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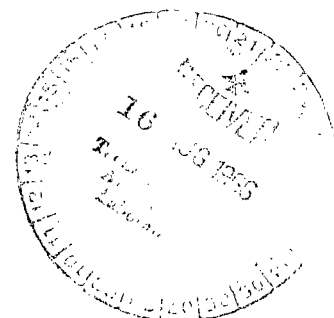$c.1$

INVERSE OF THE VANDERMONDE MATRIX WITH APPLICATIONS

by L. Richard Turner

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Cleveland, Ohio

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SUMMARY

The inverse of the Vandermonde matrix is given in the form of the product $U^{-1}L^{-1}$ of two triangular matrices by the display of generating formulas from which the elements of U^{-1} and L^{-1} may be directly computed. From these, it is directly possible to generate special formulas for the approximation of linear transforms such as integrals, interpolants, and derivates for a variety of functions that behave as $P(x)f(x)$, where $P(x)$ is a polynomial and $f(x)$ is generally not a polynomial and may have localized singularities.

INTRODUCTION

The Vandermonde matrix, sometimes called an alternant matrix, comes from the approximation by a polynomial of degree $n - 1$ of a function $f(x)$ with known values at n distinct values of the independent variable x.

Many important functions of applied analysis cannot be well represented by polynomials, but these functions are accurately represented as products of polynomials by other functions that may not be analytic in the sense of function theory but can be effectively manipulated.

To facilitate both the generation of working formulas for integration, interpolation, and differentiation and the calculation of other linear transformations, the inverse of the Vandermonde matrix is presented in a simple matrix product form in which the elements are all computed by elementary explicit or recursive formulas. A few examples of applications are given.

ANALYSIS

The Vandermonde matrix A arises as the matrix of coefficients required to

evaluate the coefficients a_i in any polynomial approximation, as, for example,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{1}$$

to a function y given at the n distinct points $x_1, x_2, x_3, \cdots, x_n$. The matrix A has the form

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \tag{2}$$

If A^{-1} is known, the value of the coefficients is formally given by the expression

$$\left[a_0, a_1, a_2, \cdots, a_{n-1} \right] = A^{-1} \left[y_1, y_2, y_3, \cdots, y_n \right] \tag{3}$$

where the brackets denote column matrices and y_i is equal to $y(x_i)$.

A simple form of the inverse matrix A^{-1} is described in terms of the product $U^{-1} L^{-1}$, where U^{-1} is an upper triangular matrix and L^{-1} is a lower triangular matrix.

The Vandermonde matrix A has the determinant equal to $\prod_{j>1} (x_j - x_i)$ (ref. 1, p. 9) and is nonsingular if all values of x_i are distinct. It can, therefore, be factored into a lower triangular matrix L and an upper triangular matrix U where A is equal to LU. The factorization is unique if no row or column interchanges are made and if it is specified that the diagonal elements of U are unity.

The upper triangular factor U and the inverse L^{-1} of the lower triangular factor L are developed in reference 2, but the authors were content to depend on the evaluation of the numerical values for the coefficients a_i of equation (1) by the solutions of the equation

$$U \left[a_0, a_1, a_2, \cdots, a_{n-1} \right] = L^{-1} \left[y_1, y_2, \cdots, y_n \right] \tag{4}$$

in each case.

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It is seen in reference 2 that, with a translation to matrix notation, the elements ℓ_{ij} of L^{-1} are given by the relations

$$\left.\begin{array}{c} \ell_{ij} = 0 \qquad i < j \\[2mm] \ell_{11} = 1 \\[2mm] \displaystyle \ell_{ij} = \prod_{\substack{k=1 \\ k \neq j}}^{i} \frac{1}{x_j - x_k} \qquad \text{otherwise} \end{array}\right\} \tag{5}$$

The explicit form of L^{-1} for a few rows and columns is

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\[3mm] \dfrac{1}{x_1 - x_2} & \dfrac{1}{x_2 - x_1} & 0 & \cdots \\[4mm] \dfrac{1}{(x_1 - x_2)(x_1 - x_3)} & \dfrac{1}{(x_2 - x_1)(x_2 - x_3)} & \dfrac{1}{(x_3 - x_1)(x_3 - x_2)} & \cdots \\[2mm] \vdots & \vdots & \vdots & \cdots \\ & & & \cdots \\ & & & \cdots \end{pmatrix} \tag{6}$$

It is asserted and proved herein that the elements u_{ij} of U^{-1} are given by the definition

$$\left.\begin{array}{c} u_{ii} = 1 \\[2mm] u_{i1} = 0 \\[2mm] u_{ij} = u_{i-1, j-1} - u_{i, j-1} x_{j-1} \qquad \text{otherwise} \end{array}\right\} \tag{7}$$

where

$$u_{0j} = 0$$

The first few rows and columns of the asserted inverse U^{-1} are

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$$
U^{-1} =
\begin{pmatrix}
1 & -x_1 & x_1 x_2 & -x_1 x_2 x_3 & \cdots \\
0 & 1 & -(x_1 + x_2) & x_1 x_2 + x_2 x_3 + x_3 x_1 & \cdots \\
0 & 0 & 1 & -(x_1 + x_2 + x_3) & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots
\end{pmatrix}
\tag{8}
$$

It is noted that the jth column of U^{-1} does not depend on x_j but only on $x_1, x_2, \cdots, x_{j-1}$. A proof that U^{-1} is as described by definition (7) is developed by showing that the product AU^{-1} is lower triangular and, therefore, equal to L.

By direct computation, this is true for the Vandermonde matrix involving the two coordinates x_1 and x_2:

$$
\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}
\begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}
=
\begin{pmatrix} 1 & 0 \\ 1 & x_2 - x_1 \end{pmatrix}
\tag{9}
$$

At this point, it can be assumed that, for the coordinates $x_1, x_2, x_3, \cdots, x_n$, the product of AU^{-1} is lower triangular, and the effect of adding the ordinate x_{n+1} and the corresponding rows and columns of A and U^{-1} can be considered. It is recalled that the nth column of U^{-1} involves only the ordinates $x_1, x_2, x_3, \cdots, x_{n-1}$, and that the inner products of the rows $(1, x_i, x_i^2, \cdots)$ of A and column n of U^{-1} are zero except for the diagonal element, which has the value $(x_n - x_1)(x_n - x_2)(x_n - x_3)$. Because column $n + 1$ of U^{-1} defined by equation (7) is a linear combination of the elements of column n, the inner products of the first $n - 1$ rows of A and the $n + 1$ column of U^{-1} are all zero. It remains to show only that the element $p_{n, n+1}$ of the product matrix vanishes.

From the recursive definition of columns of U^{-1}, it is seen that the $n + 1$ column can be represented as the weighted sum of the two column matrices $[0, u_{1n}, u_{2n}, \cdots, u_{nn}] - x_n[u_{1n}, u_{2n}, u_{3n}, \cdots, u_{nn}, 0]$. The inner product of the column by the nth row of A, which is $[1, x_n, x_n^2, \cdots, x_n^n]$, produces two terms that are exactly equal but opposite in sign; hence, their sum is zero and $p_{n, n+1} = u_{n, n-1}$

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(of U^{-1}) is zero. Therefore, the product AU^{-1} of the augmented matrix is lower triangular, which was to have been proved.

PROPERTIES OF FACTOR MATRICES

In general, the premultiplication by L^{-1} of a matrix of values of the function $f(x)$, whose values are known at the ordinates $\left[x_1, \ x_2, \ x_3, \ \cdot \ \cdot \ \cdot \right]$, produces the divided differences used in Newton's interpolation formula (ref. 1, p. 3).

It is also noted that the matrix U^{-1} does not involve the last ordinate. Because the ordinates X_i at which the data Y are known may be arbitrarily identified, any single ordinate may be left unspecified, and the elements of U^{-1} and all but the last row of L^{-1} can be computed independently of the location of the unspecified ordinate.

It should also be mentioned that the ordinates x_i may be complex numbers as long as their values are distinct.

For any evenly spaced set of ordinates x_i where $x_{i+1} = 1 + x_i$, the matrix L^{-1} has the specific numerical form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
-1 & 1 & 0 & 0 & \cdots \\
\dfrac{1}{2} & -1 & \dfrac{1}{2} & 0 & \cdots \\
-\dfrac{1}{6} & \dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{6} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{pmatrix}
\tag{10}
$$

of which the elements have the values

$$
\ell_{ij} = \frac{(-1)^{i+j}}{(i-1)!} \binom{i-1}{j-1}
$$

where $\binom{i-1}{j-1}$ are the binomial coefficients.

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For the set of ordinates $x_i = i$, the upper factor U^{-1} has the form

$$
\begin{pmatrix}
1 & -1 & 2 & -6 & \cdots \\
0 & 1 & -3 & 11 & \cdots \\
0 & 0 & 1 & -6 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{pmatrix}
\tag{11}
$$

where, in this special case,

$$
u_{ij} = S_i^{(j-1)} \qquad j > 1
$$

where $S_p^{(k)}$ are the Stirling numbers of the first kind (ref. 3, p. 135).

APPLICATIONS

Because the Vandermonde matrix arises from the basic problem of passing a polynomial function through a given set of data, the results may be thought of as the approximation of a function $y(x)$ by a polynomial $\hat{y}(x)$. To the extent that $\hat{y}(x)$ is a reasonable approximation of $y(x)$, it is possible to approximate linear transformations of $y(x)$ by the corresponding transformation of $\hat{y}(x)$.

Most of the classical formulas for numerical integration, interpolation, or differentiation can be generated directly. Exceptions are cases such as Gaussian and Chebysheff integrations, which require that the ordinates be especially selected by other considerations.

Few examples of standard results will be displayed, but the techniques of their generation should be clear from the special cases. Two cases are considered by which special formulas of high accuracy may be generated.

Formulas for integration of products of functions and other related linear transforms may be developed as follows: Consider an integral or other linear transform of the product $y(x)f(x)$, which is herein designated as $T[y(x)f(x)]$. The coefficients of the approximation to $y(x)$, namely,

$$
\hat{y}(x) = a_o + a_1 x + a_2 x^2 + \cdots
$$

are given by the relation

$$\left[a_0, \ a_1, \ a_2, \ \cdot \ \cdot \ \cdot \right] = U^{-1}L^{-1}\left[y(x_1), \ y(x_2), \ y(x_3), \ \cdot \ \cdot \ \cdot \right] \tag{12}$$

Then, if it is possible to develop suitably computable expressions for $T(x^n f(x))$, the transform $T(\hat{y}(x)f(x))$ is given by

$$\left(T(f(x)), \ T(xf(x)), \ \cdot \ \cdot \ \cdot \right) U^{-1}L^{-1}\left[y_1, \ y_2, \ y_3, \ \cdot \ \cdot \ \cdot \right]$$

and, if $y(x)$ is very nearly a polynomial, $T(y(x)f(x))$ is reasonably approximated by $T(\hat{y}(x)f(x))$.

Because the matrices U^{-1} and L^{-1} exist, an array of Lagrangian coefficients may be computed by the evaluation of

$$\left(T(f(x)), \ T(xf(x)), \ T(x^2 f(x)), \ \cdot \ \cdot \ \cdot \right) U^{-1}L^{-1} \tag{13}$$

independently of the actual values of $y(x_i)$.

A second situation arises when $y(x)$ is known, perhaps for analytical reasons, to be expressible by $y(x) = p(x)f(x)$, where $p(x)$ is a polynomial and $f(x)$ is some known function. For example, $f(x)$ and $x^n f(x)$ may be Lesbegue integrable but cannot be accurately approximated by a polynomial. In this case, $y(x)$ is approximated by

$$\hat{y}(x) = \left(a_0 + a_1 x + a_2 x^2 + \cdot \ \cdot \ \cdot \right) f(x) \tag{14}$$

and the coefficients a_i are computable by the relation

$$\left[a_0, \ a_1, \ a_2, \ \cdot \ \cdot \ \cdot \right] = U^{-1}L^{-1}\left[\frac{y(x_1)}{f(x_1)}, \ \frac{y(x_2)}{f(x_2)}, \ \cdot \ \cdot \ \cdot \right] \tag{15}$$

The desired Lagrangian coefficients to compute an approximate transform are generated by evaluating

$$\left(T(f(x)), \ T(xf(x)), \ \cdot \ \cdot \ \cdot \right) U^{-1}L^{-1}\left\{ \frac{1}{f(x_1)}, \ \frac{1}{f(x_2)}, \ \frac{1}{f(x_3)}, \ \cdot \ \cdot \ \cdot \right\} \tag{16}$$

where the braces denote a diagonal matrix. The resulting matrix of coefficients operates directly on the data

$$\left[y_1, \ y_2, \ y_3, \ \cdot \ \cdot \ \cdot \right]$$

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NUMERICAL EXAMPLES

(1) Start with a classical example of generating the Adams-Moulton predictor corrector formulas, with $f(x) = 1$. For this example, the basis set x_i is $\begin{bmatrix} -3, & -2, & -1, & 0 \end{bmatrix}$, and U^{-1} has the form

$$U^{-1} = \begin{pmatrix} 1 & 3 & 6 & 6 \\ 0 & 1 & 5 & 11 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

and L^{-1} is given by definition (10).

The predictor formula is

$$h \int_0^1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 \right) dx = h \left[a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} \right]_0^1 \tag{18}$$

The corrector formula is found from

$$h \int_{-1}^0 \hat{y}(x) dx = h \left[a_0 x - \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} - \frac{a_3 x^4}{4} \right]_0^1 \tag{19}$$

where h is the unit of step size. Both sets of coefficients are therefore generated by evaluating the expression

$$h \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} \end{pmatrix} U^{-1} L^{-1}$$

It is convenient, at least for manual effort, to evaluate the expressions from left to right,

in particular,

$$h \begin{pmatrix} 1 & \dfrac{1}{2} & \dfrac{1}{3} & \dfrac{1}{4} \\ 1 & -\dfrac{1}{2} & \dfrac{1}{3} & -\dfrac{1}{4} \end{pmatrix} U^{-1} = h \begin{pmatrix} 1 & \dfrac{7}{2} & \dfrac{53}{6} & \dfrac{55}{4} \\ 1 & \dfrac{5}{2} & \dfrac{23}{6} & \dfrac{9}{4} \end{pmatrix} \tag{20}$$

and

$$h \begin{pmatrix} 1 & \dfrac{7}{2} & \dfrac{53}{6} & \dfrac{55}{4} \\ 1 & \dfrac{5}{2} & \dfrac{23}{6} & \dfrac{9}{4} \end{pmatrix} L^{-1} = \dfrac{h}{24} \begin{pmatrix} -9 & 37 & -59 & 55 \\ 1 & -5 & 19 & 9 \end{pmatrix} \tag{21}$$

The leading term of the error series for these integrations is obtained by applying the resulting operators to the data $x_i^4 \dfrac{h^4 y''''}{24}$ and subtracting the correct integral for this term, which is $\dfrac{h^5 y''''}{120}$. The data apart from the multiplicative constant are $[81, 16, 1, 0]$, and the results are $-\dfrac{251}{720} h^5 y''''$ for the predictor and $\dfrac{19}{720} h^5 y''''$ for the corrector.

(2) As an example of a more general process, an integral of the form

$$\int_0^x \frac{y(\xi) d(\xi)}{\sqrt{(\xi)}}$$

should be considered, where $y(x)$ is assumed to be regular and known and capable of being adequately approximated by a polynomial. This is an example of equation (14) with the data set $x_i = [0, 1, 2, \cdots]$, with a scale h, and with $f(x) = 1/\sqrt{x}$. The transform

$$T(\xi^n f(x)) = \int_0^x \frac{\xi^n \, d\xi}{\sqrt{\xi}} = \frac{2}{2n + 1} (x)^{(2n+1)/2}$$

Hence, the appropriate set of Lagrangian integration coefficients is given by

$$h\left(2x^{1/2}, \frac{2}{3}x^{3/2}, \frac{2}{5}x^{5/2}, \frac{2}{7}x^{7/2}, \cdots\right)U^{-1}L^{-1}$$

where U^{-1} has the form

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 2 & \cdots \\ 0 & 0 & 1 & -3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \qquad (22)$$

and L^{-1} has the form of definition (10).

Because integrals for each ordinate x_i are desired, the first matrix becomes

$$h\begin{pmatrix} 2 & \dfrac{2}{3} & \dfrac{2}{5} & \dfrac{2}{7} & \cdots \\[2mm] 2\sqrt{2} & \dfrac{4}{3}\sqrt{2} & \dfrac{8}{5}\sqrt{2} & \dfrac{16}{7}\sqrt{2} & \cdots \\[2mm] 2\sqrt{3} & 2\sqrt{3} & \dfrac{18}{5}\sqrt{3} & \dfrac{54}{7}\sqrt{3} & \cdots \end{pmatrix} = C \qquad (23)$$

$$CU^{-1} = h\begin{pmatrix} 2 & \dfrac{2}{3} & \dfrac{-4}{15} & \dfrac{44}{105} & \cdots \\[2mm] 2\sqrt{2} & \dfrac{4}{3}\sqrt{2} & \dfrac{4}{15}\sqrt{2} & \dfrac{16}{105}\sqrt{2} & \cdots \\[2mm] 2\sqrt{3} & 2\sqrt{3} & \dfrac{8}{5}\sqrt{3} & \dfrac{32}{35}\sqrt{3} & \cdots \end{pmatrix} \qquad (24)$$

and the final result $CU^{-1}L^{-1}$ computed for three data points only is

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$$CU^{-1}L^{-1} = \frac{h}{15} \begin{pmatrix} 18 & 14 & -2 \\ 12\sqrt{2} & 16\sqrt{2} & 2\sqrt{2} \\ 12\sqrt{3} & 6\sqrt{3} & 12\sqrt{3} \end{pmatrix} \qquad (25)$$

The coefficients operate on the data $y(0)$, $y(h)$, and $y(2h)$.

(3) If the data for an integration are $y(x)$, which is known to be of the form $P(x)/\sqrt{x}$, and $\lim y(x)\sqrt{x} = P(0)$ is known, a form may be derived to compute $\displaystyle\int_0^x y(\xi)d\xi$, which can be obtained from example (2) by postmultiplying the general result of the form $CU^{-1}L^{-1}$ by the diagonal matrix $\left(1, 1, \sqrt{2}, \sqrt{3}, \cdots\right)$. The resulting matrix operates on the data $[y(0), y(h), y(2h), \cdots]$. The matrix for three data points is

$$\frac{h}{15} \begin{pmatrix} 18 & 14 & -2\sqrt{2} \\ 12\sqrt{2} & 16\sqrt{2} & 4 \\ 12\sqrt{3} & 6\sqrt{3} & 12\sqrt{6} \end{pmatrix} \qquad (26)$$

(4) If, in the preceding example, it is known that $y(x)$ has the form $P(x)/\sqrt{x}$ but the limit at zero of $y(x)\sqrt{x}$ is not known, similar formulas can be generated by considering the point set for x_i of $[h, 2h, 3h, \cdots]$. In this case, U^{-1} has the form given in equation (22), L^{-1} has the form of definition (10), the matrix C has the form of equation (23), and CU^{-1} has the form

$$CU^{-1} = h \begin{pmatrix} 2 & \dfrac{-4}{3} & \dfrac{12}{5} & \dfrac{-712}{105} \\ 2\sqrt{2} & \dfrac{-2\sqrt{2}}{3} & \dfrac{8}{5}\sqrt{2} & \dfrac{-488}{105}\sqrt{2} \\ 2\sqrt{3} & 0 & \dfrac{8}{5}\sqrt{3} & \dfrac{-393}{105}\sqrt{3} \end{pmatrix} \qquad (27)$$

For three points of data at h, $2h$, and $3h$,

$$CU^{-1}L^{-1} = h \begin{pmatrix} \dfrac{68}{15} & \dfrac{-56}{15} & \dfrac{6}{5} \\[2ex] \dfrac{52}{15}\sqrt{2} & \dfrac{-34}{15}\sqrt{2} & \dfrac{4}{5}\sqrt{2} \\[2ex] \dfrac{14}{5}\sqrt{3} & \dfrac{-8}{5}\sqrt{3} & \dfrac{4}{5}\sqrt{3} \end{pmatrix} \qquad (28)$$

This matrix is finally postmultiplied by the diagonal matrix $\left\{1, \sqrt{2}, \sqrt{3}\right\}$. The result of this postmultiplication is

$$\frac{h}{15} \begin{pmatrix} 68 & -56\sqrt{2} & 18\sqrt{3} \\[1ex] 52\sqrt{2} & -68 & 12\sqrt{6} \\[1ex] 42\sqrt{3} & -24\sqrt{6} & 36 \end{pmatrix} \qquad (29)$$

It is easy to generate weight coefficients for various processes of integration and differentiation or for the calculation of other transforms that are disadvantageous from the standpoint of error propagation. This may lead to computational instability if these weight coefficients are used within a larger problem, such as the numerical solution of a differential equation.

While there are no simple rules for the analysis of the problem of stability, a crude measure of stability can be obtained by computing $\sqrt{n \sum w_i^2 \Big/ \left(\sum w_i\right)^2}$ where w_i denotes the weighting coefficients of the numerical transform.

As an example, the measure $\sqrt{n \sum (w_i)^2 \Big/ \left(\sum w_i\right)^2}$ has been computed for each of the expository computations in this report. The expository computations are identified in table I by equation and row.

The mere magnitude of the stability measure is not an adequate guide to the effectiveness of a given quadrature formula but, in general, large values of the stability measure are viewed unfavorably.

It may be noted that the repeated use of the corrector formula of the Adams-Bashforth process (table I, row 2 of eq. (21)) leads to stability in the integration of systems of differential equations if the step size is adequately small (ref. 4). Note that the stability factor for equation (21) is 1.803.

TABLE I. - STABILITY FACTORS FOR

SELECTED EXAMPLES

Equation	Row	Stability factor, $\sqrt{n\sum(w_i)^2\big/\left(\sum w_i\right)^2}$
(21)	1	7.433
	2	1.803
(25)	1	1.322
	2	1.156
	3	1.039
(26)	1	1.368
	2	1.144
	3	1.072
(29)	1	9.443
	2	5.177
	3	3.473

On the other hand, the alternate use of the open three-point Newton-Cotes formula with a stability factor of 1.732 and Simpson's rule (closed three-point Newton-Cotes formula) with a stability factor of 1.225, as is done in Milne's schedule for the integration of systems of differential equations, is almost always unstable.

In general, it will be observed that when strong inferences concerning the data such as the use of very many points in a simple quadrature formula or inferences concerning values of data outside the interval actually spanned by the data as in equation (29) or row 1 of equation (21) are made, the stability factors are large, and the resulting method may be a poor one.

CONCLUSION

The inverse A^{-1} of the Vandermonde matrix is given in the form of the product of two triangular matrices $U^{-1}L^{-1}$ by the display of generating formulas from which the elements of U^{-1} and L^{-1} may be directly computed. From these, it is directly possible to generate special formulas for the approximation of linear transforms, such as integrals, interpolants, and derivates, for a variety of functions that behave as $P(x)f(x)$, where $P(x)$ is a polynomial and $f(x)$ is generally not a polynomial and may be singular.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 9, 1966,
129-04-06-02-22.

REFERENCES

1. Milne-Thompson, L. M.: The Calculus of Finite Differences. Macmillan Co., 1933.

2. Frenkiel, F. N.; and Polachek, H.: An Algorithm for Fitting a Polynomial Through n Given Points. Proc. Computation Seminar, Dec. 1949, Int. Business Machines Corp., cc. 1951, pp. 71-73.

3. Hildebrand, F. B.: Introduction to Numerical Analysis. McGraw-Hill Book Co., Inc., 1956.

4. Hull, T. E.; and Creemer, A. L.: Efficiency of Predictor-Corrector Procedures. Assoc. Comput. Mach. J., vol. 10, no. 3, July 1963, pp. 291-301.

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