# Bijections with composition ($S_A$)

Miliyon T.

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Let $A$ be a non empty set and $S_A$ denotes *the set of all bijections of $A$ onto itself* i.e.

$$S_A = \{f : A \to A \text{ and } f \text{ is 1-1 \& onto}\}$$

Claim $S_A$ with composition, $(S_A, \circ)$, is a group.

*Proof.* First of all $S_A \neq \emptyset$ because $\mathrm{Id}(x) = x$ for all $x \in A$. Thus, $\mathrm{Id} \in S_A$.

1. **Closed**: Let $f, g \in S_A$. Then $f : A \to A$ and $g : A \to A$.

   $$f \circ g : A \to A \text{ is 1-1 correspondence}$$

   Thus, $f \circ g \in S_A$. Hence $S_A$ is closed under $\circ$.

2. **Associative**: Let $f, g, h \in S_A$

   $$(f \circ g) \circ h = f \circ (g \circ h) \qquad \text{(Composition of function is associative)}$$

3. **Identity**: Let $f \in S_A$. Then $f : A \to A$ is 1-1 and onto. Define

   $$\mathrm{Id} : A \to A \quad \text{ by } \mathrm{Id}(x) = x$$

   then

   $$f \circ \mathrm{Id} : A \to A$$

   Let $x \in A$,

   $$(f \circ \mathrm{Id})(x) = f(\mathrm{Id}(x)) = f(x) \Rightarrow f \circ \mathrm{Id} = f$$
   $$(\mathrm{Id} \circ f)(x) = \mathrm{Id}(f(x)) = f(x) \Rightarrow \mathrm{Id} \circ f = f$$

   Hence $f \circ \mathrm{Id} = f = \mathrm{Id} \circ f$. Thus, the identity element of $S_A$ is $\mathrm{Id} : A \to A$ defined by $\mathrm{Id}(x) = x$ for all $x$ in $A$.

4. **Inverse**: Let $f \in S_A$. Then $f : A \to A$ is 1-1 and onto. Then $f^{-1} : A \to A$ is 1-1 and onto. This implies $f^{-1} \in S_A$.

Therefore $(S_A, \circ)$ is a group. □

**Remark 0.1.** If $A = \{1, 2, 3, \ldots, n\}$, then instead of $S_A$ we write $S_n$.