Counting Using Bijections

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We will only count elements of sets. Whenever we are faced with a combinatorial problem, we will put it in the form "How many elements does the set S have?"
One of the most widely used facts in combinatorics is that two sets have the same number of elements if and only if there is a bijection between them. Let us see how we can use this fact in solving problems.

Example 0.1. In how many ways can we distribute 15 identical apples to 4 distinct students. Not all students have to get an apple.

Solution. Let S be the set of all distributions of 15 apples to 4 students. We can represent S as the collection of all ordered sequences of length 4 whose components add up to 15. For example, the sequence (7,0,4,4) means that the first student gets 7 apples, the second gets 0, the third and fourth get 4 apples each. We may now write:

$$S = \{(a, b, c, d) : a, b, c, d \in \mathbb{N}_0, a + b + c + d = 15\}.$$

Consider the set T of all sequences of length 18 that consist of 15 zeroes and 3 ones. For each $(a, b, c, d) \in S$ we define $f(a, b, c, d)$ in the following way:

$$f(a, b, c, d) = \underbrace{00\ldots0}_{a}1\underbrace{00\ldots0}_{b}1\underbrace{00\ldots0}_{c}1\underbrace{00\ldots0}_{d}.$$

Then $f : S \to T$ is a bijection and this is very easy to verify. Therefore $|S| = |T|$. However, it is easy to find the number of elements of T. This is the same problem as choosing 3 elements from the set of 18 zeroes and turning them into ones. This can be done in $\binom{18}{3}$ ways. Therefore there are $\binom{18}{3}$ ways to distribute 15 apples to 4 students. $\square$

Example 0.2. Determine the number of subsets of 1, 2, 3, 4, ..., 50 whose sum of elements is larger than or equal to 638.

Solution. Let $U = \{1, 2, \ldots, 50\}$. Denote by S the set of all subsets of U whose sum of elements is larger than or equal to 638. Let T be the set of those subsets whose sum of elements is at most 637. Then we have $U = S \cup T$ and $S \cap T = \emptyset$. Consider the following function f on the set S. If $A \subseteq S$ we define $f(A) = A^C$. Since the total sum of all elements of U is $1+2+\cdots+50 = \frac{50\cdot51}{2} = 25\cdot51 = 1275$ we see that the sum of elements of A^C is smaller than or equal to $1275-638 = 637$ therefore $A^C \subseteq T$. Clearly f is a bijection, which means that $|S| = |T|$ and from $U = S \cup T$ we get $|U| = 2|S|$. We know that $|U| = 2^{50}$ and this implies that $|S| = 2^{49}$. $\square$