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# THE HARMONIC SERIES AND EULER'S CONSTANT

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If you study series, one of the first divergent series you will meet is the harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \cdots\ =\sum_{k=1}^{\infty}\frac{1}{k}.$$

There are many ways to see that the harmonic series diverges. I will show you two, completely different ways. To begin with I will show you a proof by contradiction, i.e., I will assume that the series converges and then deduce a contradiction showing that the assumption that the series converges is false. Suppose the series converges, and its sum is $S$. Then

$$\begin{aligned}
S &= 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6})+(\frac{1}{7}+\frac{1}{8})+\ \cdots\\
&> \frac{1}{2}+(\frac{1}{2}+\frac{1}{2})+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{6}+\frac{1}{6})+(\frac{1}{8}+\frac{1}{8})+\ \cdots\\
&= \frac{1}{2}+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \cdots\\
&= \frac{1}{2}+S,
\end{aligned}$$

where I have replaced each fraction with an odd denominator greater than or equal to 3 with the smaller fraction with denominator increased by 1, and in the next step I have used the fact that

$$\frac{1}{2n}+\frac{1}{2n}=\frac{1}{n}.$$

Thus we obtain $S>\frac{1}{2}+S,$ which you will agree is impossible. The conclusion is that the harmonic series does not add up to a number, $S$. That is, the series diverges.

My next demonstration that the harmonic series diverges uses integration. Thus it is rather more sophisticated than my first demonstration.
Draw the graph of $y=f(x)=1/x$ for $x$ from $1$ to $\infty$. Above each interval $[n,n+1]$, $n=1,\ 2,\ 3,\ \cdots$, draw the line that joins the point $(n,1/n)$ to the point $(n+1,1/(n+1))$. This forms a series of trapezia and the area of a general trapezium on the interval $[n,n+1]$ is simply $\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{n+1}=\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$.

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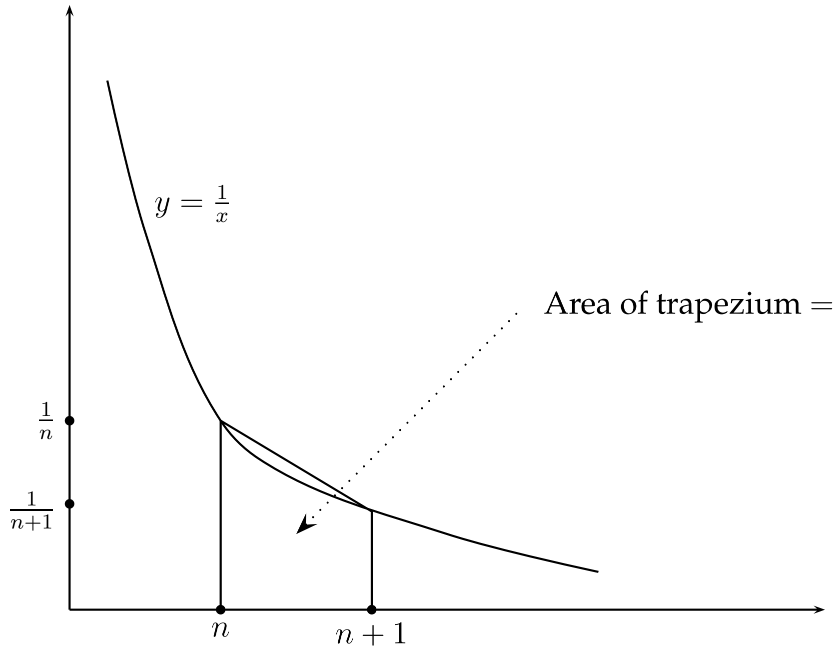


From the figure above it is clear that $f(x)$ lies below the tops of the trapezia so

$$\int_1^N f(x)\, dx < \text{ sum of the areas of the trapezia } = S_N$$

if $N$ is an integer greater than 1.
That is,

$$\log_e N < \frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) + \cdots + \frac{1}{2}\left(\frac{1}{N-1}+\frac{1}{N}\right),$$

or, if we add both $\frac{1}{2}$ and $\frac{1}{2N}$ to both sides,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > \log_e N + \frac{1}{2} + \frac{1}{2N}.$$

Since $\log_e N \to \infty$ as $N \to \infty$, we see that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \to \infty \text{ as } N \to \infty,$$

that is, the harmonic series diverges.

We saw above that the difference between $S_N$ and $\int_1^N f(x)\, dx$ is pretty small (the two graphs are close together), and indeed that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \log_e N$$

is a little more than $\frac{1}{2}$. What I would like to do now is study that quantity, so let me define

$$\delta(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n.$$

We saw that

$$\delta(n) > \frac{1}{2}.$$

What I will show now is that $\delta(n)$ is a decreasing function of $n$, and so $\delta(n)$ is never more than $1$, its value when $n = 1$.

Let us examine the quantity

$$\begin{aligned}\delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e n - \frac{1}{n+1} \\ &= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx \\ &> 0\end{aligned}$$

since the integrand is greater than 0 for $n \leq x < n+1$ (and equals 0 at $x = n+1$). So $\delta(n)$ is a decreasing quantity and is bounded below by $\frac{1}{2}$. It follows that $\delta(n)$ approaches a limit as $n \to \infty$, and that limit is greater than or equal to $\frac{1}{2}$. The limit as $n \to \infty$ of $\delta(n)$ is called "Euler's constant", and is denoted by $\gamma$'

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log_e n\right) = \gamma \geq \frac{1}{2}.$$

Euler introduced the constant $\gamma$ in 1735 and calculated it to sixteen digits in 1781. There are many remarkable mathematical relations involving $\gamma$. One such relation discovered recently is $e^\gamma = \left(\frac{2}{1}\right)^{1/2}\left(\frac{2^2}{1.3}\right)^{1/3}\left(\frac{2^3.2^4}{1.3^3}\right)^{1/4}\cdots$ Sondow (2003).

The remainder of this article is devoted to getting a really good approximation to $\gamma$, and a really good estimate of the difference between $\delta(n)$ and $\gamma$. First I want to re-examine the quantity $\delta(n) - \delta(n+1)$. We have

$$\begin{aligned}\delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e(n) - \frac{1}{n+1} \\ &= \int_n^{n+1} f(x)\,dx - \frac{1}{n+1} < \int_n^{n+1} g(x)\,dx - \frac{1}{n+1} \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) - \frac{1}{n+1} \\ &= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right).\end{aligned}$$

If we write this down for $n,\ n+1,\ n+2,\ \cdots,N-1$, we get

$$\begin{aligned}
\delta(n)-\delta(n+1) &< \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right),\\
\delta(n+1)-\delta(n+2) &< \frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}\right),\\
\delta(n+2)-\delta(n+3) &< \frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+3}\right),\\
&\vdots\\
\delta(N-1)-\delta(N) &< \frac{1}{2}\left(\frac{1}{N-1}-\frac{1}{N}\right).
\end{aligned}$$

If we add all these up, we get

$$\delta(n)-\delta(N) < \frac{1}{2}\left(\frac{1}{n}-\frac{1}{N}\right).$$

If we now hold $n$ fixed and let $N\to\infty$, and remember that $\lim_{N\to\infty}\delta(N)=\gamma$, we find that

$$\delta(n)-\gamma \le \frac{1}{2n}.$$

That is,

$$\gamma \ge \left(1+\frac{1}{2}+\frac{1}{3}+\ \cdots\ +\frac{1}{n}-\log_e n\right)-\frac{1}{2n}.$$

If we put $n=1$, we get $\gamma\ge\frac{1}{2}$, which is no more than we knew before, but if we put $n=100$, and use the computer to calculate the right hand side, we get

$$\gamma \ge 0.577207332,$$

which, as we shall see, is not a bad start! To do more, it seems to me that we have to know more. The first thing we have to know is the series for $\log_e(1+x)$,

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+-\ \cdots\ ,$$

valid for $0\le x\le 1$. (It's actually also valid for $-1<x<0$, and in the complex domain for $|x|<1$, but we don't need that.) This is a remarkable formula, which, as you will see, is not hard to prove. We start with the sum to $n$ terms of the geometric series

$$1-t+t^2-\ \cdots\ +(-1)^{n-1}t^{n-1}=\frac{1-(-t)^n}{1+t}.$$

We can rearrange this as follows.

$$\frac{1}{1+t}=1-t+t^2-\ \cdots\ +(-1)^{n-1}t^{n-1}+(-1)^n\frac{t^n}{1+t}.$$

If we now integrate this from $0$ to $x$, we get

$$\begin{aligned}\log_e(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \ \cdots \ + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}\,dt \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \ \cdots \ + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \epsilon_n\end{aligned}$$

where $$0 \le \epsilon_n = \int_0^x \frac{t^n}{1+t}\,dt \le \int_0^x t^n\,dt = \frac{x^n}{n}.$$

If we fix $x$ in the interval $0 \le x \le 1$ and let $n \to \infty$, we see that $\epsilon_n \to 0$, and so

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \cdots \ ,$$

as I claimed above. We found above that

$$\begin{aligned}\delta(n) - \delta(n+1) &= \log_e(n+1) - \log_e n - \frac{1}{n+1} \\ &= \log_e\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + - \ \cdots \right) \\ &\quad - \left(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + - \ \cdots \right) \\ &= \frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - + \ \cdots \ ,\end{aligned}$$

this being valid for $n > 1$. **Indeed, from now on, I shall assume that $n > 1$.** On the other hand, we have that

$$\begin{aligned}\frac{1}{2}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) &= \frac{1}{2}\Big(\frac{1}{n} - \Big(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + - \ \cdots \Big)\Big) \\ &= \frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4} - + \ \cdots \right) \\ &= \frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - + \ \cdots \ .\end{aligned}$$

The difference between the two series is

$$\begin{aligned}&\Big(\frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - \frac{1}{2n^5} - + \ \cdots \Big) \\ &\quad - \left(\frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - \frac{4}{5n^5} + - \ \cdots \right) \\ &\quad = \frac{1}{6n^3} - \frac{2}{8n^4} + \frac{3}{10n^5} - \frac{4}{12n^6} + - \ \cdots \ .\end{aligned}$$

That is,

$$\delta(n) - \delta(n+1) = \frac{1}{2}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) - \Big(\frac{1}{6n^3} - \frac{2}{8n^4} + \frac{3}{10n^5} \\ - \frac{4}{12n^6} + - \ \cdots \ \Big).$$

Now, somewhat out of the blue, consider the series for

$$\frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{1}{12}\left(\frac{1}{n^2} - \left(\frac{1}{n^2} - \frac{2}{n^3} + \frac{3}{n^4} - \frac{4}{n^5} + - \ \cdots \right)\right) \\ = \frac{2}{12n^3} - \frac{3}{12n^4} + \frac{4}{12n^5} - \frac{5}{12n^6} + - \ \cdots \ .$$

From the last two series, it is clear that

$$\delta(n) - \delta(n+1) = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ + \left(\frac{4}{12} - \frac{3}{10}\right)\frac{1}{n^5} - \left(\frac{5}{12} - \frac{4}{12}\right)\frac{1}{n^6} \\ + \left(\frac{6}{12} - \frac{5}{14}\right)\frac{1}{n^7} - + \ \cdots \ .$$

A careful analysis of this enables us to say that

$$\delta(n) - \delta(n+1) > \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right).$$

If we write this for $n, \ n+1, \ \cdots \ , N-1$ and add, we get

$$\delta(n) - \delta(N) > \frac{1}{2}\left(\frac{1}{n} - \frac{1}{N}\right) - \frac{1}{12}\left(\frac{1}{n^2} - \frac{1}{N^2}\right).$$

If we fix $n$ and let $N \to \infty$, we get

$$\delta(n) - \gamma \geq \frac{1}{2n} - \frac{1}{12n^2},$$

or,

$$\gamma \leq \left(1 + \frac{1}{2} + \frac{1}{3} + \ \cdots \ + \frac{1}{n} - \log_e n\right) - \frac{1}{2n} + \frac{1}{12n^2}.$$

Putting $n = 100$ gives

$$\gamma \leq 0.577215665$$

We can continue in the same way, and find that

$$\delta(n) - \gamma \geq \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6},$$

and

$$\delta(n) - \gamma \leq \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8}$$

and so on.

If we put $n = 100$ in the last two of these we find

$$0.57721566490153286\underline{06} \leq \gamma \leq 0.57721566490153286\underline{11}.$$

There are more things that can be said.

(a) Although we can calculate $\gamma$ to any large finite number of decimal places, no–one knows whether $\gamma$ is irrational or rational. (b) The pattern above can be continued, and involves numbers called the Bernoulli numbers. I had never seen this before, but I guess someone found it before me.

(c) $$S_n - \int_1^n f(x)\,dx = \delta(n) - \frac{1}{2} - \frac{1}{2n}$$

is an increasing quantity, approaches $\gamma - \frac{1}{2}$ as $n \to \infty$, and from the work we have done above can be put between bounds involving only even powers of $n$.