Midterm Exam – Calculus (II)

100%

(4%) 1. Suppose f is continuous on $[a, b]$.

(a) If $g(x) =$ __________, then $g'(x) = f(x)$.

(b) $\int_a^b F'(x)dx =$ __________, where $F' = f$.

Sol：

(a) $\int_a^x f(t)dt$

(b) $F(b) - F(a)$

(3%) 2. Determine a region where area is equal to $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$. Do not evaluate the limit.

Sol：

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \frac{\pi/4 - 0}{n} = \frac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \frac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

(3%) 3. Express $\int_1^3 \sqrt{4 + x^2}dx$ as a limit of Riemann sums. Do not evaluate the limit.

Sol：

$f(x) = \sqrt{4 + x^2}$, $a = 1$, $b = 3$, and $\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$. Using Theorem 4, we get $x_i^* = x_i = 1 + i\,\Delta x = 1 + \frac{2i}{n}$, so

$$\int_1^3 \sqrt{4 + x^2}\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \left(1 + \frac{2i}{n}\right)^2} \cdot \frac{2}{n}.$$

(5%) 4.	A particle moves along a line to that its velocity at time t is $v(t) = t^2 - 2t - 3$ (in meters per second). Find the distance traveled during the time period $2 \le t \le 4$.

Sol：

$v(t) = t^2 - 2t - 3 = (t+1)(t-3)$, so $v(t) < 0$ for $-1 < t < 3$, but on the interval $[2, 4]$, $v(t) < 0$ for $2 \le t < 3$.

$$\text{Distance traveled} = \int_2^4 |t^2 - 2t - 3|\, dt = \int_2^3 -(t^2 - 2t - 3)\, dt + \int_3^4 (t^2 - 2t - 3)\, dt$$
$$= \left[-\tfrac{1}{3}t^3 + t^2 + 3t\right]_2^3 + \left[\tfrac{1}{3}t^3 - t^2 - 3t\right]_3^4$$
$$= (-9 + 9 + 9) - \left(-\tfrac{8}{3} + 4 + 6\right) + \left(\tfrac{64}{3} - 16 - 12\right) - (9 - 9 - 9) = 4 \text{ m}$$

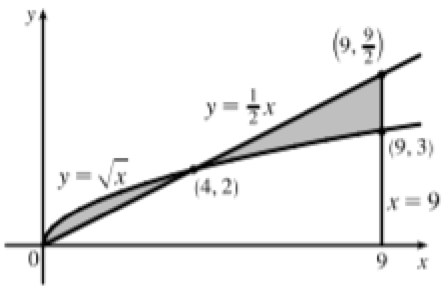
(10%) 5.	Sketch the region enclosed by the given curves and find its area. (a) $y = \sqrt{x}$, $y = x/2$, $x = 9$. (b) $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$.

Sol：

(a)

$\frac{1}{2}x = \sqrt{x} \;\Rightarrow\; \frac{1}{4}x^2 = x \;\Rightarrow\; x^2 - 4x = 0 \;\Rightarrow\; x(x-4) = 0 \;\Rightarrow\; x = 0$ or 4, so

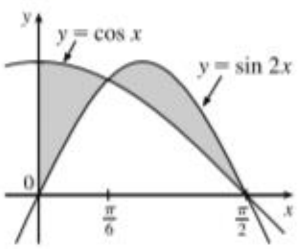
$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx$$
$$= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$
$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right]$$
$$= \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$



(b)

Notice that $\cos x = \sin 2x = 2\sin x \cos x \;\Leftrightarrow\; 2\sin x \cos x - \cos x = 0 \;\Leftrightarrow\; \cos x\,(2\sin x - 1) = 0 \;\Leftrightarrow\; 2\sin x = 1$ or $\cos x = 0 \;\Leftrightarrow\; x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$
$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$



(5%) 6. Find a formula for the inverse of the function $f(x) = 1 + \sqrt{2+3x}$.

Sol：

$y = f(x) = 1 + \sqrt{2+3x}$ $(y \geq 1)$ $\Rightarrow$ $y - 1 = \sqrt{2+3x}$ $\Rightarrow$ $(y-1)^2 = 2 + 3x$ $\Rightarrow$ $(y-1)^2 - 2 = 3x$ $\Rightarrow$ $x = \frac{1}{3}(y-1)^2 - \frac{2}{3}$. Interchange x and y: $y = \frac{1}{3}(x-1)^2 - \frac{2}{3}$. So $f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$. Note that the domain of f^{-1} is $x \geq 1$.

(10%) 7. Find the volume of the solid obtained by the region bounded by the given curves about the specified line. Sketch the region, the solid and a typical disk or washer. (a) $y = x^3$, $y = 1$, $x = 2$; about $y = -3$. (b) $x = y^2$, $x = 1$; about $x = 1$.

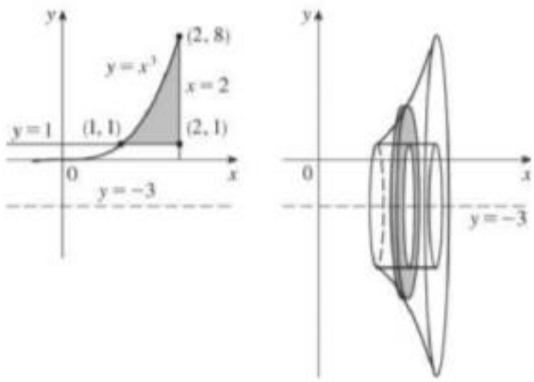
Sol：

(a)

A cross-section is a washer with inner radius $1 - (-3) = 4$ and outer radius $x^3 - (-3) = x^3 + 3$, so its area is

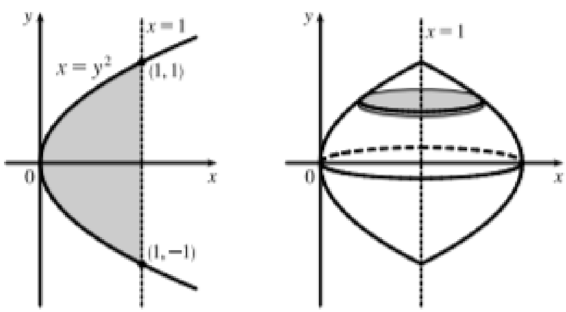
$$A(x) = \pi(x^3+3)^2 - \pi(4)^2 = \pi(x^6 + 6x^3 - 7).$$

$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi(x^6 + 6x^3 - 7)\,dx$$
$$= \pi\left[\tfrac{1}{7}x^7 + \tfrac{3}{2}x^4 - 7x\right]_1^2$$
$$= \pi\left[\left(\tfrac{128}{7} + 24 - 14\right) - \left(\tfrac{1}{7} + \tfrac{3}{2} - 7\right)\right] = \tfrac{471\pi}{14}$$



(b)

$$V = \int_{-1}^{1} \pi(1-y^2)^2\,dy = 2\int_0^1 \pi(1-y^2)^2\,dy$$
$$= 2\pi \int_0^1 (1 - 2y^2 + y^4)\,dy$$
$$= 2\pi\left[y - \tfrac{2}{3}y^3 + \tfrac{1}{5}y^5\right]_0^1$$
$$= 2\pi \cdot \tfrac{8}{15} = \tfrac{16}{15}\pi$$



(5%) 8. $f(x) = 3 + x^2 + \tan(\pi x/2)$, $-1 < x < 1$. Find $(f^{-1})'(3)$.

Sol：

$f(0) = 3 \quad \Rightarrow \quad f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \quad \Rightarrow \quad f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and $f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

(15%) 9. Find the derivative of the function.

(a) $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$. $g'(x) =$ ________.

(b) $y = \int_{1}^{3x+2} \frac{t}{1+t^3}\,dt$. $y' =$ ________.

(c) $h(x) = \int_{1}^{\frac{1}{x}} \sin^4 t\,dt$. $h'(x) =$ ________.

Sol：

(a)

$$g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_{1}^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} \quad \Rightarrow$$

$$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}}\frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}}\frac{d}{dx}(x^2) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$$

(b)

Let $u = 3x + 2$. Then $\frac{du}{dx} = 3$. Also, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{1}^{3x+2} \frac{t}{1+t^3}\,dt = \frac{d}{du}\int_{1}^{u} \frac{t}{1+t^3}\,dt \cdot \frac{du}{dx} = \frac{u}{1+u^3}\frac{du}{dx} = \frac{3x+2}{1+(3x+2)^3} \cdot 3 = \frac{3(3x+2)}{1+(3x+2)^3}$$

(c)

Let $u = \frac{1}{x}$. Then $\frac{du}{dx} = -\frac{1}{x^2}$. Also, $\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_{2}^{1/x} \sin^4 t\,dt = \frac{d}{du}\int_{2}^{u} \sin^4 t\,dt \cdot \frac{du}{dx} = \sin^4 u\,\frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

(40%) 10. Evaluate the integral.

(a) $\int (2 + tan^2\theta)\ d\theta$ = ________________.

(b) $\int \frac{dt}{cos^2 t \cdot \sqrt{1+\tan t}}$ = ________________.

(c) $\int \sin t \cdot sec^2(\cos t)\, dt$ = ________________.

(d) $\int x \cdot \sqrt{x+2}\ dx$ = ________________.

(e) $\int_{-1}^{2} (3u - 2)(u + 1)\ du$ = ________________.

(f) $\int_0^{\frac{3\pi}{2}} |\sin x|\, dx$ = ________________.

(g) $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\ dx$ = ________________.

(h) $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$ = ________________.

Sol：

(a)

$\int (2 + \tan^2\theta)\, d\theta = \int [2 + (\sec^2\theta - 1)]\, d\theta = \int (1 + \sec^2\theta)\, d\theta = \theta + \tan\theta + C$

(b)

Let $u = 1 + \tan t$. Then $du = \sec^2 t\, dt$, so

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int \frac{\sec^2 t\, dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

(c)

Let $u = \cos t$. Then $du = -\sin t\, dt$ and $\sin t\, dt = -du$, so

$\int \sin t\, \sec^2(\cos t)\, dt = \int \sec^2 u \cdot (-du) = -\tan u + C = -\tan(\cos t) + C.$

(d)

Let $u = x + 2$. Then $du = dx$ and $x = u - 2$, so

$\int x\sqrt{x+2}\, dx = \int (u-2)\sqrt{u}\, du = \int (u^{3/2} - 2u^{1/2})\, du = \frac{2}{5}u^{5/2} - 2 \cdot \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$

(e)

$\int_{-1}^{2} (3u-2)(u+1)\, du = \int_{-1}^{2} (3u^2 + u - 2)\, du = \left[u^3 + \frac{1}{2}u^2 - 2u\right]_{-1}^{2} = (8 + 2 - 4) - (-1 + \frac{1}{2} + 2) = 6 - \frac{3}{2} = \frac{9}{2}$

(f)

$\int_0^{3\pi/2} |\sin x|\ dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi/2} (-\sin x)\,dx = \big[-\cos x\big]_0^{\pi} + \big[\cos x\big]_{\pi}^{3\pi/2} = [1-(-1)] + [0-(-1)] = 2+1 = 3$

(g)

EXAMPLE 9 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

(h)

Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3-1) = 3.$$