# Graph Theory

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### Basic Definitions

## 1 Basic Facts

**Theorem 1.1.** *Let $G$ be a $(p, q)$ graph, which may be a multigraph or a loop-graph, or both.*
*Let $V = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of $G$.*
*Then* $\sum_{i=1}^{p} \deg_G(v_i) = 2q$ *where* $\deg_G(v_i)$ *is the degree of vertex* $v_i$.
*That is, the sum of all the degrees of all the vertices of a graph is equal to twice the total number of its edges. This result is known as the* $\boxed{\textit{Handshaking Lemma}}$.

*Proof.* In the notation $(p, q)$ graph, $p$ is its order and $q$ its size.
That is, $p$ is the number of vertices in $G$, and $q$ is the number of edges in $G$.
Each edge is incident to exactly two vertices.
The degree of each vertex is defined as the number of edges to which it is incident.
So when we add up the degrees of all the vertices, we are counting all the edges of the graph twice. □

**Corollary 1.2.** *The number of odd vertices(vertices of odd degree) in a graph must be even.*

### 1.1 Historical Note

This result was first given by Leonhard Paul Euler in his 1736 paper "Solutio problematis ad geometriam situs pertinentis", widely considered as the first ever paper in the field of graph theory.

## 2 Tree

**Definition 2.1.** A tree is a connected graph with no cycles.

**Lemma 2.2.** *For any tree $T$ of order $n$, $|E| = n - 1$*

*Proof.* By Induction
For $n = 1$, $E = 1 - 1 = 0$ True!( trivial graph).
Suppose it is true for any order of vertices which are less than $n$. Now let's remove an edge $vu$ from $T$. Since every edge on tree is a bridge we will get two components $T_1$ and $T_2$. Clearly, the order of $T_1$ and $T_2$ are less than $n$. Hence by induction assumption $|T_1| = v(T_1) - 1$ and $|T_2| = v(T_2) - 1$
Now,

$$\begin{aligned}|E| &= v(T_1) - 1 + v(T_2) - 1 + 1 \\ &= v(T_1) + v(T_2) - 1 \\ &= v(T) - 1\end{aligned}$$

□

**Theorem 2.3.** *Every non trivial tree has at least two pendant vertices.*

*Proof.* By the Handshake Lemma,

$$\sum_{v \in V(T)} d(v) = 2n - 2.$$

Since every vertex except the one leaf has degree at least 2, we get

$$2 \sum_{v \in V(T)} d(v) \geq 2[2(n-1)+1] = 4n - 2 > 2(n-1),$$

a contradiction. $\square$