AN ALGORITHM FOR A MINIMUM COVER
OF A GRAPH

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Introduction. This paper contains two similar theorems giving conditions for a *minimum cover* and a *maximum matching* of a graph. Both of these conditions depend on the concept of an alternating path, due to Petersen [2]. These results immediately lead to algorithms for a minimum cover and a maximum matching respectively.

The first part of the paper contains a noninductive proof of the minimum cover theorem (Theorem 1) and the resultant algorithm. Next we define a set of "level transformations" on the set of minimum covers; any minimum cover can be obtained from any other minimum cover by a finite sequence of level transformations (Theorem 2). Thus we have a procedure for finding all minimum covers: First, reduce any cover to a minimum cover by the algorithm; second, apply the level transformations to obtain all other minimum covers. The treatment of the maximum matching theorem and its corresponding algorithm, in the following section, closely parallels that given for minimum covers. The final section establishes a relationship between minimum covers and maximum matchings.

Both the minimum cover theorem and the maximum matching theorem were first proved by induction. The inductive proof of the maximum matching theorem, together with a discussion of its relevance to two similar theorems dealing with point-covers and point-matchings, are given in Berge [1].

The problem of finding a simple algorithm for obtaining a minimum cover of a graph was proposed by Paul Roth as the one-dimensional instance of the more general question of finding a minimum cover for a cubical complex. This problem is a topological formulation of the synthesis of a switching system with minimum cost (cf. Roth [3]).

1. **General definitions.** Let G be a graph. A set C of edges of G is a *cover* of G if every vertex of G is incident to an edge of C. We shall present a procedure for reducing a given cover to a minimum cover,

using the concept of an alternating path first introduced by Petersen [2].

By a *path* of G we mean a sequence of pairwise distinct successively adjacent edges of G. If E is a set of edges of G, an *alternating path* of (G, E) is a path whose edges are alternately in E and not in E. The first and last edges of a path are its *terminal edges*. Its *terminal vertices* are the vertex incident to the first but not the second edge, and the vertex incident to the last but not the preceding edge.

Let C be a cover of G. An alternating path of (G, C) is a *reducing path* if

(1) its terminal edges are in C;

(2) its terminal vertices are incident to edges of C which are not terminal edges of the path.

If (G, C) possesses no reducing path, C is called an *irreducible cover* of G. A cover with the fewest possible edges is a *minimum cover*.

2. Irreducible and minimum covers. It is easy to see that *every minimum cover is irreducible*. The converse of this statement is the main topic of this section.

THEOREM 1. *Every irreducible cover is a minimum cover.*

This theorem gives rise to an algorithm for finding a minimum cover. For if C is a nonminimum cover of G, it has a reducing path π. We can form a new cover C' from C by replacing the edges of π in C with those edges of π not in C. Then since π begins and ends with edges of C, C' has one fewer edge than C. This process is repeated until an irreducible cover is formed; such a cover is a minimum cover.

To prove the theorem we shall make use of the following distance function. Let E and F be two sets of edges of a graph G. The distance $d(E, F)$ is defined as the number of edges of E not in F.

LEMMA 1. *Let C be an irreducible cover and let C_1 be a cover for which $d(C_1, C) = 0$. Then $C_1 = C$.*

PROOF. The assumption $d(C_1, C) = 0$ asserts that *every edge of C_1 is in C*. Since C_1 is a cover, both vertices of any edge e in $C - C_1$ would be covered by edges of C_1; thus e is a reducing path of length 1 of (G, C), contradicting the irreducibility of C.

In the following let C be an irreducible cover of a graph G and let M be a minimum cover chosen so that $d(M, C)$ is a minimum. We contend that $M = C$.

LEMMA 2. *If e is in $M - C$, then e is not adjacent to any edge of M.*

PROOF. Let $e = \{v_1, v_2\}$ and assume that v_1 is incident to another

edge e_1 of M. Let e_2 be an edge of C incident to v_2. The set M' $= M \cup \{e_2\} - \{e\}$ is a cover. Since e is in M, M' has no more edges than M, and hence it too is a minimum cover. Therefore e_2 is not in M and $d(M', C) = d(M, C) - 1$, contrary to the hypothesis that $d(M, C)$ is a minimum.

PROOF OF THEOREM 1. Assume $d(M, C) > 0$, so that $M - C$ is not empty. Consider the collection of all simple paths (i.e., those without repeated vertices) satisfying the conditions: (a) the edges of the path are alternately in $M - C$ and in C; (b) the terminal edges of the path are in $M - C$. The collection is nonempty since a single edge of $M - C$ is a path satisfying (a) and (b). Let $\pi = (e_0, \cdots, e_n)$, $e_i = \{v_i, v_{i+1}\}$, be a path of maximal length in the collection. The vertex v_0 is covered by an edge $e_0' = \{u_0, v_0\}$ in C. Now $u_0 = v_n$ is impossible because then $M' = M \cup \{e_0', e_1, e_3, \cdots, e_{n-1}\} - \{e_0, e_2, \cdots, e_n\}$ would be a minimum cover with $d(M', C) < d(M, C)$. Similarly, there is an edge $e_n' = \{u_n, v_n\}$ in C such that $u_n \neq v_0$.

We contend that u_0 and u_n are covered by edges of C other than e_0' and e_n'. If $u_0 = v_k$ for some k, then $k < n$, and one of the edges e_{k-1} or e_k is in C. Otherwise u_0 is incident to no edge of π and we proceed as follows: Let e_0'' be an edge of M incident to u_0. By Lemma 2, $e_0'' \neq e_0'$. But now e_0'' must be in C, for otherwise the path $(e_0'', e_0', e_0, e_1, \cdots, e_n)$ is a path longer than π satisfying (a) and (b). Hence again u_0 is covered by an edge of C different from e_0' and e_n'. A similar argument holds for u_n.

The path $(e_0', e_0, e_1, \cdots, e_n, e_n')$ is therefore a reducing path of (G, C), contradicting the irreducibility of C. Thus $d(M, C) = 0$, and by Lemma 1, $M = C$.

3. **Level transformations and minimum covers.** Motivated by the previous proof, we are led to define two operations, T_1 and T_2, called *level transformations* which, when applied to a minimum cover, transform it into a new minimum cover. Application of a T_1-transformation to any cover C consists of picking a path (e_1, e_2, e_3) of three edges such that e_2 and e_3 are in C, and forming the cover $T_1(C) = C \cup \{e_1\}$ $- \{e_2\}$. Application of a T_2-transformation to a cover C consists of picking a circuit $(e_1, e_2, \cdots, e_{2n})$ such that for all k, e_{2k-1} is in C and e_{2k} is not in C, and forming the cover

$$T_2(C) = C \cup \{e_2, e_4, \cdots, e_{2n}\} - \{e_1, e_3, \cdots, e_{2n-1}\}.$$

A T_1-transformation may decrease the number of edges of a cover by one; a T_2-transformation always leaves the number of edges invariant. When applied to a *minimum* cover, M, however, both $T_1(M)$ and $T_2(M)$ have the same cardinality as M, and are thus both mini-

mum covers. Since the most interesting applications of T_1 and T_2 are to minimum covers, the use of the term "level transformation" is justified.

Examining the proofs of Lemmata 1 and 2 and the proof of the main theorem, one immediately sees that they actually entail the following statement: Let C be an irreducible cover and let Γ be a collection of covers closed with respect to T_1- and T_2-transformations. *If M in Γ minimizes $d(M', C)$ for M' in Γ, then $M = C$.* From this remark we immediately derive two results.

THEOREM 2. *If M_1 and M_2 are minimum covers, then M_2 can be obtained by applying a (finite) sequence of level transformations to M_1.*

PROOF. Let Γ be the collection of all minimum covers obtainable from M_1 by a finite sequence of level transformations; Γ is certainly closed with respect to level transformations. As M_2 is a minimum cover, it must be irreducible; hence $M_2 = M$ for some M in Γ.

THEOREM 3. *The collection $\mathfrak{M}$ of all minimum covers of a graph G can be characterized as the nonempty collection of covers of G closed with respect to T_1- and T_2-transformations in which all elements have the same cardinality.*

PROOF. Let Γ be such a collection. Let M be any minimum cover. Then by the closure of Γ, M is in Γ. Since all elements of Γ have the same cardinality, this implies $\Gamma \subset \mathfrak{M}$. That $\Gamma = \mathfrak{M}$ now follows from Theorem 2.

4. **Unaugmentable and maximum matchings.** A set P of edges of a graph G is a *matching* if no two of its edges are adjacent. An *augmenting path* of (G, P) is an alternating (G, P) path whose terminal vertices are incident to no edges of P. Such a path is easily seen to be a simple path. If (G, P) has no augmenting path, P is called *unaugmentable*. A matching with the greatest possible number of edges is *maximum*. Of course every maximum matching is unaugmentable. The converse of this theorem is a theorem due to Berge [1] which gives rise to an algorithm similar to the one just discussed.

THEOREM 4 (BERGE). *Every unaugmentable matching is maximum.*

PROOF. Let P be an unaugmentable matching and let M be a maximum matching for which $d(P, M)$ is a minimum. Clearly, if $d(P, M) = 0$ then $P = M$. We construct a path of maximum length whose edges are alternately in $P - M$ and M. Since M is also unaugmentable, it cannot begin and end with edges of $P - M$. It has, there-

fore, equally many edges of $P - M$ and M. Now from M we can form a new maximum matching M' by exchanging the edges of the path in M for those in P. Then $d(P, M') < d(P, M)$, contrary to the minimality of $d(P, M)$.

5. Relations between covers and matchings. Let C be a cover of a graph. We shall say that C is minimal if no proper subset of it is a cover. Similarly, a matching is maximal if no proper superset of it is a matching. Let C be a minimal cover of G. For each vertex v_i of G we define its *C-degree* a_i as the number of edges of C incident to v_i. Let A be the set of multiply covered vertices of G, i.e., those for which $a_i > 1$. For each vertex v_i of A, we delete from C $a_i - 1$ edges incident to v_i. The resulting set P is a matching, since the P-degree of each point is at most 1. However, P need not be a maximal matching.

Let P be a maximal matching. Then for each vertex of P-degree 0, add to P any incident edge of the graph. The result is, of course, a cover, but it need not be minimal. However, if we begin with a minimum cover C, this process produces a maximum matching, while if we start with a maximum matching, the process yields a minimum cover. To demonstrate, let C be a minimum cover of a graph G, and let P result from C by elimination of $a_i - 1$ edges incident to each multiply covered vertex v_i. Let c denote the number of edges of C, and let $a = \sum (a_i - 1)$ be the number of edges deleted in forming P. Thus a is the number of vertices of P-degree 0. If P is not maximum there is a matching P' with $c - a + 1$ edges, and the number of vertices of P'-degree 0 is $a - 2$. Form a cover by adding one edge incident to each vertex of P-degree 0. This cover has $c - 1$ edges, contrary to the hypothesis that C is a minimum cover. The dual assertion is proved similarly.

REFERENCES

1. C. Berge, *Two theorems in graph theory*, Proc. Nat. Acad. Sci. U.S.A. vol. 43 (1957) pp. 842–844.

2. J. Petersen, *Die Theorie der regulären Graphen*, Acta Math. vol. 51 (1891) pp. 193–220.

3. J. P. Roth, *Algebraic topological methods for the synthesis of switching systems* I, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 301–326.

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