EXAM 1

Graph Theory

Miliyon Tilahun (NSR/4137/05)
October 21, 2015

SOLVE THE FOLLOWING PROBLEMS BY SHOWING ALL THE NECESSARY STEPS.

1. Show that either a graph or its complement is connected.
2. Show that at least two vertices in a graph of order 2 or more have the same degree.
3. Show that any non trivial tree has at least two pendant(degree 1) vertices.

SOLUTION

1. Let G be a graph(finite)
   Suppose G is disconnected and assume $G_1, G_2, \ldots, G_n$ the connected components of G.
   Now, $G = G_1 + G_2 + \ldots + G_n$ where + is the disjoint union operator.
   Thus,
   $$\begin{aligned}\overline{G} &= \overline{G_1 + G_2 + \ldots + G_n} \\ &= \overline{G_1} \vee \overline{G_2} \vee \ldots \vee \overline{G_n}\end{aligned}$$
   where $\vee$ is the join operator.
   Hence $\overline{G}$ is connected. i.e. whenever G is disconnected $\overline{G}$ is connected.
   That was to be shown!

2. Let G be a graph of order $n \geq 2$.
   The possible values for $\deg(v_i)$ are $n-1, n-2, \ldots, 2, 1$
   It is a kind of pigeonhole principle; we have n vertices(pigeons) and $n-1$ possible values of degree(hole).
   Which implies there has to be at least two vertices(pigeons) with the same degree(for pigeons in the same hole).

3. Suppose not(there is at most one vertex of degree one(pendant)) and assume T is order $n \geq 2$.
We know that $||T|| = n-1$ (which can be proved using induction).
Thus, handshaking lemma tells us

$$\sum_{v \in V(T)} \deg(v) = 2(n-1) \tag{1}$$

But from our assumption there is at most one pendant vertex. This implies

$$\sum_{v \in V(T)} \deg(v) \geq 2(n-1) = 2n-2+1 = 2n-1 \tag{2}$$

Putting (1) and (2) together

$$2(n-1) \geq 2n-1 \qquad \text{which is a contradiction!}$$

Therefore, our assumption was false hence.