Short Resolution proofs

Exercise 1. Write the negation of the property that every linear order on n elements has the least element as a family of propositional CNFs F_n. What is the size of the formula with respect to n?

Definition 2. We will use the following notation:

- $A_n := \bigwedge_{\substack{1 \leq i,j,k \leq n \\ i \neq j \neq k}} (\neg P_{i,j} \vee \neg P_{j,k} \vee P_{i,k})$,
- $B_n := \bigwedge_{\substack{1 \leq i,j \leq n \\ i \neq j}} (\neg P_{i,j} \vee \neg P_{j,i})$,
- $C_n := \bigwedge_{1 \leq j \leq n} \bigvee_{\substack{1 \leq i \leq n \\ i \neq j}} P_{i,j}$,

and, moreover,

- $A(i,j,k) := \neg P_{i,j} \vee \neg P_{j,k} \vee P_{i,k}$,
- $B(i,j) := \neg P_{i,j} \vee \neg P_{j,i}$,
- $C_m(j) := \bigvee_{\substack{1 \leq i \leq m \\ i \neq j}} P_{i,j}$.

Theorem 3. For $m < n$, C_m can be derived from C_{m+1}, A_n and B_n by introducing at most $O(n^2)$ new clauses.

Exercise 4. Derive an empty clause from C_2 and some clause in B_n.

Exercise 5. Using Theorem 3 and Exercise 4, what is the size of the proof of F_n?