Algebraic Algorithms

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1 Theoretical part

0. * Recall that you have already proven that an oriented graph G is *normal* iff it is confluent. Can we strengthen this theorem and state that an oriented graph G is *convergent* iff it is confluent? What if G is assumed to be finite?

1. Prove that $\ll$ is a strict and terminating ordering assuming $<$ is an admissible ordering of terms. Is $\ll$ linear?

2. * Notice that $\ll$ in general is not linear (previous exercise). But assuming that there is some natural ordering of the underlying coefficient field (e.g. $\mathbb{Q}$ or $\mathbb{R}$) it is clear how to *linearize* $\ll$ into $\ll'$. What will be the problem with such new ordering $\ll'$?

3. Order given polynomials $xy^2 + x^2y$, $-2xy^2 + x^2 + x^2y$, $-x^2y + 2x^2y^2 + 2$, $x^2y + x + 2y^2$ using $\ll$-ordering induced by:

   (a) $<_{LEX}, x > y$;

   (b) $<_{LEX}, x < y$;

   (c) $<_{GLEX}, x > y$;

   (d) $<_{GLEX}, x < y$.

4. Prove that $<_{LEX}, <_{GLEX}$ and weighted orderings are admissible.

5. Assume $R = \{xy - x^3 - x^2, x^4 + x^3 + x\}$ is a Gröbner basis for $<_{LEX}, x < y$. Decide whether $x^4y^4 + x^3y^3$ belongs to $<R>$.

6. Show that the above R is not a Gröbner basis for $<_{LEX}, x > y$.

7. * Prove that R is a Gröbner basis of the ideal $I = <R>$ if and only if $<lt(R)> = <lt(I)>$, where $lt(U)$ denotes $\{lt(u) \mid u \in U\}$.

8. ** Show that the problem of deciding whether a polynomial $p(\overline{x})$ belongs to a given ideal $I \leq \mathbb{Q}[\overline{x}]$ is NP-hard (and is co-NP-hard, as well).

2 Computational part

1. Verify 3 using sage (follow this link to see how to introduce ordering to polynomial rings in sage).
2. Verify 5 using sage.