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Theory and Methodology

Sensitivity analysis in linear programming: just be careful! [1]

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Abstract

In this paper we review the topic of sensitivity analysis in linear programming. We describe the problems that may occur when using standard software and advocate a framework for performing complete sensitivity analysis. Three approaches can be incorporated within it: one using bases, an approach using the optimal partition and one using optimal values. We elucidate problems and solutions with an academic example and give results from an implementation of these approaches to a large practical linear programming model of an oil refinery. This shows that the approaches are viable and useful in practice. © 1997 Elsevier Science B.V.

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1. Introduction

The merits of Linear Programming (LP) are nowadays well-established and LP is widely accepted as a useful tool in Operations Research and Management Science. A large number of companies is using this way of modelling to solve various kinds of practical problems. Applications include transportation problems, production planning, investment decision problems, blending problems, location and allocation problems, among many other.

In most of cases, for solving the LP-problem the *simplex method* is used. The method, due to Dantzig in 1946, has been implemented in a large variety of codes which are successfully used in practice. Most of these packages do not only solve the LP-problem, but also provide the option to ask for information on the sensitivity of the solution to certain changes in the data. This is referred to as *sensitivity analysis* or *postoptimal analysis*. This information can be of tremendous importance in practice, where parameter values may be estimates, questions of type "What if..." are frequently encountered, and implementation of a specific solution may be difficult. Sensitivity analysis serves as a tool for obtaining information about the bottlenecks and degrees of freedom in the problem. The sensitivity analysis performed by most commercial packages returns two types of information, which are of particular interest: it tells something about the behavior of the optimal value and about the optimal solution itself. To be more specific, first, for each of the constraints

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a *dual solution* (called *shadow price, dual price or shadow cost*) is given, which is often interpreted as the rate at which the objective function value changes when the value of the right hand side (RHS) coefficient changes. Second, for each of the coefficients in the objective function (COST) and for each of the right hand side coefficients of the constraints a certain *interval* (*range*) is given. By way of example, we will show that one should be careful in interpreting the information given by a package.

Example 1. We consider a problem of transporting goods from three suppliers to three markets at minimal cost. Each supplier can serve each of the markets at a transportation cost of 1 per unit. The capacity of the suppliers is equal to 2, 6 and 5 units respectively. The markets each require at least 3 units. It is easy to see that due to the special cost structure any solution that exactly transports the required demand is optimal. We can formulate this problem as an LP-problem using the variables

x_{ij}: the amount of units transported from supplier i to market j,

σ_i: excess supply at supplier i,

d_j: shortage demand at market j,

where i and j run from 1 to 3. The LP-formulation is then

$$\min \sum_{i=1}^{3}\sum_{j=1}^{3} x_{ij},$$

subject to

$$x_{11} + x_{12} + x_{13} + \sigma_1 = 2, \tag{1}$$

$$x_{21} + x_{22} + x_{23} + \sigma_2 = 6, \tag{2}$$

$$x_{31} + x_{32} + x_{33} + \sigma_3 = 5, \tag{3}$$

$$x_{11} + x_{21} + x_{31} - d_1 = 3, \tag{4}$$

$$x_{12} + x_{22} + x_{32} - d_2 = 3, \tag{5}$$

$$x_{13} + x_{23} + x_{33} - d_3 = 3, \tag{6}$$

$$x_{ij},\ \sigma_i,\ d_j \geq 0 \quad i,j = 1,2,3.$$

We solved this problem with five commercially available (simplex-) LP-packages and asked for sensitivity information. The results are summarized in Tables 1 and 2.

Even in this simple example, it is striking to notice that no two packages yield the same result. Of course the optimal value obtained is the same. However, four different optimal solutions are given and all packages return different ranges for the coefficients in the objective function. Note that although CPLEX and OSL give the same optimal solution, the ranges for the objective coefficients are not all the same. Even though the shadow prices are equal in all packages, the ranges for the right hand side coefficients are not the same.

The first important observation we make is very fundamental and (should be) well-known: shadow prices and ranges never go apart. The ranges given in Table 2 exactly say where the corresponding number obtained in Table 1 is valid in the sense to be discussed later.

Unfortunately, practitioners and users of LP are often unaware of the phenomenon encountered in the example. Rubin and Wagner [15, p. 150] state: "Managers who build their own microcomputer linear programming models are apt to misuse the resulting shadow prices and shadow costs. Fallacious interpretations of these values can lead to expensive mistakes, especially unwarranted capital investments". Even the use of high standard LP-codes does not prevent the user from these problems. On the other hand, there is a vast amount of (scientific) literature on this phenomenon, on accompanying problems and on possible solutions. Many papers try to adapt the standard (simplex) approach (e.g., Gal [4], Knolmayer [13], Greenberg [8]). For a survey we refer to Gal [5]. Another approach is the use of strictly complementary solutions (e.g., Adler and Monteiro [1], Jansen et al. [10], Greenberg [9], and Mehrotra and Monteiro [14]), which is natural in the context of interior point methods for LP.

In this practice-oriented paper we try to bridge (part of) the gap between practice and theory. We will address the question of what is important to know in sensitivity analysis, and to what extent this information is provided by various approaches. An important part of this paper gives results of the application of sensitivity approaches to a linear model of an oil refinery (with production and distribution).

This paper is built up as follows. In Section 2 we give some fundamental aspects of linear programming and sensitivity analysis. We introduce the *optimal value function* and explain its importance regarding

Table 1
Optimal solution and shadow prices in Example 1

LP-package	Optimal solution									Shadow prices					
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	(1)	(2)	(3)	(4)	(5)	(6)
CPLEX	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
LINDO	2	0	0	0	0	2	1	3	1	0	0	0	1	1	1
OSL	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
PC-PROG	0	0	0	0	3	1	3	0	2	0	0	0	1	1	1
XMP	0	0	2	3	3	0	0	0	1	0	0	0	1	1	1

Table 2
Ranges in Example 1

LP-package	COST-ranges								
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
CPLEX	$[1,\infty)$	$(-\infty,1]$	$[1,\infty)$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,\infty)$	$[1,\infty)$
LINDO	$(-\infty,1]$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$
OSL	$[1,\infty)$	$[1,1]$	$[1,\infty)$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,\infty)$	$[1,\infty)$
PC-PROG	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,\infty)$	$[1,1]$
XMP			$(-\infty,1]$	$[1,1]$	$[0,1]$		$[1,1]$		$[1,1]$

LP-package	RHS-ranges					
	(1)	(2)	(3)	(4)	(5)	(6)
CPLEX	$[0,3]$	$[4,7]$	$[1,\infty)$	$[2,7]$	$[2,5]$	$[2,5]$
LINDO	$[1,3]$	$[2,\infty)$	$[4,7]$	$[2,4]$	$[1,4]$	$[1,7]$
OSL	$[0,3]$	$[4,7]$	$[1,\infty)$	$[2,7]$	$[2,5]$	$[2,5]$
PC-PROG	$[0,\infty)$	$[4,\infty)$	$[3,6]$	$[2,5]$	$[0,5]$	$[2,5]$
XMP	$[0,3]$	$[3,6]$	$[1,\infty)$	$[3,7]$	$[3,6]$	$[2,7]$

questions arising in sensitivity analysis. In Section 3 we develop a framework for sensitivity analysis in LP based on the use of optimal sets. In Section 4 we give three approaches that fit in the general framework. The first uses the optimal partition of the problem, the second the optimal value, and the third is based on the use of (simplex) bases. We address the question to what extent standard LP-packages provide the information that should be obtained. In Section 5 we apply sensitivity analysis to an LP model of a refinery, and show the differences between using a commercial LP-code and performing the complete sensitivity analysis. Finally, some concluding remarks follow.

2. Principles of linear programming and sensitivity analysis

Any LP-problem can be written in the following standard form:

$$\text{(P)} \quad \min_x \left\{ c^T x \colon Ax = b,\ x \geq 0 \right\},$$

which is the *primal problem*. Here x is the vector with n variables, A is the $m \times n$ constraint matrix, c the n-vector with objective coefficients, while b is the right-hand side vector. Associated with (P) is the *dual problem* (D):

$$\text{(D)} \quad \max_{y,s} \left\{ b^T y \colon A^T y + s = c,\ s \geq 0 \right\}.$$

For each x feasible in (P) and y feasible in (D) it holds $c^T x \geq b^T y$; for optimal solutions equality holds, $c^T x = b^T y$. A different way of expressing optimality is using the *complementary slackness conditions*:

$$x_j s_j = x_j (c_j - a_j^T y) = 0, \quad j = 1, \ldots, n,$$

where we denote by a_j the jth column of A.

In standard sensitivity analysis one element in the right hand side vector b or the cost vector c is changed. Assume for the moment that just c_j is varied for some j. The behavior of the optimal value as c_j varies can be investigated with the *optimal value function*. In the case of varying c_j this function is piecewise linear, concave and monotonically increasing; the function may look like the one given in Fig. 1. As indicated in the figure, there are several intervals for c_j on which this function is linear; these intervals are called *linearity intervals*. The points where the slope of the optimal value function changes are called *breakpoints*.

Assume that we have solved the LP-problem for a certain value of c_j. What would we like to know? We give here four questions a typical user could ask.

Question 1. How is the optimal value affected by a change in c_j?

Question 2. In what interval may c_j be varied such that this rate of change is constant?

Question 3. In what interval may c_j be varied such that the optimal solution of (P) obtained from our solution procedure remains optimal?

Question 4. What happens to the optimal solution of (D) obtained from our solution procedure in the interval from Question 3?

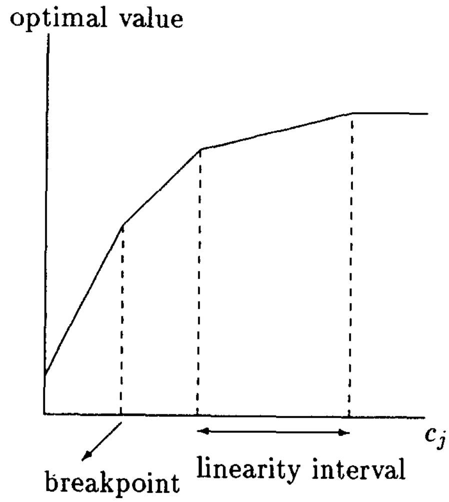

Fig. 1. Optimal value function for c_j.

All these questions appear to have a clear interpretation in connection with the optimal value function. The first two questions can be answered in a straightforward way. The last two are much more involved, since the answer depends on the type of solution that is computed by our solution procedure.

As can be seen from Fig. 1, if c_j is not a breakpoint of the optimal value function, then the answer to Question 1 must clearly be that the derivative (slope) of the optimal value function is the rate at which the optimal value changes. This rate of change is called the *shadow cost* or *shadow price*. However, if c_j is a breakpoint, then we must distinguish between increasing and decreasing c_j, since the rate of change (slope) is different in both cases. We note that a standard LP-package will never notify the user that he has to distinguish between decreasing and increasing parameter values, since the package returns just one "shadow price". We refer to Gauvin [6], Akgül [2] and Greenberg [8] for a theoretical analysis of left and right shadow prices. From Fig. 1 it is clear that the rate of change is constant on a linear piece of the optimal value function. Hence the answer to Question 2 must be a linearity interval. Obviously, in a breakpoint two answers are possible, depending on the sign of change of c_j. Note that in Example 1 the LP-packages do not provide the same intervals, hence they typically do not give the complete linearity interval.

In the next section we will show how breakpoints, linearity intervals and shadow prices can be computed, by solving auxiliary LP problems defined over the *optimal sets* of (P) and (D).

3. A framework for sensitivity analysis

We consider the primal and dual LP problems (P) and (D). The sets of feasible solutions of (P) and (D) are denoted by $\mathcal{P}$ and $\mathcal{D}$, whereas the sets of optimal solutions are given by $\mathcal{P}^*$ and $\mathcal{D}^*$. Assuming A to have full rank we can identify any feasible $s \geq 0$ with a unique y such that $A^T y + s = c$, and vice versa; hence we will sometimes just use $y \in \mathcal{D}^*$ or $s \in \mathcal{D}^*$ instead of $(y, s) \in \mathcal{D}^*$.

We will study the pair of LP problems (P) and (D) as their right-hand sides b and c change. Therefore, we index the problems as $(\mathrm{P}(b,c))$ and $(\mathrm{D}(b,c))$. We denote the optimal value function by $z(b,c)$. Specifically we are interested in the behavior of the optimal value function as one parameter changes. Although this is a severe restriction, it is both common from a theoretical and a computational point of view, since the multi-parameter case is very hard (see e.g. Ward and Wendell [16] for a practical approximative approach). So, let Δb and Δc be given perturbation vectors and define

$$b(\beta) := b + \beta\Delta b, \quad f(\beta) := z(b(\beta), c),$$
$$c(\gamma) := c + \gamma\Delta c, \quad g(\gamma) := z(b, c(\gamma)).$$

In the next lemma we prove a well-known elementary fact on the optimal value function.

Lemma 2. *The optimal value function $f(\beta)$ is convex and piecewise linear in β, while $g(\gamma)$ is concave and piecewise linear in γ.*

Proof. By definition

$$f(\beta) = \max_y \{b(\beta)^T y : y \in \mathcal{D}\}.$$

If $f(\beta)$ has a finite value, the optimal value is attained at one of the vertices of $\mathcal{D}$. Since this number is finite it holds

$$f(\beta) = \max_y \{b(\beta)^T y : y \in \mathcal{S}\},$$

where $\mathcal{S}$ is a finite set, viz. the set of vertices of $\mathcal{D}$. For each $y \in \mathcal{S}$ we have

$$b(\beta)^T y = b^T y + \beta \Delta b^T y,$$

which is linear in β. So $f(\beta)$ is the maximum of a finite set of linear functions, which implies the first statement. The second can be shown similarly. □

The next two lemmas show that the set of optimal solutions for $(\mathrm{D}(b(\beta), c))$ (being denoted by $\mathcal{D}^*_\beta$) is constant on a linearity interval of $f(\beta)$ and changes in its breakpoints. Similar results can be obtained for variations in c and are therefore omitted.

Lemma 3. *If $f(\beta)$ is linear on the interval $[\beta_1, \beta_2]$ then the optimal set $\mathcal{D}^*_\beta$ is constant on (β_1, β_2).*

Proof. Let $\overline{\beta} \in (\beta_1, \beta_2)$ be arbitrary and let $\overline{y} \in \mathcal{D}^*_{\overline{\beta}}$ be arbitrary as well. Then

$$f(\overline{\beta}) = b^T \overline{y} + \overline{\beta} \Delta b^T \overline{y},$$

and, since y is feasible for all β

$$b(\beta_1)^T \overline{y} = b^T \overline{y} + \beta_1 \Delta b^T \overline{y} \leq f(\beta_1), \text{ and}$$
$$b(\beta_2)^T \overline{y} = b^T \overline{y} + \beta_2 \Delta b^T \overline{y} \leq f(\beta_2).$$

Using the linearity of $f(\beta)$ on $[\beta_1, \beta_2]$ yields

$$\Delta b^T \overline{y} \leq \frac{f(\beta_2) - f(\overline{\beta})}{\beta_2 - \overline{\beta}} = \frac{f(\overline{\beta}) - f(\beta_1)}{\overline{\beta} - \beta_1} \leq \Delta b^T \overline{y}.$$

So all the above inequalities are equalities and we obtain $f'(\overline{\beta}) = \Delta b^T \overline{y}$, which in turn implies

$$f(\tilde{\beta}) = b^T \overline{y} + \tilde{\beta} \Delta b^T \overline{y} = b(\tilde{\beta})^T \overline{y}, \quad \forall \tilde{\beta} \in [\beta_1, \beta_2]. \tag{7}$$

Hence $\overline{y} \in \mathcal{D}^*_\beta$ for all $\beta \in [\beta_1, \beta_2]$. From this we conclude that the sets $\mathcal{D}^*_\beta$ are constant for $\beta \in (\beta_1, \beta_2)$. □

Corollary 4. *Let $f(\beta)$ be linear on the interval $[\beta_1, \beta_2]$ and denote $\overline{\mathcal{D}}^* := \mathcal{D}^*_\beta$ for arbitrary $\beta \in (\beta_1, \beta_2)$. Then $\overline{\mathcal{D}}^* \subseteq \mathcal{D}^*_{\beta_1}$ and $\overline{\mathcal{D}}^* \subseteq \mathcal{D}^*_{\beta_2}$.*

Observe that the proof of the lemma reveals that $\Delta b^T y$ must have the same value for all $y \in \mathcal{D}^*_\beta$ for all $\beta \in (\beta_1, \beta_2)$. We will next deal with the converse implication.

Lemma 5. *Let β_1 and β_2 be such that $\mathcal{D}^*_{\beta_1} = \mathcal{D}^*_{\beta_2} =: \overline{\mathcal{D}}^*$. Then $\mathcal{D}^*_\beta = \overline{\mathcal{D}}^*$ for $\beta \in [\beta_1, \beta_2]$ and $f(\beta)$ is linear on this interval.*

Proof. Let $\bar{y} \in \overline{\mathcal{D}}^*$ be arbitrary. Then

$$f(\beta_1) = b(\beta_1)^T\bar{y}, \text{ and } f(\beta_2) = b(\beta_2)^T\bar{y}.$$

Consider the linear function $h(\beta) := b(\beta)^T\bar{y}$. Note that $h(\beta_1) = f(\beta_1)$ and $h(\beta_2) = f(\beta_2)$. Since f is convex it thus holds $f(\beta) \le h(\beta)$ for $\beta \in [\beta_1, \beta_2]$. On the other hand, since $\bar{y}$ is feasible for all β we have

$$f(\beta) \ge b(\beta)^T\bar{y} = h(\beta).$$

Hence $f(\beta)$ is linear on $[\beta_1, \beta_2]$ and $\bar{y} \in \mathcal{D}^*_\beta$ for all $\beta \in [\beta_1, \beta_2]$. Hence $\overline{\mathcal{D}}^*$ is a subset of the optimal set on (β_1, β_2). From Corollary 4 we know the reverse also holds, hence for all $\beta \in (\beta_1, \beta_2)$ the optimal set equals $\overline{\mathcal{D}}^*$. □

As we have seen in the proof of Lemma 3 the quantity $\Delta b^T y$ is the same for all $y \in \mathcal{D}^*_\beta$ for β in a linearity interval. The next lemma shows that this property distinguishes a linearity interval from a breakpoint. Gauvin [6] was one of the first[3] to show this result and to emphasize the need to discriminate between *left and right shadow prices*, i.e., between decreasing and increasing the parameter.

Lemma 6. *Let $f'_-(\beta)$ and $f'_+(\beta)$ be the left and right derivative of $f(\cdot)$ in β. Then*

$$f'_-(\beta) = \min_y\{\Delta b^T y : y \in \mathcal{D}^*_\beta\},$$
$$f'_+(\beta) = \max_y\{\Delta b^T y : y \in \mathcal{D}^*_\beta\}.$$

Proof. We give the proof for $f'_+(\beta)$; the one for $f'_-(\beta)$ is similar. Let $\overline{\beta}$ be in the linearity interval just to the right of β and let $\bar{y} \in \mathcal{D}^*_{\overline{\beta}}$. Then

$$f(\overline{\beta}) = b(\overline{\beta})^T\bar{y} \ge (b + \overline{\beta}\Delta b)^T y, \quad \forall y \in \mathcal{D}^*_\beta.$$

Since $\bar{y} \in \mathcal{D}^*_\beta$ by Corollary 4 we also have $b^T y = b^T\bar{y}$, $\forall y \in \mathcal{D}^*_\beta$. Hence

$$\Delta b^T y \le \Delta b^T \bar{y}, \quad \forall y \in \mathcal{D}^*_\beta.$$

Since $\bar{y} \in \mathcal{D}^*_\beta$ and $f'_+(\beta) = f'(\overline{\beta}) = \Delta b^T\bar{y}$ this implies the result. □

Note that the shadow prices are computed by solving auxiliary (LP) problems defined over the optimal set of (D). We now show that a linearity interval can be obtained with a similar procedure.

Lemma 7. *Let β_1, β_2 be two consecutive breakpoints of the optimal value function $f(\beta)$. Let $\overline{\beta} \in (\beta_1, \beta_2)$ and define $\overline{\mathcal{D}}^* := \mathcal{D}^*_{\overline{\beta}}$. Then*

$$\beta_1 = \min_{\beta, x}\{\beta : Ax - \beta\Delta b = b,\ x \ge 0,\ x^T s = 0,\ \forall s \in \overline{\mathcal{D}}^*\},$$

$$\beta_2 = \max_{\beta, x}\{\beta : Ax - \beta\Delta b = b,\ x \ge 0,\ x^T s = 0,\ \forall s \in \overline{\mathcal{D}}^*\}.$$

Proof. We will only give the proof for the minimization problem. By Lemma 3 $\overline{\mathcal{D}}^*$ is the optimal set for all $\beta \in (\beta_1, \beta_2)$. Observe that the minimization problem is convex; let (β^*, x^*) be a solution to it. Obviously x^* is also optimal in $(P(b(\beta^*), c))$ with optimal value $(b + \beta^*\Delta b)^T y$ for arbitrary $y \in \overline{\mathcal{D}}^*$. Hence $\beta^* \ge \beta_1$. On the other hand, let $x^{(1)}$ be optimal in $(P(b(\beta_1), c))$. By Corollary 4 it holds $(x^{(1)})^T s = 0$, $\forall s \in \overline{\mathcal{D}}^*$. Hence the pair $(\beta_1, x^{(1)})$ is feasible in the minimization problem and we have $\beta^* \le \beta_1$. This completes the proof. □

4. Three approaches

Considering the results in Section 3, we see that computation of linearity intervals and shadow prices can be done unambiguously using optimal sets, contrarily to what is usually done by using just one optimal solution. Next we give three approaches based on the use of optimal sets, motivated by three different but equivalent ways of describing the optimal set. The first uses optimal partitions, the second optimal values and the third (primal/dual) optimal bases.

[3] Personal communication 1992; Gauvin's paper is not mentioned in the historical survey by Gal [5].

4.1. Using optimal partitions

In 1956 Goldman and Tucker [7] showed that in each primal-dual pair of LP problems a strictly complementary solution (x^*, s^*) exists. Such a solution satisfies $x_i^* + s_i^* > 0$, for all i, hence exactly one of x_i^* and s_i^* is zero. A strictly complementary solution uniquely determines the *optimal partition* of the LP problem, denoted as $\pi = (B, N)$, where

$$B := \{i: x_i > 0 \text{ for some } x \in \mathcal{P}^*\},$$
$$N := \{i: s_i > 0 \text{ for some } (y, s) \in \mathcal{D}^*\}.$$

Using the optimal partition we may rewrite the primal and dual optimal sets as

$$\mathcal{P}^* = \{x: Ax = b,\ x_B \geq 0,\ x_N = 0\},$$
$$\mathcal{D}^* = \{(y, s): A^{\mathrm{T}}y + s = c,\ s_B = 0,\ s_N \geq 0\}.$$

Here x_B denotes the part of the vector x corresponding to the indices in B, etc. Using this description of the optimal sets in the auxiliary LP problems of Section 3 we may conclude that the optimal partition remains constant on a linearity interval and changes in a breakpoint. For further details we refer to Adler and Monteiro [1], and Jansen et al. [10]. For some examples of practical LP problems where the knowledge of the optimal partition is essential we refer to Greenberg [9].

It is important to observe that the optimal partition enables to divide the variables in important variables (index in B) and unimportant variables (index in N), since for *any* optimal solution of (P) none of the variables with indices in N can have a positive value. Similarly, in the dual problem, all constraints with index in B are binding in *any* optimal solution of (D).

Example 8. Using the optimal partition we obtain the sensitivity output of Example 1 as given in Table 3. The solution given is strictly complementary. For the cost coefficients we note that they are all in a breakpoint of their optimal value functions. Hence left and right shadow prices are of interest. For the right hand side coefficients complete linearity intervals are given. Observe that the commercial LP-codes all gave subintervals of the complete linearity intervals for the RHS coefficients. Note that the shadow prices for the constraints are equal to the dual solution.

4.2. Using optimal values

Once knowing the optimal value of the LP problem, we can describe the optimal faces as follows:

$$\mathcal{P}^* = \{x: Ax = b,\ c^{\mathrm{T}}x = z^*,\ x \geq 0\},$$
$$\mathcal{D}^* = \{y, s: A^{\mathrm{T}}y + s = c,\ b^{\mathrm{T}}y = z^*,\ s \geq 0\},$$

where z^* is the (known!) optimal value. Given a pair of optimal solutions (x^*, y^*, s^*), we can also use $z^* = c^{\mathrm{T}}x^* = b^{\mathrm{T}}y^*$. To obtain linearity intervals and shadow prices using this approach implies solving auxiliary LP-problems. For example, to find the linearity interval when the objective coefficient c_j is changed, one needs to solve the two LP-problems:

$$\gamma_{\min} = \min\{\gamma: A^{\mathrm{T}}y + s = c + \gamma e_j,\ b^{\mathrm{T}}y = c^{\mathrm{T}}x^* + \gamma x_j^*,\ s \geq 0\},$$
$$\gamma_{\max} = \max\{\gamma: A^{\mathrm{T}}y + s = c + \gamma e_j,\ b^{\mathrm{T}}y = c^{\mathrm{T}}x^* + \gamma x_j^*,\ s \geq 0\}.$$

For right-hand side coefficients analogous problems using the primal feasible set need to be considered. Note also, that the validity of the resulting ranges stems from the concavity, respectively convexity of the optimal value functions. An advantage of this approach is that we do not need to know the optimal partition.

4.3. Using optimal bases

When the simplex method is used for solving a LP problem an optimal basic solution is obtained. A basis of A is a set of m indices, denoted by $\mathcal{B}$, such that the submatrix $A_{\mathcal{B}}$ of A is nonsingular. The corresponding variables are called the basic variables. The indices of the remaining nonbasic variables are in $\mathcal{N}$. Given a basis $\mathcal{B}$, the associated primal basic solution x is given by

$$x = \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{N}} \end{pmatrix} := \begin{pmatrix} A_{\mathcal{B}}^{-1}b \\ 0 \end{pmatrix},$$

and the dual basic solution by

$$y = A_{\mathcal{B}}^{-\mathrm{T}}c_{\mathcal{B}}, \quad s = \begin{pmatrix} s_{\mathcal{B}} \\ s_{\mathcal{N}} \end{pmatrix} := \begin{pmatrix} 0 \\ c_{\mathcal{N}} - A_{\mathcal{N}}^{\mathrm{T}}y \end{pmatrix}.$$

Table 3
Sensitivity output using the optimal partition in Example 1

	primal								
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}
str. comp. solution	0.62	0.62	0.62	1.45	1.45	1.45	0.93	0.93	0.93
ranges	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]	[1,1]
shadow prices	2 0	2 0	2 0	3 0	3 0	3 0	3 0	3 0	3 0

	dual					
	y_1	y_2	y_3	y_4	y_5	y_6
str. comp. solution	0	0	0	1	1	1
ranges	$[0,\infty)$	$[2,\infty)$	$[1,\infty)$	[0,7]	[0,7]	[0,7]
shadow prices	0	0	0	1	1	1

If $x_{\mathcal{B}} \geq 0$ then $\mathcal{B}$ is a primal feasible basis; if $s_{\mathcal{N}} \geq 0$ then $\mathcal{B}$ is dual feasible. We call the basis *optimal* if it is both primal and dual feasible; a basis is called *primal optimal* if the associated primal basic solution is optimal for (P); analogously, a basis is called *dual optimal* if the associated dual basic solution is optimal for (D). Note that a primal (dual) optimal basis need not be dual (primal) feasible.

As follows immediately from the definition, to describe the primal (resp. dual) optimal set with bases we need to consider the primal (resp. dual) optimal bases. Hence, the linearity interval can be obtained by taking the union of the ranges provided by each of the primal (resp. dual) optimal bases. The reason that different answers are obtained from different LP packages in Example 1 is explained by the degeneracy apparent in the problem, whence the optimal basis might not be unique and/or the optimal primal or dual solution might not be unique. Moreover, it is not sufficient to consider (all) optimal bases, as suggested by, e.g., [3–5,8,13].

We emphasize once more that the ranges and the shadow prices provided by commercial LP-codes can only be given a valid interpretation in combination with each other. A shadow price without a range is of no practical value and can lead to fallable decisions.

5. Application to a refinery model

5.1. Introduction

In this section we will apply the complete sensitivity approach as discussed in Section 4 to a practical LP-model of a refinery. In the model production, transportation and product exchange are involved, during three periods. The model has 2110 variables and 1101 constraints; there are equality as well as inequality constraints. For the computation of the simplex-ranges we used the package CPLEX. For the new approach we made an implementation in OSL. OSL is a library of routines for various optimization purposes. We solved the initial LP-problem by an interior point method in OSL and for each of the coefficients in the right hand side and the objective function we solved the auxiliary LP-problems by a simplex-subroutine. Although these are quite a number of problems, computer techniques are so well-developed nowadays, that we can get an optimal solution along with *all* the sensitivity information for the refinery model within 15 minutes on an HP720-workstation.

In this section we summarize the results of our experiments. We first consider the coefficients in the objective function, then we look at the right hand side.

Finally, we give some examples to show what the managerial implications may be from incorrect interpretation of sensitivity information given by a standard LP-package.

5.2. Changes in the objective coefficients

We compare the sensitivity information from CPLEX with the correct numbers. We first consider the shadow prices, or the rate at which the objective function changes. Recall from Section 2 that different situations may occur: either the number given by CPLEX will be the shadow price, or a left or a right shadow price (a one-sided shadow price) or it has no meaning as a shadow price at all. In the last case, the value is in between the left and the right shadow price.

In the refinery model, we found that for 4.8% of the coefficients the number returned by CPLEX was just a one-sided shadow price or no shadow price at all. Moreover, for 39.3% of the coefficients the range CPLEX reports is not equal to a complete linearity interval. Only a minor percentage of these were due to the fact that the coefficient was in a breakpoint. These results are summarized in Table 4.

To see whether the differences in the ranges are significant, we split them up into six categories, depending on the ratio of the length of the linearity interval and the length of the range computed by CPLEX. A histogram of this is given in Fig. 2. Here, the category *breakp* means that the coefficient is in a breakpoint while CPLEX reports a range. If the linearity interval has infinite length then we use the category $100 - inf$. For example, the category $1.1 - 10$ means, that the linearity interval is 1.1 to 10 times larger than the CPLEX-range.

The same we did for the shadow prices. We compared the CPLEX outcome with both the left and right shadow price and categorized the greatest difference (that is, either the difference between the CPLEX outcome and the left shadow price or between CPLEX and the right shadow price). If the current value of the objective coefficient is not a breakpoint, then the CPLEX number will be the unique shadow price. See Fig. 3; the category *inf* means that the LP-problem gets unbounded as soon as the objective coefficient is either increased or decreased.

We conclude from the tables above that the two approaches to sensitivity analysis give really significant

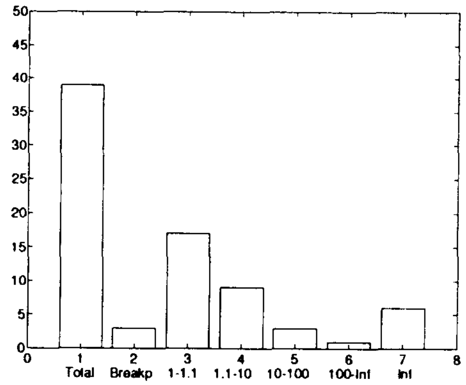

Fig. 2. Differences in objective coefficient ranges (%).

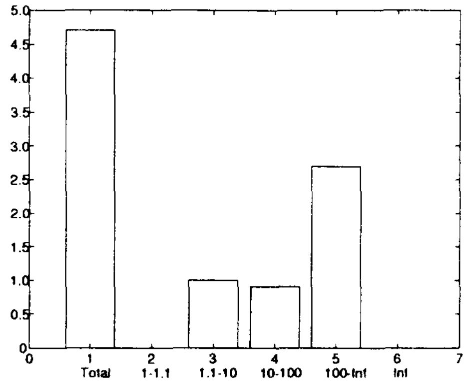

Fig. 3. Differences in shadow prices for the objective coefficients (%).

differences, and that a commercial LP-code only gives partial sensitivity information.

Let us examine the LP-model of the refinery more closely. The variables in the problem are divided into different groups, according to their physical meaning. For example, there are variables connected to transportation, to supplies, to product exchange, etc. It is interesting to see whether there is a relation between groups of variables and the completeness of the sensitivity information. In Table 5 we have made a split-up of the differences. When the differences are of a specific order or due to a specific reason, a comment can be found in the last column.

It is clear that the ranges and shadow costs returned by CPLEX are quite similar to the correct values for the cost coefficients dealing with transport to and exchange with other refineries and markets. Those deal-

Table 4
Objective coefficient ranges and shadow costs. (Different answers given by CPLEX)

Different ranges (%)			Different shadow prices (%)		
Total	In breakpoint	Other	Total	One side	No shadow price
39.3	2.9	36.4	4.7	4.0	0.7

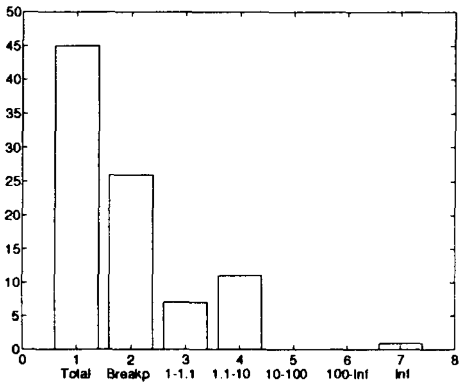

Fig. 4. Differences in RHS ranges (%).

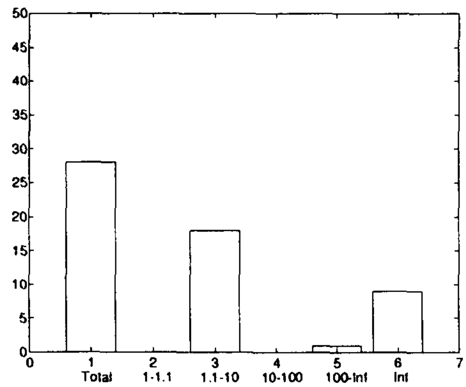

Fig. 5. Differences in shadow prices for RHS coefficients (%).

ing with processes in the refinery are showing great differences. For the transport costs and import costs we observe that differences depend on the period under consideration. We see that the shadow prices are different for the Local Supply, Product Blend, Total Blended and Volume Loss/Gain. This means in fact that there are different optimal solutions for the LP-problem which have different values assigned to the variables in these groups.

5.3. Changes in the RHS

For the RHS elements of the refinery model we made the same comparison as we did for the objective coefficient case. The percentage of RHS elements at a breakpoint is 45.5% now. The breakdown is given in Table 6. Note that the number of cases in which just a one-sided shadow price is given by CPLEX is quite large: 25.4%.

Again, we made histograms to split the differences with respect to their magnitude, see Fig. 4 for the ranges and Fig. 5 for the shadow prices.

Furthermore, we divided the constraints into several groups, each dealing with specific aspects of the refinery problem. For each group, the differences are listed in Table 7.

A great deal of the differences in the ranges are caused by breakpoints. In many cases CPLEX gives an allowable interval for the RHS coefficient to vary, where it should give just one point (the current value). Except for the constraints dealing with Unit Capacity and Fuel, differences between CPLEX and IPM are significant and very frequent.

5.4. Specific examples

To show what the impact might be of fallacious interpretation of shadow prices and intervals from LP-packages, we highlight two coefficients in the refinery-model. We computed (part of) their optimal value function and showed what is obtained by CPLEX as compared to the complete information.

In Fig. 6 the optimal value function is drawn for the RHS element of the constraint that specifies the maximum availability of a certain crude in Period 3. The current value is 80.00.

As can be seen from the figure, the slope of the function changes for several values of the maximum-

Table 5
Ranges and shadow costs per group of objective coefficients

Description	Example	%-age of different ranges	%-age of different shadow prices	Severity of differences of the ranges
Different ranges and shadow prices for objective coefficients				
Transportcosts per product from market X to market Y	021PR04.	per. 1: 8 per. 2,3: 0	0	cat. 1 – 1.1
Product exchange	H.1..01 021PR.02	0	0	
Global supply	021PRS1 021PRSS	1	0	cat. 1 – 1.1
Importcosts per product for each refinery. TIER 1	021PRI1	per. 1: 10 per. 2,3: 0	0	cat. 1 – 1.1
Refinery				
Purchase prices crudes	AA1000	50	0	cat. inf
Util. purchase electricity (1 variable per period)	UA102P	100	0	CPLEX gives a point, IPM an interval
Imports TIER 1	IA1PR.1	0	0	
Local supply	SA1GP..1	20	7	all cat.
Product exchange	HA1PR01	0	0	
Stock transfercosts per product (6 variables)	OA1BI FA1BI	0	0	
Process costs per crude	PA1O001	60	0	all cat.
Maximum unit days per period	CA1CDMO	70	0	all cat.
Internal transfer	DA1001	50	0	all cat.
Product blend	YA1GP605	50	13	all cat.
Refinery total blended per product weight/volume	XA1GPW XA1GPV	60	3	cat. 1.1 – 100
Volume loss / gain bonus / penalty (10 var.)	XA1PRBON XA1PRPEN	90	100	all at breakpoint; CPLEX gives intervals
REFU	XA120C MA120	100	0	all cat.
Electricity costs (2 variables)	UA101Y UA102Y	0	0	cat. 10 – 100
Transportcosts per product from refinery to market	TA1PR02 JA1IG03	per. 1,2: 5 per. 3: 0	0	cat. 1 – 1.1

Table 6
RHS ranges and shadow prices. (Different answers given by CPLEX)

Different ranges (%)			Different shadow prices (%)		
Total	In breakpoint	Other	Total	One side	No shadow price
45.5	26.3	19.2	28.2	25.4	2.8

Table 7
Ranges and shadow prices per group of RHS coefficients. (Different ranges and shadow prices for the RHS coefficients)

Description	Example	%-age of different ranges	%-age of different shadowprices	Severity of the ranges
Maximum and minimum availability per crude	A.1000M A.1000N	60	60	many breakp, CPLEX gives intervals
Product exchange	G.1PR01	30	30	per. 1,2: all breakp, per. 3: cat. 1.1 – 10
Product equation	.021PR	per. 1,2: 10 per. 3: 50	8	per. 1,2: all breakp, per. 3: cat. 1.1 – 10
Maximum process unit capacity	CA1CDM	15	0	cat. 1 – 1.1
Component balance per product	BA1000	60	30	30% breakp, 20% cat. 1 – 1.1 rest cat. 1.1 – 10
Volume and weight totals, min and max.	SA../QA.. RA1PR1MV	30	10	all cat.
Min gas to fuel, sulphur, refu balance (11 var)	RA1REFN RA3REFU	9	0	cat. 1 – 1.1

availability. Since the current value (80) is not at a breakpoint, CPLEX will return the correct shadow price 2.80. However, the range that CPLEX gives, is not the complete linearity interval, but [74.5, 80.5]. This means that a manager using the sensitivity result of CPLEX can only say what happens to the optimal value in this interval, and does not know what happens outside of this interval. In fact, in this case, he almost has complete information, although he does not know he has. For the maximum availability of the same crude in Period 2 we found that CPLEX gave the interval [77.01, 89.25], only a portion of the linearity interval [77.01, 94.6].

For the second example we consider one of the coefficients in the objective function, namely one which is the profit per unit supply for a certain crude. The current value is 325. A part of the corresponding optimal value function is drawn in Fig. 7.

The current value (325) is a breakpoint of the optimal value function. In this case CPLEX gave the shadow price to the right, which is 4.29. However, the interval connected to this number was reported to be [325, 334], while the complete linearity interval is much larger, namely [325, 505]. Also, the CPLEX-user can conclude nothing when the profit per unit would decrease, whereas from the figure it is clear that

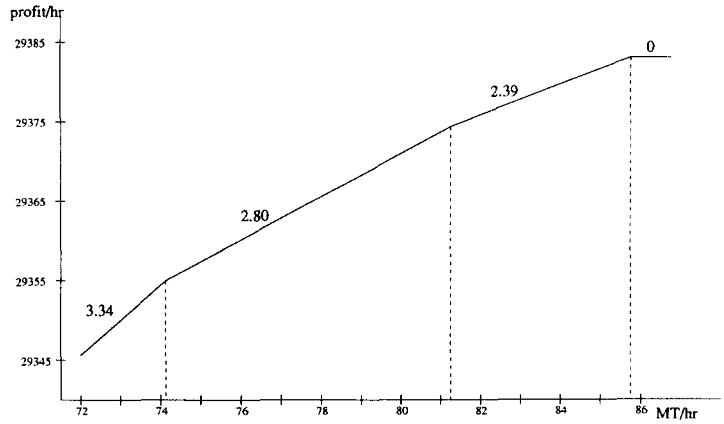

Fig. 6. Optimal value function for a RHS-coefficient.

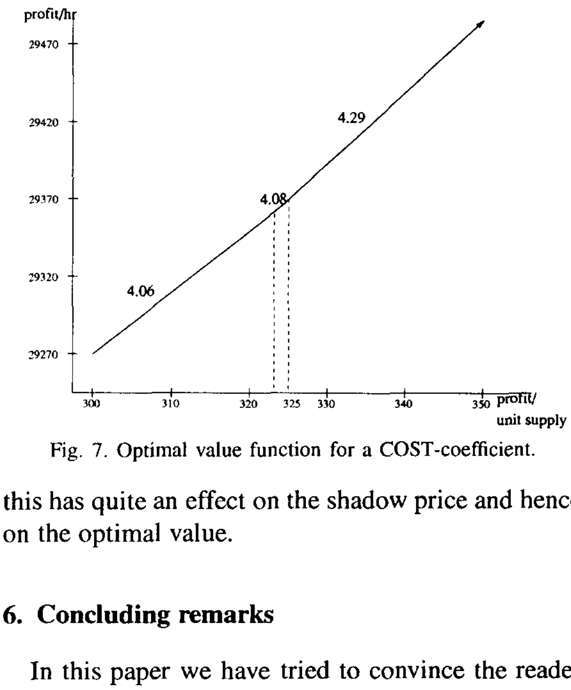

Fig. 7. Optimal value function for a COST-coefficient.

this has quite an effect on the shadow price and hence on the optimal value.

6. Concluding remarks

In this paper we have tried to convince the reader that he should be careful in interpreting the sensitivity output of standard LP-packages. We stress that the reader should always be prudent in having clearly in mind in what he would like to know and to what extent the information provided by the package does give an answer to his question. In particular, we stress that shadow prices and ranges always go together, in the sense that one only knows that they are valid in the ranges specified and does not know anything outside the ranges.

On the other hand, we showed that it is possible to obtain complete sensitivity information for a practical LP-problem of a refinery in moderate computational effort. Of course, full information is more costly than partial information. Therefore we suggest the following compromise. Since the computation of the simplex-information using a standard LP-package is very cheap, we propose to do this for all the variables, while keeping clearly in mind that the shadow prices obtained are only valid in the intervals that are computed along with them. Then the full analysis can be done for some user-specified coefficients. We mention, that this is quite viable in practice, since many coefficients are just coupled to internal variables in the model and not directly interesting for managerial purposes. Even better it is to compute (part of) the optimal value function for important coefficients, since this gives the user a global view of the impact of the changes he considers.

Finally, we mention that all approaches to sensitivity analysis are indeed independent of the method used to solve the initial problem. However, each method for linear programming suggests its own way performing the analysis. The simplex method suggests using bases, interior point methods naturally suggest optimal partitions. De Jong [11] made an implementation of both the optimal partition approach and the optimal value approach. From this research it can be concluded that for many LP problems the differences with ordinary sensitivity analysis, performed by commercial software, is considerable. From a computational point of view, it appears that the optimal partition approach is often faster and more accurate than the optimal value approach. On the other hand, in some problems it is hard to determine the optimal partition, since in an optimal solution a variable may have a positive but very small value. In most cases, it appeared to be necessary to scale the problem. In both approaches, the subproblems itself can be solved by either a simplex routine or an interior point method. Depending on the actual size, a choice between them should be made.

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