Linear Algebra Final Practice (範圍:3-1~3-4, 4-1~4-6, 5-1~5-2)

1. Find the transition matrix P from the basis $B = \{(1, 2), (3, -1)\}$ of $\mathbf{R}^2$ to the basis $B' = \{(3, 1), (5, 2)\}$. If $\mathbf{u}$ is a vector such that $\mathbf{u}_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find $\mathbf{u}_{B'}$.

2. Consider the linear transformation T: $\mathbf{R}^3 \to \mathbf{R}^2$, defined by $T(x, y, z) = (x + y, 2z)$. Find the matrix of T with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u}'_1, \mathbf{u}'_2\}$ of $\mathbf{R}^3$ and $\mathbf{R}^2$, where $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 4)$, $\mathbf{u}_3 = (1, 2, 3)$, and $\mathbf{u}'_1 = (1, 0)$, $\mathbf{u}'_2 = (0, 2)$. Use this matrix to find the image of the vector $\mathbf{u} = (2, 3, 5)$.

3. Determine whether the vector $(3, -1, 11)$ lies in the subspace span $\{(-1, 5, 3), (2, -3, 4)\}$ of $\mathbf{R}^3$.

4. Determine whether the set $\{(1, 2, 0), (0, 1, -1), (1, 1, 2)\}$ is linearly independent in $\mathbf{R}^3$.

5. Find a basis for the column space of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -1 \\ -1 & -4 & 6 \end{bmatrix}$, and determine its rank.

6. Determine the projection of the vector $\mathbf{v} = (6, 7)$ onto the vector $\mathbf{u} = (1, 4)$.

7. The set $\{(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)\}$ is linearly independent in $\mathbf{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbf{R}^4$. Construct an orthonormal basis for V.

8. The following vectors $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, \frac{3}{5}, \frac{4}{5})$, and $\mathbf{u}_3 = (0, \frac{4}{5}, -\frac{3}{5})$ form an orthonormal basis for $\mathbf{R}^3$. Express the vector $\mathbf{v} = (7, -5, 10)$ as a linear combination of these vectors.

9. Solve the system of linear equations using Cramer's rule.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= -3 \\ 2x_1 + 5x_2 + x_3 &= -5 \\ x_1 + 2x_2 + 3x_3 &= 6 \end{aligned}$$

10. Find the characteristic polynomials, eigenvalues, and corresponding eigenspaces of the matrix $\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

11. Use the formula for the inverse of a matrix to compute the inverse of the matrix $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & 3 & 5 \end{bmatrix}$.

12. If B is a 2 * 2 matrix with $|B| = 5$, use the properties of determinants to compute the following determinants.
(a) $|3B|$ (b) $|B^2|$ (c) $|BB^tB^{-1}|$, assuming B^{-1} exist.

13. Show that the matrix $\begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{bmatrix}$ is singular.

14. Determine values of λ for which the following system of equations has nontrivial solutions. Find the solutions for each value of λ.

$$\begin{aligned} (\lambda + 2)x_1 + (\lambda + 4)x_2 &= 0 \\ 2x_1 + (\lambda + 1)x_2 &= 0 \end{aligned}$$