Final Exam

1. Find the limit.

(a) $\lim_{x\to 3^+} \log_{10}(x^2 - 5x + 6)$

(b) $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos 2\theta}$

(c) $\lim_{x\to\infty} \frac{\ln(\ln x)}{x}$

(d) $\lim_{x\to 1}(\frac{x}{x-1} - \frac{1}{\ln x})$

(e) $\lim_{x\to 0}(\cot 2x)(\sin 6x)$

(f) $\lim_{x\to 0^+}(1+\sin 3x)^{\frac{1}{x}}$

(g) $\lim_{x\to\infty} \tan^{-1}(e^x)$

sol：

(a)

Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim_{x\to 3^+} \log_{10}(x^2 - 5x + 6) = \lim_{t\to 0^+} \log_{10} t = -\infty$

[analogous to (4) in Section 6.2*].

(b)

This limit has the form $\frac{0}{0}$. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos 2\theta} \overset{\text{H}}{=} \lim_{\theta\to\pi/2} \frac{-\cos\theta}{-2\sin 2\theta} \overset{\text{H}}{=} \lim_{\theta\to\pi/2} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{4}$

(c)

This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \frac{\ln\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0$

(d)

This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1-(1/x)+\ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to 1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} = \lim_{x\to 1} \frac{x}{1+x} = \frac{1}{1+1} = \frac{1}{2}$$

(e)

This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$\lim_{x\to 0} \cot 2x \sin 6x = \lim_{x\to 0} \frac{\sin 6x}{\tan 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{6\cos 6x}{2\sec^2 2x} = \frac{6(1)}{2(1)^2} = 3$

(f)

$$y = (1 + \sin 3x)^{1/x} \quad \Rightarrow \quad \ln y = \frac{1}{x} \ln(1 + \sin 3x) \quad \Rightarrow$$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 3x)}{x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{[1/(1 + \sin 3x)] \cdot 3 \cos 3x}{1} = \lim_{x \to 0^+} \frac{3 \cos 3x}{1 + \sin 3x} = \frac{3 \cdot 1}{1 + 0} = 3 \quad \Rightarrow$$

$$\lim_{x \to 0^+} (1 + \sin 3x)^{1/x} = \lim_{x \to 0^+} e^{\ln y} = e^3$$

(g)

Let $t = e^x$. As $x \to \infty$, $t \to \infty$. $\lim\limits_{x \to \infty} \arctan(e^x) = \lim\limits_{t \to \infty} \arctan t = \frac{\pi}{2}$ by (8).

2. Prove that $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

sol：

Since tan is differentiable, $\tan^{-1}$ is also differentiable. To find its derivative, let $y = \tan^{-1}x$. Then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

3. Differentiate the function.

(a) $y = \log_2 (x \log_5 x)$

(b) $y = x \sin^{-1} x + \sqrt{1 - x^2}$

(c) $y = \sqrt{\tan^{-1} x}$

sol：

(a)

$y = \log_2(x \log_5 x) \quad \Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)} \frac{d}{dx}(x \log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x \cdot \frac{1}{x \ln 5} + \log_5 x\right) = \frac{1}{(x \log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \frac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base formula. Thus, $y' = \frac{1}{x \ln x \ln 2} + \frac{1}{x \ln 2} = \frac{1 + \ln x}{x \ln x \ln 2}$.

(b)

$y = x \sin^{-1} x + \sqrt{1-x^2} \quad \Rightarrow$

$$y' = x \cdot \frac{1}{\sqrt{1-x^2}} + (\sin^{-1} x)(1) + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$$

(c)

$y = \sqrt{\tan^{-1} x} = (\tan^{-1} x)^{1/2} \quad \Rightarrow$

$$y' = \tfrac{1}{2}(\tan^{-1} x)^{-1/2} \cdot \frac{d}{dx}(\tan^{-1} x) = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1+x^2} = \frac{1}{2\sqrt{\tan^{-1} x}\,(1+x^2)}$$

4. Evaluate the integral.

(a) $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{6}{\sqrt{1-x^2}}\,dx$

(b) $\int \frac{x}{x^4+9}\,dx$

(c) $\int \tan^{-1}(2x)\ dx$

(d) $\int_4^9 \frac{\ln x}{\sqrt{x}}\,dx$

(e) $\int x^3\, e^x\, dx$

(f) $\int e^x\ \sin x\ dx$

(g) $\int \tan x\ \sec^3 x\, dx$

(h) $\int_0^{\pi/2} \sin^7\theta\, \cos^5\theta\, d\theta$

(i) $\int \sin^4 x\, dx$

(j) $\int \tan^3 x\, dx$

(k) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

(l) $\int_0^{(3\sqrt{3})/2} \frac{x^3}{(4x^2+9)^{3/2}}\ dx$

(m) $\int \frac{x}{\sqrt{3-2x-x^2}}\ dx$

sol：

(a)

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{6}{\sqrt{1-p^2}}\,dp = 6\left[\sin^{-1} p\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = 6\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right] = 6\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 6\left(\frac{\pi}{2}\right) = 3\pi$$

(b)

SOLUTION We substitute $u = x^2$ because then $du = 2x\,dx$ and we can use Equation 14 with $a = 3$:

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

(c)

Let $u = \tan^{-1} 2y$, $dv = dy \quad \Rightarrow \quad du = \dfrac{2}{1+4y^2}\,dy$, $v = y$. Then by Equation 2,

$$\int \tan^{-1} 2y\,dy = y\tan^{-1} 2y - \int \frac{2y}{1+4y^2}\,dy = y\tan^{-1} 2y - \int \frac{1}{t}\left(\frac{1}{4}\,dt\right) \qquad \begin{bmatrix} t = 1+4y^2, \\ dt = 8y\,dy \end{bmatrix}$$

$$= y\tan^{-1} 2y - \tfrac{1}{4}\ln|t| + C = y\tan^{-1} 2y - \tfrac{1}{4}\ln(1+4y^2) + C$$

(d)

Let $u = \ln y$, $dv = \dfrac{1}{\sqrt{y}}\,dy = y^{-1/2}\,dy \quad \Rightarrow \quad du = \dfrac{1}{y}\,dy$, $v = 2y^{1/2}$. Then

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[2\sqrt{y}\,\ln y\right]_4^9 - \int_4^9 2y^{-1/2}\,dy = (6\ln 9 - 4\ln 4) - \left[4\sqrt{y}\right]_4^9 = 6\ln 9 - 4\ln 4 - (12-8)$$

$$= 6\ln 9 - 4\ln 4 - 4$$

(e)

First let $u = z^3$, $dv = e^z dz \quad \Rightarrow \quad du = 3z^2 dz$, $v = e^z$. Then $I_1 = \int z^3 e^z dz = z^3 e^z - 3\int z^2 e^z dz$. Next let $u_1 = z^2$, $dv_1 = e^z dz \quad \Rightarrow \quad du_1 = 2z\,dz$, $v_1 = e^z$. Then $I_2 = z^2 e^z - 2\int z e^z dz$. Finally, let $u_2 = z$, $dv_2 = e^z dz \quad \Rightarrow \quad du_2 = dz$, $v_2 = e^z$. Then $\int z e^z dz = z e^z - \int e^z dz = z e^z - e^z + C_1$. Substituting in the expression for I_2, we get $I_2 = z^2 e^z - 2(z e^z - e^z + C_1) = z^2 e^z - 2z e^z + 2e^z - 2C_1$. Substituting the last expression for I_2 into I_1 gives $I_1 = z^3 e^z - 3(z^2 e^z - 2z e^z + 2e^z - 2C_1) = z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C$, where $C = 6C_1$.

(f)

SOLUTION Neither e^x nor $\sin x$ becomes simpler when differentiated, but we try choosing $u = e^x$ and $dv = \sin x\,dx$ anyway. Then $du = e^x\,dx$ and $v = -\cos x$, so integration by parts gives

$$\boxed{4} \qquad \int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. This time we use $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\boxed{5} \qquad \int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x\,dx$, which is where we started. However, if we put the expression for $\int e^x \cos x\,dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x\,dx$ to both sides, we obtain

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$

(g)

$$\int \tan x \sec^3 x\,dx = \int \tan x \sec x \sec^2 x\,dx = \int u^2\,du \qquad [u = \sec x,\ du = \sec x \tan x\,dx]$$
$$= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C$$

(h)

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta\,d\theta = \int_0^{\pi/2} \sin^7\theta \cos^4\theta \cos\theta\,d\theta = \int_0^{\pi/2} \sin^7\theta\,(1-\sin^2\theta)^2 \cos\theta\,d\theta$$
$$\stackrel{s}{=} \int_0^1 u^7(1-u^2)^2\,du = \int_0^1 u^7(1-2u^2+u^4)\,du = \int_0^1 (u^7 - 2u^9 + u^{11})\,du$$
$$= \left[\frac{1}{8}u^8 - \frac{1}{5}u^{10} + \frac{1}{12}u^{12}\right]_0^1 = \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) - 0 = \frac{15-24+10}{120} = \frac{1}{120}$$

(i)

SOLUTION We could evaluate this integral using the reduction formula for $\int \sin^n x\,dx$ (Equation 8.1.7) together with Example 3 (as in Exercise 43 in Section 8.1), but a better method is to write $\sin^4 x = (\sin^2 x)^2$ and use a half-angle formula:

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

Since $\cos^2 2x$ occurs, we must use another half-angle formula

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x)$$

This gives

$$\int \sin^4 x\,dx = \tfrac{1}{4}\int [1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]\,dx$$
$$= \tfrac{1}{4}\int (\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x)\,dx$$
$$= \tfrac{1}{4}(\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x) + C$$

(j)

SOLUTION Here only $\tan x$ occurs, so we use $\tan^2 x = \sec^2 x - 1$ to rewrite a $\tan^2 x$ factor in terms of $\sec^2 x$:

$$\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$$
$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$
$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

(k)

EXAMPLE 1 Evaluate $\int \quad x^2$

SOLUTION Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

(Note that $\cos\theta \ge 0$ because $-\pi/2 \le \theta \le \pi/2$.) Thus the Inverse Substitution Rule gives

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C$$

Since this is an indefinite integral, we must return to the original variable x. This can be done either by using trigonometric identities to express $\cot\theta$ in terms of $\sin\theta = x/3$ or by drawing a diagram, as in Figure 1, where θ is interpreted as an angle of a right triangle. Since $\sin\theta = x/3$, we label the opposite side and the hypotenuse as having lengths x and 3. Then the Pythagorean Theorem gives the length of the adjacent side as $\sqrt{9 - x^2}$, so we can simply read the value of $\cot\theta$ from the figure:

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

(Although $\theta > 0$ in the diagram, this expression for $\cot\theta$ is valid even when $\theta < 0$.) Since $\sin\theta = x/3$, we have $\theta = \sin^{-1}(x/3)$ and so

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

(l)

SOLUTION First we note that $(4x^2 + 9)^{3/2} = (\sqrt{4x^2+9}\,)^3$ so trigonometric substitution is appropriate. Although $\sqrt{4x^2+9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$. When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\,\sin\theta\, d\theta$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\, d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$. Therefore

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du = \frac{3}{16}\int_1^{1/2} (1 - u^{-2})\, du$$

$$= \frac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} = \frac{3}{16}\left[\left(\tfrac{1}{2} + 2\right) - (1 + 1)\right] = \frac{3}{32}$$

(m)

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$\begin{aligned} 3 - 2x - x^2 &= 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) \\ &= 4 - (x + 1)^2 \end{aligned}$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\,du$$

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4 - u^2} = 2\cos\theta$, so

$$\begin{aligned} \int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C \\ &= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C \end{aligned}$$

□