Theorem of Paris and Wilkie

We consider $\mathbb{N}$ as a first order structure over the language L_{PA} containing basic arithmetic symbols: $0, 1, +, \cdot$ and $\leq$. Denote T as all L_{PA}-sentences (i.e. formulas without free variables) true in $\mathbb{N}$.

We augment L_{PA} by a single unspecified binary relation symbol R (resulting language is denoted as $L(R)$).

Exercise 1. Express formally the statement "if R is the graph of a function with domain $[0, \ldots, n]$ and range $[0, \ldots, n-1]$, then the function is not injective".

The above expression is known as the (injective) pigeonhole principle (for n) and is denoted as $\mathrm{PHP}_n^{n+1}(R)$. Denote $\mathrm{PHP}(R)$ as $\forall n\, \mathrm{PHP}_n^{n+1}(R)$.

Note that $\mathrm{PHP}(R)$ holds no matter the interpretation of R in $\mathbb{N}$.

Exercise 2. Show that $\mathrm{PHP}(R)$ is not provable in T (i.e. no specific axioms regarding R are given). The simplest way to prove this is to construct a model. Notice, however, that you cannot simply interpret R suitably in $\mathbb{N}$.

One possible solution to the above exercise is to first pick a non-standard model $\mathbb{M}$ of T, then pick a non-standard number n, and define R as a graph of the function mapping $[0, \ldots, n]$ to $[0, \ldots, n-1]$ such that each standard number is mapped to itself, while non-standard numbers are shifted by one.

While the above is a viable solution, the model constructed is not extremely useful, since it does not satisfy induction for formulas involving R.

Exercise 3. Give an example of an $L(R)$-formula $\varphi(x)$ violating the principle of induction. In other words, there must be a number m so that

- $\varphi(0)$
- $\neg\varphi(m)$
- $\forall p < m\ \varphi(p) \to \varphi(p+1)$

all hold true in $\mathbb{M}$ for R as above.

Try to come up with as simple a formula, as possible.

Exercise 4. A particular solution to the above exercise is just $\varphi(x) := R(x, x)$.

For such a formula, try to redefine R so that the corresponding function is still an injective mapping from $[0, \ldots, n]$ to $[0, \ldots, n-1]$, while the induction principle for the formula above holds true (for any value of m).

Exercise 5. ** Show that it is not possible to define R violating $\mathrm{PHP}(R)$ while at the same time satisfying induction for all $L(R)$-formulas simultaneously.

The goal for today is to prove the following

Theorem 6 (Paris, J. and Wilkie, A.)**.** It is possible to define R violating $\mathrm{PHP}(R)$ while at the same time satisfying induction for existential formulas, i.e. formulas $\varphi(x)$ of the form $\exists y_1, \ldots, \exists y_k\ \psi(y_1, \ldots, y_k, x)$ with ψ a quantifier-free $L(R)$-formula ($\varphi(x)$ may contain free variables).

We start with a countable non-standard model $\mathbb{M}$ of T. Let n be a non-standard number. Denote an interval $[0, \ldots, m]$ as $[m]$.

We build R iteratively. At each step, we are given some finite relation $\sigma \subset [n] \times [n-1]$ defining the graph of a partial injective function. We then extend it to a finite $\tau \supseteq \sigma$ which is the graph of a partial injection, as well.

How exactly we extend depends on which step of the construction we currently are at. In our case, we discriminate between odd and even steps.

Exercise 7. Explain a way to take action at each even step so that, no matter what happens at any odd step, the limit R is the graph of an injective function from $[n]$ to $[n-1]$ (in fact, you can easily make it be a bijection).

One can then say, that the $\neg\mathrm{PHP}^{n+1}_n(R)$ is *forced* (by the above construction and specification of actions at even steps). We can go further and actually define forcing as follows: for a σ as above and an $L(R)$-sentence φ, say σ forces φ if, no matter how the construction proceeds, assuming the limit R extends σ implies the resulting interpretation satisfies φ.

The above exercise can then be solved as follows. First note that any σ forces R to be the graph of an injective partial function (i.e. no pigeon is mapped to more than one hole and no hole is occupied by more than one pigeon). The totality of such function is not forced by any condition. However, given an arbitrary $p \in [n]$ and σ, it is always possible to extend σ to τ so that τ is now defined for p. This makes a property of being definable on a particular p to be *dense*.

By specifying the even steps of the construction, we are using such density to ensure the totality of the resulting function. Note that we can make sure that any particular dense property is being satisfied by the limit R (and also countably many of them simultaneously) by the exact same argument.

We are left with the induction principle. We show how to deal with a single existential formula of the form $\exists y\, \varphi(x, y)$, where $\varphi(x, y)$ is quantifier-free and does not contain other free variables than x and y. However, the argument is easily adaptable to a general case.

Exercise 8. Assume that for any σ and any number a it is always possible to find a number b and τ extending σ so that τ forces $\varphi(a, b)$.

Show that one can then ensure that the limit R satisfies $\forall x\, \exists y\, \varphi(x, y)$.

So we assume the existence of σ and a so that no τ extending σ forces $\varphi(a, b)$.

Exercise 9. * Assuming the above, argue that, starting from σ, one can actually ensure that the limit R satisfies $\forall y\, \neg\varphi(a, y)$.

Furthermore, argue that the set of all as as above is definable in $\mathbb{M}$ and so has the smallest element.

Assuming the smallest a is non-zero, argue that one can ensure that the limit R satisfies $\exists y\, \varphi(a-1, y)$ and $\forall y\, \neg\varphi(a, y)$.