Calculus (I) – Final Exam

100%

(5%)1.	Use the definition of limit to prove that if $f(x)$ and $g(x)$ are both differentiable, then $\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Sol:

PROOF Let $F(x) = f(x)g(x)$. Then

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

In order to evaluate this limit, we would like to separate the functions f and g as in the proof of the Sum Rule. We can achieve this separation by subtracting and adding the term $f(x+h)g(x)$ in the numerator:

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x)$$

(5%)2.	Prove that $\frac{d}{dx}(\sec x) = \sec x\cdot\tan x$.

Sol:

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\ \tan x$$

(5%)3.	Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

Sol:

Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and

differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

(5%)4.	Find the critical numbers of $g(y) = \frac{y-1}{y^2-y+1}$.

Sol:

$$g(y) = \frac{y-1}{y^2-y+1} \quad \Rightarrow$$

$$g'(y) = \frac{(y^2-y+1)(1)-(y-1)(2y-1)}{(y^2-y+1)^2} = \frac{y^2-y+1-(2y^2-3y+1)}{(y^2-y+1)^2} = \frac{-y^2+2y}{(y^2-y+1)^2} = \frac{y(2-y)}{(y^2-y+1)^2}.$$

$g'(y) = 0 \quad \Rightarrow \quad y = 0, 2$. The expression $y^2 - y + 1$ is never equal to 0, so $g'(y)$ exists for all real numbers. The critical numbers are 0 and 2.

(5%)5.	Find the absolute maximum and absolute minimum values of $f(t) = 2\cos t + \sin(2t)$ on $[0, \pi/2]$.

Sol:

$f(t) = 2\cos t + \sin 2t, \ [0, \pi/2]$.

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1)$.

$f'(t) = 0 \quad \Rightarrow \quad \sin t = \frac{1}{2}$ or $\sin t = -1 \quad \Rightarrow \quad t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

(5%)6.	Find the slant asymptotes of $y = \frac{x^3}{(x+1)^2}$.

Sol:

$$y = f(x) = \frac{x^3}{(x+1)^2} = x - 2 + \frac{3x+2}{(x+1)^2}$$

$\lim_{x \to \pm\infty} [f(x) - (x-2)] = \lim_{x \to \pm\infty} \frac{3x+2}{(x+1)^2} = 0$, so $y = x - 2$ is a SA.

(20%)7.	Find the limit (a) $\lim_{\theta \to 0} \frac{cos\theta - 1}{2\theta^2}$ = ________. (b) $\lim_{x \to 0} \frac{sin3x}{5x^3 - 4x}$ = ________. (c) $\lim_{x \to -\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$ = ________. (d) $\lim_{x \to \infty} \sqrt{x}\, sin\frac{1}{x}$ = ________.

Sol:

(a)

$$\begin{aligned}\lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta^2} &= \lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta^2} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{2\theta^2(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{2\theta^2(\cos\theta + 1)} \\ &= -\frac{1}{2} \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta + 1} = -\frac{1}{2} \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta + 1} \\ &= -\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1+1} = -\frac{1}{4}\end{aligned}$$

(b)

$$\lim_{x \to 0} \frac{\sin 3x}{5x^3 - 4x} = \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4} \right) = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{3}{5x^2 - 4} = 1 \cdot \left(\frac{3}{-4} \right) = -\frac{3}{4}$$

(c)

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\frac{\lim\limits_{x\to-\infty}-\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty}(2/x^3-1)}\qquad[\text{since } x^3=-\sqrt{x^6}\text{ for } x<0]$$

$$=\frac{\lim\limits_{x\to-\infty}-\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty}(1/x^3)-\lim\limits_{x\to-\infty}1}=\frac{-\sqrt{\lim\limits_{x\to-\infty}(1/x^6)+\lim\limits_{x\to-\infty}4}}{2(0)-1}$$

$$=\frac{-\sqrt{0+4}}{-1}=\frac{-2}{-1}=2$$

(d)

If $t=\dfrac{1}{x}$, then $\displaystyle\lim_{x\to\infty}\sqrt{x}\sin\frac{1}{x}=\lim_{t\to0^+}\frac{1}{\sqrt{t}}\sin t=\lim_{t\to0^+}\frac{t}{\sqrt{t}}\frac{\sin t}{t}=\lim_{t\to0^+}\sqrt{t}\cdot\lim_{t\to0^+}\frac{\sin t}{t}=0\cdot1=0.$

(5%)8.	Find the most general antiderivative of the function $f(t) = 8\sqrt{t} - \sec t{\cdot}\tan t$.

Sol:

$f(t)=8\sqrt{t}-\sec t\tan t \quad\Rightarrow\quad F(t)=8\cdot\frac{2}{3}t^{3/2}-\sec t+C=\frac{16}{3}t^{3/2}-\sec t+C$

(5%)9.	$f'(x) = \sqrt{x}(6+5x)$, $f(1) = 10$. Find $f(x)$.

Sol:

$f'(x)=\sqrt{x}(6+5x)=6x^{1/2}+5x^{3/2} \quad\Rightarrow\quad f(x)=4x^{3/2}+2x^{5/2}+C.$

$f(1)=6+C$ and $f(1)=10 \quad\Rightarrow\quad C=4$, so $f(x)=4x^{3/2}+2x^{5/2}+4.$

(30%)10.	Find the derivative of the function. (a) $y = \frac{t \cdot \sin t}{1+t}$. $y' =$ ________. (b) $y = \cos\sqrt{\sin(\tan \pi x)}$. $y' =$ ________. (c) $g(u) = (\frac{u^3-1}{u^3+1})^8$. $g'(u) =$ ________. (d) $y = \sin(t + \cos\sqrt{t})$. $y' =$ ________. (e) $\sqrt{xy} = 1 + x^2y$. $\frac{dy}{dx} =$ ________. (f) $\frac{x^2}{x+y} = y^2 + 1$. $\frac{dy}{dx} =$ ________.

Sol:

(a)

$$y = \frac{t \sin t}{1+t} \quad \Rightarrow$$

$$y' = \frac{(1+t)(t\cos t + \sin t) - t\sin t(1)}{(1+t)^2} = \frac{t\cos t + \sin t + t^2\cos t + t\sin t - t\sin t}{(1+t)^2} = \frac{(t^2+t)\cos t + \sin t}{(1+t)^2}$$

(b)

$$y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$$

$$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx}\tan \pi x = \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi\cos(\tan \pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$$

(c)

$$g(u) = \left(\frac{u^3-1}{u^3+1}\right)^8 \quad \Rightarrow$$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$

$$= 8\frac{(u^3-1)^7}{(u^3+1)^7}\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

(d)

$$y = \sin(t+\cos\sqrt{t}) \quad \Rightarrow$$

$$y' = \cos(t+\cos\sqrt{t})\cdot\frac{d}{dt}(t+\cos\sqrt{t}) = \cos(t+\cos\sqrt{t})\cdot\left(1-\sin\sqrt{t}\cdot\frac{1}{2\sqrt{t}}\right) = \cos(t+\cos\sqrt{t})\,\frac{2\sqrt{t}-\sin\sqrt{t}}{2\sqrt{t}}$$

(e)

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(1+x^2y) \quad\Rightarrow\quad \tfrac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1) = 0 + x^2y' + y\cdot 2x \quad\Rightarrow$$

$$\frac{x}{2\sqrt{xy}}\,y' + \frac{y}{2\sqrt{xy}} = x^2y' + 2xy \quad\Rightarrow\quad y'\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}} \quad\Rightarrow$$

$$y'\left(\frac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right) = \frac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \quad\Rightarrow\quad y' = \frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$$

(f)

$$\frac{d}{dx}\left(\frac{x^2}{x+y}\right) = \frac{d}{dx}(y^2+1) \quad\Rightarrow\quad \frac{(x+y)(2x)-x^2(1+y')}{(x+y)^2} = 2y\,y' \quad\Rightarrow$$

$$2x^2+2xy-x^2-x^2\,y' = 2y(x+y)^2\,y' \quad\Rightarrow\quad x^2+2xy = 2y(x+y)^2\,y' + x^2\,y' \quad\Rightarrow$$

$$x(x+2y) = [2y(x^2+2xy+y^2)+x^2]\,y' \quad\Rightarrow\quad y' = \frac{x(x+2y)}{2x^2y+4xy^2+2y^3+x^2}$$

Or: Start by clearing fractions and then differentiate implicitly.

(20%)11.	Use the following guidelines to sketch the curve $y = \frac{x}{x^2-4}$. (1%) (i) Domain. (1%) (ii) Intercepts. (2%) (iii) Symmetry. (2%) (iv) Asymptotes. (4%) (v) Intervals of increase or decrease. (2%) (vi) Local maximum and minimum value. (4%) (vii) Concavity and point of inflection. (4%) (viii) Sketch the curve.

Sol:

$y = f(x) = \dfrac{x}{x^2-4} = \dfrac{x}{(x+2)(x-2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x \to 2^+} \dfrac{x}{x^2-4} = \infty$, $\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to -2^+} f(x) = \infty$, $\lim\limits_{x \to -2^-} f(x) = -\infty$, so $x = \pm 2$ are VAs.

$\lim\limits_{x \to \pm\infty} \dfrac{x}{x^2-4} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2} < 0$ for all x in D, so f is decreasing on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

F. No local extrema

G.
$$f''(x) = -\frac{(x^2-4)^2(2x) - (x^2+4)2(x^2-4)(2x)}{[(x^2-4)^2]^2} = -\frac{2x(x^2-4)[(x^2-4) - 2(x^2+4)]}{(x^2-4)^4} = -\frac{2x(-x^2-12)}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x+2)^3(x-2)^3}.$$

$f''(x) < 0$ if $x < -2$ or $0 < x < 2$, so f is CD on $(-\infty, -2)$ and $(0, 2)$, and CU on $(-2, 0)$ and $(2, \infty)$. IP at $(0, 0)$

H.