Theorem of Paris and Wilkie - II

Recall, that $\mathbb{M}$ is a countable non-standard model of T (the true theory of arithmetic), and n is a non-standard number. Our goal is to interpret binary relation R on $\mathbb{M}$ so that the resulting structure preserves induction for existential formulas, while R becomes the graph of an injective function mapping $[n]$ to $[n-1]$.

We are constructing R step-by-step, each time extending a given finite partial injection σ to another finite partial injection $\tau \supseteq \sigma$. We call σ and τ - conditions.

We have defined a notion of forcing $\Vdash$ semantically: given σ a condition, and φ an $L(R)$-formula, we say σ forces φ (written $\sigma \Vdash \varphi$), iff, no matter how the construction proceeds starting from σ, the resulting interpretation of R in $\mathbb{M}$ satisfies φ. In other words, for $R^{\mathbb{M}}$ an arbitrary interpretation of R on $\mathbb{M}$, so that $R^{\mathbb{M}}$ is the graph of a partial injection from $[n]$ to $[n-1]$ (not necessary finite) extending σ, the resulting structure $(\mathbb{M}, R^{\mathbb{M}})$ satisfies φ.

We have argued that any condition forces R to be the graph of a partial injective function. Moreover, given any $p \in [n]$ and any σ, it is possible to extend σ to τ which is definable on p. In other words, for any particular p, the set of all conditions forcing R to be definable on p is dense.

We then leveraged this density to ensure that the resulting interpretation of R is the graph of a total function.

We are left with the induction principle. However, we will actually work with the least number principle.

Exercise 1. The least number principle LNP for $\varphi(x)$ is an axiom defined as

$$\exists x\, \varphi(x) \to \exists x \forall y < x\, (\varphi(x) \land \neg\varphi(y)).$$

Show that LNP for all existential formulas implies induction for the same class of formulas. (In fact, the two axiom schemes are equivalent for the mentioned class, although this is not true in general for arbitrary formula classes.)

We now consider an existential formula $\varphi(x)$ of the form $\exists y\, \psi(x, y)$, where $\psi(x, y)$ is quantifier-free (the case with more quantifiers is completely analogous. We have also dropped possible parameters appearing in $\varphi(x)$.)

Exercise 2. Let σ be condition. Assume $\forall x, y$ it holds that $\sigma \Vdash \neg\psi(x, y)$. Show that this implies $\sigma \Vdash \forall x \neg\varphi(x)$, thus implying LNP for $\varphi(x)$.

Assume now $\exists x, y$, so that $\sigma \nVdash \neg\psi(x, y)$. Argue that there is $\tau \supseteq \sigma$ such that $\tau \vDash \exists x \varphi(x)$. Does this τ forces LNP for $\varphi(x)$?

Exercise 3. * Show that the set of all x, y, so that $\sigma \nVdash \neg\psi(x, y)$, is definable in $\mathbb{M}$. In particular, there is the smallest x such that $\exists y$ for which $\sigma \nVdash \neg\psi(x, y)$.

Argue that for x as above and τ as in the previous exercise, $\tau \Vdash \mathsf{LNP}(\varphi(x))$.