Farkas' Lemma

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Optimization in $\mathbb{R}^n$, lecture 2

Today's Lecture

Theorem (Farkas' Lemma, 1894)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then either:

1. *There is an $x \in \mathbb{R}^n$ such that $Ax \leq b$; or*
2. *There is a $y \in \mathbb{R}^m$ such that $y \geq 0$, $yA = 0$ and $yb < 0$.*

- Farkas' Lemma is at the core of linear optimization.
- There are extremely efficient proofs of Farkas' Lemma. We'll do a slow, geometrical proof.
- Farkas' Lemma is augmented by *Carathéodory's theorem*.

Convex sets

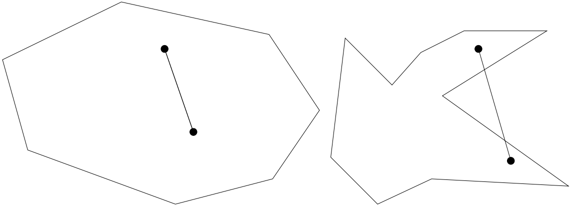
Definition

Let $x, y \in \mathbb{R}^n$. The *line segment between x and y* is

$$[x, y] := \{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\}.$$

Definition

A set $C \subseteq \mathbb{R}^n$ is *convex* if $[x, y] \subseteq C$ for all $x, y \in C$.



Lemma

If $C_\alpha \subseteq \mathbb{R}^n$ is convex for each α, then $\bigcap_\alpha C_\alpha$ is convex.

A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *affine* if $f : x \mapsto Ax + b$ for some A, b.

Lemma

If $C \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^m \to \mathbb{R}^n$ is affine, then $f^{-1}[C]$ is convex.

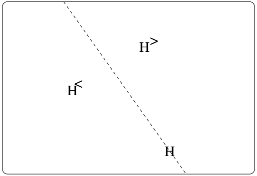
Example

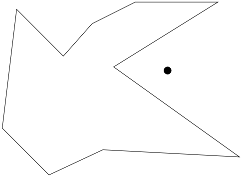
- any *affine space* $\{x \in \mathbb{R}^n \mid Ax = b\}$ is convex
- any *halfspace* $\{x \in \mathbb{R}^n \mid ax \leq \beta\}$ is convex
- any *polyhedron* $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ is convex
- the *unit ball* $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is convex
- any *ellipsoid* $\{Ax + b \mid \|x\| \leq 1\}$ $(\det(A) \neq 0)$ is convex

Separation

Definition

A set $H \subseteq \mathbb{R}^n$ is a *hyperplane* if $H = H_{d,\delta} := \{x \in \mathbb{R}^n \mid dx = \delta\}$ for some nonzero rowvector $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$.





Definition

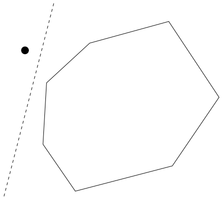
Let $X, Y \subseteq \mathbb{R}^n$. Then $H_{d,\delta}$ *separates* X *from* Y if

$$dx < \delta \text{ for all } x \in X, \text{ and } dy > \delta \text{ for all } y \in Y.$$

The Separation Theorem

Theorem

Let $C \subseteq \mathbb{R}^n$ be a closed, convex set and let $x \in \mathbb{R}^n$.
If $x \notin C$, then there is a hyperplane separating $\{x\}$ from C.



Proof.

1. $\min\{\|y - x\| \mid y \in C\}$ is attained; let $z \in C$ attain the minimum.
2. Let $d := (z - x)^t$ and $\delta := \frac{1}{2}d(z + x)$.
3. $H_{d,\delta}$ separates $\{x\}$ from C: $dx < \delta$ and $dy > \delta$ for all $y \in C$

□

Separation — variations

Theorem

Let C be an open convex set and let $x \in \mathbb{R}^n \setminus C$.
Then there exists a hyperplane H such that $x \in H$ and $C \cap H = \emptyset$.

Proof.

Let $x_i \in \mathbb{R}^n \setminus \overline{C}$ be such that $\lim_{i \to \infty} x_i = x$.
Apply the Separation theorem to $\overline{C}, x_i$. Construct H. □

Theorem

Let $C, D \subseteq \mathbb{R}^n$ be disjoint open convex sets.
Then there is a hyperplane H separating C from D.

Proof.

.. up to you. □

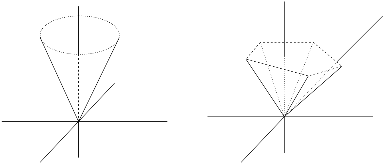
Cones

Definition

A set $C \subseteq \mathbb{R}^n$ is a *cone* if $\alpha x + \beta y \in C$ for all $x, y \in C$ and $\alpha, \beta > 0$.

Example

The *Lorenz cone* is $L^n := \{x \in \mathbb{R}^n \mid \sqrt{x_1^2 + \cdots + x_{n-1}^2} \leq x_n\}$.



Example

Let $a_1, \ldots, a_m \in \mathbb{R}^n$. Then the *cone spanned by* $a_1, \ldots, a_m$ is

$$\text{cone}\{a_1, \ldots, a_m\} := \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid \lambda_i \geq 0 \text{ for all } i\}.$$

Farkas' Lemma

Theorem

Let $C \subseteq \mathbb{R}^n$ be a closed cone and let $x \in \mathbb{R}^n$. Either

1. $x \in C$, *or*
2. *there is a* $d \in \mathbb{R}^n$ *such that* $dy \geq 0$ *for all* $y \in C$ *and* $dx < 0$.

Theorem (Farkas' Lemma, 1894)

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$. Then either

1. $b \in cone\{a_1, \ldots, a_m\}$; *or*
2. *there is a* $d \in \mathbb{R}^n$ *such that* $da_i \geq 0$ *for all* i *and* $db < 0$.

Lemma

Let $a_1, \ldots, a_m \in \mathbb{R}^n$. Then $cone\{a_1, \ldots, a_m\}$ is a closed set.

Farkas' Lemma — variations

Theorem (Farkas' Lemma, variant 1)

Let A be an $m \times n$ matrix, let $b \in \mathbb{R}^m$. Then either:

1. *there is an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$; or*
2. *there is a $y \in \mathbb{R}^m$ such that $yA \geq 0$ and $yb < 0$.*

Theorem (Farkas' Lemma, variant 2)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then either:

1. *there is an $x \in \mathbb{R}^n$ such that $Ax \leq b$; or*
2. *there is a $y \in \mathbb{R}^m$ such that $y \geq 0$, $yA = 0$ and $yb < 0$.*

Proof.

Apply the previous theorem to $A' := \left[\begin{array}{c|c|c} A & -A & I \end{array}\right]$, $b' := b$. $\square$

Farkas' Lemma — variations

Definition

If $L \subseteq \mathbb{R}^n$ is a linear space, then

$$L^\perp := \{y \in \mathbb{R}^n \mid y \perp x \text{ for all } x \in L\}$$

is the *orthogonal complement* of L.

Theorem (Farkas' Lemma, coordinate-free variant)

Let $L \subseteq \mathbb{R}^n$ be a linear space. Exactly one of the following holds:

1. *there exists an $x \in L$ such that $x \geq 0$ and $x_n > 0$; or*
2. *there exists an $y \in L^\perp$ such that $y \geq 0$ and $y_n > 0$.*

Proof.

Without loss of generality $L = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid Ax = bt \right\}$. Apply 'Variant 1'. □

Carathéodory's theorem

Theorem (Carathéodory, 1911)

Let $S \subseteq \mathbb{R}^n$ be a finite set, and let $x \in \mathbb{R}^n$. If $x \in \text{cone}\, S$ then there is a linearly independent set $T \subseteq S$ such that $x \in \text{cone}\, T$.

Proof.

1. Let $T \subseteq S$ be the smallest set such that $x \in \text{cone}\, T$.
2. T is linearly independent: if not,
   - there are $\mu_t \in \mathbb{R}$ (not all 0) such that $\sum_{t \in T} \mu_t t = 0$.
   - there are $\lambda_t \geq 0$ such that $\sum_{t \in T} \lambda_t t = x$.
   - there is an $\alpha \in \mathbb{R}$ such that $\lambda_t + \alpha\mu_t \geq 0$ for all t, and $\lambda_{t_0} + \alpha\mu_{t_0} = 0$ for some t_0.
   - then $x \in \text{cone}(T \setminus \{t_0\})$; contradiction.

□

Few linear inequalities suffice for inconsistency

Theorem (Carathéodory, 1911)

Let $S \subseteq \mathbb{R}^n$ be a finite set, and let $x \in \mathbb{R}^n$. If $x \in \text{cone}\, S$ then there is a linearly independent set $T \subseteq S$ such that $x \in \text{cone} T$.

Corollary

If the system of linear inequalities

$$a_1 x \leq b_1, \ldots, \; a_m x \leq b_m,$$

has no solution $x \in \mathbb{R}^n$, then there is a set $J \subseteq \{1, \ldots, m\}$ with at most $n + 1$ members such that the subsystem

$$a_i x \leq b_i \text{ for all } i \in J$$

has no solution $x \in \mathbb{R}^n$.

Linear inequalities have 'extreme' solutions

Theorem (Carathéodory, 1911)

Let $S \subseteq \mathbb{R}^n$ be a finite set, and let $x \in \mathbb{R}^n$. If $x \in$ cone S then there is a linearly independent set $T \subseteq S$ such that $x \in$ cone$T$.

Corollary

If the system of linear inequalities

$$a_1 x \leq b_1, \ldots, \; a_m x \leq b_m,$$

has a solution $x \in \mathbb{R}^n$, then there also is a solution x^* such that

$$lin.hull\{a_1, \ldots, a_m\} = lin.hull\{a_i \mid a_i x^* = b_i\}.$$

The 'fundamental theorem' of linear inequalities

Theorem

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$. Then exactly one of the following holds:

1. *there is a linearly independent subset $T \subseteq \{a_1, \ldots, a_m\}$ such that $b \in \text{cone } T$; and*
2. *there is a nonzero $d \in \mathbb{R}^n$ such that $da_i \geq 0$ for all i, $db < 0$, and $\text{rank}\{a_i \mid da_i = 0\} = \text{rank}\{a_1, \ldots, a_m, b\} - 1$.*