Calculus (I) – Midterm Exam

 # 1

$f(x) = \sqrt{x}$, $g(x) = \sqrt[3]{1-x}$.

(a) Find the function $g \circ f$

(b) Find the domain of $g \circ f$

[Solution]

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1-\sqrt{x}}$.

The domain of $g \circ f$ is $\{x \mid x \text{ is in the domain of } f \text{ and } f(x) \text{ is in the domain of } g\}$. This is the domain of f, that is, $[0, \infty)$.

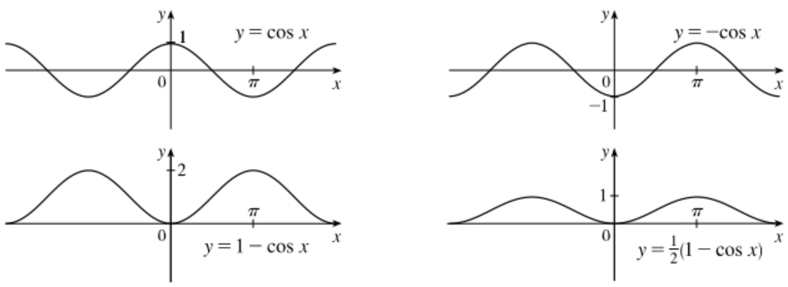
 # 2

Graph the function by hand,not by plotting ponts,but by starting with the graph of one of the standard functions given in sec1.2,and then applying the appropriate transfortions.

$y = \frac{1}{2}(1 - cosx)$

[Solution]

$y = \frac{1}{2}(1 - \cos x)$: Start with the graph of $y = \cos x$, reflect about the x-axis, shift 1 unit upward, and then shrink vertically by a factor of 2.



 # 3

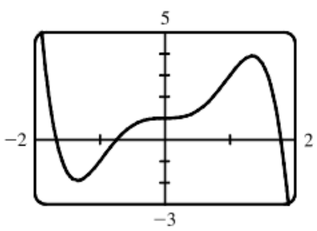
$f(x) = 1 + 3x^3 - x^5$.Determine whether f is even,odd,or neither.

[Solution]

$f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5)$$
$$= 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



4

Find the vertical asymptotes of the function

$y = \frac{x^2+1}{3x-2x^2}$

[Solution]

The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when

$x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are

vertical asymptotes of the function.

5

Find the limit.

(a) $\lim\limits_{x\to 2^-} \frac{x^2-2x}{x^2-4x+4} =$ ________.

(b) $\lim\limits_{h\to 0} \frac{\sqrt{9+h}-3}{h} =$ ________.

(c) $\lim\limits_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) =$ ________.

(d) $\lim\limits_{x\to \pi} \sin(x + sinx) =$ ________.

[Solution]

(a)

$\lim\limits_{x\to 2^-} \dfrac{x^2-2x}{x^2-4x+4} = \lim\limits_{x\to 2^-} \dfrac{x(x-2)}{(x-2)^2} = \lim\limits_{x\to 2^-} \dfrac{x}{x-2} = -\infty$ since the numerator is positive and the denominator

approaches 0 through negative values as $x \to 2^-$.

(b)

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{\left(\sqrt{9+h}\right)^2 - 3^2}{h\left(\sqrt{9+h}+3\right)} = \lim_{h\to 0} \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{3+3} = \frac{1}{6}$$

(c)

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)} = \lim_{t\to 0} \frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$$

$$= \lim_{t\to 0} \frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)} = \frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)} = -\frac{1}{2}$$

(d)

Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function

$f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

6

Prove that $\lim_{x \to 0} 3x^5 cos\frac{4}{x} = 0$.

[Solution]

-1≤$cos(4/x)$ ≤1 ⇒ -$3x^5$ ≤ $3x^5 cos(4/x)$ ≤ $3x^5$.Since $\lim_{x \to 0}(-3x^5) = 0$ and $\lim_{x \to 0}(3\,x^5) = 0$,we have

$\lim_{x \to 0}\left[3x^5 \cos\left(\frac{4}{x}\right)\right] = 0$ *by the Squeeze Theorem.*

7

Prove the statement using the ε, σ definition of a limit $\lim_{x \to 2}(14 - 5x) = 4$.

[Solution]

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(14 - 5x) - 4| < \varepsilon$. But $|(14 - 5x) - 4| < \varepsilon \Leftrightarrow$ $|-5x + 10| < \varepsilon \Leftrightarrow |-5|\,|x - 2| < \varepsilon \Leftrightarrow |x - 2| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $0 < |x - 2| < \delta \Rightarrow$ $|(14 - 5x) - 4| < \varepsilon$. Thus, $\lim_{x \to 2}(14 - 5x) = 4$ by the definition of a limit.

8

Show that there is a root of the equation.

$2sinx = 3 - 2x$

[Solution]

Let $f(x) = 2\sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2\sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2\sin x = 3 - 2x$ has a root in $(0, 1)$.

9

(a) A function *f* is differentiable at a,if __________exists.

(b) Show the relationship between continuous and differentiable.

If *f* is _________ at a,then *f* is _________ at a.

[Solution]

(a) $f'(a)$

(b) differentiable , continuous

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Suppose $f(x)$ and $g(x)$ are both differentiable.Let $F(x) = f(x)g(x)$.

Prove that $F'(x) = f(x)g'(x) + f'(x)g(x)$

[Solution]

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)] \cdot g(x+\Delta x) + f(x) \cdot [g(x+\Delta x) - g(x)]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \lim_{\Delta x \to 0} g(x+\Delta x) + \lim_{\Delta x \to 0} f(x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

11

Differentiate

(a) $y = \sqrt[3]{x}\ (2+x)$, $y' =$ ________.

(b) $y = \frac{x^2+1}{x^3-1}$, $y' =$ ________.

[Solution]

(a)

$$y = \sqrt[3]{x}\,(2+x) = 2x^{1/3} + x^{4/3} \quad \Rightarrow \quad y' = 2\left(\tfrac{1}{3}x^{-2/3}\right) + \tfrac{4}{3}x^{1/3} = \tfrac{2}{3}x^{-2/3} + \tfrac{4}{3}x^{1/3} \text{ or } \frac{2}{3\sqrt[3]{x^2}} + \frac{4}{3}\sqrt[3]{x}$$

(b)

$$y = \frac{x^2+1}{x^3-1} \overset{\text{QR}}{\Rightarrow}$$

$$y' = \frac{(x^3-1)(2x) - (x^2+1)(3x^2)}{(x^3-1)^2} = \frac{x\left[(x^3-1)(2) - (x^2+1)(3x)\right]}{(x^3-1)^2} = \frac{x(2x^3 - 2 - 3x^3 - 3x)}{(x^3-1)^2} = \frac{x(-x^3 - 3x - 2)}{(x^3-1)^2}$$

12

Find the equations of the tangent line andnormal line to curve $y = x + \sqrt{x}$ at point (1,2).

[Solution]

$y = x + \sqrt{x} \;\Rightarrow\; y' = 1 + \frac{1}{2}x^{-1/2} = 1 + 1/(2\sqrt{x})$. At $(1, 2)$, $y' = \frac{3}{2}$, and an equation of the tangent line is $y - 2 = \frac{3}{2}(x - 1)$, or $y = \frac{3}{2}x + \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$, so an equation of the normal line is $y - 2 = -\frac{2}{3}(x - 1)$, or $y = -\frac{2}{3}x + \frac{8}{3}$.