Bernoulli Numbers

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Abstract

Bernoulli numbers[1] are a key figure in number theory. They are extremely useful in understanding Riemann Zeta function.

1 Introduction

Definition 1.1. The Bernoulli numbers[2] are defined by the recursive formula

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad \text{where} \quad B_0 = 1 \tag{1}$$

Expanding (1) shows their relation to the Pascal's triangle.

$$\begin{aligned}
B_0 &= 1\\
B_2 + 2B_1 + 1 &= B_2\\
B_3 + 3B_2 + 3B_1 + 1 &= B_3\\
B_4 + 4B_3 + 6B_2 + 4B_1 + 1 &= B_4\\
B_5 + 5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 &= B_5\\
B_6 + 6B_5 + 15B_4 + 20B_3 + 15B_2 + 6B_1 + 1 &= B_6\\
B_7 + 7B_6 + 21B_5 + 35B_4 + 35B_3 + 21B_2 + 7B_1 + 1 &= B_7\\
&\vdots
\end{aligned}$$

[1] Discovered by Jacob Bernoulli$(1654 - 1705)$ and discussed by him in a posthumous work *Ars Conjectandi*(1713).

[2] By convention if $B_1 = \frac{-1}{2}$, the given Bernoulli sequence is called **first Bernoulli numbers** and **second Bernoulli numbers** if $B_1 = \frac{1}{2}$.

2 How to Compute B_n's

Using Definition 1.1, we can easily show the first few Bernoulli numbers to be

$$B_0 = 1, \quad B_1 = \frac{-1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{-1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \cdots$$

Note that B_3 and B_5 are both zero. In fact, this is a well known result about Bernoulli numbers; $B_n = 0$ whenever n is odd with exception when $n = 1$.

3 Some Facts

Lemma 3.1 (Binomial Convolution)**.** *Let* $f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$*,* $g(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n$ *and* $h(z) = f(z)g(z)$*. Then there exists* d_n *such that*

$$h(z) = \sum_{n\geq 0} \frac{d_n}{n!} z^n, \quad \textit{where} \quad d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Proof. Multiplying $f(z)$ and $g(z)$ term by term gives us

$$\left(\frac{a_0}{0!}z^0 + \frac{a_1}{1!}z^1 + \frac{a_2}{2!}z^2 + \cdots\right)\left(\frac{b_0}{0!}z^0 + \frac{b_1}{1!}z^1 + \frac{b_2}{2!}z^2 + \cdots\right)$$

After expanding and regrouping we will get

$$\frac{a_0b_0}{0!0!}z^0 + \left(\frac{a_0b_1}{0!1!} + \frac{a_1b_0}{1!0!}\right)z^1 + \left(\frac{a_0b_2}{0!2!} + \frac{a_1b_1}{1!1!} + \frac{a_2b_0}{2!0!}\right)z^2 + \cdots \tag{2}$$

If we let c_n to be the coefficient of z^n. For example $c_0 = \frac{a_0b_0}{0!0!}$. Then we have the following formula

$$c_n = \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \tag{3}$$

Using (2) and (3) we write $h(z)$ as the following sum

$$h(z) = \sum_{n\geq 0} c_n z^n \tag{4}$$

Now let's define a value d_n such that

$$\begin{aligned} d_n = n!c_n &= n! \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \\ &= \sum_{k=0}^{n} \frac{n! a_k b_{n-k}}{k!(n-k)!} \\ &= \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \end{aligned}$$

This gives us that

$$\frac{d_n}{n!} = c_n \tag{5}$$

Substituting (5) into (4) completes the proof. ∎

4 Generating function

The following theorem provides a generating function for the Bernoulli numbers.[3]

Theorem 4.1. *The function*

$$G(z) = \frac{z}{e^z - 1}$$

is a generating function for Bernoulli numbers.

Proof. Let

$$G(z) = \sum_{n \geq 0} \frac{B_n}{n!} z^n,$$

where B_n stands for the n^{th} Bernoulli number. Multiplying both sides of the above equation by e^z gives

$$\begin{aligned} e^z G(z) &= \sum_{n \geq 0} \frac{z^n}{n!} \cdot \sum_{n \geq 0} \frac{B_n}{n!} z^n \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} B_k \right) \frac{z^n}{n!} \qquad \text{by Lemma 3.1} \end{aligned} \tag{6}$$

By our definition of the Bernoulli number in (1)

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad \text{where} \quad B_0 = 1$$

If we add B_n to both sides, then we have

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k + B_n = \sum_{j=0}^{n} \binom{n}{j} B_j = B_n, \tag{7}$$

or we would get $B_n + 1$ in the case where $n = 1$. This enables us to simplify the result in (6) to get

$$e^z G(z) = z + \sum_{n \geq 0} B_n \frac{z^n}{n!} = z + G(z) \tag{8}$$

Note that the z at the right side of (8) comes from the fact that at $n = 1$, our result is

$$(B_1 + 1)\frac{z^1}{1!} = \frac{B_1}{1!} z^1 + z$$

If we subtract $G(z)$ from both sides, we get

$$\begin{aligned} e^z G(z) - G(z) &= z \\ (e^z - 1)G(z) &= z \end{aligned}$$

Dividing the last equation by $e^z - 1$ yields the desired result.

$$G(z) = \frac{z}{e^z - 1} \tag{9}$$

□

[3] They were invented in 1718 by French mathematician Abraham De Moivre (16671754)[5].

5 Why the odds vanish?

Corollary 5.1. *For odd integer n different from $n = 1$, $B_n = 0$. i.e.*

$$B_{2i+1} = 0 \quad \textit{if} \quad i \geq 1.$$

Proof. From Theorem 4.1 we have

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!} = \sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!} + \frac{B_1 z}{1!}$$

Then using the fact that $B_1 = -1/2$, we have the following equation

$$\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1} + \frac{z}{2} \tag{10}$$

Now, look closely at the following algebraic simplification

$$\begin{aligned}
\frac{z}{e^z - 1} + \frac{z}{2} &= \frac{2z + z(e^z - 1)}{2(e^z - 1)} \\
&= \left(\frac{z}{2}\right) \frac{(e^z + 1)}{(e^z - 1)} \\
&= \left(\frac{z}{2}\right) \frac{(e^z + 1)}{(e^z - 1)} * \left(\frac{e^{-z/2}}{e^{-z/2}}\right) \\
&= \left(\frac{z}{2}\right) \left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right)
\end{aligned}$$

If we replace z by $-z$ in the last equation, the expression remain unchanged i.e.

$$\left(\frac{-z}{2}\right)\left(\frac{e^{-z/2} + e^{-(-z/2)}}{e^{-z/2} - e^{-(-z/2)}}\right) = \left(\frac{-z}{2}\right)\left(\frac{e^{-z/2} + e^{z/2}}{e^{-z/2} - e^{z/2}}\right) = \left(\frac{z}{2}\right)\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right)$$

Hence, the function $\frac{z}{e^z - 1} - \frac{z}{2}$ is an even function. But this means that the same must hold true for $\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!}$. Since in (10), we showed that they are equal functions. So that we have

$$\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!} = \sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{(-z)^n}{n!} = \sum_{\substack{n \geq 1 \\ n \neq 1}} (-1)^n B_n \frac{z^n}{n!}$$

Matching up each of the terms according to powers of z^n, this gives us

$$B_n = (-1)^n B_n$$

Thus, $B_n = 0$, whenever n is odd. Since we have excluded the case $n = 1$, we conclude that

$$B_{2i+1} = 0 \text{ if } i \geq 1.$$

□

6 Simple but Elegant Application

6.1 Faulhaber's Formula

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k m^{n+1-k} \tag{11}$$

Acknowledgments

It is a pleasure to thank my mentor.

References

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[3] G. H. Hardy and M. Wright *An Introduction to the Theory of Numbers*, 6th ed, 2008.

[4] John H. Conway and Richard K. Guy The Book of Numbers (1996).

[5] [Thomas Koshy] Catalan numbers with Application pp. 382.