The models $K(F)$

Remark 1. Assume that $\mathcal{M}$ is a model of T_{ALL} satisfying the properties (1) and (2) and Ω, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{I}$ are as we defined them in earlier sheets. Moreover, assume that $L \subseteq L_{all}$ is some fixed language.

Definition 2. We say that a non-empty set of functions F is an L-closed family of random variables if

1. $F \subseteq \mathcal{M}$,
2. every $\alpha \in F$ is a function $\alpha : \Omega \to \mathcal{M}$,
3. F contains all L-constants and for every k-ary L-function f and every $\alpha_1, \ldots, \alpha_k$ the function

$$f(\alpha_1, \ldots, \alpha_n)(\omega) = f(\alpha_1(\omega), \ldots, \alpha_k(\omega))$$

   is in F.

Note that F itself may not be $\mathcal{M}$-definable.

Definition 3 (The structures $K(F)$)**.** Assume that F is an L-closed family. The structure $K(F)$ is a Boolean-valued structure, whose universe is F. The Boolean valuation of L-sentences with parameters from F has values in $\mathcal{B}$ and is given by the following inductive conditions:

- $[\![\alpha = \beta]\!] = \{\omega \in \Omega; \alpha(\omega) = \beta(\omega)\}/\mathcal{I}$
- $[\![R(\alpha_1, \ldots, \alpha_k)]\!] = \{\omega \in \Omega; R(\alpha_1(\omega), \ldots, \alpha_k(\omega))\}/\mathcal{I}$, $R \in L$,
- $[\![\, - \,]\!]$ commutes with $\neg$, $\vee$ and $\wedge$.
- $[\![(\forall x)(A(x))]\!] = \bigwedge_{\alpha \in F} [\![A(\alpha)]\!]$
- $[\![(\exists x)(A(x))]\!] = \bigvee_{\alpha \in F} [\![A(\alpha)]\!]$.

We say an L-sentence A is valid in $K(F)$ if $[\![A]\!] = 1_{\mathcal{B}}$.

Remark 4. An implication $B \to A$ is valid in $K(F)$ if and only if $[\![B]\!] \leq [\![A]\!]$.

Fact 5. Let T be a set of L-sentences and A an L-sentence. If $T \vdash A$, then there is a finite $T_0 \subseteq T$ such that

$$\bigwedge_{B \in T_0} [\![B]\!] \leq [\![A]\!].$$

Definition 6. Let $n \in \mathcal{M} \setminus \mathbb{N}$ be even, and $\Omega = \{m < n; m \in \mathcal{M}\}$. We let $F_{bit} = \{0, 1, \alpha, \beta\}$, where α is the function $\omega \mapsto \omega \bmod 2$ and β is $\omega + 1 \mapsto \omega \bmod 2$. Also, let $L_{bit} \subseteq L_{all}$ be the language consisting of the constants 0, 1, all relations and all functions with range contained in $\{0, 1\}$.

Exercise 7. Show that F_{bit} is L_{bit}-closed.

Exercise 8. Show that in $K(F_{bit})$ we have $[\![0 \leq \alpha \wedge \alpha \leq 1]\!] = 1_{\mathcal{B}}$.

Exercise 9. Show that in $K(F_{bit})$ we have

- $[\![\alpha = 0]\!] \neq 1_{\mathcal{B}}$
- $[\![\alpha = 1]\!] \neq 1_{\mathcal{B}}$
- $[\![(\alpha = 0) \vee (\alpha = 1)]\!] = 1_{\mathcal{B}}$
- $[\![\alpha = \beta]\!] = 0_{\mathcal{B}}$.

Compute the values of $\mu([\![\alpha = 0]\!])$ and $\mu([\![\alpha = 1]\!])$.

Exercise 10 (*). Show that in $K(F_{bit})$ for any L_{bit}-sentence A we have

$$\mu([\![A]\!]) \in \{0, 1/2, 1\}.$$

Definition 11. We say $K(F)$ *witnesses existential sentences* if: For every quantifier free L-sentence A, potentially with F-parameters, we have that there is $\gamma \in F$ such that

$$[\![(\exists x)A(x)]\!] = [\![A(\gamma)]\!].$$

Definition 12. Let $n \in \mathcal{M} \setminus \mathbb{N}$ be even and $\Omega = \{m < n; m \in \mathcal{M}\}$. We let $F'_{bit} = \{\alpha, \beta\}$, where α is the function $\omega \mapsto \omega \bmod 2$ and β is $\omega + 1 \mapsto \omega \bmod 2$. And let $L'_{bit} \subseteq L_{all}$ be the language consisting of the constants 0, 1 and all relations.

Exercise 13. Show that F'_{bit} is L'_{bit}-closed but not L_{bit}-closed.

Exercise 14. Show that $K(F'_{bit})$ does not witness existential sentences.

Exercise 15 (*). Show that $K(F_{bit})$ does witness existential sentences.

Exercise 16 (*). Prove or disprove: Every family of random variables F which has exactly one non-constant function does witness existential quantifiers.

Definition 17. For a prefix class of formulas Γ and a theory T, let $\Gamma(T)$ be the set of all formulas from T which are in the prefix class Γ. E.g. $\forall(T)$ denotes the set of universal formulas from T.

Theorem 18. Let F be an L-closed family. Then, $\forall(\mathrm{Th}(\mathbb{N}))$ is valid in $K(F)$. Moreover, if F contains all elements of $\mathbb{N}$ as constant functions, then $\exists\forall(\mathrm{Th}(\mathbb{N}))$ is valid in $K(F)$.