# Calculus (I) – Final Exam.

110%

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| ( 5%) 1. | Find equations of the tangent line and normal line to the curve $\frac{\sqrt{x}}{1+x^2}$ at the point (1, 1/2). |

sol：

**SOLUTION** According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}(2x)}{(1+x^2)^2}$$

$$= \frac{(1+x^2) - 4x^2}{2\sqrt{x}(1+x^2)^2} = \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$$

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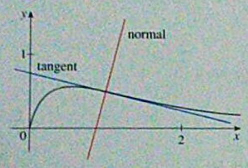


FIGURE 5

So the slope of the tangent line at $(1, \frac{1}{2})$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1 - 3 \cdot 1^2}{2\sqrt{1}(1+1^2)^2} = -\frac{1}{4}$$

We use the point-slope form to write an equation of the tangent line at $(1, \frac{1}{2})$:

$$y - \tfrac{1}{2} = -\tfrac{1}{4}(x-1) \quad \text{or} \quad y = -\tfrac{1}{4}x + \tfrac{3}{4}$$

The slope of the normal line at $(1, \frac{1}{2})$ is the negative reciprocal of $-\frac{1}{4}$, namely 4, so an equation is

$$y - \tfrac{1}{2} = 4(x-1) \quad \text{or} \quad y = 4x - \tfrac{7}{2}$$

The curve and its tangent and normal lines are graphed in Figure 5.

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| ( 5%) 2. | At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$? |

sol：

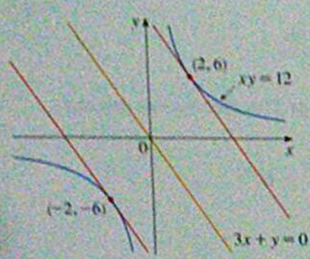


FIGURE 6

**SOLUTION** Since $xy = 12$ can be written as $y = 12/x$, we have

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{-1}) = 12(-x^{-2}) = -\frac{12}{x^2}$$

Let the $x$-coordinate of one of the points in question be $a$. Then the slope of the tangent line at that point is $-12/a^2$. This tangent line will be parallel to the line $3x + y = 0$, or $y = -3x$, if it has the same slope, that is, $-3$. Equating slopes, we get

$$-\frac{12}{a^2} = -3 \quad \text{or} \quad a^2 = 4 \quad \text{or} \quad a = \pm 2$$

Therefore the required points are $(2, 6)$ and $(-2, -6)$. The hyperbola and the tangents are shown in Figure 6.

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| ( 5%) 3. | Prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$. |

sol：

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\ \cot x$$

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| (27%) 4. | Differentiate.<br>(a) $y = \frac{t^3+3t}{t^2-4t+3}$. $y' =$ ________.<br>(b) $f(x) = \frac{x}{x+\frac{c}{x}}$. $f' =$ ________.<br>(c) $f(\theta) = \theta \cdot \cos\theta \cdot \sin\theta$. $f' =$ ________.<br>(d) $y = \sin(t + \cos\sqrt{t})$. $y' =$ ________.<br>(e) $f(t) = \tan(\sec(\cos t))$. $f' =$ ________.<br>(f) $y = (\frac{1-\cos 2x}{1+\cos 2x})^4$. $y' =$ ________.<br>(g) $\sin(xy) = \cos(x+y)$. $\frac{dy}{dx} =$ ________.<br>(h) $x^4(x+y) = y^2(3x-y)$. $\frac{dy}{dx} =$ ________.<br>(i) $xy = \sqrt{x^2+y^2}$. $\frac{dy}{dx} =$ ________. |

sol：

(a)

$$y = \frac{t^3+3t}{t^2-4t+3} \overset{QR}{\Rightarrow}$$

$$y' = \frac{(t^2-4t+3)(3t^2+3) - (t^3+3t)(2t-4)}{(t^2-4t+3)^2}$$

$$= \frac{3t^4+3t^2-12t^3-12t+9t^2+9-(2t^4-4t^3+6t^2-12t)}{(t^2-4t+3)^2} = \frac{t^4-8t^3+6t^2+9}{(t^2-4t+3)^2}$$

(b)

$$f(x) = \frac{x}{x+c/x} \quad \Rightarrow \quad f'(x) = \frac{(x+c/x)(1) - x(1-c/x^2)}{\left(x+\frac{c}{x}\right)^2} = \frac{x+c/x-x+c/x}{\left(\frac{x^2+c}{x}\right)^2} = \frac{2c/x}{\frac{(x^2+c)^2}{x^2}}\cdot\frac{x^2}{x^2} = \frac{2cx}{(x^2+c)^2}$$

(c)

Using Exercise 2.3.87(a), $f(\theta) = \theta \cos\theta \, \sin\theta \quad \Rightarrow$

$$f'(\theta) = 1\cos\theta\,\sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta\,\sin\theta - \theta\sin^2\theta + \theta\cos^2\theta$$

$$= \sin\theta\,\cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta\cos 2\theta \quad \text{[using double-angle formulas]}$$

(d)

$y = \sin(t + \cos\sqrt{t}\,) \quad \Rightarrow$

$$y' = \cos(t + \cos\sqrt{t}\,) \cdot \frac{d}{dt}(t + \cos\sqrt{t}\,) = \cos(t + \cos\sqrt{t}\,) \cdot \left(1 - \sin\sqrt{t} \cdot \frac{1}{2\sqrt{t}}\right) = \cos(t + \cos\sqrt{t}\,)\,\frac{2\sqrt{t} - \sin\sqrt{t}}{2\sqrt{t}}$$

(e)

$f(t) = \tan(\sec(\cos t)) \quad \Rightarrow$

$$f'(t) = \sec^2(\sec(\cos t)) \cdot \frac{d}{dt}\,\sec(\cos t) = \sec^2(\sec(\cos t)) \cdot \sec(\cos t)\tan(\cos t) \cdot \frac{d}{dt}\,\cos t$$

$$= -\sin t\,\sec^2(\sec(\cos t))\sec(\cos t)\tan(\cos t)$$

(f)

$y = \left(\dfrac{1 - \cos 2x}{1 + \cos 2x}\right)^4 \quad \Rightarrow$

$$y' = 4\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^3 \cdot \frac{(1 + \cos 2x)(2\sin 2x) + (1 - \cos 2x)(-2\sin 2x)}{(1 + \cos 2x)^2}$$

$$= 4\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^3 \cdot \frac{2\sin 2x\,(1 + \cos 2x + 1 - \cos 2x)}{(1 + \cos 2x)^2} = \frac{4(1 - \cos 2x)^3}{(1 + \cos 2x)^3}\,\frac{2\sin 2x\,(2)}{(1 + \cos 2x)^2} = \frac{16\sin 2x\,(1 - \cos 2x)^3}{(1 + \cos 2x)^5}$$

(g)

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}\cos(x + y) \quad \Rightarrow \quad \cos(xy) \cdot (xy' + y \cdot 1) = -\sin(x + y) \cdot (1 + y') \quad \Rightarrow$$

$$x\cos(xy)\,y' + y\cos(xy) = -\sin(x + y) - y'\sin(x + y) \quad \Rightarrow$$

$$x\cos(xy)\,y' + y'\sin(x + y) = -y\cos(xy) - \sin(x + y) \quad \Rightarrow$$

$$[x\cos(xy) + \sin(x + y)]\,y' = -1\,[y\cos(xy) + \sin(x + y)] \quad \Rightarrow \quad y' = -\frac{y\cos(xy) + \sin(x + y)}{x\cos(xy) + \sin(x + y)}$$

(h)

$$\frac{d}{dx}\left[x^4(x+y)\right]=\frac{d}{dx}\left[y^2(3x-y)\right] \quad\Rightarrow\quad x^4(1+y')+(x+y)\cdot 4x^3=y^2(3-y')+(3x-y)\cdot 2y\,y' \quad\Rightarrow$$

$$x^4+x^4y'+4x^4+4x^3y=3y^2-y^2y'+6xy\,y'-2y^2y' \quad\Rightarrow\quad x^4y'+3y^2y'-6xy\,y'=3y^2-5x^4-4x^3y \quad\Rightarrow$$

$$(x^4+3y^2-6xy)\,y'=3y^2-5x^4-4x^3y \quad\Rightarrow\quad y'=\frac{3y^2-5x^4-4x^3y}{x^4+3y^2-6xy}$$

(i)

$$\frac{d}{dx}(xy)=\frac{d}{dx}\sqrt{x^2+y^2} \quad\Rightarrow\quad xy'+y(1)=\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\,y') \quad\Rightarrow$$

$$xy'+y=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\,y' \quad\Rightarrow\quad xy'-\frac{y}{\sqrt{x^2+y^2}}\,y'=\frac{x}{\sqrt{x^2+y^2}}-y \quad\Rightarrow$$

$$\frac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}\,y'=\frac{x-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad\Rightarrow\quad y'=\frac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$$

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| (18%) 5. | Find the limit<br>(a) $\lim_{\theta\to 0}\frac{cos\theta-1}{\theta}$ = ________.<br>(b) $\lim_{x\to 0}\frac{sin3x}{5x^3-4x}$ = ________.<br>(c) $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$ = ________.<br>(d) $\lim_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$ =________.<br>(e) $\lim_{x\to\infty}\sqrt{x}\,sin\frac{1}{x}$ = ________.<br>(f) $\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)$ = ________. |

sol：

(a)

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=\lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right)=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}$$

$$=\lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}=-\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right)$$

$$=-\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1}$$

$$=-1\cdot\left(\frac{0}{1+1}\right)=0 \qquad \text{(by Equation 2)}$$

(b)

$$\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{3}{5x^2-4}\right)=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{3}{5x^2-4}=1\cdot\left(\frac{3}{-4}\right)=-\frac{3}{4}$$

(c)

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=\tfrac{1}{3}\cdot 1=\tfrac{1}{3}$$

(d)

$$\lim_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}=\lim_{t\to\infty}\frac{\left(t-t\sqrt{t}\right)/t^{3/2}}{\left(2t^{3/2}+3t-5\right)/t^{3/2}}=\lim_{t\to\infty}\frac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}}=\frac{0-1}{2+0-0}=-\frac{1}{2}$$

(e)

If $t=\dfrac{1}{x}$, then $\displaystyle\lim_{x\to\infty}\sqrt{x}\sin\frac{1}{x}=\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\sin t=\lim_{t\to 0^+}\frac{t}{\sqrt{t}}\frac{\sin t}{t}=\lim_{t\to 0^+}\sqrt{t}\cdot\lim_{t\to 0^+}\frac{\sin t}{t}=0\cdot 1=0.$

(f)

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)&=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}\\&=\lim_{x\to\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}\\&=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}\end{aligned}$$

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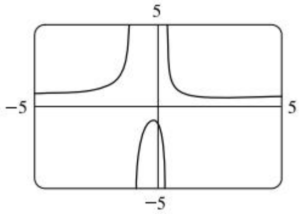
| ( 5%) 6. | Find the horizontal and vertical asymptotes of $y=\dfrac{2x^2+1}{3x^2+2x-1}$. |

sol：

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{2x^2+1}{3x^2+2x-1}&=\lim_{x\to\pm\infty}\frac{(2x^2+1)/x^2}{(3x^2+2x-1)/x^2}\\&=\lim_{x\to\pm\infty}\frac{2+1/x^2}{3+2/x-1/x^2}=\frac{2}{3}\end{aligned}$$




so $y=\dfrac{2}{3}$ is a horizontal asymptote. $y=f(x)=\dfrac{2x^2+1}{3x^2+2x-1}=\dfrac{2x^2+1}{(3x-1)(x+1)}$.

The denominator is zero when $x=\frac{1}{3}$ and $-1$, but the numerator is nonzero, so $x=\frac{1}{3}$ and $x=-1$ are vertical asymptotes.

The graph confirms our work.

| (5%) 7. | Find the absolute maximum and absolute minimum value of $f(t) = t + \cot(t/2)$ on interval $[\pi/4, 7\pi/4]$. |
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sol：

$f(t) = t + \cot(t/2)$, $[\pi/4, 7\pi/4]$. $f'(t) = 1 - \csc^2(t/2) \cdot \frac{1}{2}$.

$f'(t) = 0 \quad \Rightarrow \quad \frac{1}{2}\csc^2(t/2) = 1 \quad \Rightarrow \quad \csc^2(t/2) = 2 \quad \Rightarrow \quad \csc(t/2) = \pm\sqrt{2} \quad \Rightarrow \quad \frac{1}{2}t = \frac{\pi}{4}$ or $\frac{1}{2}t = \frac{3\pi}{4}$

$\left[\frac{\pi}{4} \le t \le \frac{7\pi}{4} \quad \Rightarrow \quad \frac{\pi}{8} \le \frac{1}{2}t \le \frac{7\pi}{8}\right.$ and $\csc(t/2) \ne -\sqrt{2}$ in the last interval$\Big] \quad \Rightarrow \quad t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$.

$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \cot\frac{\pi}{8} \approx 3.20$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cot\frac{\pi}{4} = \frac{\pi}{2} + 1 \approx 2.57$, $f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} + \cot\frac{3\pi}{4} = \frac{3\pi}{2} - 1 \approx 3.71$, and $f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} + \cot\frac{7\pi}{8} \approx 3.08$. So $f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$ is the absolute maximum value and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ is the absolute minimum value.

| ( 5%) 8. | Show that $2x - 1 - \sin x = 0$ has exactly one real root. |
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sol：

Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. $f$ is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so $f$ is continuous (and differentiable) for all $x$. By the Intermediate Value Theorem, there is a number $c$ in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots $a$ and $b$ with $a < b$, then $f(a) = f(b) = 0$. Since $f$ is continuous on $[a, b]$ and

differentiable on $(a, b)$, Rolle's Theorem implies that there is a number $r$ in $(a, b)$ such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \le 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

| (10%) 9. | Find $f$.<br>(a) $f''(x) = x^{2/3} + x^{-2/3}$.<br>(b) $f''(\theta) = \sin\theta + \cos\theta$. $f(0) = 3$, $f'(0) = 4$. |
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sol：

(a)

$f''(x) = x^{2/3} + x^{-2/3}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$f'(x) = \begin{cases} \frac{3}{5}x^{5/3} + 3x^{1/3} + C_1 & \text{if } x < 0 \\ \frac{3}{5}x^{5/3} + 3x^{1/3} + C_2 & \text{if } x > 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \frac{9}{40}x^{8/3} + \frac{9}{4}x^{4/3} + C_1x + D_1 & \text{if } x < 0 \\ \frac{9}{40}x^{8/3} + \frac{9}{4}x^{4/3} + C_2x + D_2 & \text{if } x > 0 \end{cases}$$

(b)

$f''(\theta) = \sin\theta + \cos\theta \quad \Rightarrow \quad f'(\theta) = -\cos\theta + \sin\theta + C.$ $f'(0) = -1 + C$ and $f'(0) = 4 \quad \Rightarrow \quad C = 5$, so

$f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D.$ $f(0) = -1 + D$ and $f(0) = 3 \quad \Rightarrow \quad D = 4,$

so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4.$

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| (25%) 10. | Use the following guidelines to sketch the curve $y = \frac{x^3}{(x+1)^2}$. | |
| | (1%) | (i) Domain. |
| | (2%) | (ii) Intercepts. |
| | (1%) | (iii) Symmetry. |
| | (4%) | (iv) Asymptotes. |
| | (4%) | (v) Intervals of increase or decrease. |
| | (2%) | (vi) Local maximum and minimum value. |
| | (6%) | (vii) Concavity and point of inflection. |
| | (5%) | (viii)Sketch the curve. |

sol：

$y = f(x) = \dfrac{x^3}{(x+1)^2} = x - 2 + \dfrac{3x+2}{(x+1)^2}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** $x$-intercept: 0; $y$-intercept: $f(0) = 0$

**C.** No symmetry **D.** $\lim\limits_{x \to -1^-} f(x) = -\infty$ and $\lim\limits_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (x-2)] = \lim\limits_{x \to \pm\infty} \dfrac{3x+2}{(x+1)^2} = 0$, so $y = x - 2$ is a SA.

**E.** $f'(x) = \dfrac{(x+1)^2(3x^2) - x^3 \cdot 2(x+1)}{[(x+1)^2]^2} = \dfrac{x^2(x+1)[3(x+1) - 2x]}{(x+1)^4} = \dfrac{x^2(x+3)}{(x+1)^3} > 0 \quad \Leftrightarrow \quad x < -3$ or

$x > -1$ $[x \neq 0]$, so $f$ is increasing on $(-\infty, -3)$ and $(-1, \infty)$, and $f$ is decreasing on $(-3, -1)$.

**F.** Local maximum value $f(-3) = -\frac{27}{4}$, no local minimum

**G.** 
$$f''(x) = \frac{(x+1)^3(3x^2+6x) - (x^3+3x^2) \cdot 3(x+1)^2}{[(x+1)^3]^2}$$
$$= \frac{3x(x+1)^2[(x+1)(x+2) - (x^2+3x)]}{(x+1)^6}$$
$$= \frac{3x(x^2+3x+2-x^2-3x)}{(x+1)^4} = \frac{6x}{(x+1)^4} > 0 \quad \Leftrightarrow$$

$x > 0$, so $f$ is CU on $(0, \infty)$ and $f$ is CD on $(-\infty, -1)$ and $(-1, 0)$. IP at $(0, 0)$

**H.**