Calculus (I) – Midterm Exam

100%

(2%)1.	The derivative of a function f at a number a, denoted by $f'(a)$, is $f'(a) =$ ________ if the limit exists.

Sol:

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ or $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

(2%)2.	A function f is continuous at a number a, if $f(a) =$ ________.

Sol:

$\lim_{x \to a} f(x)$

(2%)3.	A function f is differentiable at a number a, if ________ exists.

Sol:

$f'(a)$

(4%)4.	State the relationship between a differentiable and a continuous function; that is, if f is ________ at a, then f is ________ at a.

Sol:

differentiable; continuous

(5%)5.	Find two functions $f(x)$, $g(x)$, and a such that $\lim_{x\to a}((f+g)(x))$ exists but $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist.

Sol:

The solution is not unique. One could be:

$f(x) = \sin x$, $g(x) = -\sin x$, $a = \infty$

(5%)6.	$H(x) = \frac{1}{\sqrt[4]{x^2-5x}}$ is given. (3%) (a) Find functions f, g and h such that $H = f \circ g \circ h$. (2%) (b) On what intervals is $H(x)$ continuous ?

Sol:

(a) $f(x) = \frac{1}{x}$, $g(x) = \sqrt[4]{x}$, $h(x) = x^2 - 5x$.

(b)

$h(x) = 1 / \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

(5%)7.	Determine whether $f(x) = x \cdot \lvert 3x \rvert$ is even, odd, or neither.

Sol:

$f(x) = x \cdot |3x|$

$f(-x) = (-x)|-3x| = (-x)|3x| = -(x|3x|) = -f(x)$

Since $f(-x) = -f(x)$. f is odd function.

(5%)8.	If $f(x) = \sin(x)$, $g(x) = x^2 + 1$, find $f \circ g$ and its domain.

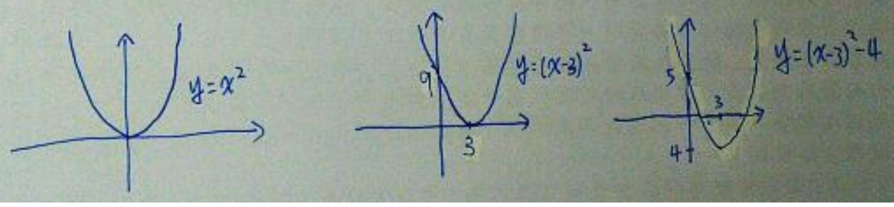
Sol:

$$(f \circ g)(x) = f(g(x)) = f(x^2+1) = \sin(x^2+1), \;\; D = \mathbb{R}.$$

(6%)9.	Sketch the graph of $f(x) = x^2 - 6x + 5$ by hand, not plotting points, but starting with a fundamental function and then applying the appropriate transformations.

Sol:

$y = x^2 - 6x + 5 = (x^2 - 6x + 9) - 4 = (x-3)^2 - 4$: Start with the graph of $y = x^2$, shift 3 units to the right, and then shift downward 4 unit.



(6%)10.	Show that $\lim\limits_{x\to 4}\frac{2x}{x-4}$ does not exist.

Sol:

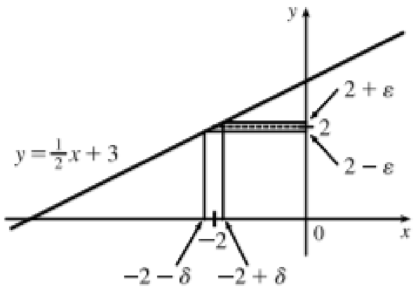
Since $\lim\limits_{x\to 4^+}\frac{2x}{x-4} = \infty$ and $\lim\limits_{x\to 4^-}\frac{2x}{x-4} = -\infty$, we know that $\lim\limits_{x\to 4^+}\frac{2x}{x-4} \neq \lim\limits_{x\to 4^-}\frac{2x}{x-4}$

$\because \lim\limits_{x\to 4^+}\frac{2x}{x-4} \neq \lim\limits_{x\to 4^-}\frac{2x}{x-4}$ $\therefore \lim\limits_{x\to 4}\frac{2x}{x-4}$ does not exist.

(8%)11.	Prove $\lim_{x \to -2} \frac{x}{2} + 3 = 2$ using the ε, δ definition of a limit.

Sol:

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon$. But $\left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon \;\Leftrightarrow\; \left|\frac{1}{2}x + 1\right| < \varepsilon \;\Leftrightarrow\; \frac{1}{2}|x + 2| < \varepsilon \;\Leftrightarrow\; |x - (-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \;\Rightarrow\; \left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon$. Thus, $\lim_{x \to -2}(\frac{1}{2}x + 3) = 2$ by the definition of a limit.



(5%)12.	Prove that $\lim_{x \to 0^+} \sqrt{x} \cdot \left[1 + \sin^2(\frac{2\pi}{x})\right] = 0$.

Sol:

$-1 \le \sin(2\pi/x) \le 1 \;\Rightarrow\; 0 \le \sin^2(2\pi/x) \le 1 \;\Rightarrow\; 1 \le 1 + \sin^2(2\pi/x) \le 2 \;\Rightarrow\; \sqrt{x} \le \sqrt{x}\left[1 + \sin^2(2\pi/x)\right] \le 2\sqrt{x}$. Since $\lim_{x \to 0^+} \sqrt{x} = 0$ and $\lim_{x \to 0^+} 2\sqrt{x} = 0$, we have $\lim_{x \to 0^+} \left[\sqrt{x}\,(1 + \sin^2(2\pi/x))\right] = 0$ by the Squeeze Theorem.

(5%)13.	Find the vertical asymptotes of the function $y = \frac{x-1}{x^2(x+2)}$.

Sol:

The denominator of $y = \frac{x-1}{x^2(x+2)}$ is equal to zero when $x=0$ and $x=-2$.
So $x=0$ and $x=-2$ are vertical asymptotes of the function.

(5%)14.	Prove that the equation $\sqrt{x-5} = \frac{1}{x+3}$ has at least one real root.

Sol:

Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c+3} = \sqrt{c-5}$.

(5%)15.	Find the tangent line to $y = x^2 - 8x + 9$ at the point (2, -3).

Sol:

The derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$.

Therefore the slope of the tangent line at $(2, -3)$ is $f'(2) = -4$

Thus an equation of tangent line is $y - (-3) = -4(x-2)$

or $y = -4x + 5$.

(5%)16.	Given $g(x) = \sqrt{9-x}$, find $g'(x)$ using the definition of derivative.

Sol:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$$

$$= \lim_{h\to 0}\frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0}\frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

(25%)17.	Find the limit. (a) $\lim_{x\to -6}\frac{2x+12}{\lvert x+6\rvert} =$ ____________. (b) $\lim_{x\to 4}\frac{x^2+3x}{x^2-x-12} =$ ____________. (c) $\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} =$ ____________. (d) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right) =$ ____________. (e) $\lim_{x\to \pi/4} x^2\cdot\tan(x) =$ ____________.

Sol:

(a)

$$|x+6| = \begin{cases} x+6 & \text{if } x+6\ge 0 \\ -(x+6) & \text{if } x+6<0 \end{cases} = \begin{cases} x+6 & \text{if } x\ge -6 \\ -(x+6) & \text{if } x<-6 \end{cases}$$

We'll look at the one-sided limits.

$$\lim_{x\to -6^+}\frac{2x+12}{|x+6|} = \lim_{x\to -6^+}\frac{2(x+6)}{x+6} = 2 \quad\text{and}\quad \lim_{x\to -6^-}\frac{2x+12}{|x+6|} = \lim_{x\to -6^-}\frac{2(x+6)}{-(x+6)} = -2$$

The left and right limits are different, so $\lim_{x\to -6}\frac{2x+12}{|x+6|}$ does not exist.

(b)

$\lim_{x\to 4} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to 4} \frac{x(x+3)}{(x-4)(x+3)} = \lim_{x\to 4} \frac{x}{x-4}$. The last limit does not exist since $\lim_{x\to 4^-} \frac{x}{x-4} = -\infty$ and $\lim_{x\to 4^+} \frac{x}{x-4} = \infty$.

(c)

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} = \lim_{t\to 0} \frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0} \frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)} = \lim_{t\to 0} \frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)} = \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$

$$= \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1$$

(d)

$$\lim_{t\to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)} = \lim_{t\to 0} \frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$

$$= \lim_{t\to 0} \frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)} = \frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)} = -\frac{1}{2}$$

(e)

The function $f(x) = x^2 \tan x$ is continuous throughout its domain because it is the product of a polynomial and a trigonometric function. The domain of f is the set of all real numbers that are not odd multiples of $\frac{\pi}{2}$; that is, domain $f = \{x \mid x \neq n\pi/2,\ n \text{ an odd integer}\}$. Thus, $\frac{\pi}{4}$ is in the domain of f and

$$\lim_{x\to \pi/4} x^2 \tan x = f\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^2 \tan\frac{\pi}{4} = \frac{\pi^2}{16} \cdot 1 = \frac{\pi^2}{16}$$