Calculus (II) – Midterm Exam 1

100%

(5%)1.Use the form of limit of Riemann sums to prove that $\int_a^b x^2 dx = (b^3 - a^3)/3$.

sol：

$$\int_a^b x^2\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left[a + \frac{b-a}{n} i\right]^2 = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left[a^2 + 2a\frac{b-a}{n} i + \frac{(b-a)^2}{n^2} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{n^3} \sum_{i=1}^{n} i^2 + \frac{2a\,(b-a)^2}{n^2} \sum_{i=1}^{n} i + \frac{a^2\,(b-a)}{n} \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{n^3} \frac{n\,(n+1)\,(2n+1)}{6} + \frac{2a\,(b-a)^2}{n^2} \frac{n\,(n+1)}{2} + \frac{a^2\,(b-a)}{n} n\right]$$

$$= \lim_{n\to\infty} \left[\frac{(b-a)^3}{6} \cdot 1 \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + a\,(b-a)^2 \cdot 1 \cdot \left(1 + \frac{1}{n}\right) + a^2\,(b-a)\right]$$

$$= \frac{(b-a)^3}{3} + a\,(b-a)^2 + a^2\,(b-a) = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{3} + ab^2 - 2a^2b + a^3 + a^2b - a^3$$

$$= \frac{b^3}{3} - \frac{a^3}{3} - ab^2 + a^2b + ab^2 - a^2b = \frac{b^3 - a^3}{3}$$

(5%)2.Express $\lim_{n\to\infty} \sum_{i=1}^{n} i^4/n^5$ as a definite integral.

sol：

$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{i^4}{n^5} = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{i^4}{n^4} \cdot \frac{1}{n} = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\frac{i}{n}\right)^4 \frac{1}{n}$. At this point, we need to recognize the limit as being of the form $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral is $\int_0^1 x^4\,dx$.

3. Suppose f is continuous on $[a, b]$.

(2%)(a) If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.

(3%)(b) $\int_a^b f(x)\,dx = \int f(x)]_a^b$ (in terms of $f(x)$.)

(20%)4. Find the derivative of the function.

(a) $h(u) = \int_0^u (\sqrt{t}/(t + 1))\ dt$

(b) $h(x) = \int_2^{1/x} sin^4 t\ dt$

(c) $f(x) = \int_{\sqrt{x}}^{\pi/4} \theta\ tan\theta\ d\theta$

(d) $g(x) = \int_{\tan x}^{x^2} 1/\sqrt{2 + t^4}\ dt$

sol：

(a)

$f(t) = \dfrac{\sqrt{t}}{t+1}$ and $h(u) = \displaystyle\int_0^u \frac{\sqrt{t}}{t+1}\,dt$, so by FTC1, $h'(u) = f(u) = \dfrac{\sqrt{u}}{u+1}$.

(b)

Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \sin^4 t\,dt = \frac{d}{du}\int_2^u \sin^4 t\,dt \cdot \frac{du}{dx} = \sin^4 u\,\frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

(c)

Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan\theta\,d\theta = -\frac{d}{du}\int_{\pi/4}^{\sqrt{x}} \theta\tan\theta\,d\theta \cdot \frac{du}{dx} = -u\tan u\,\frac{du}{dx} = -\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$$

(d)

$$g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_1^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_1^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_1^{x^2} \frac{dt}{\sqrt{2+t^4}} \Rightarrow$$

$$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}}\frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}}\frac{d}{dx}(x^2) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$$

(50%)5. Evaluate the integral.

(a) $\int_{-1}^{2}(3u-2)(u+1)du$

(b) $\int_{0}^{\pi/4} sec\ \theta\ tan\ \theta\ d\theta$

(c) $\int_{0}^{3\pi/2}|sin\ x|$ dx

(d) $\int_{0}^{\pi/4}((1+\ cos^2\theta)/cos^2\theta)\ d\theta$

(e) $\int_{0}^{\pi/6}(sin\ t/cos^2 t)\ dt$

(f) $\int_{-\pi/4}^{\pi/4}(x^3\ +\ x^4 tan\ x)\ dx$

(g) $\int sin\ t\ sec^2(cos\ t)\ dt$

(h) $\int(1/cos^2 t\sqrt{1+tan\ t}\ )\ dt$

(i) $\int(x^3/(x^4-5)^2)\ dx$

(j) $\int(sin\ 2x/sin\ x)\ dx$

sol：

(a)

$\int_{-1}^{2}(3u-2)(u+1)\,du=\int_{-1}^{2}(3u^2+u-2)\,du=\left[u^3+\frac{1}{2}u^2-2u\right]_{-1}^{2}=(8+2-4)-\left(-1+\frac{1}{2}+2\right)=6-\frac{3}{2}=\frac{9}{2}$

(b)

$\int_{0}^{\pi/4}\sec\theta\,\tan\theta\,d\theta=[\sec\theta]_{0}^{\pi/4}=\sec\frac{\pi}{4}-\sec 0=\sqrt{2}-1$

(c)

$\int_{0}^{3\pi/2}|\sin x|\,dx=\int_{0}^{\pi}\sin x\,dx+\int_{\pi}^{3\pi/2}(-\sin x)\,dx=\left[-\cos x\right]_{0}^{\pi}+\left[\cos x\right]_{\pi}^{3\pi/2}=[1-(-1)]+[0-(-1)]=2+1=3$

(d)

$$\int_{0}^{\pi/4}\frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta=\int_{0}^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta=\int_{0}^{\pi/4}(\sec^2\theta+1)\,d\theta$$
$$=\left[\tan\theta+\theta\right]_{0}^{\pi/4}=\left(\tan\frac{\pi}{4}+\frac{\pi}{4}\right)-(0+0)=1+\frac{\pi}{4}$$

(e)

Let $u=\cos t$, so $du=-\sin t\,dt$. When $t=0$, $u=1$; when $t=\frac{\pi}{6}$, $u=\sqrt{3}/2$. Thus,

$$\int_{0}^{\pi/6}\frac{\sin t}{\cos^2 t}\,dt=\int_{1}^{\sqrt{3}/2}\frac{1}{u^2}(-du)=\left[\frac{1}{u}\right]_{1}^{\sqrt{3}/2}=\frac{2}{\sqrt{3}}-1.$$

(f)

$\int_{-\pi/4}^{\pi/4}(x^3+x^4\tan x)\,dx=0$ by Theorem 6(b), since $f(x)=x^3+x^4\tan x$ is an odd function.

(g)

Let $u = \cos t$. Then $du = -\sin t\, dt$ and $\sin t\, dt = -du$, so

$\int \sin t \, \sec^2(\cos t)\, dt = \int \sec^2 u \cdot (-du) = -\tan u + C = -\tan(\cos t) + C$.

(h)

Let $u = 1 + \tan t$. Then $du = \sec^2 t\, dt$, so

$$\int \frac{dt}{\cos^2 t\,\sqrt{1+\tan t}} = \int \frac{\sec^2 t\, dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

(i)

Let $u = x^4 - 5$. Then $du = 4x^3\, dx$ and $x^3\, dx = \frac{1}{4}\, du$, so

$$\int \frac{x^3}{(x^4-5)^2}\, dx = \int \frac{1}{u^2}\left(\frac{1}{4}\, du\right) = \frac{1}{4}\int u^{-2}\, du = \frac{1}{4}\frac{u^{-1}}{-1} + C = -\frac{1}{4u} + C = -\frac{1}{4\,(x^4-5)} + C.$$

(j)

$$\int \frac{\sin 2x}{\sin x}\, dx = \int \frac{2\sin x\, \cos x}{\sin x}\, dx = \int 2\cos x\, dx = 2\sin x + C$$

(15%)6. Sketch the region enclosed by the given curves and find its area.
(a) $x = 2y^2$, $x = 4 + y^2$.

(b) $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$.

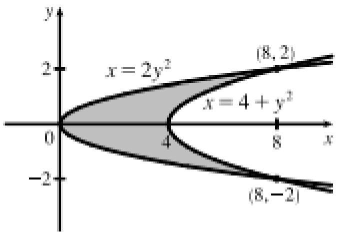
(c) $4x + y^2 = 12$, $x = y$.

sol：

(a)

$2y^2 = 4 + y^2 \quad \Leftrightarrow \quad y^2 = 4 \quad \Leftrightarrow \quad y = \pm 2$, so

$$A = \int_{-2}^{2} \left[(4 + y^2) - 2y^2\right] dy$$
$$= 2\int_{0}^{2} (4 - y^2)\, dy \qquad \text{[by symmetry]}$$
$$= 2\left[4y - \tfrac{1}{3}y^3\right]_0^2 = 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3}$$

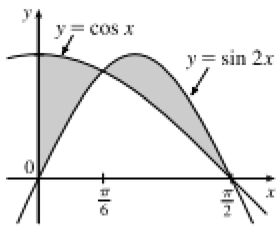


(b)

Notice that $\cos x = \sin 2x = 2\sin x \cos x \quad \Leftrightarrow \quad 2\sin x\cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$

$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$
$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$



(c)

$4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow$

$x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy$$
$$= \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2}$$
$$= \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18)$$
$$= 22 - \tfrac{2}{3} = \tfrac{64}{3}$$