Calculus (I) – Midterm Exam

 # 1

(a) A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = \underline{\ f(a)\ }.$$

(b) the derivative of a function $f(x)$ at a number a , that is, $f'(a)$,

is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

(c) the tangent line to $y = f(x)$ at $(a, f(a))$ is the line through

$(a, f(a))$ whose slope is equal to $\underline{\ f'(a)\ }$.

(d) A function f is differentiable at a if $\underline{\ f'(a)\ }$ exists.

(e) If f is <u>differentiable</u> at a, then f is <u>continuous</u> at a.

(continuous/differentiable)

 # 2

Let $f(x) = \sqrt{x+1}$, $g(x) = 4x - 3$. Find the function $f \circ g$ and its domain.

[Solution]

$(f \circ g)(x) = f(g(x)) = f(4x-3) = \sqrt{(4x-3)+1} = \sqrt{4x-2}$

The domain of $f \circ g$ is $\{x \mid 4x-3 \geq -1\} = \{x \mid 4x \geq 2\} = \left\{x \mid x \geq \frac{1}{2}\right\} = \left[\frac{1}{2}, \infty\right)$.

 # 3

Determine whether $f(x) = x|\,x\,|$ is even, odd, or neither.

[Solution]

$f(x) = x\,|x|$.

$$\begin{aligned} f(-x) &= (-x)\,|-x| = (-x)\,|x| = -(x\,|x|) \\ &= -f(x) \end{aligned}$$

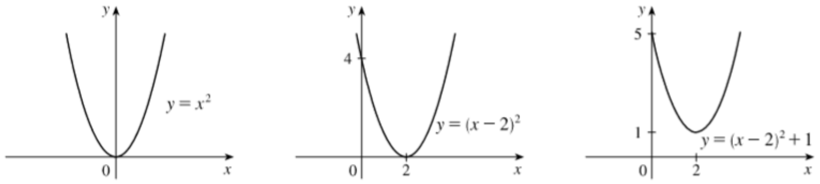
Since $f(-x) = -f(x)$, f is an odd function.

4

Graph $y = x^2 - 4x + 5$ by hand, not by plotting points, but by starting with the graph of one of the standard function, and then applying the appropriate transformations.

[Solution]

$y = x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 = (x-2)^2 + 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift upward 1 unit.



5

Determine the vertical asymptotes of the function $y = x \csc x$.

[Solution]

vertical asymptotes : $\lim_{x\to a} x\csc x = \pm\infty$

$y = x\csc x = \frac{x}{\sin x}$

∴ when $\sin x = 0$. $x = n\pi$. $n \in \mathbb{Z}$

Note that when $n=0$. $x=0$. $\lim_{x\to 0} \frac{x}{\sin x} \neq +\infty$ or $-\infty$

∴ the vertical asymptotes is $x = n\pi$. $n \in \mathbb{Z}\setminus\{0\}$

6

Determine the limit.

(a) $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$ = ________.

(b) $\lim_{t\to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$ = ________.

(c) $\lim_{x\to \pi^-} \cot x$ = ________.

(d) $\lim_{x\to 0} x^2 \cos(20\pi x)$ = ________.

(e) $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$ = ________.

(f) $\lim_{x\to 0^+} (\frac{1}{x} - \frac{1}{|x|})$ = ________.

[Solution]

(a)

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = \lim_{h\to 0} \frac{\left(8 + 12h + 6h^2 + h^3\right) - 8}{h} = \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h}$$
$$= \lim_{h\to 0} \left(12 + 6h + h^2\right) = 12 + 0 + 0 = 12$$

(b)

$$\lim_{t\to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} = \lim_{t\to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t} + \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}} = \lim_{t\to 0} \frac{\left(\sqrt{1+t}\right)^2 - \left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t} + \sqrt{1-t}\right)}$$
$$= \lim_{t\to 0} \frac{(1+t) - (1-t)}{t\left(\sqrt{1+t} + \sqrt{1-t}\right)} = \lim_{t\to 0} \frac{2t}{t\left(\sqrt{1+t} + \sqrt{1-t}\right)} = \lim_{t\to 0} \frac{2}{\sqrt{1+t} + \sqrt{1-t}}$$
$$= \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{2} = 1$$

(c)

$\lim_{x\to\pi^-} \cot x = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

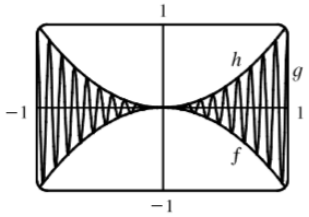
(d)

Let $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x) = x^2$. Then

$-1 \le \cos 20\pi x \le 1 \quad \Rightarrow \quad -x^2 \le x^2 \cos 20\pi x \le x^2 \quad \Rightarrow \quad f(x) \le g(x) \le h(x)$.

So since $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, by the Squeeze Theorem we have

$\lim_{x\to 0} g(x) = 0$.



(e)

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geq -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x\to\pi} \frac{\sin x}{2 + \cos x} = \lim_{x\to\pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0 \quad \square$$

(f)

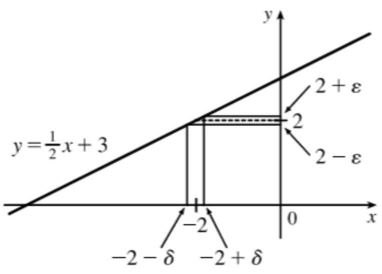
Since $|x| = x$ for $x > 0$, we have $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \to 0+} 0 = 0$.

7

Prove that $\lim_{x \to -2} (\frac{1}{2}x + 3) = 2$ using the ε, δ definition of a limit.

[Solution]

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon$. But $\left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon \quad \Leftrightarrow$ $\left|\frac{1}{2}x + 1\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{2}|x + 2| < \varepsilon \quad \Leftrightarrow \quad |x - (-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \quad \Rightarrow$ $\left|(\frac{1}{2}x + 3) - 2\right| < \varepsilon$. Thus, $\lim_{x \to -2} (\frac{1}{2}x + 3) = 2$ by the definition of a limit.



8

Prove that $\sqrt{x - 5} = \frac{1}{x + 3}$ has at least one real root.

[Solution]

Let $f(x) = \sqrt{x - 5} - \dfrac{1}{x + 3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c + 3} = \sqrt{c - 5}$.

9

Find an equation of the tangent line to $y = \frac{2x + 1}{x + 2}$ at (1, 1).

[Solution]

Using (1) with $f(x) = \dfrac{2x + 1}{x + 2}$ and $P(1, 1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x + 1}{x + 2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x + 1 - (x + 2)}{x + 2}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 2)}$$
$$= \lim_{x \to 1} \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

10

$f(x) = \sqrt{9 - x}$.

(a) Find $f'(x)$ using the definition of derivative.

(b) State the domain of $f(x)$.

(c) State the domain of $f'(x)$.

[Solution]

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$$

$$= \lim_{h\to 0} \frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

11

Where is the function $f(x) = |x|$ differentiable?

[Solution]

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

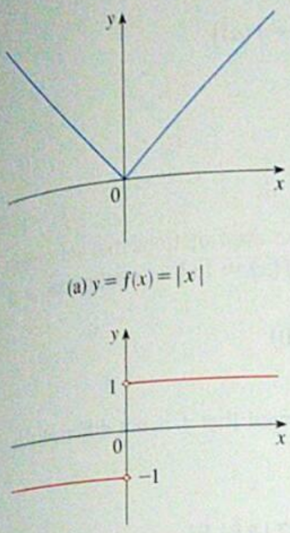
$$= \lim_{h \to 0} \frac{-(x + h) - (-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} \qquad \text{(if it exists)}$$



(a) $y = f(x) = |x|$

(b) $y = f'(x)$

FIGURE 5

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and $$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at $(0, 0)$. [See Figure 5(a).] □

12

Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

[Solution]

SOLUTION

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.