Primary Decomposition of Ideals

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The decomposition of an ideal into primary ideals is a traditional pillar of ideal theory. From another perspective primary decomposition provides a generalization of the factorization of an integer as a product of prime-powers [Atiyah]. In these presentation we have compiled some of the basic definitions and results for the theory of primary decomposition of ideals.

1 Definitions

Definition 1.1 (Ideal)**.** An ideal $\mathfrak{a}$ of a ring R is a subset of R which is an additive subgroup and is such that $R\mathfrak{a} \subset \mathfrak{a}$ (i.e., $x \in R$ and $y \in \mathfrak{a}$ imply $xy \in \mathfrak{a}$).

Definition 1.2 (Ideal quotient)**.** If $\mathfrak{a}, \mathfrak{b}$ are ideals in a ring R, their Ideal quotient is

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in R : x\mathfrak{b} \subset \mathfrak{a}\}$$

which is an ideal. If $\mathfrak{b}$ is a principal ideal (x), we shall write $(\mathfrak{a} : x)$ in place of $(\mathfrak{a} : (x))$.

Definition 1.3 (Radical)**.** If $\mathfrak{a}$ is any ideal of R, the radical of $\mathfrak{a}$ is

$$r(\mathfrak{a}) = \{x \in R : x^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

Definition 1.4 (Nilradical)**.** The set $\mathfrak{N}$ of all nilpotent[1] elements in a ring R is called nilradical of R.

Definition 1.5 (Prime Ideal)**.** An ideal $\mathfrak{p}$ in R is prime if $\mathfrak{p} \neq (1)$ and if $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Definition 1.6 (Primary Ideal)**.** An ideal $\mathfrak{q}$ in a ring R is primary if $\mathfrak{q} \neq R$ and if $xy \in \mathfrak{q} \Rightarrow$ either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n > 0$.

Equivalently, an ideal $\mathfrak{q}$ is primary if $R/\mathfrak{q} \neq 0$ and every zero-divisor[2] in $R/\mathfrak{q}$ is nilpotent.

Remark 1.7. Trivially every prime ideal is primary.

Definition 1.8 (Noetherian Ring)**.** A ring R is said to be Noetherian if every ideal I of R is finitely generated.

[1] An element a in a ring R is said to be nilpotent if $a^n = 0$ for some $n \geq 1$.
[2] A zero-divisor is a nonzero element a of a ring R such that there is a nonzero element $b \in R$ with $ab = 0$.

1.1 Examples

Example 1.9. The primary ideals in $\mathbb{Z}$ are (0) and (p^n), where p is prime.
If $xy \in \langle p^n \rangle$ then some power of p divides x or y. Let $p|x$, then $x^n \in \langle p^n \rangle$.

Example 1.10. Let $A = k[x, y]$, $\mathfrak{q} = \langle x, y^2 \rangle$. Then

$$A/\mathfrak{q} = \frac{k[x, y]}{\langle x, y^2 \rangle} = \frac{k[y]}{\langle y^2 \rangle}$$

in which the zero divisors are all the multiples of y, hence are nilpotent. Hence $\mathfrak{q}$ is primary, and it radical $r(\mathfrak{q}) = \langle x, y \rangle = \mathfrak{p}$. Note that $\mathfrak{p}$ is prime since $A/\mathfrak{p} = k$ is a field and $\mathfrak{p}^2 = \langle x^2, xy, y^2 \rangle$. We have the strict inclusions $\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$. Thus, a primary ideal is not necessarily a prime power.

Example 1.11. Conversely, a prime power $\mathfrak{p}^n$ is not necessarily primary, although its radical is the prime ideal $\mathfrak{p}$. For example, let $A = k[x, y, z]/\langle xy - z^2 \rangle$ and let $\overline{x}, \overline{y}, \overline{z}$ denote the images of x, y, z respectively in A. Then $\mathfrak{p} = \langle \overline{x}, \overline{z} \rangle$ is prime (since $A/\mathfrak{p} = k[y]$, an integral domain); we have $\overline{xy} = \overline{z}^2 \in \mathfrak{p}^2$ but $\overline{x} \notin \mathfrak{p}^2$ and $\overline{y} \notin r(\mathfrak{p}^2) = \mathfrak{p}$; hence $\mathfrak{p}^2$ is not primary.

1.2 Basic results

An alternative definition of $\mathfrak{N}$ is given by the following lemma.

Lemma 1.12. *The nilradical of R is the intersection of all the prime ideals of R.*

Proposition 1.13. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$.*

Proof. Lemma (1.12) applied to $R/\mathfrak{a}$ tells us that the nilradical of $R/\mathfrak{a}$ is the intersection of all prime ideals of $R/\mathfrak{a}$ which is in correspondence with the set of all prime ideals containing $\mathfrak{a}$. $\square$

Lemma 1.14. *If $\mathfrak{q}$ is primary, then $r(\mathfrak{q})$ is a prime ideal.*

Proof. Let $r = st \in r(\mathfrak{q})$. If $s \notin r(\mathfrak{q})$ we must show that $t \in r(\mathfrak{q})$. If $s \notin r(\mathfrak{q})$, then no power of s belongs to $\mathfrak{q}$. Then $t \in \mathfrak{q} \subset r(\mathfrak{q})$ since $\mathfrak{q}$ is primary. $\square$

If $\mathfrak{p} = r(\mathfrak{q})$, then $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary.

Remark 1.15. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary. This means if $ab \in \mathfrak{q}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{q}$.

Lemma 1.16. *If $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a $\mathfrak{p}$-primary ideal.*

Proof. $r(\mathfrak{q}) = r\big(\bigcap_{i=1}^n \mathfrak{q}_i\big) = \bigcap_{i=1}^n r(\mathfrak{q}_i) = \mathfrak{p}$. Let $xy \in \mathfrak{q}, y \notin \mathfrak{q}$. Then for some i we have $xy \in \mathfrak{q}_i$ and $y \notin \mathfrak{q}_i$ hence $x \in \mathfrak{p}$, since $\mathfrak{q}_i$ is primary. $\square$

Let R be a ring and I an ideal of R. We say that I is *irreducible* if for any two ideals J, K of R such that $I = J \cap K$ we have either $I = J$ or $J = K$.

Proposition 1.17. *(a) A prime ideal is irreducible. (b) An irreducible ideal in a Noetherian ring is primary.*

Lemma 1.18. *Let R be ring, let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal, and let $x \in R$.*
(a) If $x \in \mathfrak{q}$ then $(\mathfrak{q} : (x)) = R$.
(b) If $x \notin \mathfrak{q}$ then $(\mathfrak{q} : (x))$ is $\mathfrak{p}$-primary.
(c) If $x \notin \mathfrak{p}$ then $(\mathfrak{q} : (x)) = \mathfrak{q}$.

Proof. (a) If $x \in \mathfrak{q}$ then $1 \cdot (x) = (x) \subset \mathfrak{q}$ so $1 \in (\mathfrak{q} : (x))$.
(b) If $y \in (\mathfrak{q} : (x))$, then $xy \in \mathfrak{q}$. By assumption $x \notin \mathfrak{q}$, so $y^n \in \mathfrak{q}$ for some n and thus $y \in r(\mathfrak{q}) = \mathfrak{p}$. So $\mathfrak{q} \subseteq (\mathfrak{q} : (x)) \subseteq \mathfrak{p}$; taking radicals we get $r((\mathfrak{q} : (x))) = \mathfrak{p}$. Moreover, if $yz \in (\mathfrak{q} : (x))$ with $y \notin r(\mathfrak{q} : (x)) = \mathfrak{p}$, then $xyz = y(xz) \in \mathfrak{q}$, so $xz \in \mathfrak{q}$, thus $z \in (\mathfrak{q} : (x))$. We get that $(\mathfrak{q} : (x))$ is primary.
(c) In any case $\mathfrak{q} \subseteq (\mathfrak{q} : (x))$. If $x \notin \mathfrak{p} = r(\mathfrak{q})$ and $y \in (\mathfrak{q} : (x))$, then $xy \in \mathfrak{q}$; since no power of x is in $\mathfrak{q}$, we must have $y \in \mathfrak{q}$. $\square$

2 Main Topic

2.1 Primary Decomposition of Ideals

Definition 2.1 (Primary Decomposition)**.** A primary decomposition of an ideal $\mathfrak{a}$ in R is an expression of $\mathfrak{a}$ as a finite intersection of primary ideals, say

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i. \tag{1}$$

where each $\mathfrak{q}_i$ is primary ideal.

In general such a primary decomposition need not exist. We shall say that $\mathfrak{a}$ is **decomposable** if it has a primary decomposition [Atiyah].

We say the above primary decomposition is minimal(irredundant) if
(i) The prime ideals $r(\mathfrak{q}_i)$ are distinct,
(ii) No $\mathfrak{q}_i$ can be omitted i.e. $\bigcap_{j\neq i} \mathfrak{q}_j \nsubseteq \mathfrak{q}_i$ $(1 \leq i \leq n)$.

By lemma (1.16) we can achieve (i) and then we can omit any superfluous terms to achieve (ii); thus any primary decomposition can be reduced to a minimal one.

Definition 2.2. Associated prime ideal of $\mathfrak{a}$ can be written as $r(\mathfrak{a} : x)$ for some $x \in R$. The set of such prime ideals of $\mathfrak{a}$ is called associated prime ideals of $\mathfrak{a}$ and denoted by $\mathrm{Ass}(\mathfrak{a})$.

Example 2.3. Here are some examples of primary decomposition

1. $\langle 12\rangle = \langle 4\rangle \cap \langle 3\rangle$ in $R = \mathbb{Z}$. Here $\mathfrak{q}_1 = \langle 4\rangle = \mathfrak{p}_1^2$, $\mathfrak{p}_1 = \langle 2\rangle$ and $\mathfrak{q}_2 = \mathfrak{p}_2 = \langle 3\rangle$.

2. $\langle x^2, y\rangle = \langle x\rangle \cap \langle x, y\rangle^2$ in $R = k[x, y]$ where k is a field. Here $\mathfrak{q}_1 = \langle x\rangle = \mathfrak{p}_1$ and $\mathfrak{q}_2 = \mathfrak{p}_2^2$ with $\mathfrak{p}_2 = \langle x, y\rangle$.

3. For $I = \langle xy, xz, yz\rangle \subset k[x, y, z]$ there is a primary decomposition $I = \langle x, y\rangle \cap \langle x, z\rangle \cap \langle y, z\rangle$ and $\mathrm{Ass}(I) = \{\langle x, y\rangle, \langle x, z\rangle, \langle y, z\rangle\}$.

4. Consider $\mathfrak{a} = \langle x^3, x^2y, xy^2z\rangle$

$$\begin{aligned}\mathfrak{a} &= \langle x^3, x^2, x\rangle \cap \langle x^3, x^2, y^2\rangle \cap \langle x^3, x^2, z\rangle \cap \langle x^3, y, x\rangle \cap \langle x^3, y, y^2\rangle \cap \langle x^3, y, z\rangle \\ &= \langle x\rangle \cap \langle x^2, y^2\rangle \cap \langle x^2, z\rangle \cap \langle y, x\rangle \cap \langle x^3, y\rangle \cap \langle x^3, y, z\rangle\end{aligned}$$

   We have $\langle x^3, y\rangle \subset \langle x^3, y, z\rangle$ and $\langle x^2, y^2\rangle \subset \langle x, y\rangle$, so we can delete $\langle x^3, y, z\rangle$ and $\langle x, y\rangle$. We have $\langle x^2, y^2\rangle \cap \langle x^3, y\rangle = \langle x^3, x^2y, x^3y^2, y^2\rangle = \langle x^3, x^2y, y^2\rangle$. Thus $\mathfrak{a} = \langle x\rangle \cap \langle x^3, x^2y, y^2\rangle \cap \langle x^2, z\rangle$ is a primary decomposition of $\mathfrak{a}$. The associated prime ideals are $\langle x\rangle, \langle x, y\rangle$ and $\langle x, z\rangle$

Theorem 2.4. *Let $\mathfrak{a}$ be a decomposable ideal and let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$. Let $\mathfrak{p}_i = r(\mathfrak{q}_i)$, $1 \leq i \leq n)$. Then the $\mathfrak{p}_i$ are precisely the prime ideals which occur in the set of ideals $r(\mathfrak{a} : x)$, $x \in A$, and hence are independent of the particular decomposition of $\mathfrak{a}$.*

Proof. Since the decomposition is minimal we can find $x \in \bigcap_{j\neq i} \mathfrak{q}_j$ with $x \notin \mathfrak{q}_i$ and we ave

$$r(\mathfrak{a} : x) = r(\mathfrak{q}_1 : x \cap \cdots \cap \mathfrak{q}_n : x) = r(\mathfrak{q}_1 : x) \cap \cdots \cap r(\mathfrak{q}_n : x) = R \cap \cdots \cap \mathfrak{p}_i \cap \cdots \cap R = \mathfrak{p}_i$$

Hence $\mathfrak{p}_i \in \mathrm{Ass}(\mathfrak{a})$

Let $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})$, thus, $r(\mathfrak{a} : x) = \mathfrak{p}$ for some $x \in R$. Then

$$\mathfrak{p} = r(\mathfrak{a} : x) = r(\mathfrak{q}_1 : x) \cap \cdots \cap r(\mathfrak{q}_n : x)$$

Since $\mathfrak{p}$ is prime it contains $r(\mathfrak{q}_i : x)$ for some i. Also $\mathfrak{p} \subset r(\mathfrak{q}_i : x)$. Thus, $\mathfrak{p} = r(\mathfrak{q}_i : x)$. $\square$

Theorem 2.5 (Noether). *Any proper ideal in a Noetherian ring admits a primary decomposition.*

Proof. Let I be a proper ideal in the Noetherian ring R. We claim I is a finite intersection of irreducible ideals; by part (b) of Proposition 1.17 this gives the desired result. To see this: suppose that the set of proper ideals which cannot be written as a finite intersection of irreducible ideals is nonempty, and choose a maximal element I. Then I is reducible, so we may write $I = J \cap K$ where each of J and K is strictly larger than I. But being strictly larger than I each of J and K can be written as a finite intersection of irreducible ideals, and hence so can I, which is a contradiction. $\square$

References

[1] [David Cox, John Little & Donal O'Shea] Ideals, Varieties, and Algorithms (1997).

[2] [M. F. Atiyah & I. G. MacDonald] Introduction to Commutative Algebra (1969).

[3] [Ralf Fröberg] An Introduction to Gröbner Bases (1997).