BINET'S FORMULA

October 7, 2014

Miliyon T.
Addis Ababa University
Department of Mathematics
miliyon@ymail.com

Contents

1 Introduction

1.1 Golden ratio

The golden ratio (golden section) is defined as follows

$$\frac{a}{b} = \frac{a+b}{a}$$

or we can define the golden ratio using continued fraction

$$\phi := 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}} \tag{1}$$

The other way to define the golden ratio would be this

$$\phi := \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}} \tag{2}$$

1.2 Interesting Identities

The identity that we get from (1) directly

$$\phi = 1 + \frac{1}{\phi} \tag{3}$$

The following identity is derived from (2)

$$\phi^2 = 1 + \phi \tag{4}$$

1.3 Some Linear Algebra facts

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \& \qquad \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} \tag{5}$$

We are going to see the importance of (5) in proving the following lemma.

Lemma 1.1. *For any $n \in \mathbb{N}$*

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Proof. [Induction]

For $n = 1$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = I_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $n = k$. This step is called Induction hypothesis (IH).

$$\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $n = k + 1$. But from (5) we have

$$\begin{aligned}
\begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (by\ IH) \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence by using principle of mathematical induction we can conclude that

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \forall n \in \mathbb{N}.$$

$\square$

1.4 Generating Functions

Definition 1.2. The (ordinary) **generating function** for the sequence $a_0, a_1, a_2, \cdots$ of real numbers is the formal power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

or any equivalent closed form expression.

Note:

1. Generating functions can be used to solve recurrence relations. As we will see soon.

2. Each sequence a_n defines a unique generating function $f(x)$, and conversely.

3. Generating functions are considered as algebraic forms and can be manipulated as such, without regard to actual convergence of the power series.

Example 1.1. *The generating function for the sequence $1, 1, 1, 1, 1, \cdots$ is*

$$f(x) = 1 + x + x^2 + x^3 + \cdots \tag{6}$$

As we know from geometric series the closed form of (6) is given by

$$f(x) = \frac{1}{1 - x}$$

Example 1.2. *The generating function for the sequence* $1, 1, 1, 1, 1, ...$ *is*

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Differentiating both sides of this expression produces

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

Thus, $\frac{1}{(1-x)^2}$ *is a closed form expression for the generating function of the sequence* $1, 2, 3, 4, ...$

1.5 Recurrence Relation

Definition 1.3. A **recurrence relation** for the sequence $a_0, a_1, a_2, \cdots$ is an equation relating the term a_n to certain of the preceding terms a_i, $i \leq n$, for each $n \geq n_0$.

2 Binet's formula

2.1 Combinatorial proof

Theorem 2.1. *For any* $n \in \mathbb{N}$ *the* n^{th} *term of the Fibonacci sequence is given by*

$$F_n = \frac{1}{\sqrt{5}} \left[\phi^n - \left(-\frac{1}{\phi} \right)^n \right]$$

where ϕ *is a golden ratio.*

Proof. The Fibonacci sequence is defined by the following recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

This can be rewritten as follows

$$a_n - a_{n-1} - a_{n-2} = 0 \tag{7}$$

Which is clearly a homogeneous equation. The characteristic[1](χ) equation of (7) is given by

$$\lambda^2 - \lambda - 1 = 0 \tag{8}$$

Thus using quadratic formula the solutions to (8) are

$$\lambda = \frac{1 + \sqrt{5}}{2} \qquad \& \qquad \lambda = \frac{1 - \sqrt{5}}{2}$$

Hence the general solution to the recurrence relation (7) is

$$a_n = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n \tag{9}$$

But from Fibonacci sequence we know that the values of a_0 and a_1 are 0 and 1 respectively. So by using this we are going to find the values of α and β
Substitute $n = 1$ in (9)

$$a_1 = \alpha \left(\frac{1 + \sqrt{5}}{2} \right)^1 + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^1$$

---

[1]Dr. Yirgalem

But $a_1 = 1$, then we have

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1 \tag{10}$$

Substitute $n = 2$ in (9)

$$a_1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^2 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^2$$

But $a_2 = 1$, then we have

$$\alpha\left(\frac{3+\sqrt{5}}{2}\right) + \beta\left(\frac{3-\sqrt{5}}{2}\right) = 1 \tag{11}$$

From (10) and (11) we have two equations with two variable. Hence by using simultaneous equation or other method we will get the values of α and β to be $\frac{1}{\sqrt{5}}$ and $\frac{-1}{\sqrt{5}}$ respectively. Therefore we have

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\square$

2.2 Proof by Induction

Theorem 2.2. *For any $n \in \mathbb{N}$ the nth term of the Fibonacci sequence is given by*

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right]$$

where ϕ is a golden ratio.

Proof. [2]For $n = 1$

$$\begin{aligned}
F_1 + F_2 &= \frac{1}{\sqrt{5}}\left[\phi - \frac{1}{\phi}\right] + \frac{1}{\sqrt{5}}\left[\phi^2 - \frac{1}{\phi^2}\right] \\
&= \frac{1}{\sqrt{5}}\left[2\phi - 1\right] + \frac{1}{\sqrt{5}}\left[\frac{\phi^4 - 1}{\phi^2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\sqrt{5}\right] + \frac{1}{\sqrt{5}}\left[\frac{(\phi^2 - 1)(\phi^2 + 1)}{\phi^2}\right] \\
&= 1 + \frac{1}{\sqrt{5}}\left[\frac{\phi(\phi^2 - 1)}{\phi^2}\right] \qquad (\phi^2 - 1 = \phi \text{ by } (4)) \\
&= 1 + \frac{1}{\sqrt{5}}\left[\frac{\phi^2 - 1}{\phi}\right] = 1 + \frac{1}{\sqrt{5}}\left[\phi - \frac{1}{\phi}\right] \\
&= 1 + \frac{1}{\sqrt{5}}\left[\sqrt{5}\right] \\
&= 2 = F_3
\end{aligned}$$

For $n = k$

$$F_k + F_{k-1} = F_{k-2}$$

---
[2]Alfred and Ingmar

Now, for $n = k + 1$

$$F_{k+1} + F_{k+2} = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - \left(-\frac{1}{\phi}\right)^{k+1}\right] + \frac{1}{\sqrt{5}}\left[\phi^{k+2} - \left(-\frac{1}{\phi}\right)^{k+2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+2} + \phi^{k+1} - \left(-\frac{1}{\phi}\right)^{k+2} - \left(-\frac{1}{\phi}\right)^{k+1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+1}(\phi + 1) - \left(-\frac{1}{\phi}\right)^{k+1}\left(\left(-\frac{1}{\phi}\right) + 1\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+1}(\phi^2) - \left(-\frac{1}{\phi}\right)^{k+1}\left(-\frac{1}{\phi}\right)^2\right] \qquad \left(\because \left(-\frac{1}{\phi}\right)^2 = \left(-\frac{1}{\phi}\right) + 1\right)$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+3} - \left(-\frac{1}{\phi}\right)^{k+3}\right]$$

$$= F_{k+3}$$

Hence, by principles of mathematical induction it follows that

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right] \qquad \forall n \in \mathbb{N}.$$

$\square$

2.3 Linear algebra proof

Theorem 2.3 (Binet). *A closed form of a Fibonacci sequence is given by*

$$F_n = \frac{1}{\sqrt{5}}\left[\phi_1^n - \phi_2^n\right] \tag{12}$$

Where $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$.

Proof. From (1.3) we have

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

If some how we could diagonalize matrix A (i.e to write A in the form $A = PDP^{-1}$), taking any powers of A would be simple. Because we know that $A^n = PD^nP^{-1}$ and we would get such a simple formula for F_n. So let's start diagonalize A.

First let's find the eigenvalues. Which can be found as follows

$$|A - \lambda I_2| = \begin{vmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix}$$

$$= -\lambda(1 - \lambda) - 1$$

$$= \lambda^2 - \lambda - 1$$

Thus, by using quadratic formula we would get

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} = \phi_1 \qquad and \qquad \lambda_2 = \frac{1 - \sqrt{5}}{2} = \phi_2$$

6

Now, let's find the corresponding eigenvectors
For $\lambda_1 = \phi_1$

$$(A - \lambda_1 I_2)V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$(A - \lambda_1 I_2)V = (A - \phi_1 I_2)V = \begin{bmatrix} -\phi_1 & 1 \\ 1 & 1-\phi_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After multiplying the matrices in the left side and equating with the right side we will get the following system of equation

$$-\phi_1 v_1 + v_2 = 0 \tag{13}$$
$$v_1 + (1-\phi_1)v_2 = 0 \tag{14}$$

Multiplying (14) by ϕ_1

$$\phi_1 v_1 - v_2 = 0 \tag{15}$$

Using (13) and (15)

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ \phi_1 v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} v_1$$

Hence the corresponding eigenvector for λ_1 is $\begin{bmatrix} 1 \\ \phi_1 \end{bmatrix}$.

Similarly, one can find that the eigenvector for λ_2 is $\begin{bmatrix} 1 \\ \phi_2 \end{bmatrix}$. Now we are going to write matrix in a form $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$

Finally, let's compute

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}^{n-1} \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1} & 0 \\ 0 & \phi_2^{n-1} \end{bmatrix} \begin{bmatrix} \phi_2 - 1 \\ -\phi_1 + 1 \end{bmatrix}$$

$$= \frac{1}{-\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1}(\phi_2 - 1) \\ \phi_2^{n-1}(-\phi_1 + 1) \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}} \begin{bmatrix} \phi_1^{n-1}(-\phi_1) + \phi_2^{n-1}(\phi_2) \\ \phi_1^n(-\phi_1) + \phi_2^n(\phi_2) \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \phi_1^n - \phi_2^n \\ \phi_1^{n+1} - \phi_2^{n+1} \end{bmatrix}$$

□

That was to be shown!

2.4 Proof using generating function

Theorem 2.4. *The nth term of a Fibonacci sequence is given by*

$$F_n = \frac{1}{\sqrt{5}}\left[\phi_1^n - \phi_2^n\right]$$

Where $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$.

Proof. Consider a Fibonacci sequence which is given by the following recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2. \tag{16}$$

And F_0 and F_1 are defined to be 0 and 1 respectively. The generating function for Fibonacci sequence is

$$f(x) = F_0 + F_1 x + F_2 x^2 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$

When we assign their respective numbers for each F_i's $f(x)$ becomes

$$f(x) = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots \tag{17}$$

The generating function (17) wont help us to solve the recurrence relation in (16). So, let's find the closed form. In order to find the closed form we are going to do some trick, which indeed based at how the Fibonacci sequence recursively defined. Here is what we are going to do, first write (17) as it is,

$$
\begin{array}{rclccccccc}
f(x) & = & 0 & + 1x & + 1x^2 & + 2x^3 & + 3x^4 & + 5x^5 & + \cdots \\
xf(x) & = & 0x & + 1x^2 & + 1x^3 & + 2x^4 & + 3x^5 & + 5x^6 & + \cdots \\
x^2 f(x) & = & 0x^2 & + 1x^3 & + 1x^4 & + 2x^5 & + 3x^6 & + 5x^7 & + \cdots \\
\hline
f(x) - xf(x) - x^2 f(x) & = & 0 & +x & +x^2\{1-1\} & +0\cdot x^3 & +0\cdot x^4 & +0\cdot x^5 & + \cdots
\end{array}
$$

Clearly, on the right side we are left with x because the summands goes to 0. Hence

$$f(x) - xf(x) - x^2 f(x) = x$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2} \tag{18}$$

So far, we have found a generating function (18) for (16). Now, let's use method partial fraction

$$f(x) = \frac{-x}{x^2 + x - 1} = \frac{-x}{(x + \phi_1)(x + \phi_2)} = \frac{A}{(x + \phi_1)} + \frac{B}{(x + \phi_2)}$$

To find A and B we are going to solve the following system of equation

$$A + B = -1$$
$$\phi_2 A + \phi_1 B = 0$$

8

Which results $A = \frac{-\phi_1}{\sqrt{5}}$ and $B = \frac{\phi_2}{\sqrt{5}}$. Thus,

$$
\begin{aligned}
f(x) &= -\frac{1}{\sqrt{5}} \frac{\phi_1}{(x + \phi_1)} + \frac{1}{\sqrt{5}} \frac{\phi_2}{(x + \phi_2)} \\
&= \frac{1}{\sqrt{5}} \left[\frac{\phi_2}{(x + \phi_2)} - \frac{\phi_1}{(x + \phi_1)} \right] \\
&= \frac{1}{\sqrt{5}} \left[\frac{1}{(\frac{x}{\phi_2} + 1)} - \frac{1}{(\frac{x}{\phi_1} + 1)} \right] = \frac{1}{\sqrt{5}} \left[\frac{1}{(-\phi_1 x + 1)} - \frac{1}{(-\phi_2 x + 1)} \right] \\
&= \frac{1}{\sqrt{5}} \left[\frac{1}{(1 - \phi_1 x)} - \frac{1}{(1 - \phi_2 x)} \right]
\end{aligned}
$$

But we know that $\frac{1}{1-ax} = 1 + ax + a^2 x^2 + a^3 x^3 + \cdots$. Hence

$$
\frac{1}{1 - \phi_1 x} = 1 + \phi_1 x + \phi_1^2 x^2 + \phi_1^3 x^3 + \cdots
$$

$$
\frac{1}{1 - \phi_2 x} = 1 + \phi_2 x + \phi_2^2 x^2 + \phi_2^3 x^3 + \cdots
$$

Then, we will get

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{5}} \left[\frac{1}{(1 - \phi_1 x)} - \frac{1}{(1 - \phi_2 x)} \right] \\
&= \frac{1}{\sqrt{5}} \left[1 + \phi_1 x + \phi_1^2 x^2 + \phi_1^3 x^3 + \cdots - (1 + \phi_2 x + \phi_2^2 x^2 + \phi_2^3 x^3 + \cdots) \right] \\
&= \frac{1}{\sqrt{5}} \left[(\phi_1 - \phi_2)x + (\phi_1^2 - \phi_2^2)x^2 + (\phi_1^3 - \phi_2^3)x^3 + \cdots \right] \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\phi_1^n - \phi_2^n) x^n
\end{aligned}
$$

Therefore,

$$
F_n = \frac{1}{\sqrt{5}} \left[\phi_1^n - \phi_2^n \right]
$$

$\square$

References

[1] [Mrio Livio] The Golden Ratio; *The Story of Phi,the World's Most Astonishing Number.*

[2] [Alfred and Ingmar] The Fabulous Fibonacci Numbers.

[3] [Ron Larson] Elementary Linear Algebra, 6th ed.