As in the scalar theory, the question arises as to whether or not $\lim X_i(n)$ exists as $n \to \infty$ and if so, the dependence of this quantity upon the initial state $X(0)$.

The following result holds.

THEOREM. *If A_{ijk} is positive definite for all i, j, and k, in general we will have $\lim_{n \to \infty} X_i(n) = X_i$, where X_i is independent of $X(0)$.*

By the phrase "in general" we mean apart from situations in which certain special relations exist among the A_{ijk} which will permit $\sum_{j,k} A_{ijk}X_jA_{ijk}^T$ to be semidefinite for semidefinite X_i.

There are several different proofs of this result, extensions of the proofs used in the scalar case. Each is a slight variation since the one-dimensional concept of positivity has to be modified in an appropriate fashion. An essential tool is the concept of the adjoint transformation represented by

$$Z_i = \sum_{k=1}^{M} \sum_{i=1}^{N} A_{ijk}X_iA_{ijk}^T.$$

SOME FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING

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1. *Introduction.*—The purpose of this paper is to present some results concerning representative types of functional equations occurring in the theory of dynamic programming. Since considerations of space forbid our entering into any discussion of the origin and interpretation of all of the equations appearing below, we shall discuss one, the "gold-mining" equation, in some detail and refer to previous notes[2–4] and a forthcoming monograph, "An Introduction to the Theory of Dynamic Programming," soon to be published by The RAND Corporation, for further details.

2. *Functional Equations.*—In this section we state some results whose proof is contained in the monograph cited above.

THEOREM 1. (*The Gold-Mining Equation.*) *Consider the equation*

$$f(x, y, z) = \operatorname{Max} \begin{bmatrix} \text{A}: & p_1f(0, y, x + z) + p_2f(c_2x, y, z + c_1x) + p_3\phi(z) \\ \text{B}: & q_1f(x, 0, y + z) + q_2f(x, d_2y, z + d_1y) + q_3\phi(z) \end{bmatrix}, \tag{1.1}$$

$x, y, z \geq 0$, with $f(0, 0, z) = \phi(z)$, and

$$\begin{array}{lll} \text{(a)} & p_1, p_2, p_3, q_1, q_2, q_3 \geq 0, & q_1 + q_2 + q_3 = p_1 + p_2 + p_3 = 1 \\ \text{(b)} & c_1, c_2, d_1, d_2 \geq 0, & c_1 + c_2 = d_1 + d_2 = 1. \end{array} \tag{1.2}$$

Define

$$F(x, y, z) = p_1q_3(\phi(x + z) - \phi(z)) + p_2q_3(\phi(z + c_1x) - \phi(z)) - q_1p_3(\phi(y + z) - \phi(z)) - q_2p_3(\phi(z + d_1y) - \phi(z)). \tag{1.3}$$

If $\phi(z)$ is strictly increasing and continuous, $\phi(0) \geq 0$, $F(x, y, z) \geq 0$ implies $F(c_2x, y, z + c_1x) \geq 0$, and $F(x, y, z) \leq 0$ implies $F(x, d_2y, z + d_1y) \leq 0$, then the unique solution to (1.1) is given by

$$f(x, y, z) = p_1f(0, y, x + z) + p_2f(c_2x, y, z + c_1x) + p_3\phi(z), \tag{1.4}$$

for $F(x, y, z) \geq 0$, and

$$f(x, y, z) = q_1f(x, 0, y + z) + q_2f(x, d_2y, z + d_1y) + q_3\phi(z), \tag{1.5}$$

for $F(x, y, z) \leq 0$.

COROLLARY. *If $\phi(z) = e^{bz}$, $b > 0$, the solution is given by: For*

$$\frac{p_1(e^{bx} - 1) + p_2(e^{bc_1x} - 1)}{p_3} > \frac{q_1(e^{by} - 1) + q_2(e^{bd_1y} - 1)}{q_3} \tag{1.6}$$

choose A; if the reverse inequality holds, employ B; if equality, use either.

The limiting inequality obtained as $b \to 0$ yields the result contained in reference 2, Theorem 4, corresponding to $\phi(z) = z$. The asymptotic behavior of f may be obtained in the case above where $\phi(z) = e^{bz}$.

THEOREM 2. (*Optimal Allocation.*) *Consider the equation*

$$f(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y) + f(ay + b(x - y))], \tag{1.7}$$

$x \geq 0$ with $f(0) = 0$. Let us assume

$$\begin{array}{lll} \text{(a)} & g(0) = h(0) = 0 & \\ \text{(b)} & g'(x), h'(x) \geq 0, & \text{for } x \geq 0 \\ \text{(c)} & g''(x), h''(x) \leq 0, & \text{for } x \geq 0, \end{array} \tag{1.8}$$

and consider the sequence of approximations to f defined by

$$f_0(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y)]$$

$$f_{n+1}(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y) + f_n(ay + b(x - y))], \quad n = 0, 1, 2, \ldots. \tag{1.9}$$

For each n, there is a unique $y_n = y_n(x)$ which yields the maximum. If $b \leq a$, we have $y_1 \leq y_2 \leq y_3 \ldots$, and the reverse inequalities for $b \geq a$. In particular, if $y_n(x) = x$ for some n, in the case $b \leq a$, then $y_m(x) = x$ for $m \geq n$, and the solution of the original equation in (7) will be furnished by $y = x$.

A result concerning the same equation was presented in reference 2 for the case where g and h were both convex.

THEOREM 3. *(An Optimal Inventory Equation.) Consider the equation*

$$u(x) = \operatorname*{Min}_{y \geq x} \left[g(y - x) + a((M + u(0))e^{-by} + b \int_0^y e^{-bv} u(y - v)\, dv)\right] \quad (1.10)$$

where we assume

$$\begin{array}{llll} \text{(a)} & g(0) = 0, & g'(y) \geq 0, & g''(y) \geq 0 \\ \text{(b)} & b,\ M > 0, & 0 < a < 1. & \end{array} \quad (1.11)$$

The solution is given by the rule:

$$\begin{aligned} y &= x_\infty, \quad \text{for } 0 \leq x \leq x_\infty, \\ y &= x, \quad \text{for } x \geq x_\infty, \end{aligned} \quad (1.12)$$

where x_∞ is a constant determined by the above data by means of a set of, in general, transcendental equations.

The case where $g(y)$ is concave is much more difficult, and, unfortunately, the more interesting case for applications. Using the same techniques that yield Theorems 2 and 3 it can be shown that if g is a concave function of the form

$$\begin{aligned} g(x) &= c_1 x, \quad 0 \leq x \leq x_1 \\ g(x) &= c_1 x_1 + c_2(x - x_1), \quad x \geq x_1 \end{aligned} \quad (1.13)$$

where $c_2 < c_1$, the conclusion of Theorem 3 is still valid.

The optimal inventory equation was first studied by Arrow, Harris, and Marschak,[1] and then in great detail, together with extensions and generalizations, by Dvoretzky, Kiefer, and Wolfowitz.[6]

THEOREM 4. *(Games of Survival.) Consider the equation*

$$\begin{aligned} f(x) &= \operatorname*{Min}_{q} \operatorname*{Max}_{p} [p_1q_1f(x - 1) + p_1q_2f(x + a) + p_2q_1f(x + c) + \\ &\qquad p_2q_2f(x - b)] \\ &= \operatorname*{Max}_{p} \operatorname*{Min}_{q} [p_1q_1f(x - 1) + p_1q_2f(x + a) + p_2q_1f(x + c) + \\ &\qquad p_2q_2f(x - b)] \end{aligned} \quad (1.14)$$

for $x = 1, 2, \ldots, d - 1$, with

$$\begin{aligned} f(x) &= 0, \quad x \leq 0 \\ &= 1, \quad x \geq d, \end{aligned} \quad (1.15)$$

where $p_1, p_2, q_1, q_2 \geq 0$, $p_1 + p_2 = q_1 + q_2 = 1$, and a, b and c are positive integers.

There is a unique bounded solution.

This formulation of "games of survival," of which the above represents a very simple case, was obtained by the author and J. P. LaSalle in 1949 in an unpublished RAND paper. An alternate analysis of the problem, due to M. Peisakoff, will be found in the monograph mentioned previously.

THEOREM 5. (*Probability of Success.*) *Consider the equation*

$$u(n) = \underset{\leq 1 i \leq M}{\text{Max}} \left[\sum_{j=1}^{R} a_{ij} u(n-j) \right] \tag{1.16}$$

where we assume

(a) $a_{ij} \geq 0$,

(b) *there is one equation* $r^R = \sum_{j=1}^{R} a_{kj} r^{R-j}$ *whose largest positive root is greater than the corresponding roots of the other equations of this type,*

(c) *for this index* k, $a_{k1} \neq 0$. (1.17)

Under these circumstances the solution of (16) is given by

$$u(n) = \sum_{j=1}^{R} a_{kj} u(n-j) \tag{1.18}$$

for n sufficiently large.

A similar result can be obtained for the inhomogeneous equation

$$u(n) = \underset{1 \leq i \leq M}{\text{Max}} \left(\sum_{j=1}^{R} a_{ij} u(n-j) + g_i \right), \tag{1.19}$$

under the assumptions

$$\begin{aligned} &\text{(a)} \quad a_{ij} \geq 0, \qquad \sum_{j=1}^{R} a_{ij} = 1 \\ &\text{(b)} \quad g_i \geq 0. \end{aligned} \tag{1.20}$$

In this case the solution is determined for large n by the index for which $ng_i / \sum_{j=1}^{R} j a_{ij}$ assumes its maximum.

A problem which arises in production planning and seems of some difficulty is the following: "Given a finite set of A_i of non-negative square matrices, determine for each N, the matrix $C_N = B_1 B_2 \ldots B_N$, where each B_i is an A_j, which possesses the largest characteristic root."

The following partial result has been obtained:

THEOREM 6. *Let us define*

$$\phi(A) = \textit{characteristic root of } A \textit{ of largest absolute value} \tag{1.21}$$

Then

$$\lambda = \lim_{N \to \infty} \phi(C_N)^{1/N} \tag{1.22}$$

exists.

Let M_N *denote the smallest majorant of all the* 2^N *products,* $B_1, B_2 \ldots B_N$, *i.e., the ij element in M_N is the maximum of the 2^N ij*th *elements of the C_N. Then*

$$\mu = \lim_{N \to \infty} \phi(M_N)^{1/N} \tag{1.23}$$

exists.

This problem leads inevitably to the question of determining the limiting distribution of the elements of $\prod_{i=1}^{N} Z_i$ in the case where the Z_i are random matrices with a given distribution. Some preliminary results are given in reference 5.

3. *The Gold-Mining Equation.*—The equation occurring in Theorem 1 may be considered to arise in the following way: We possess two gold mines, Anaconda with a vein of value x, and Bonanza with a vein of value y, and one gold-mining machine. If the machine is used in Anaconda, there is a probability p_1 that all the gold is mined and that the machine is undamaged, a probability p_2 that a fraction c_1 of the gold is mined and that the machine is undamaged, and finally a probability p_3 that no gold is mined and the machine damaged beyond repair. Similarly the Bonanza mine has the associated probabilities q_1, q_2, q_3 and the ratio d_1. This mining operation continues until the machine is damaged. If at any stage we let z be the amount already mined and $f(x, y, z)$ be the expected value of $\phi(z + w)$, w the amount obtained from that stage on using an optimal sequence of choices, the functional equation for f is (1) of §2.

The case where more than two decisions are possible becomes very difficult if a decision affects the state of both mines at the same time. A counter-example due to Karlin and Shapiro,[7] shows that no immediate generalization of Theorem 1 exists.

We may, however, remedy this by considering a continuous version of (1) of §2, utilizing the technique discussed in reference 4. For these more tractable problems it turns out that the policy of maximizing expected gain over expected cost at each stage is sometimes optimal. The translation of this policy into mathematical terms is the substance of (3) and (6) of §2. A treatment of the continuous problems will be given subsequently.

[1] Arrow, K. J., Blackwell, D., and Girshick, M. A., "Bayes and Minimax Solutions of Sequential Decision Problems," *Econometrica*, **17**, 214–244 (1949).

[2] Bellman, R., "On the Theory of Dynamic Programming," PROC. NATL. ACAD. SCI., **38**, 716–719 (1952).

[3] Bellman, R., Glicksberg, I., and Gross, O., "On Some Variational Problems Occurring in the Theory of Dynamic Programming," *Ibid.*, **39**, 298–301 (1953).

[4] Bellman, R., "Bottleneck Problems and Dynamic Programming," RAND Paper No. 407 (May 1953); PROC. NATL. ACAD. SCI., **39**, 947–951 (1953).

[5] Bellman, R., "On Limit Theorems for Non-commutative Operations—I," RAND Paper No. 398 (April 1953).

[6] Dvoretzky, A. J., Kiefer, J., and Wolfowitz, J., "The Inventory Problem—I: Case of Known Distributions of Demand" and "The Inventory Problem—II: Case of Unknown Distributions of Demand," *Econometrica*, **20**, 187–222 (April 1952).

[7] Karlin, S., and Shapiro, H. N., "Decision Processes and Functional Equations," RAND Research Memorandum No. 933 (1952).

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*CLOSURE CLASSES ORIGINATING IN THE THEORY OF PROBABILITY**

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We will operate in Euclidean $E_k: (\alpha_j)$, $k \geqq 1$, but other group and similar spaces could also be envisaged.

Definition.—A sequence of continuous functions $\{\varphi_n(\alpha_j)\}$ will be called *P-convergent*, if it converges at all points,

$$\lim_{n \to \infty} \varphi_n(\alpha) = \varphi(\alpha),$$

and uniformly so in every compact set

$$|\alpha| = (\alpha_1^2 + \ldots + \alpha_k^2)^{1/2} \leqq A_0.$$

We will also speak of P-limits, P-closed families, the (smallest) P-closure of a family, etc.

It is known that the class of positive-definite functions

$$\begin{aligned} \varphi(\alpha_j) &= \int_{E_k} e^{i(\alpha, x)} \, dF(x_j), \\ (\alpha, x) &= (\alpha_1 x_1 + \ldots + \alpha_k x_k) \\ F(A) \geqq 0, \; F(E_k) &= \int_{E_k} dF(x) < \infty, \end{aligned} \tag{1}$$

is a P-closed family, and this fact is the basis for all central limit theorems in the theory of probability.

The differences

$$\psi(\alpha) = \varphi(0) - \varphi(\alpha_j) \equiv \int_{E_k} (1 - e^{i(\alpha, x)}) \, dF(x)$$

are not P-closed, but it is known from the theory of infinitely subdivisible stochastic processes that their P-closure are the functions

$$iL(\alpha_j) + Q_2(\alpha_j) + \int_{|x| > 0} \left(1 - e^{i(\alpha, x)} + \frac{i(\alpha, x)}{1 + |x|^2}\right) dF(x),$$