hull of S. Here T is also compact. Let

$$v = \min_{t \in T} \max_i t_i = \max_i t_i^* . \qquad \square$$

The compact convex set T and the open convex set $G = \{s\colon \max_i s_i < v\}$ are disjoint. By the weak separation theorem for convex sets there exists a $\xi \neq 0$ and constant c such that

$$\begin{aligned} &\text{for all } s \in G\,, \quad (\xi\ , s) \leq c \quad \text{and} \\ &\text{for all } t \in T\,, \quad (\xi\ , t) \geq c\,. \end{aligned}$$

Using the property that $\mathbf{v} = (v, v, \ldots, v) \in \bar{G}$ and $t^* \in T \cap \bar{G}$, we have $(\xi\ , t^*) = c$. For any $u \geq 0, t^* - u \in \bar{G}$. Thus $(\xi\ , t^* - u) \leq c$. That is $(\xi, u) \geq 0$. We can assume ξ is a probability vector, in which case

$$(\xi\ , \mathbf{v}) = v \leq c = (\xi\ , t^*) \leq \max_i t_i^* = v\,.$$

Now Player II has $t^* = \sum_j \mu_j x^j$ a mixed strategy which chooses x^j with probability μ_j. Since t^* is a boundary point of $T = \operatorname{con} S$, by the Caratheodary theorem for convex hulls, the convex combination above involves at most m points of S. Hence the theorem. See [50].

Geometric Consequences

Many geometric theorems can be derived by using the minimax theorem for S-games. Here we give as an example the following theorem of Berge [6] that follows from the theorem on S-games.

Theorem 27 *Let $S_i,\ i = 1 \ldots, m$ be compact convex sets in $\mathbf{R}^n$. Let them satisfy the following two conditions.*

1. *$S = \bigcup_{i=1}^m S_i$ is convex.*
2. *$\bigcap_{i \neq j} S_i \neq \emptyset\,,\ j = 1, 2 \ldots, m$.*

Then $\bigcap_{i=1}^m S_i \neq \emptyset$.

Proof Suppose Player I secretly chooses one of the sets S_i and Player II secretly chooses a point $x \in S$. Let the payoff to I be $d(S_i, x)$ where d is the distance of the point x from the set S_i. By our S-game arguments we have for some probability vector ξ, and mixed strategy $\mu = (\mu_1, \ldots, \mu_m)$

$$\sum_i \xi_i d(S_i, x) \geq v \quad \text{for all } x \in S \tag{20}$$

$$\sum_j \mu_j d(S_i, x^j) \leq v \quad \text{for all } i = 1 \ldots, m\,. \tag{21}$$

Since $d(S_i, x)$ are convex functions, the second inequality (1) implies

$$d\left(S_i, \sum_j \mu_j x^j\right) \leq v\,. \tag{22}$$

$\square$

The game admits a pure optimal $x^\circ = \sum_j \mu_j x^j$ for Player II. We are also given

$$\bigcap_{i \neq j} S_i \neq \emptyset\,, \quad j = 1, 2 \ldots, m\,.$$

For any optimal mixed strategy $\xi = (\xi_1, \xi_2, \xi_m)$ of Player I, if any $\xi_i = 0$ and we choose an $x^* \in \bigcap_{i \neq 1} S_i$, then from (20) we have $0 \geq v$ and thus $v = 0$. When the value v is zero, the third inequality (22) shows that $x^\circ \in \bigcap_i S_i$. If $\xi > 0$, the second inequality (21) will become an equality for all i and we have $d(S_i,\ x^\circ) = v$. But for $x^\circ \in S$, we have $v = 0$ and $x^\circ \in \bigcap_{i=1}^m S_i$. (See Raghavan [53] for other applications.)

Ky Fan–Sion Minimax Theorems

General minimax theorems are concerned with the following problem: Given two arbitrary sets X, Y and a real function $K\colon X \times Y \to \mathbf{R}$, under what conditions on K, X, Y can one assert

$$\sup_{x\in X}\ \inf_{y\in Y} K(x,y) = \inf_{y\in Y}\ \sup_{x\in X} K(x,y)\,.$$

A standard technique for proving general minimax theorems is to reduce the problem to the minimax theorem for matrix games. Such a reduction is often possible with some form of compactness of the space X or Y and a suitable continuity and convexity or quasi-convexity of the kernel K.

Definition 28 A function $f\colon X \to R$ is upper-semi-continuous on X if and only if for any real c, $\{x\colon f(x) < c\}$ is open in X. A function $f\colon X \to R$ is lower semi-continuous in X if and only if for any real c, $\{x\colon f(x) > c\}$ is open in X.

Definition 29 Let X be a convex subset of a topological vector space. A function $f\colon X \to \mathbf{R}$ is quasi-convex if and only if for each real c, the set $\{x\colon f(x) < c\}$ is convex. A function g is quasi-concave if and only if $-g$ is quasi-convex. Clearly any convex function (concave function) is quasi-convex (conceive).

The following minimax theorem is a special case of more general minimax theorems due to Ky Fan [21] and Sion [61].

Theorem 30 *Let X, Y be compact convex subsets of linear topological spaces. Let $K\colon X \times Y \to \mathbf{R}$ be upper semi continuous (u.s.c) in x (for each fixed y) and lower semi continuous (l.s.c) in y (for each x). Let $K(x, y)$ be quasi-concave in x and quasi-convex in y. Then*

$$\max_{x\in X}\min_{y\in Y} K(x,y) = \min_{y\in Y}\max_{x\in X} K(x,y)\,.$$

Proof The compactness of spaces and the $u.s.c$, $l.s.c$ conditions guarantee the existence of $\max_{x\in X}\min_{y\in Y} K(x,y)$ and $\min_{y\in Y}\max_{x\in X} K(x,y)$. We always have

$$\max_{x\in X}\min_{y\in Y} K(x,y) \le \min_{y\in Y}\max_{x\in X} K(x,y)\,. \qquad \square$$

If possible let $\max_x \min_y K(x,y) < c < \min_y \max_x K(x,y)$. Let $A_x = \{y\colon K(x,y) < c\}$ and $B_y = \{x\colon K(x,y) > c\}$. Therefore we have finite subsets $X_1 \subseteq X$, $Y_1 \subseteq Y$ such that for each $y \in Y$ and hence for each $y \in \text{Con } Y_1$, there is an $x \in X_1$ with $K(x,y) > c$ and for each $x \in X$ and hence for each $x \in \text{Con } X_1$, there is a $y \in Y_1$, with $K(x,y) < c$. Without loss of generality let the finite sets X_1, Y_1 be with *minimum* cardinality m and n satisfying the above conditions. The minimum cardinality conditions have the following implications. The sets $S_i = \{y\colon K(x_i,y) \le c\} \bigcap \text{Con } Y_1$ are non-empty and convex.

Further $\bigcap_{i\neq j} S_i \neq \emptyset$ for all $j = 1,\dots,n$, but $\bigcap_{i=1}^n S_i = \emptyset$. Now by Berge's theorem (Subsect. "Geometric Consequences"), the union of the sets S_i cannot be convex. Therefore there exists $y_0 \in \text{Con } Y_1$, with $K(x,y_0) > c$ for all $x \in X_1$. Since $K(.,y_0)$ is quasi-concave we have $K(x,y_0) > c$ for all $x \in \text{Con } X_1$. Similarly there exists an $x_0 \in \text{Con } X_1$ such that $K(x_0,y) < c$ for all $y \in Y_1$ and hence for all $y \in \text{Con } Y_1$ (by quasi-convexity of $K(x_0,y)$). Hence $c < K(x_0,y_0) < c$, and we have a contradiction.

When the sets X, Y are mixed strategies (probability measures) for suitable Borel spaces one can find value for some games with optimal strategies for one player but not for both. See [50], Alpern and Gal [1988].

Applications of Infinite Games

S-games and Discriminant Analysis

Motivated by Fisher's enquiries [25] into the problem of classifying a randomly observed human skull into one of several known populations, discriminant analysis has exploded into a major statistical tool with applications to diverse problems in business, social sciences and biological sciences [33,54].

Example 31 A population Π has a probability density which is either f_1 or f_2 where f_i is multivariate normal with mean vector μ_i, $i = 1,2$ and variance covariance matrix, Σ, the same for both f_1 and f_2. Given an observation X from population Π the problem is to classify the observation into the proper population with density f_1 or f_2. The costs of misclassifications are $c(1/2) > 0$ and $c(2/1) > 0$ where $c(i/j)$ is the cost of misclassifying an observation from Π_j to Π_i. The aim of the statistician is to find a suitable decision procedure that minimizes the worst risk possible.

This can be treated as an S-game where the pure strategy for Player I (nature) is the secret choice of the population and a pure strategy for the statistician (Player II) is to partition the sample space into two disjoint sets (T_1, T_2) such that observations falling in T_1 are classified as from Π_1 and observations falling in T_2 are classified as from Π_2. The payoffs to Player I (Nature) when the observation is chosen from Π_1, Π_2 is given by the risks (expected costs):

$$r(1,(T_1,T_2)) = c(2/1)\int_{T_2} f_1(x)\mathrm{d}x\,,$$

$$r(2,(T_1,T_2)) = c(1/2)\int_{T_1} f_2(x)\mathrm{d}x\,.$$

The following theorem, based on an extension of Lyapunov's theorem for non-atomic vector measures [43], is due to Blackwell [10], Dvoretsky-Wald and Wolfowitz [20].

Theorem 32 *Let*

$$S = \left\{(s_1,s_2)\colon s_1 = c(2/1)\int_{T_2} f_1(x)dx\,,\ s_2 = c(1/2)\int_{T_1} f_2(x)dx\,,\ (T_1,T_2)\in\mathcal{T}\right\}$$

where $\mathcal{T}$ is the collection of all measurable partitions of the sample space. Then S is a closed bounded convex set.

We know from the theorem on S games (Theorem 22) that Player II (the statistician has a minimax strategy which is pure. If v is the value of the game and if (ξ_1^*, ξ_2^*) is an optimal strategy for Player I then we have:

$$\xi_1^*.c(2/1)\int_{T_2} f_1(x)\mathrm{d}x + \xi_2^*.c(1/2)\int_{T_1} f_2(x)\mathrm{d}x \geq v$$

for all measurable partitions $\mathcal{T}$. For any general partition $\mathcal{T}$, the above expected payoff to I simplifies to:

$$\xi_1^* c(2/1) + \int_{T_1}[\xi_2^*.c(1/2) f_2(x) - \xi_1^* c(2/1)] f_1(x)\mathrm{d}x\,.$$

It is minimized whenever the integrand is ≤ 0 on T_1. Thus the optimal pure strategy (T_1^*, T_2^*) satisfies:

$$T_1^* = \left\{x\colon \xi_2^*\, c(1/2) f_2(x) - \xi_1^* c(2/1) f_1(x) \leq 0\right\}$$

This is equivalent to

$$T_1^* = \left\{x\colon U(x) = (\mu_1-\mu_2)^{\mathrm{T}}\sum^{-1} x - \frac{1}{2}(\mu_1-\mu_2)^{\mathrm{T}}\sum^{-1}(\mu_1+\mu_2) \geq k\right\}$$

and

$$T_2^* = \left\{U(x) = (\mu_1-\mu_2)^{\mathrm{T}}\sum^{-1} x - \frac{1}{2}(\mu_1-\mu_2)^{\mathrm{T}}\sum^{-1}(\mu_1+\mu_2) < k\right\}$$

for some suitable k. Let $\alpha = (\mu_1-\mu_2)^{\mathrm{T}}\sum^{-1}(\mu_1-\mu_2)$.

The random variable U is univariate normal with mean $\frac{\alpha}{2}$, and variance α if $x\in\Pi_1$. The random variable U has mean $\frac{-\alpha}{2}$ and variance α if $x\in\Pi_2$. The minimax strategy for the statistician will be such that

$$c(2/1)\int_{-\infty}^{\frac{1}{\sqrt{\alpha}}(k-\frac{\alpha}{2})}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y = c(1/2)\int_{\frac{1}{\sqrt{\alpha}}(k+\frac{\alpha}{2})}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y\,.$$

The value of k can determined by trial and error.

General Minimax Theorem and Statistical Estimation

Example 33 A coin falls heads with unknown probability θ. Not knowing the true value of θ a statistician wants to estimate θ based ona single toss of the coin with squared error loss.

Of course, the outcome is either heads or tails. If heads he can estimate $\hat{\theta}$ as x and if the outcome is tails, he can estimate $\hat{\theta}$ as y. To keep the problem zero-sum, the statistician pays a penalty $(\theta-x)^2$ when he proposes x as the estimate and pays a penalty $(\theta-y)^2$ when he proposes y as the estimate. Thus the *expected loss or risk* to the statistician is given by

$$\theta(\theta-x)^2 + (1-\theta)(\theta-y)^2\,.$$

It was Abraham Wald [68] who pioneered this game theoretic approach. The problem of estimation is one of discovering the true state of nature based on partial knowledge about nature revealed by experiments. While nature reveals in part, the aim of nature is to conceal its secrets. The statistician has to make his decisions by using observed information about nature. We can think of this as an ordinary zero sum two person game where the pure strategy for nature is to choose any $\theta\in[0,\ 1]$ and a pure strategy for Player II (statistician) is any point in the unit square $I=\{(x,y)\colon 0\leq x\leq 1, 0\leq y\leq 1\}$ with the payoff given above. Expanding the payoff function we get

$$K(\theta,(x,y)) = \theta^2(-2x+1+2y)+\theta(x^2-2y-y^2)+y^2.$$

The statistician may try to choose his strategy in such a way that no matter what θ is chosen by nature, it has no effect on his penalty given the choice he makes for x, y. We call such a strategy an *equalizer*. We have an equalizer strategy if we can make the above payoff independent of θ. In fact we have used this trick earlier while dealing with behavior strategies. For example by choosing x and y such that the coefficient of θ^2 and θ are zero we can make the payoff expression independent of θ. We find that $x = \frac{3}{4}, y = \frac{1}{4}$ is the solution. While this may guarantee the statistician a constant risk of $\frac{1}{16}$, one may wonder whether this is the best? We will show that it is best by finding a mixed strategy for mother nature that guarantees an expected payoff of $\frac{1}{16}$ no matter what (x, y) is proposed by the statistician. Let $F(\theta)$ be the cumulative distribution function that is optimal for mother nature. Integrating the above payoff with respect to F we get

$$K(F,(x,y) = (-2x+1+2y)\int_0^1 \theta^2 \mathrm{d}F(\theta) + (x^2-2y-y^2)\int_0^1 \theta \mathrm{d}F(\theta) + y^2 .$$

Let

$$m_1 = \int_0^1 \theta \mathrm{d}F(\theta) \quad \text{and} \quad m_2 = \int_0^1 \theta^2 \mathrm{d}F(\theta) .$$

In terms of m_1, m_2, x, y the expected payoff can be rewritten as

$$L((m_1,m_2),(x,y)) = m_2(-2x+1+2y) + m_1(x^2-2y-y^2) + y^2 .$$

For fixed values of m_2, m_1 the minimum value of $m_2(-2x+1+2y) + m_1(x^2-2y-y^2) + y^2$ must satisfy the first order conditions

$$\frac{m_2}{m_1} = x^* \text{ , and } \quad \frac{m_2 - m_1}{m_1 - 1} = y* \text{ .}$$

If $m_1 = 1/2$ and $m_2 = 3/8$ then $x* = 3/4$ and $y* = 1/4$ is optimal. In fact, a simple probability distribution can be found by choosing the point $1/2 - \alpha$ with probability $1/2$ and $1/2 + \alpha$ with probability $1/2$. Such a distribution will have mean $m_1 = 1/2$ and second moment $m_2 = (1/2\,(1/2-\alpha)^2 + 1/2\,(1/2-\alpha)^2) = 1/4 + \alpha^2$. When $m_2 = 3/8$, we get $\alpha^2 = 1/8$. Thus the optimal strategy for nature also called the *least favorable distribution*, is to toss an ordinary unbiased coin and if it is heads, select a coin which falls heads with probability $(\sqrt{2}-1)/(2\sqrt{2})$ and if the ordinary coin toss is tails, select a coin which falls heads with probability $(\sqrt{2}+1)/(2\sqrt{2})$. For further applications of zero-sum two person games to statistical decision theory see Ferguson [23].

In general it is difficult to characterize the nature of optimal mixed strategies even for a C^∞ payoff function. See [37]. Another rich source for infinite games do appear in many simplified parlor games. We give a simplified poker model due to Borel (See the reference under Ville [65]).

Borel's Poker Model

Two risk neutral players I and II initiate a game by adding an ante of $1 each to a pot. Players I and II are dealt a *hand*, a random value u of U and a random value v of V respectively. Here U, V are independent random variables uniformly distributed on $[0, 1]$.

The game begins with Player I. After seeing his hand he either *folds* losing the pot to Player II, or *raises* by adding $1 to the pot. When Player I raises, Player II after seeing his hand can either *fold*, losing the pot to Player I, or *call* by adding $1 to the pot. If Player II calls, the game ends and the player with the better hand wins the entire pot. Suppose players I and II restrict themselves to using only strategies g, h respectively where

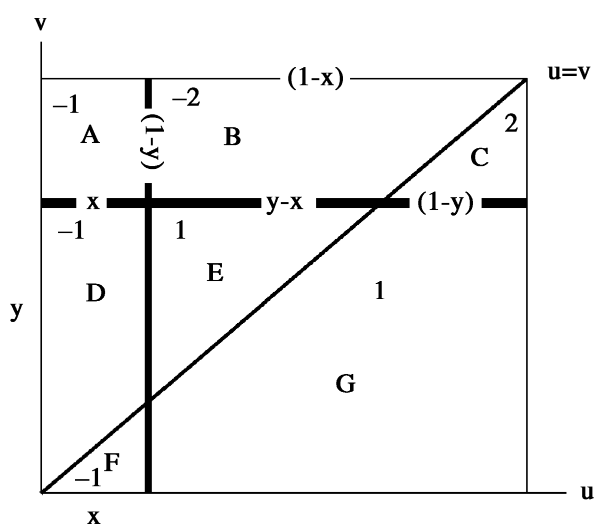
$$\text{Player I:} \quad g(u) = \begin{cases} \text{fold if } u \le x \\ \text{raise if } u > x \text{ ,} \end{cases}$$

$$\text{Player II:} \quad h(v) = \begin{cases} \text{fold if } v \le y \\ \text{raise if } v > y \text{ .} \end{cases}$$

The computational details of the expected payoff $K(x, y)$ based on the above partitions of the u, v space is given below in Table 3.

The payoff $K(x, y)$ is simply the sum of each area times the local payoff given by

$$K(x,y) = 2y^2 - 3xy + x - y$$
$$\text{when } x < y \text{ .}$$



Zero-Sum Two Person Games, Figure 5
Poker partition when $x < y$

Zero-Sum Two Person Games, Table 3

Outcome	Action by players:	Region	Area	Payoff
$u \le x$	I drops out	$A \cup D \cup F$	x	-1
$u > x,\ v \le y$	II drops out	$E \cup G$	$y(1-x)$	1
$u < v,\ u > x,\ v > y$	both raise	B	$\frac{(1-y)(1-2x+y)}{2}$	-2
$u > v, u > x, v > y$	both raise	C	$\frac{1}{2}(1-y)^2$	2

Zero-Sum Two Person Games, Table 4

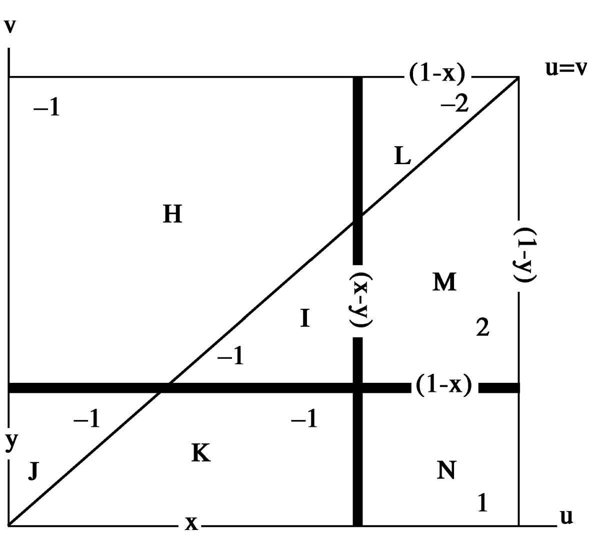
Outcome	Action by players:	Region	Area	Payoff
$u \le x$	I drops out	$H \cup I \cup J \cup K$	x	-1
$u > x,\ v \le y$	II drops out	N	$y(1-x)$	1
$u < v,\ u > x,\ v > y$	both raise	L	$\frac{(1-x)^2}{2}$	-2
$u > v, u > x, v > y$	both raise	M	$\frac{1}{2}(1-x)(1+x-2y)$	2

The payoff $K(x, y)$ is simply the sum of each area times the local payoff given by

$$K(x,y) = \begin{cases} -2x^2 + xy + x - y & \text{for } x > y \\ 2y^2 - 3xy + x - y & \text{for } x < y\,. \end{cases}$$

Also $K(x, y)$ is continuous on the diagonal and concave in x and convex in y. By the general minimax theorem of Ky Fan or Sion there exist x^*, y^* pure optimal strategies for $K(x, y)$. We will find the value of the game by explicitly computing $\min_y K(x, y)$ for each x and then taking the maximum over x.

Let $0 < \bar{x} < 1$. (Intuitively we see that in our search for saddle point, $x = 0$ or $x = 1$ are quite unlikely.) $\min_{0 \le y \le 1} K(\bar{x}, y) = \min\{\inf_{y<\bar{x}} K(\bar{x}, y), \min_{y \ge \bar{x}} K(\bar{x}, y)\}$. Observe



Zero-Sum Two Person Games, Figure 6
Poker partition when $x > y$

that $\inf_{y<\bar{x}} K(\bar{x}, y) = \min(-2\bar{x}^2 + \bar{x}y + \bar{x} - y) = y(\bar{x} - 1)$ + terms not involving y.

Clearly the infimum is attained at $y = \bar{x}$ giving a value $-\bar{x}^2$. Now for $y > \bar{x}$ we have $K(\bar{x}, y) = 2y^2 - 3\bar{x}y + \bar{x} - y$. This being a convex function in y, it has its minimum either at the boundary or at an interior point of the interval $[\bar{x}, 1]$. We have $K(\bar{x}, \bar{x}) = -\bar{x}^2, K(\bar{x}, 1) = 1 - 2\bar{x}$ or the minimum is attained at $y = (3\bar{x} + 1)/4$, the unique solution to $(\partial K(\bar{x}, y))/(\partial y) = 0$. At $y = (3\bar{x} + 1)/4$, the value of $K(\bar{x}, y) = (-9\bar{x}^2 + 2\bar{x} - 1)/8$. Clearly for $\bar{x} \neq 1$ we have $(-9\bar{x}^2 + 2\bar{x} - 1)/8 < -\bar{x}^2 < 1 - 2\bar{x}$. Thus $\min_y K(x, y) = (-9x^2 + 2x - 1)/8$. Now $\max\min_y K(x, y) = \max_x(-9x^2 + 2x - 1)/8 = -1/9$. It is attained at $x^* = 1/9$. Now $\min_y K(1/9, y)$ is attained at $y^* = 1/3$.

Thus the value of the game is $-1/9$. A good pure strategy for I is to raise when $x > 1/9$ and a good pure strategy for II is to call when $y > 1/3$.

For other poker models see von Neumann and Morgenstern [67], Blackwell and Bellman [5], Binmore [9] and Ferguson and Ferguson [22] who discuss other discrete and continuous poker models.

War Duels and Discontinuous Payoffs on the Unit Square

While general minimax theorems make some stringent continuity assumptions, rarely they are satisfied in modeling many war duels as games on the unit square. The payoffs are often discontinuous along the diagonal. The following is an example of this kind.

Example 34 Players I and II start at a distance = 1 from a balloon, and walk toward the balloon at the same speed. Each player carries a noisy gun which has been loaded with just one bullet. Player I's accuracy at x is $p(x)$ and

Player II's accuracy at x is $q(x)$ where x is the distance traveled from the starting point. Because the guns can be heard, each player knows whether or not the other player has fired. The player who fires and hits the balloon first wins the game.

Some natural assumptions are

- The functions p, q are continuous and strictly increasing
- $p(0) = q(0) = 0$ and $p(1) = q(1) = 1$.

If a player fires and misses then the opponent can wait until his accuracy $= 1$ and then fire. Player I decides to shoot after traveling distance x provided the opponent has not yet used his bullet. Player II decides to shoot after traveling distance y provided the opponent has not yet used his bullet. The following payoff reflects the outcome of this strategy choice.

$$K(x, y) = \begin{cases} (1)p(x) + (-1)(1 - p(x) = 2p(x) - 1 & \text{when } x < y \\ p(x) - q(x) & \text{when } x = y\,, \\ (-1)q(y) + (1 - q(y)(1) \ = 1 - 2q(y) & \text{when } x > y\,. \end{cases}$$

We claim that the players have optimal pure strategies and there is a saddle point. Consider $\min_y K(x, y) = \min\{2p(x) - 1, p(x) - q(x), \inf_{y<x}(1 - 2q(y))\}$. We can replace $\inf_{y<x}(1 - 2q(y))$ by $1 - 2q(x)$. Thus we get

$$\min_{0 \le y \le 1} K(x, y) = \min\left\{2p(x) - 1, p(x) - q(x), (1 - 2q(x))\right\}\,.$$

Since the middle function $p(x) - q(x)$ is the average of the two functions $2p(x)-1$ and $1-2q(x)$, the minimum of the three functions is simply the $\min\{2p(x) - 1, (1 - 2q(x))\}$. While the first function is increasing in x the second one is decreasing in x. Thus the $\max_x \min_y K(x, y)$ = the solution to the equation $2p(x) - 1 = 1 - 2q(x)$. There is a unique solution to the equation $p(x) + q(x) = 1$ as both functions are strictly increasing. Let x^* be that solution point. We get $p(x^*) + q(x^*) = 1$. The value v satisfies

$$\begin{aligned} v &= p(x^*) - q(x^*) \\ &= \min_y \max_x K(x, y) \\ &= \max_x \min_y K(x, y)\,. \end{aligned}$$

Remark 35 In the above example the winner's payoff is 1 whether player I wins or player II wins. This is crucial for the existence of optimal pure strategies for the two players.

Suppose the payoff has the following structure:

α	if I wins
β	if II wins
γ	if neither wins
0	if both shoot accurately and ends in a draw.

Depending on these values while the value exists only one player may have optimal pure strategy while the other may have only an *epsilon* optimal pure strategy, close to but not the same as the solution to the equation $p(x^*) + q(x^*) = 1$.

Example 36 (silent duel) Players I and II have the same accuracy $p(x) = q(x) = x$. However, in this duel, the players are both deaf so they do not know whether or not the opponent has fired. The payoff function is given by

$$K(x, y) = \begin{cases} (1)x + (-1)(y)(1 - x) = x - y + xy & \text{when } x < y\,, \\ 0 & \text{when } x = y, \\ (-1)y + (1)(1 - y)x = x - y - xy & \text{when } x > y\,. \end{cases}$$

This game has no saddle point. In this case, the value of the game if it exists must be zero. One can directly verify that the density

$$f(t) = \begin{cases} 0 & 0 \le t < 1/3 \\ \frac{1}{4}t^{-3} & 1/3 \le t \le 1\,. \end{cases}$$

is optimal for both players with value zero.

Remark 37 In the above game suppose exactly one player, say Player II is deaf. Treating winners symmetrically, we can represent this game with the payoff

$$K(\xi, \eta) = \begin{cases} (1)\xi - (1 - \xi)\eta & \xi < \eta \\ 0 & \xi = \eta \\ (-1)\eta + (1 - \eta)(1) & \xi > \eta\,. \end{cases}$$

Let $a = \sqrt{6} - 2$. Then the game has value $v = 1 - 2a = 5 - 2\sqrt{6}$. An optimal strategy for Player I is given by the density

$$f(\xi) = \begin{cases} 0 & \text{for } 0 \le \xi < a \\ \sqrt{2}a(\xi^2 + 2\xi - 1)^{-3/2} & \text{for } a \le \xi \le 1\,. \end{cases}$$

For Player II it is given by

$$g(\eta) = \begin{cases} 0 & \text{for } 0 \le \xi < a \\ 2\sqrt{2}.\frac{a}{2+a}(\eta^2 + 2\eta - 1)^{-3/2} & \sqrt{6} - 2 \le \eta < 1 \end{cases}$$

with an additional mass of $\frac{a}{2+a}$ at $\eta = 1$. The deaf player has to maintain a sizeable suspicion until the very end!

The study of war duels is intertwined with the study of positive solutions to integral equations. The theorems of Krein and Rutman [40] on positive operators and their positive eigenfunctions are central to this analysis. For further details see [19,37,51].

Dynamic versions of zero sum two person games where the players move among several games according to some Markovian transition law leads to the theory of stochastic games [60]. The study of value and the complex structure of optimal strategies of zero-sum two person stochastic games is an active area of research with applications to many dynamic models of competition [24].

While stochastic games study movement in discrete time, the parallel development of dynamic games in continuous time was initiated by Isaacs in several Rand reports culminating in his monograph on Differential games [31]. Many military problems of pursuit and evasion, attrition and attack lead to games where the trajectory of a moving object is being steered continuously by the actions of two players. Introducing several natural examples Isaacs explicitly tried to solve many of these games via some heuristic principles. The minimax version of Bellman's optimality principle lead to the so called Isaacs Bellman equations. This highly non-linear partial differential equation on the value function plays a key role in this study.

Epilogue

The rich theory of zero sum two person games that we have discussed so far hinges on the fundamental notions of value and optimal strategies. When either zero sum or two person assumption is dropped, the games cease to have such well defined notions with independent standing. In trying to extend the notion of optimal strategies and the minimax theorem for bimatrix games Nash [48] introduced the concept of an equilibrium point. Even more than this extension it is this concept which is simply the most seminal solution concept for non-cooperative game theory. This Nash equilibrium solution is extendable to any non-cooperative N person game in both extensive and normal form. Many of the local properties of optimal strategies and the proof techniques of zero sum two person games do play a significant role in understanding Nash equilibria and their structure [15,32,39,52,53].

When a non-zero sum game is played repeatedly, the players can peg their future actions on past history. This leads to a rich theory of equilibria for repeated games [62]. Some of them, like the repeated game model of Prisoner's dilemma impose tacitly, actual cooperation at equilibrium among otherwise non-cooperative players. This was first recognized by Schelling [58], and was later formalized by the so called folk theorem for repeated games (Aumann and Shapley [1986], [3]). Indeed Nash equilibrium per se is a weak solution concept is also one of the main messages of Folk theorem for repeated games. With a plethora of equilibria, it fails to have an appeal without further refinements. It was Selten who initiated the need for refining equilibria and came up with the notion of subgame perfect equilibria [59]. It turns out that subgame perfect equilibria are the natural solutions for many problems in sequential bargaining [57]. Often one searches for specific types of equilibria like symmetric equilibria, or Bayesian equilibria for games with incomplete information [30]. Auctions as games exhibit this diversity of equilibria and Harsanyi's Bayesian equilibria turn out to be the most appropriate solution concept for this class of games [47].

Zero sum two person games and their solutions will continue to inspire researchers in all aspects non-cooperative game theory.

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