Graph Theory

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My first triumph in mathematics was Steiner triple problem. While I was in AAU, my professor Dr. Yirgalem introduce me the problem. She thought it was "open". It took me a day to solve it. But I doubted that it might have been solved already. So, I searched on the internet. My instinct was correct. It has been solved by Kirkman.[1]

1 Definition

Definition 1.1 (Graph)**.** A Graph is a non empty set of vertices and edges. It is often denoted

$$G = (V, E) \text{ or } (V_G, E_G) \text{ or } (V(G), E(G))$$

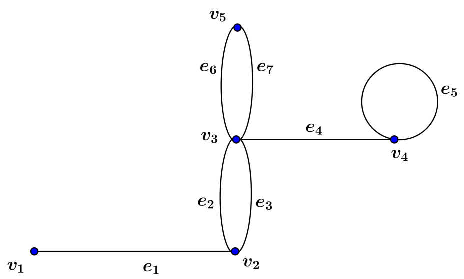

Figure 1: *A Graph*

Definition 1.2 (Vertex)**.** A point; an element of the first constituent set of a graph.

Definition 1.3 (Degree (of a vertex))**.** Given a vertex v,the number $deg(v)$ of instances of v as an endpoint; that is, the number of proper edges incident on v plus twice the number of loops at v.

Definition 1.4 (Order)**.** Given a graph G, the cardinality $|V_G|$ of the vertex set. It is denoted by $|G|$.

Definition 1.5 (Link)**.** An edge with two distinct end point.

Example 1.6. The edge e_1 in (**Figure** 1) is a link.

Definition 1.7 (Loop)**.** An edge joining a vertex to itself.

Example 1.8. The edge e_5 in (**Figure** 1) is a loop.

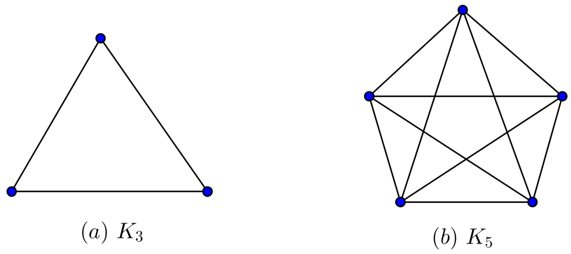

Figure 2: *A Simple Graph*

Definition 1.9 (Multi-edge)**.** A set of at least two edges,all of which have the same endpoints.

Definition 1.10 (Simple Graph)**.** A graph with no loops or multi-edge.

Definition 1.11 (Edge)**.** A line,either joining one vertex to another or joining a vertex to itself; an element of the second constituent set of a graph.

Definition 1.12 (Size)**.** Given a graph G, the cardinality $|E_G|$ of the edge set. It is denoted by $||G||$.

Definition 1.13 (Subgraph)**.** Given a graph G,a graph H whose vertices and edges are all in G.

Definition 1.14 (Walk)**.** An alternating sequence $v_0, e_1, v_1, ..., e_r, v_r$ of vertices and edges where consecutive edges are adjacent,so that each edge e_i joins vertices v_{i1} and v_i.

Definition 1.15 (Trial)**.** A *walk* in which no edge occurs more than once.

Definition 1.16 (Path)**.** A *trail* in which all of its vertices are different, except that the initial and final vertices may be the same.

Definition 1.17 (Cycle)**.** A closed *path* of positive length.

Definition 1.18. The simple graph K_n with n vertices in which every pair of vertices is adjacent.

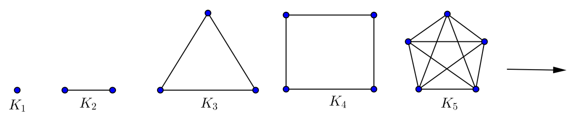

Figure 3: *Complete graph*

Definition 1.19 (Decomposition)**.** A decomposition od a graph G is a family $\mathcal{F}$ of edge-disjoint subgraph of G such that

$$\bigcup_{F \in \mathcal{F}} E(F) = E(G)$$

Definition 1.20 (Cyclic Decomposition)**.** If every subgraph of $\mathcal{F}$ is a cycle, then the decomposition is called cyclic decomposition.

[1]The British mathematician Thomas Kirkman(1806 - 1895) solved the problem in his 1847 paper.

2 Basic Results

Theorem 2.1. *(Handshaking lemma) For any graph G*

$$\sum_{v \in V(G)} deg(v) = 2||G|| \tag{1}$$

Proof. When summing the degrees of the vertices of a graph G, we count each edge of G twice, once for each of the two vertices incident with the edge. $\square$

Corollary 2.2. *The number of odd vertices(Vertices of odd degree) in a graph is even*

Proof. Let G be any graph. Partition the vertex set $V(G)$ into two

$$O - \text{the set of all odd vertices}$$
$$E - \text{the set of all odd vertices}$$

Then we have

$$\sum_{v \in V(G)} deg(v) = \sum_{v \in O} deg(v) + \underbrace{\sum_{v \in E} deg(v)}_{even} \tag{2}$$

Now, suppose the number of odd vertices in G is odd. Hence the sum

$$\sum_{v \in O} deg(v) \text{ is odd.}$$

Consequently, the sum

$$\sum_{v \in V(G)} deg(v) \text{ becomes odd.}$$

That is a contradiction to Handshaking lemma. Therefore the number of odd vertices in any graph G must be even. $\square$

Corollary 2.3. *The size of a complete graph of order n (K_n) is*

$$\binom{n}{2} = \frac{n(n-1)}{2} \tag{3}$$

3 Triumph

Lemma 3.1. *K_n can be decomposed into cycles of m iff $||K_n||$ is a multiple of m.*

Proof. $\square$

Theorem 3.2. *If K_n decomposes into K_3 (Cycles of order 3), then either $n = 6q + 1$ or $n = 6q + 3$ for some integer q.*

Proof. $\square$

References

[1] [Adrian Bondy & Murty] Graph Theory, 2007.

[2] [Gary Chartrand] Introductory Graph Theory.

[3] [Charles & Jeffrey] Handbook of combinatorial design.