Theories of arithmetic

Rock bottom

Definition 1 (Robinson's Q)**.** Let $L_Q = \{0, S, +, \cdot\}$ and let Q be an L_Q-theory with the following axioms:

$$\neg S(x) = 0 \tag{1}$$
$$S(x) = S(y) \to x = y \tag{2}$$
$$x \neq 0 \to (\exists y)(S(y) = x) \tag{3}$$
$$x + 0 = x \tag{4}$$
$$x + S(y) = S(x + y) \tag{5}$$
$$x \cdot 0 = 0 \tag{6}$$
$$x \cdot S(y) = x \cdot y + x \tag{7}$$

Exercise 2. Show that $Q \vdash S(0) + S(0) = S(S(0))$.

Exercise 3. Show that $\mathbb{N}$ as an L_Q-structure can be embedded into every model of Q.

Exercise 4. Let $Q_{\leq}$ be the $L_Q \cup \{\leq\}$-theory extending Q by

$$x \leq y \leftrightarrow (\exists z) x + z = y,$$

show that for every L_Q-sentence φ we have

$$Q \vdash \varphi \iff Q_{\leq} \vdash \varphi,$$

this property is called *conservativity* of $Q_{\leq}$ over Q.

Exercise 5. Show that $Q \nvdash x + y = y + x$.

Overshooting

Definition 6. Let $L_{\mathrm{PA}} = \{0, 1, +, \cdot, \leq\}$ and PA be an L_{PA}-theory axiomatized by $Q_{\leq}$ and the scheme of *induction*. That is, for every L_{PA}-formula $\varphi(x)$ the following is an axiom:

$$(\varphi(0) \wedge (\forall x)(\varphi(x) \to \varphi(x + 1))) \to (\forall x)(\varphi(x))$$

Exercise 7. Show that $\mathrm{PA} \vdash x + y = y + x$.

Fact 8. Let ZF_{fin} be the theory of ZF with the axiom of infinity replaced by its negation.

Then ZF_{fin} and PA are bi-interpretable, meaning that they prove the same statement if we translate the non-logical symbols using their definition in the other language.

Opinion 9 (Harvey Friedman's grand conjecture)**.** Every theorem published in the Annals of Mathematics whose statement involves only finitary mathematical objects (i.e., what logicians call an arithmetical statement) can be proved in a subsystem of PA.

Homing in

Definition 10. The set of *bounded* L_{PA}-formulas, denoted Δ_0, is the least set closed under propositional connectives, which contains open formulas, and for each $\varphi(x) \in \Delta_0$ and an L_{PA}-term t which does not contain the variable x, we have

$$(\exists x)(x \leq t \wedge \varphi(x)) \in \Delta_0,$$
$$(\forall x)(x \leq t \rightarrow \varphi(x)) \in \Delta_0.$$

These formulas are usually denoted using *bounded quantifiers*

$$(\exists x \leq t)(\varphi(x))$$
$$(\forall x \leq t)(\varphi(x)).$$

Exercise 11. Let $\psi \in \Delta_0$ and $\mathbb{N} \models (\forall x)(\exists y)\psi(x, y)$, show that there is an algorithm which given x computes y such that $\psi(x, y)$. Can you say anything about the complexity of such an algorithm?

Definition 12. $I\Delta_0$ is an L_{PA}-theory extending $Q_{\leq}$ by the scheme of *bounded induction*. That is for each $\varphi(x) \in \Delta_0$ there is an axiom:

$$(\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(x+1))) \rightarrow (\forall x)(\varphi(x)).$$

Fact 13. $I\Delta_0$ proves the (fifteen) axioms of positive parts of discretely ordered rings: $\leq$ is a linear order, commutativity of $+$ and $\cdot$, neutrality of 0 and 1, distributivity, $0 \leq x \leq 1 \rightarrow (x = 0 \vee x = 1)$, etc., which are collectively known as PA^-.

Definition 14. We say a theory $I\Delta_0$ proves the totality of a function $f(-)$ iff there is a formula $\varphi(\overline{x}, y)$ such that it holds $\mathbb{N} \models \varphi(\overline{x}, y) \leftrightarrow f(\overline{x}) = y$ and

$$I\Delta_0 \vdash (\forall x)(\exists y)\varphi(x, y).$$

Exercise 15. Prove that $I\Delta_0$ proves the totality of

$$x \dot{-} y = \begin{cases} x - y & x > y \\ 0 & x \leq y, \end{cases}$$

and

$$\lfloor x/2 \rfloor = \begin{cases} x/2 & x \text{ is even} \\ (x-1)/2 & x \text{ is odd.} \end{cases}$$

Theorem 16. Every model of $I\Delta_0$ has order type $\mathbb{N} + \mathbb{Q} \cdot \mathbb{Z}$.

Fact 17 (Tennenbaum, 'No nonstandard calculator theorem')**.** Let

$$M = (\{0,1\}^*, \oplus, \otimes, 0, 1, \leq^M)$$

be a model of $I\Delta_0$, then both $x, y \mapsto x \oplus y$ and $x, y \mapsto x \otimes y$ are not recursive, there is no algorithm that computes them.

Parikh's Theorem

Definition 18. Let $M, N \models \mathrm{PA}^-$, we say M is an *initial segment* of N, and that N is an *end extension* of M, denoted $M \subseteq_e N$, if $M \subseteq N$ and for every $a \in M, b \in N$ we have $N \models b < a$ implies $b \in M$.

Exercise 19. Let $M, N \models \mathrm{PA}^-$, $M \subseteq_e N$, then for every $\varphi(\overline{x}, \overline{y}) \in \Delta_0$ and $\overline{m} \in M$ we have

$$M \models \varphi(\overline{m}) \iff N \models \varphi(\overline{m}).$$

Exercise 20. Let $M, N \models \mathrm{PA}^-$, $M \subseteq_e N$. If $N \models I\Delta_0$ then $M \models I\Delta_0$.

Theorem 21 (Parikh)**.** Let $\varphi(x, y) \in \Delta_0$. If

$$I\Delta_0 \vdash (\forall x)(\exists y)\varphi(x, y),$$

then there is an L_{PA}-term t not containing y such that

$$I\Delta_0 \vdash (\forall x)(\exists y \leq t)\varphi(x, y).$$

Exercise 22. Assume that $I\Delta_0 \vdash (\forall x)(\exists y)\varphi(x, y)$, can you find an algorithm which given x finds a y such that $\mathbb{N} \models \varphi(x, y)$, can we bound the complexity of this algorithm?

Fact 23 (Bennet, Pudlák)**.** There is an Δ_0 formula $\exp(x, y, z)$ which has the property

$$\mathbb{N} \models x^y = z \leftrightarrow \exp(x, y, z)$$

and $I\Delta_0$ proves its recursive properties, such as

$$(\exp(x, y, z) \wedge \exp(x, y+1, z')) \rightarrow (z \cdot x = z').$$

Exercise 24. Show, that $I\Delta_0$ cannot prove that the exponential function is total.