Algebraic Algorithms

mykyta.narusevych@matfyz.cuni.cz

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1 Theoretical part

1. Find a normal reduced Gröbner basis of the ideal $< x^2y^2 + y - 1, x^2y + x > \leq \mathbb{Q}[x, y]$ for $<_{GLEX}, x < y$. Does polynomial $x^2y^3 - 2xy + 3y$ belong to the given ideal?

2. Find a normal reduced Gröbner basis of the ideal $< x^2 - 2y^2, xy - 3 > \leq \mathbb{Q}[x, y]$ for any (admissible) ordering. Does polynomial $2y^3 - x + 3$ belong to the given ideal?

3. * Is it true that $IJ = I \cap J$ in general, where I, J are some ideals over an integral domain?

4. Let $I =< f_1, ..., f_n >$, $J =< g_1, ..., g_m >$ be ideals from $T[x_1, ..., x_k]$. Prove that:

   (a) $I + J =< f_1, ..., f_n, g_1, ..., g_m >$,

   (b) $IJ =< f_i g_j \mid i \leq n, j \leq m >$,

   (c) * $I \cap J = (zI + (1 - z)J) \cap T[x_1, ..., x_k]$, where z is a new variable.

5. Let I, J be ideals. Prove that :

   (a) $V(I) \cap V(J) = V(I + J)$,

   (b) $V(I) \cup V(J) = V(IJ)$.

   Using these two observations design an algorithm which given two algebraic sets $A = V(f_1, ..., f_n)$ and $B = V(g_1, ..., g_m)$ computes $A \cap B$ and $A \cup B$.

6. How many solutions does the following system have in $\mathbb{Q}$ and in $\mathbb{C}$:

$$1 + 2x^2 + y^2 + 4x^2y^2 + 2x^2y^4 = 0, xy^2 + xy^4 = 0.$$

7. How many solutions does the following system have in $\mathbb{Q}$ and in $\mathbb{C}$:

$$x^2 - 2xy^2 + 1 = 0, xy - 2y^2 + x = 0.$$

8. ** Prove Heron's formula using Gröbner basis method. To recall, Heron's formula states that the area of a triangle with lengths a, b and c equals:

$$\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}.$$

2 Computational part

1. Verify 1 using sage.

2. Verify 2 using sage.