$$F(m) = \min_{0 \le i < m} \{F(i) + w(i,m)\}$$

| 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a | | the | | i | | and | | his | | her | | arm | | but | |

F(1) =
min{
0 + w(0,1) = 0 + 25 = 25
}

F(2) =
min{
0 + w(0,2) = 0 + 4 = 4
F(1) + w(1,2) = 25 + 9 = 34
}

F(3) =
min{
0 + w(0,3) = 0 + 1 = 1
F(1) + w(1,3) = 25 + 4 = 29
F(2) + w(2,3) = 4 + 25 = 29
}

$F(i)$

| |
|---|
| 0 |
| 25 |
| 4 |
| 1 |
| 4 |
| 1 |
| 4 |
| 1 |

w(i,m) Table

m

i

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | – | – | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | – | – | – | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | – | – | – | – | 9 | 0 | 9 | 36 | 81 |
| 4 | – | – | – | – | – | 9 | 0 | 9 | 36 |
| 5 | – | – | – | – | – | – | 9 | 0 | 9 |
| 6 | – | – | – | – | – | – | – | 9 | 0 |
| 7 | – | – | – | – | – | – | – | – | 9 |
| 8 | – | – | – | – | – | – | – | – | – |

$$F(m) = \min_{0 \le i < m} \{ F(i) + w(i,m) \}$$

$F(i)$

| |
|---|
| |
| 0 |
| 2<br>5 |
| 4 |
| 1 |
| 4 |
| 1 |
| 4 |
| 1 |

$w(i,m)$

| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | 1 | - | - | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| | 2 | - | - | - | 25 | 4 | 1 | 16 | 49 | 100 |
| | 3 | - | - | - | - | 9 | 0 | 9 | 36 | 81 |
| i | 4 | - | - | - | - | - | 9 | 0 | 9 | 36 |
| | 5 | - | - | - | - | - | - | 9 | 0 | 9 |
| | 6 | - | - | - | - | - | - | - | 9 | 0 |
| | 7 | - | - | - | - | - | - | - | - | 9 |
| | 8 | - | - | - | - | - | - | - | - | - |

$$F(m) = \min_{0 \le i < m} \{F(i) + w(i,m)\}$$

$F(i)$

| |
|---|
| 0 |
| 25 |
| 4 |
| 1 |
| 4 |
| 1 |
| 4 |
| 1 |

$w(i,m)$

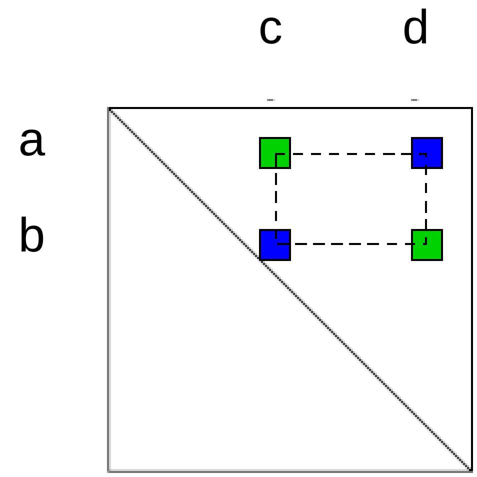
| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | - | - | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | - | - | - | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | - | - | - | - | 9 | 0 | 9 | 36 | 81 |
| 4 | - | - | - | - | - | 9 | 0 | 9 | 36 |
| 5 | - | - | - | - | - | - | 9 | 0 | 9 |
| 6 | - | - | - | - | - | - | - | 9 | 0 |
| 7 | - | - | - | - | - | - | - | - | 9 |
| 8 | - | - | - | - | - | - | - | - | - |

$OUT(i,m) = F(i) + w(i,m)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |

1. **Add F(i) to each cell (i,m) in row i of w → OUT(i,m)**
2. **Compute the column minima of the OUT matrix, such that F(i) ← the minimum of column i**

# Convexity/Concavity Properties of the tables W and OUT




**(1) Monge (concave)**

**W[b, d] - W[a, d ] < = W[b, c] - W[a, c]**

**for all a< b and c< d**

**Monge (convex)**

**W[b, d] - W[a, d ] > = W[b, c] - W[a, c]**

**for all a< b and c< d**

**(2) Total Monotonicity (concave)**

**W[b, c] < = W[a, c ] => W[b,d] ] < = W[a, d]**

**for all a< b and c< d**

**Total Monotonicity (convex)**

**W[b, c] > = W[a, c ] => W[b,d] ] > = W[a, d]**

**for all a< b and c< d**

m

i

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | - | - | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | - | - | - | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | - | - | - | - | 9 | 0 | 9 | 36 | 81 |
| 4 | - | - | - | - | - | 9 | 0 | 9 | 36 |
| 5 | - | - | - | - | - | - | 9 | 0 | 9 |
| 6 | - | - | - | - | - | - | - | 9 | 0 |
| 7 | - | - | - | - | - | - | - | - | 9 |
| 8 | - | - | - | - | - | - | - | - | - |

## W is Monge (concave) and therefore W is Totally Monotone (concave)

W[b, d] - W[a, d ] <= W[b, c] - W[a, c] for all a< b and c< d

W[b, c] <= W[a, c ] => W[b,d] ] <= W[a, d] for all a< b and c< d

0 1 2 3 4 5 6 7 8

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | the | i | and | his | her | arm | but |

i

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | – | – | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | | – | – | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | – | – | – | – | 9 | 0 | 9 | 36 | 81 |
| 4 | – | – | – | – | – | 9 | 0 | 9 | 36 |
| 5 | – | – | – | – | – | – | 9 | 0 | 9 |
| 6 | – | – | – | – | – | – | – | 9 | 0 |
| 7 | – | – | – | – | – | – | – | – | 9 |
| 8 | – | – | – | – | – | – | – | – | – |

w(i,m) =(# of characters from i to m – lineopt)$^2$

ex: W(0,4) = (8 - 6) $^2$ = 4, W(0,5) = (11 - 6) $^2$ = 25

1-4 = -3 16-25 = -9

W(1,4) = (7 - 6) $^2$ = 1, W(1,5) = (10 - 6) $^2$ = 16

# w(i,m) Table

m

i

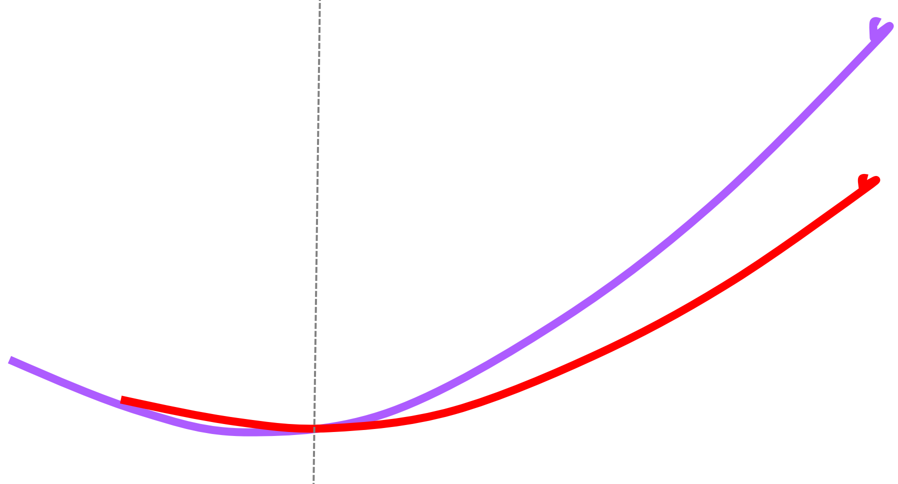
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | – | – | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | – | – | – | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | – | – | – | – | 9 | 0 | 9 | 36 | 81 |
| 4 | – | – | – | – | – | 9 | 0 | 9 | 36 |
| 5 | – | – | – | – | – | – | 9 | 0 | 9 |
| 6 | – | – | – | – | – | – | – | 9 | 0 |
| 7 | – | – | – | – | – | – | – | – | 9 |
| 8 | – | – | – | – | – | – | – | – | – |

# w(i,m) Table

m

i

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | – | – | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | – | – | – | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | – | – | – | – | 9 | 0 | 9 | 36 | 81 |
| 4 | – | – | – | – | – | 9 | 0 | 9 | 36 |
| 5 | – | – | – | – | – | – | 9 | 0 | 9 |
| 6 | – | – | – | – | – | – | – | 9 | 0 |
| 7 | – | – | – | – | – | – | – | – | 9 |
| 8 | – | – | – | – | – | – | – | – | – |



# w(i,m) Table

m

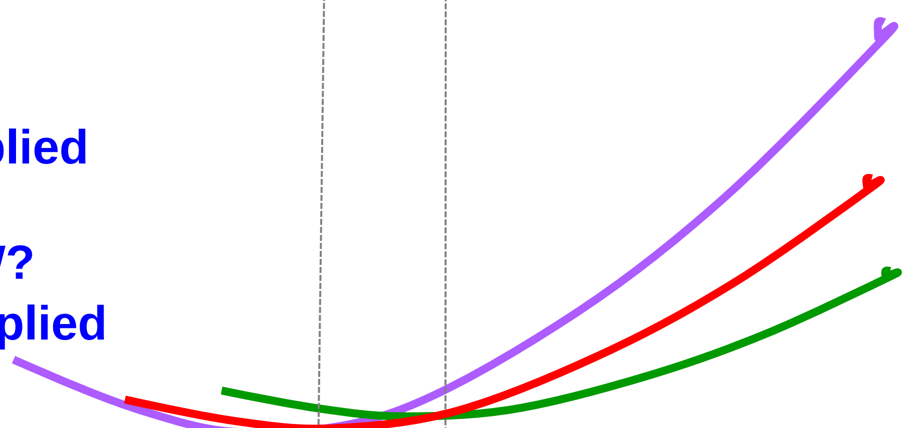
i

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | – | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | – | – | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | – | – | – | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | – | – | – | – | 9 | 0 | 9 | 36 | 81 |
| 4 | – | – | – | – | – | 9 | 0 | 9 | 36 |
| 5 | – | – | – | – | – | – | 9 | 0 | 9 |
| 6 | – | – | – | – | – | – | – | 9 | 0 |
| 7 | – | – | – | – | – | – | – | – | 9 |
| 8 | – | – | – | – | – | – | – | – | – |

**Remember SMAWK?**

**Can SMAWK be applied to find the column minima of W?**

**But can SMAWK be applied to computing F(n)?**



$$F(m) = \min_{0 \le i < m} \{F(i) + w(i,m)\}$$

$F(i)$

| |
|---|
| |
| 0 |
| 25 |
| 4 |
| 1 |
| 4 |
| 1 |
| 4 |
| 1 |

$w(i,m)$

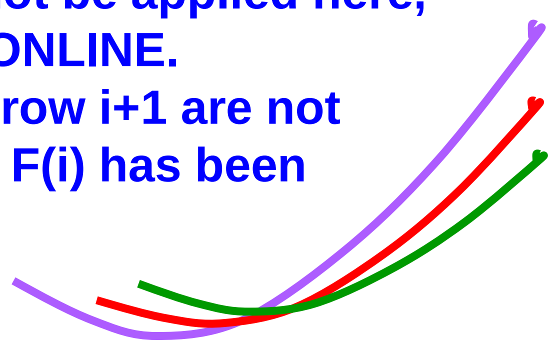
i

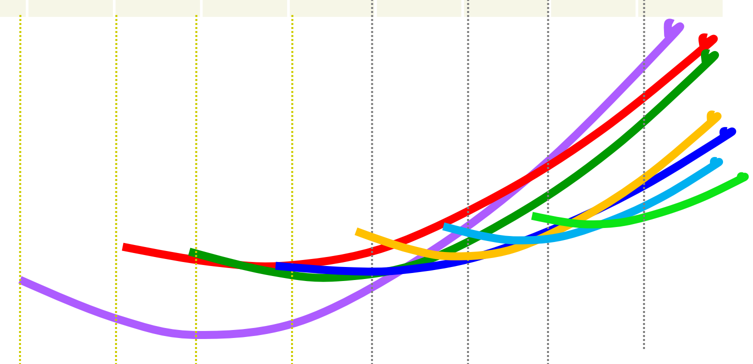
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | - | - | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | - | - | - | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | - | - | - | - | 9 | 0 | 9 | 36 | 81 |
| 4 | - | - | - | - | - | 9 | 0 | 9 | 36 |
| 5 | - | - | - | - | - | - | 9 | 0 | 9 |
| 6 | - | - | - | - | - | - | - | 9 | 0 |
| 7 | - | - | - | - | - | - | - | - | 9 |
| 8 | - | - | - | - | - | - | - | - | - |

$OUT(i,m) = F(i) + w(i,m)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |

**SMAWK can not be applied here, since OUT is ONLINE.**
**The values of row i+1 are not available until F(i) has been computed.**





# W is Monge (concave) and therefore OUT is Totally Monotone (concave)

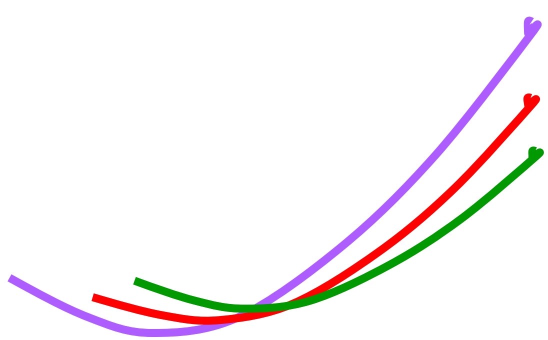
**OUT[a, c] > = OUT[b, c ] => OUT[a,d] ] > = OUT[b, d] for all a< b and c< d**

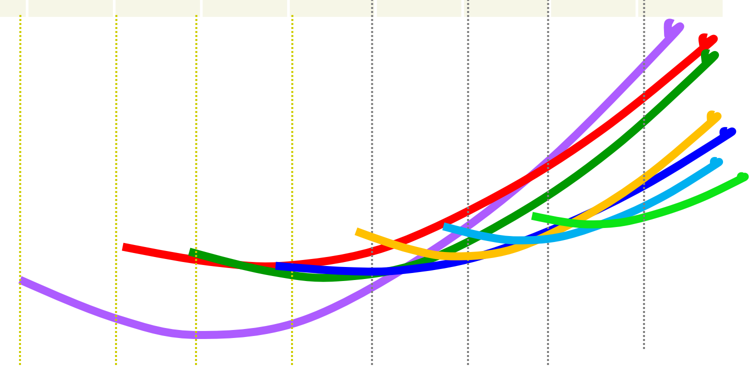
$$F(i) \qquad w(i,m) \qquad OUT(i,m) = F(i) + w(i,m)$$

| $F(i)$ |
|---|
| 0 |
| 25 |
| 4 |
| 1 |
| 4 |
| 1 |
| 4 |
| 1 |

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| 1 | - | - | 9 | 4 | 1 | 16 | 49 | 100 | 169 |
| 2 | - | - | - | 25 | 4 | 1 | 16 | 49 | 100 |
| 3 | - | - | - | - | 9 | 0 | 9 | 36 | 81 |
| 4 | - | - | - | - | - | 9 | 0 | 9 | 36 |
| 5 | - | - | - | - | - | - | 9 | 0 | 9 |
| 6 | - | - | - | - | - | - | - | 9 | 0 |
| 7 | - | - | - | - | - | - | - | - | 9 |
| 8 | - | - | - | - | - | - | - | - | - |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |





# The 1D/1D DP problem:

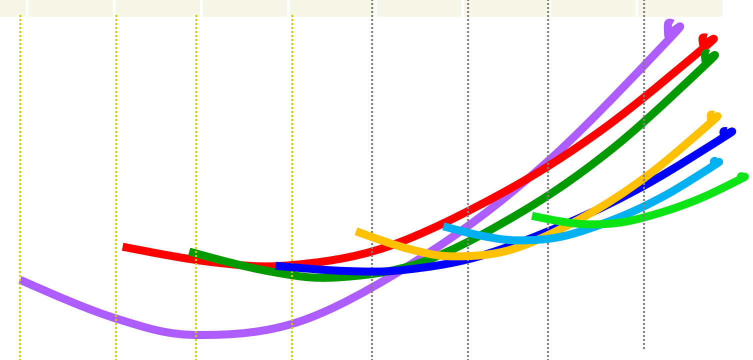
$$OUT(i,m) = F(i) + w(i,m)$$

Definition: ***The 1D/1D dynamic programming problem is to find the column minima in an***

***on-line***

***upper triangular matrix OUT.***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



# The 1D/1D DP problem:

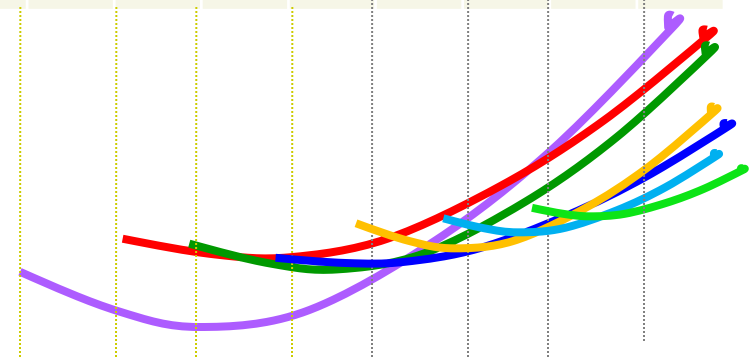
$$OUT(i,m) = F(i) + w(i,m)$$

Definition: ***The 1D/1D dynamic programming problem is to find the column minima in an***

***on-line***

***upper triangular matrix OUT.***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



# The 1D/1D DP problem:

## Procedure 1D/1D

***Initialize Queue with row 0***

***for m = 2 to n***

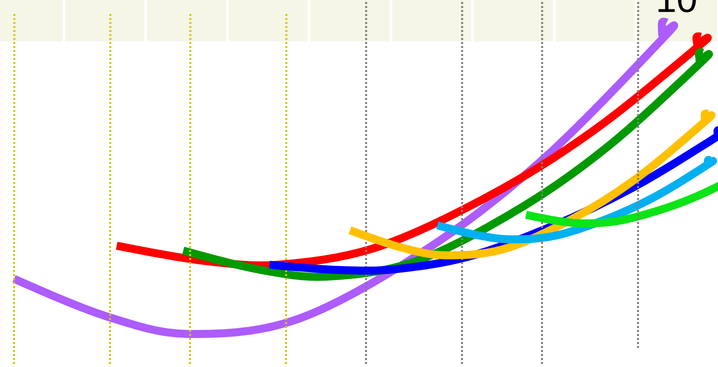
***{***

***find minimum of column m***

***update Queue with row j-1***

***}***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



# Example:

## m=1, min in row 1

$$OUT(i,m) = F(i) + w(i,m)$$

Queue:

***i=1 : OUT[1,8]***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |

# Example:

## m=2, min in row 1

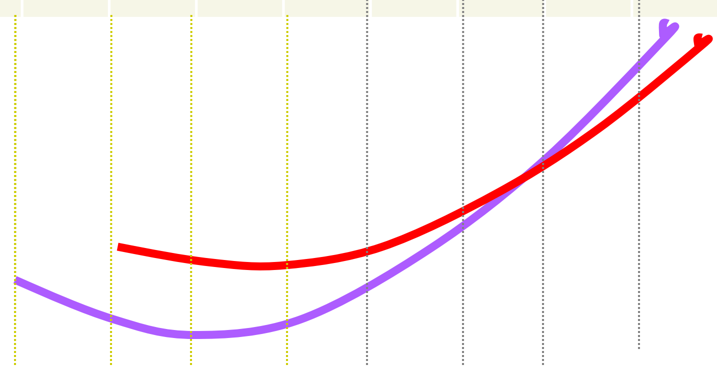
$$OUT(i,m) = F(i) + w(i,m)$$

Queue:

***i=1 :* OUT[1,8]**

***i=2 : ??***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



**Where does row 2 intersect with row 1?**

**Use binary search: intersection found in column 6…**

# Example:

## m=2, min in row 1

$$OUT(i,m) = F(i) + w(i,m)$$

Queue:

***i=1 :*** ***OUT[1,5]***

***i=2 :*** ***OUT[6,8]***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | $\boxed{25}$ | $\boxed{4}$ | $\boxed{1}$ | $\boxed{4}$ | $\boxed{25}$ | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | $\boxed{64}$ | $\boxed{125}$ | $\boxed{194}$ |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |

# Example:

Queue:

*i=1 :* ***OUT[1,5]***

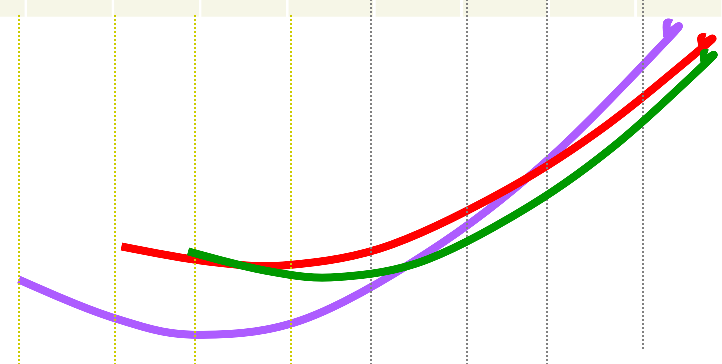
*i=2 :* ***OUT[6,8]***

*i=3:??*

**Where does row 3 intersect with row 2?**

$$OUT(i,m) = F(i) + w(i,m)$$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



**Use binary search: intersection at column 3, even before Row 2 becomes leader!**

# Example:

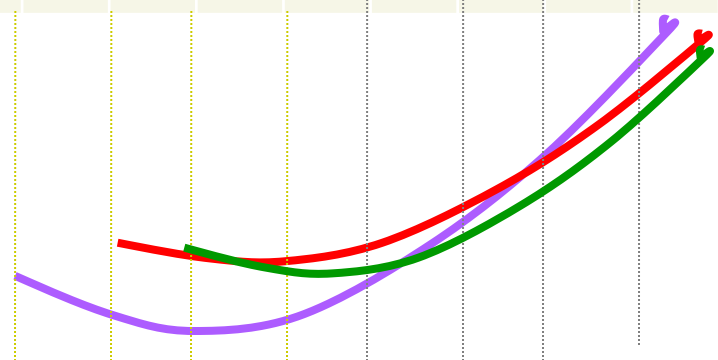
$$OUT(i,m) = F(i) + w(i,m)$$

Queue:

*i=1 :* ***OUT[1,8]***

***i=3: ??***

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



**Where does row 3 intersect with row 1?**

**Use binary search: intersection at Column 5…**

# Example:

## m=3, min in row 1

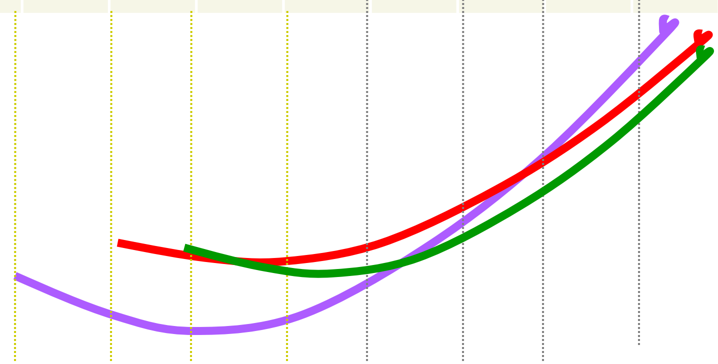
$$OUT(i,m) = F(i) + w(i,m)$$

Queue:

***i=1 :* OUT[1,4]**

***i=3:* OUT[5,8]**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | 25 | 4 | 1 | 4 | 25 | 64 | 121 | 196 |
| | | 34 | 29 | 26 | 41 | 64 | 125 | 194 |
| | | | 29 | 8 | 5 | 20 | 53 | 104 |
| | | | | 10 | 1 | 10 | 37 | 82 |
| | | | | | 10 | 1 | 10 | 37 |
| | | | | | | 10 | 1 | 10 |
| | | | | | | | 10 | 1 |
| | | | | | | | | 10 |



# Time Complexity Analysis

Each row, for m = 1..n, enters the queue at most once, when its column is reached

Each row can get removed from the queue at most once.

Altogether, O(n) rows inserted/removed from the queue.

Computing the “intersection” between the
“leadership intervals” of the new row m versus
the last row in the queue takes O(log n) time via
binary search.

For each “intersection” computation, either a row is removed from the queue, or the row comparisons for that iteration stop. Therefore, no more than O(n) “intersection” computations.

Total Time Complexity: O(n log n)