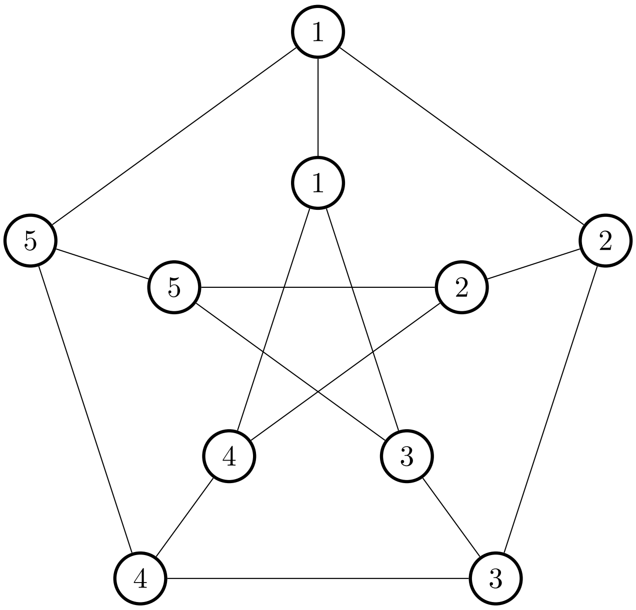
Dr. Yirgalem T.

GRAPH THEORY ASSIGNMENT

November 7, 2015



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1 Solve the following problems.

[Questions][†]

1. Find $|G|, ||G||, \delta(G)$ and $\Delta(G)$ for each of the following graphs

$$\text{a) } \theta_n(\text{Null graph}), \qquad \text{b) } K_n, \qquad \text{c) } P_n, \qquad \text{d) } C_n$$

2. Given a graph G of order n, and size m, a vertex $v \in V(G)$, and an edge $e \in E(G)$, determine the order and size of

$$\text{a) } G - v, \qquad \text{b) } G - e, \qquad \text{c) } \overline{G}$$

3. Show that a regular graph of odd degree is even order.

4. Which of the following sequences are graphic?

$$\text{a) } 5,3,2,2,2 \qquad \text{b) } 3,3,3,2,2 \qquad \text{c) } 3,3,2,2,2 \qquad \text{d) } 3,3,3,3,2 \qquad \text{e) } 4,3,3,2,2$$

5. a) Find the adjacency and incident matrix of $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 23, 24, 34, 35, 45\}$.
b) Find the adjacency matrix for

$$\text{a) } \theta_5, \qquad \text{b) } K_5, \qquad \text{c) } C_5, \qquad \text{d) } P_5$$

6. Let $G = (V, E)$, where $V = \{x, y, z, u\}$ and $E = \{xy, xz, xu, zu\}$. Draw all the subgraphs of G up to isomorphism(all non isomorphic).

7. Find two non isomorphic graph of order 5, which are of the same degree.

8. Which of the following graphs are isomorphic?

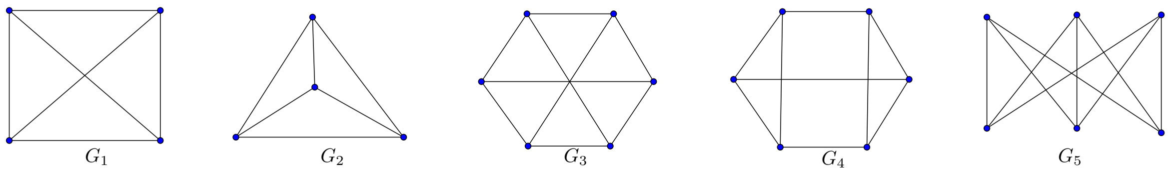


Figure 1: Isomorphism

9. Let A be a set and $\mathfrak{B}$ be a finite collection of subsets of A. Then the intersection graph $I(\mathfrak{B})$ is the graph whose vertex set is $\mathfrak{B}$, and two vertices are adjacent if they are disjoint.

   a) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $\mathfrak{B} = \{\{1, 2\}, \{2, 4\}, \{1, 2, 3\}, \{3, 4, 5\}, \{5, 6\}\}$
Use diagram to show what $G = I(\mathfrak{B})$ would look like.

   b) Let $G' = (V, E)$, where $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 34, 41\}$. Let S_i be a set consisting of the vertex i and the edges incident with i. For instance, $S_3 = \{3, 23, 34\}$
Let

$$S = \bigcup_{i=1}^{4} S_i \qquad \text{and} \qquad \mathfrak{B} = \{S_i | i = 1, 2, 3, 4\}$$

Prove that $I(\mathfrak{B}) \cong G'$

[†]Dr. Yirgalem T.

10. A graph has order 13 and 3 components. Prove that one of its components has at least 5 vertices.

11. If G is a simple graph of order n having exactly two components which are complete graph themselves. What is the minimum and maximum possible size of G(in terms of n)?

12. Given the graphs you see below, determine the smallest number of vertices that have to be removed from G to disconnect G(which vertices)?

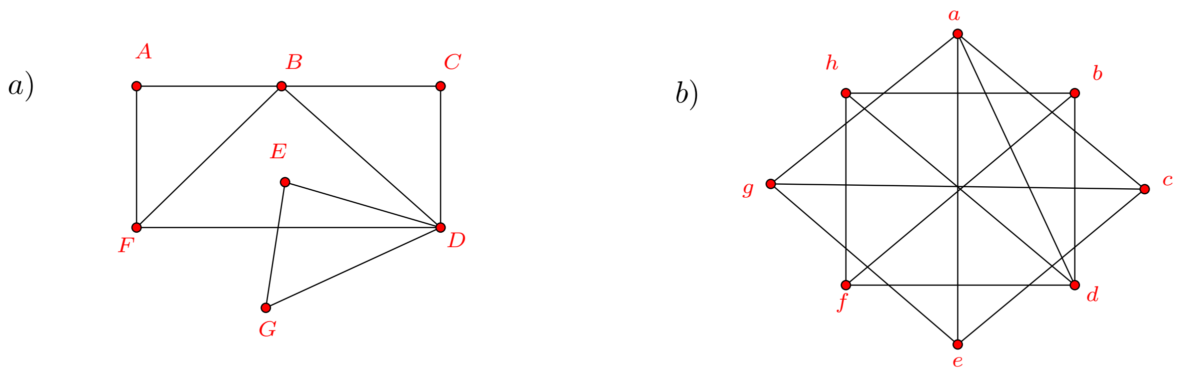


Figure 2: Disconnect

13. Let $G = (V, E)$, where $V = \{1, 2, \ldots, 15\}$, and two vertices i and j are adjacent iff greatest common divisor $\gcd(i, j) \neq 1$.

a) Draw G, b) How long is the longest path in G?

14. Find the diameters of

a) The peterson graph, b) The hypercube Q_n, c) K_n d) $K_{n,m}$ e) C_n f) P_n

15. Which of the following graphs are
a) Eulerian b) Hamiltonian

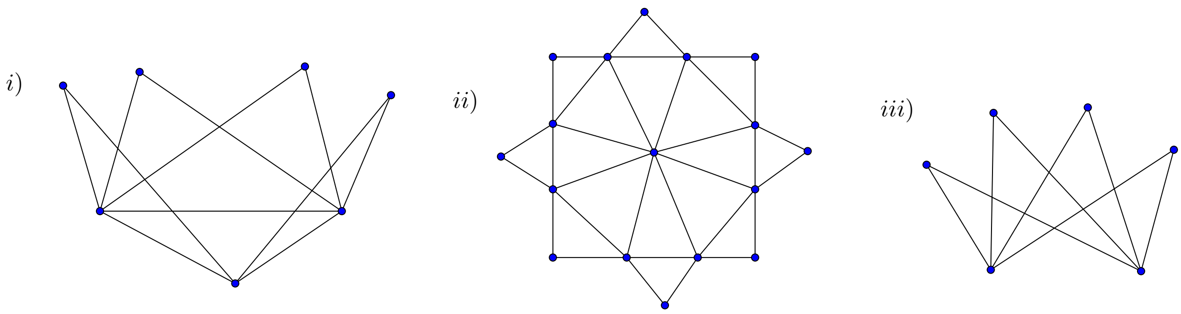


Figure 3: Eulerian and Hamiltonian

16. Show that if a graph is regular and of even order and odd size, then it is not Eulerian[1].

[1] Leonhard Euler (1707-1783) a prolific Swiss mathematician. In 1735, Euler presented a solution to the problem known as the Seven Bridges of Königsberge, which made him the father of graph theory.

17. Which of the following diagram could be drawn without lifting your pen from the paper and without repeating any line?

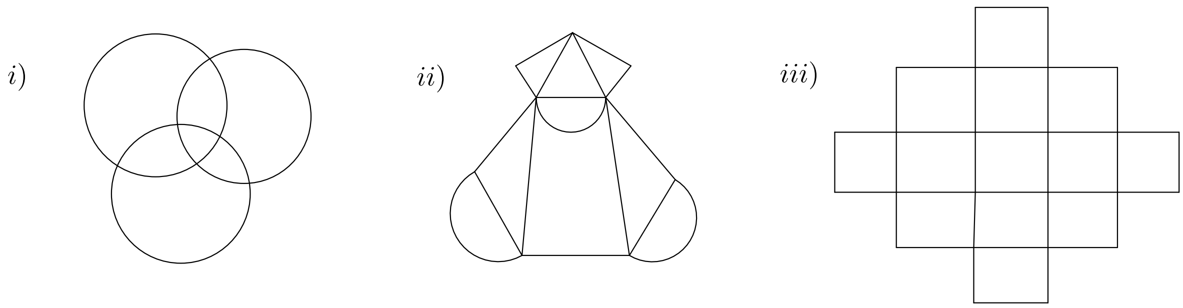


Figure 4: Euler trail

18. Take $n = 6$, and let $m \leq \frac{n}{2}, m \in \mathbb{N}$.
    a) Draw the graph $K_m \vee$[2]$(\overline{K_m} +$[3]$K_{n-2m})$, (for each $m \leq \frac{n}{2}$).
    b) Show that this graphs can not be Hamiltonian.
    c) Give the degree sequence of each of the graphs you obtained in (a) above, in a decreasing order.
    d) Is it possible to find a non-Hamiltonian graph of the same order which degree majorizes any of these graphs? why?

19. What is the minimum number of edges a simple graph G of order $n \geq 2$ should have to guarantee that it is Hamiltonian.

20. Find the degree sequence of the non-Hamiltonian simple graph with n vertices and $\binom{n-1}{2}+1$ edges.

21. Let T be a tree of order 12 that has exactly 3 vertices of degree 3 and one of degree 2. Give the degree sequence of T.

22. Show that the sequence $7, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1$ is a degree sequence of a tree.

23. Compute the number of spanning tree of K_{10}.

24. a) Find the Prüfer sequence corresponding to the tree of order 11 with $V = \{1, 2, \ldots, 11\}$ and $E = \{12, 13, 24, 25, 36, 37, 48, 49, 510, 511\}$.
    b) Construct a tree that have the Prüfer code $4, 5, 7, 2, 1, 1, 6, 6, 7$.

25. Let G be a forest[4] having m components. How many edges should be added to G to obtain a tree containing G?

26. a) What is the minimum number of leaves a tree can have if the order $n \geq 2$?(Express your answer in terms of some basic element of a graph)
    b) What is the maximum number of leaves in a tree of order n?

[2] Join operator "$\vee$"
[3] Disjoint Union operator "+"
[4] A forest is simply an acyclic graph(A disconnected graph whose components are tree).

27. Find a matching, a maximum matching or a perfect matching or explain why it does not exist in the following graph.

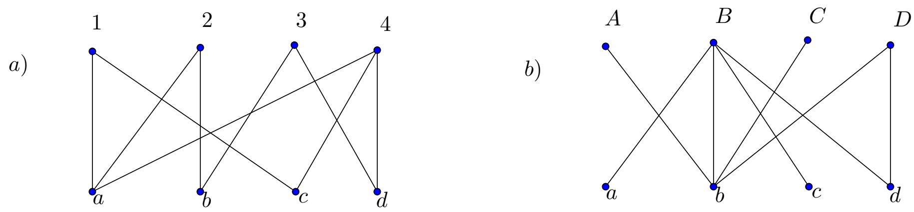


Figure 5: Matching

28. Is there a job for each of the applicant in $\{A, B, C, D, E\}$ if the jobs are a, b, c, d and e(in G below)?

G

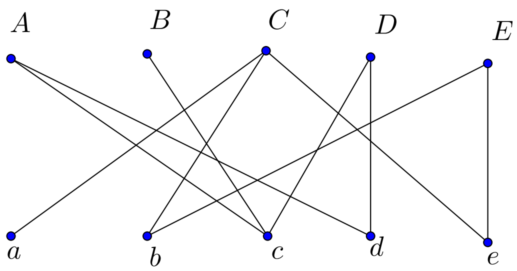


Figure 6: Applicant-Job

29. A connected plane graph of order n is 4-regular and has 10 faces(regions). What is the value of n?

30. For which values of n is K_n planar?

31. For which values of n and m is $K_{n,m}$ planar?

32. For a graph G below

G

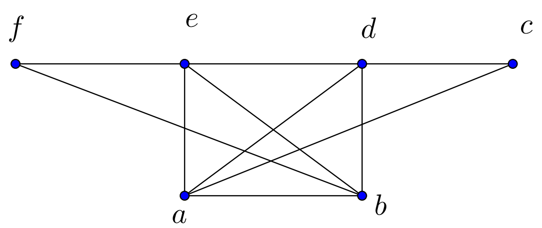


Figure 7: Dual

Draw the dual of G and show that

$$\sum_{\text{all faces}} \deg(\text{faces}) = 2|E|$$

2 Solutions

[Answers][‡]

1. For #1 look the table below

	a	b	c	d
G	θ_n	K_n	P_n	C_n
$\lvert G\rvert$	n	n	n	n
$\lVert G\rVert$	0	$\frac{n(n-1)}{2}$	$n-1$	n
$\delta(G)$	0	$n-1$	1	2
$\Delta(G)$	0	$n-1$	2	2

Table 1: Question 1

2. Given that $|G| = n$ and $\|G\| = m$

	a	b	c
G	$G - v$	$G - e$	$\overline{G}$
$\lvert G\rvert$	$n-1$	n	n
$\lVert G\rVert$	$m - \deg(v)$	$m-1$	$\lVert K_n\rVert - \lVert G\rVert = \frac{n(n-1)}{2} - m$

Table 2: Question 2

3. Let G be a regular graph of odd degree. Denote that odd degree by d(odd). We wand to show $|G|$ is even.
   Suppose not(i.e. $|G|$ is odd). For simplicity denote $|G| = k$(odd).
   As G is regular every $v \in V(G)$ is of degree d(which is odd) and we have k vertices. Thus

$$\sum_{v \in V(G)} \deg(v) = \underbrace{d}_{odd} \cdot \underbrace{k}_{odd} = \text{ odd number}$$

   Notice that both d and k are odd, then their product(which is by definition the sum of degrees in G) must be odd. But from Handshaking lemma we know that such thing(the sum of degree in any graph being odd) will never happen(ever!). Hence a contradiction! our assumption was incorrect.
   $\therefore$ $|G|$ is even.

4. The necessary and sufficient condition for any sequence of numbers to be **graphic** given by the following fact.

   Theorem 2.1. [5] *A non-increasing sequence of non-negative integers $d_1, d_2, \ldots, d_n$ is* ***graphical*** *iff its sum is even and for $k = 1, 2, \ldots, n$*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, n\} \tag{1}$$

   We use (1) to show the sequence in (a) is not **graphic**

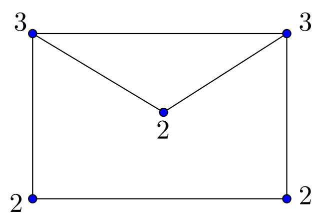
[‡] Miliyon T.
[5] This result is due to Hungarian mathematicians Erdős and Gallai

(a) $5, 3, 2, 2, 2$ $n = 5$.
For $k = 1$

$$\begin{aligned} \sum_{i=1}^{1} d_i = 5 &\leq 1(1-1) + \sum_{i=2}^{5} \min\{d_i, 1\} \\ &\leq 0 + (1+1+1+1) \\ &\leq 4 \\ \Rightarrow 5 &\leq 4 \qquad \rightarrow\leftarrow \end{aligned}$$

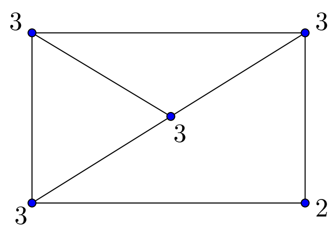
Hence the sequence given above is not **graphic**.
(b) $3, 3, 3, 2, 2$
This sequence is not **graphic** since its sum is not even on the first place.

Thanks to Havel and Hakimi, we are able to construct the corresponding simple graphs for the sequences given in (c),(d) and (e)
(c) $3, 3, 2, 2, 2$



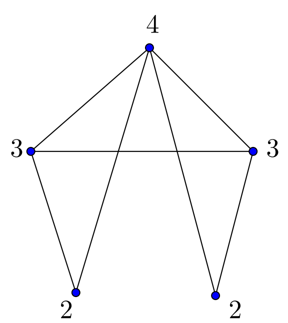
- No loops
- No multi-edges

(d) $3, 3, 3, 3, 2$



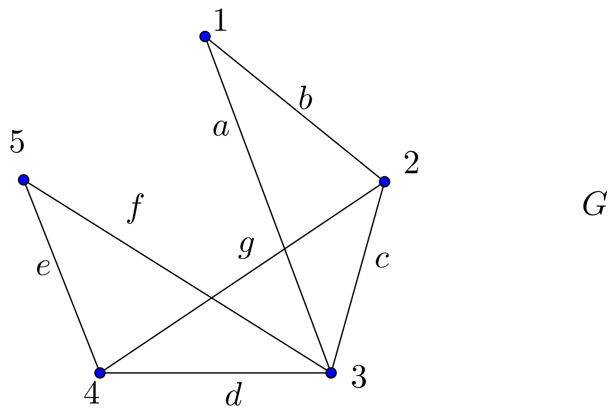
- No loops
- No multi-edges

(e) $4, 3, 3, 2, 2$



- No loops
- No multi-edges

5. (a) For graph G below



The adjacency matrix of G is

	1	2	3	4	5
1	0	1	1	0	0
2	1	0	1	1	0
3	1	1	0	1	1
4	0	1	1	0	1
5	0	0	1	1	0

$$\Rightarrow \qquad A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The incident matrix of G is

	a	b	c	d	e	f	g
1	1	1	0	0	0	0	0
2	0	1	1	0	0	0	1
3	1	0	1	1	0	1	0
4	0	0	0	1	1	0	1
5	0	0	0	0	1	1	0

$$\Rightarrow \qquad M_G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(b)

i) The adjacency matrix for θ_5 is

$$(0)_{5\times 5}, \qquad \text{null matrix of order 5.}$$

ii) The adjacency matrix for K_5

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

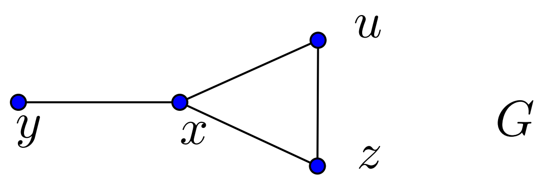
iii) The adjacency matrix for P_5

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

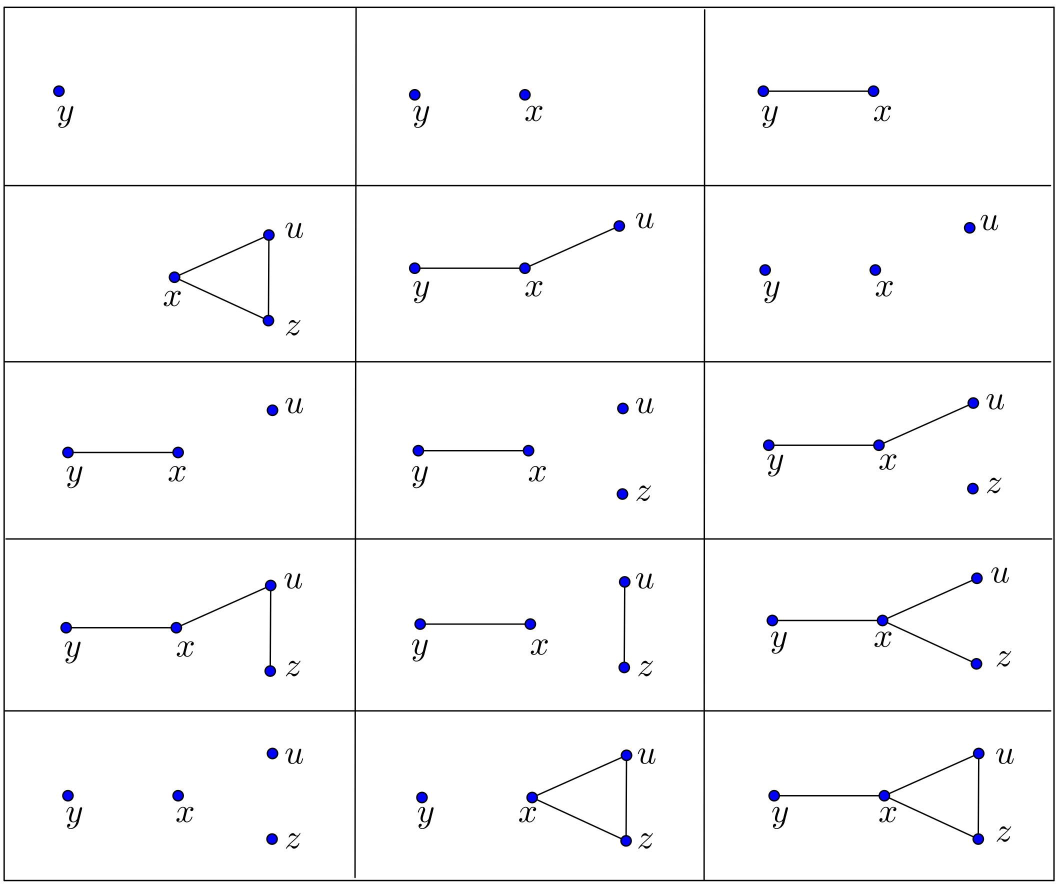
iv) The adjacency matrix for C_5

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6. Let $G = (V, E)$, where $V = \{x, y, z, u\}$ and $E = \{xy, xz, xu, zu\}$. Which can be drawn



The non isomorphic subgraphs of G are

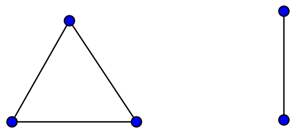


7. G_1 and G_2 below are one such example. Hence G_1 and G_2 are non-isomorphic order 5

$\mathbf{G_1}$

The degree sequence of $\mathbf{G_1}$ is; $\mathbf{1, 2, 2, 2, 1}$

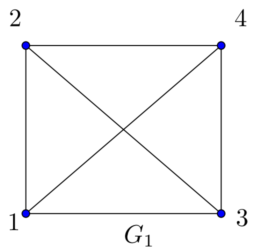
$\mathbf{G_2}$



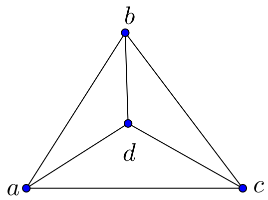
The degree sequence of $\mathbf{G_2}$ is; $\mathbf{1, 2, 2, 2, 1}$

graphs of the same degree sequence.

8. G_1 and G_2 are isomorphic.

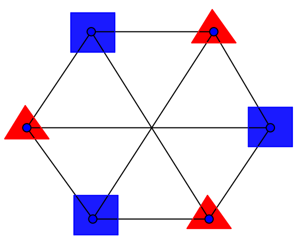


G_1

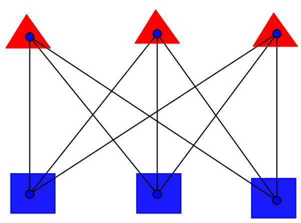


G_2

The isomorphism between G_1 and G_2 is the map $\{f : 1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$.
G_3 and G_5 are isomorphic.



G_3

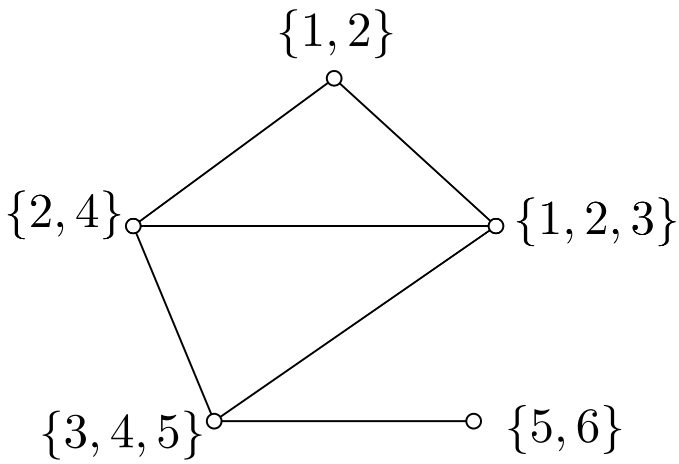


G_5

$G_3 \cong K_{3,3}$, $K_{3,3} \cong G_5$

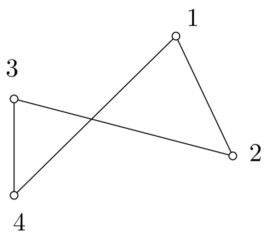
$G_3 \cong G_5$ (since isomorphism is an equivalence relation)

9. (a)

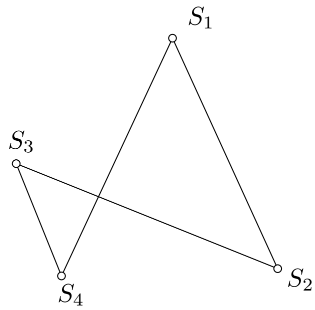


(b)

G'



$I(\mathfrak{B})$



Where

$S_1 = \{1, 12, 14\}$
$S_2 = \{2, 23, 21\}$
$S_3 = \{3, 32, 34\}$
$S_4 = \{4, 41, 43\}$

The isomorphism $I(\mathfrak{B}) \to G'$ is given by $S_i \to i$ for $i = 1, 2, 3, 4$.

10. The simplest(the best) way to approach this problem is by using **Pigeonhole Principle**[6]. So, assume components as hole and vertices as pigeon.
Now if we let each component to have 4 vertices($4 + 4 + 4 = 12 = 13 - 1$), then we will left with one extra vertex. This extra vertex belongs to one of the three components(since we've only three components). But adding that extra vertex to any of the components results a component with 5 vertices(since each component has already 4 vertices). $\square$

11. The minimum size of G is

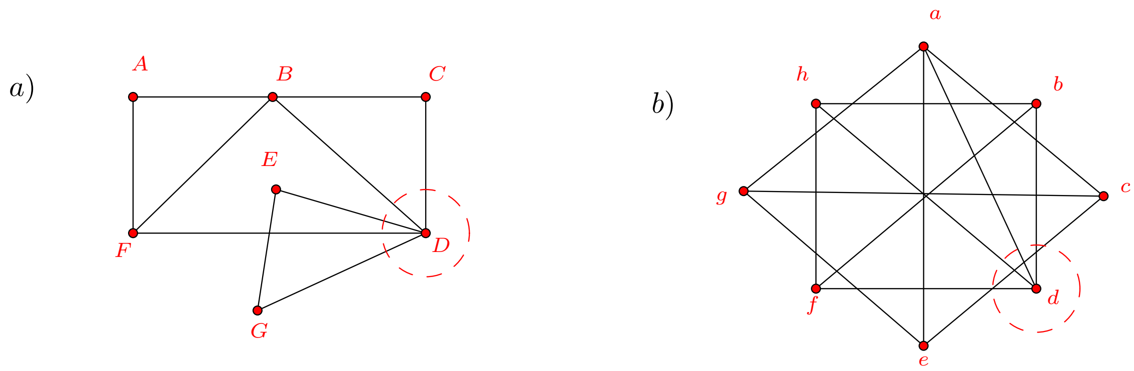
$$\begin{cases} \frac{n}{2}\left(\frac{n-2}{2}\right), & \text{for } n \text{ even.} \\ \left(\frac{n-2}{2}\right)^2, & \text{for } n \text{ odd.} \end{cases}$$

The maximum possible size of G is

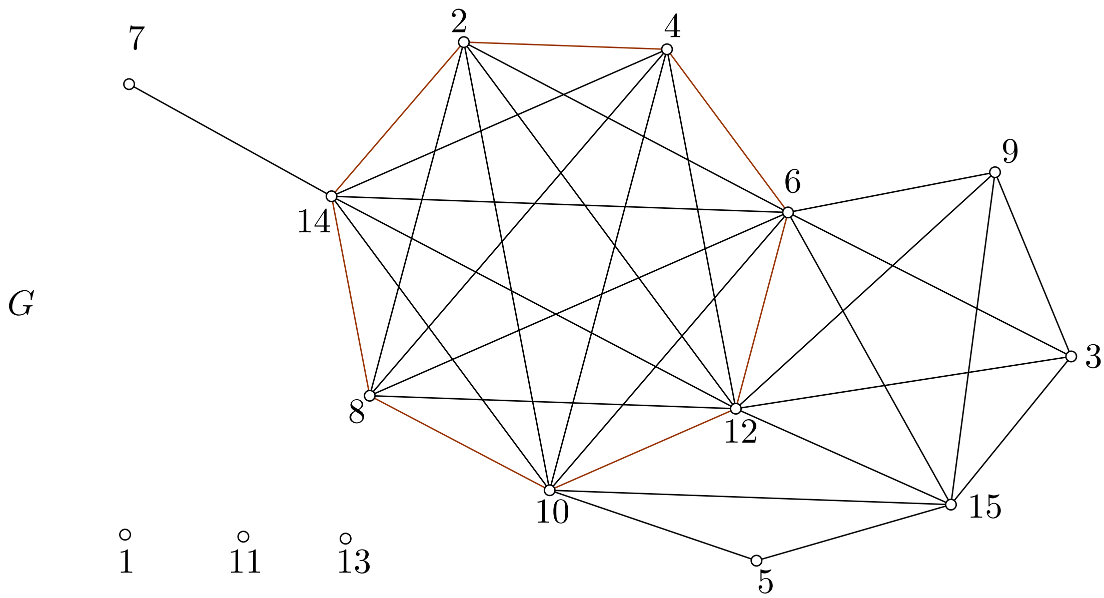
$$\frac{(n-2)(n-1)}{2}$$

[6] Peter Gustav Lejeune Dirichlet (1805 – 1859).

12. Removing just one vertex d results disconnection in both graph (a) and (b).



13. (a)



(b) The longest path in G is of length 11.

14. diam(G) to say diameter of G

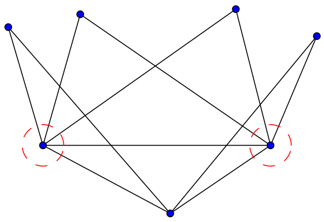
(1) $\text{diam}(\text{peterson}) = 2$ (4) $\text{diam}(K_{n,m}) = \begin{cases} 1, & \text{for } n = m = 1 \\ 2, & \text{otherwise.} \end{cases}$

(2) $\text{diam}(Q_n) = n$ (5) $\text{diam}(C_n) = \left\lfloor \dfrac{n}{2} \right\rfloor$

(3) $\text{diam}(K_n) = 1$ (6) $\text{diam}(P_n) = n - 1$

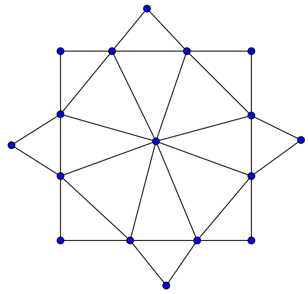
15. None of them are Hamiltonian. None of them are Eulerian except (iii).

i)



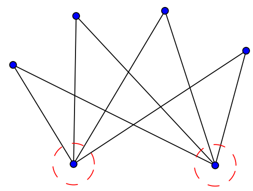
- Not Eulerian
- Not Hamiltonian

ii)



- Not Eulerian
- Not Hamiltonian

iii)



- Eulerian
- Not Hamiltonian

16. Let G be a regular graph with even order and odd size.

$$|G| = m\text{-even.}$$
$$\|G\| = n\text{-odd.}$$

Since G is regular the degree of each vertex is the same, say k.
WTS:- k is odd, hence non Eulerian follows.
Now

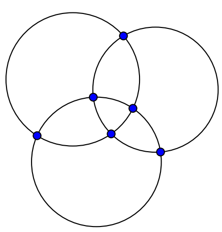
$$\sum_{v \in V(G)} \deg(v) = k \cdot \underbrace{m}_{even} = 2\|G\| = 2 \cdot \underbrace{n}_{odd}$$

$$\Rightarrow k\left(\frac{m}{2}\right) = \underbrace{n}_{odd} \tag{2}$$

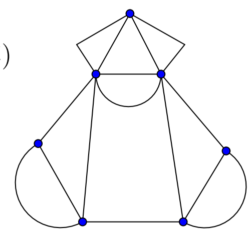
Since m is even, $\frac{m}{2} \in \mathbb{Z}^+$, and we know that $k \in \mathbb{Z}^+$.
From (2), we have the product of two integers resulting an odd number n.
Thus both k and $\frac{m}{2}$ must be odd. Hence the degree of each vertex in G is odd.
Therefore, G is not Eulerian.$\square$

17. Let's put vertices on the drawings and imagine as they are graphs.

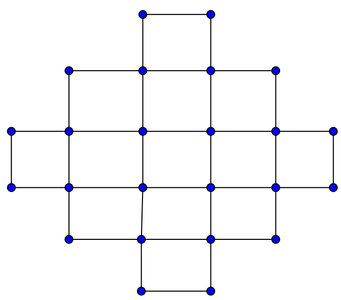
i)



ii)



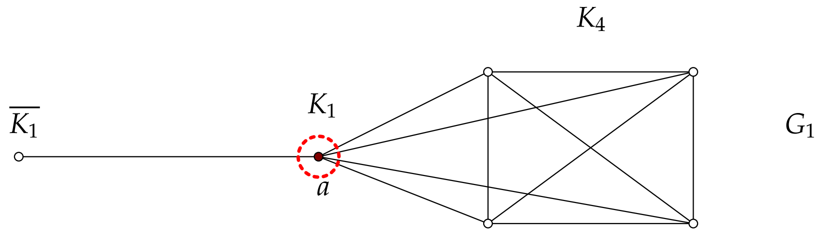
iii)



There is an **Euler tour** in (i) and (iii) since every vertex is of even degree.
Thus, the drawings in (i) and (iii) can be drawn without lifting one's pen from the paper and without repeating any line. (ii) also can be drawn in the same manner since there is an **Euler trail** that starts at a and ends at b(vise versa).

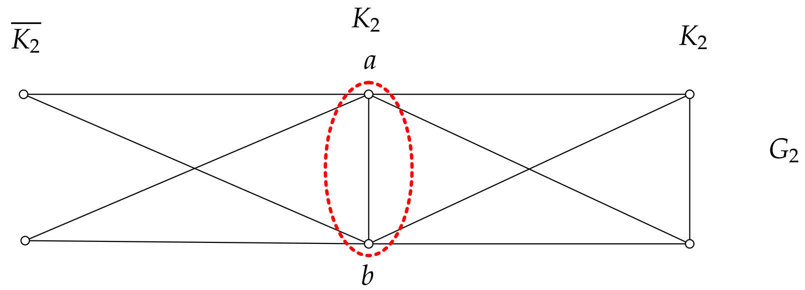
18. (a) Since $n = 6$ the values of m are 1 and 2.
For $m = 1$

$C_{1,6} = K_1 \vee (\overline{K_1} + K_4)$



For $m = 2$,

$C_{2,6} = K_2 \vee (\overline{K_2} + K_2)$



(b) The above G_1 and G_2 can't be Hamiltonian. Because removing the indicated vertices in red will result the number of component that exceed the number of vertices, which we've just removed. i.e. For G_1, take $S = \{a\}$

$$\Rightarrow |\omega(G_1 - S)| = 2 \geq |S| = 1$$

For G_2, take $S = \{a, b\}$

$$\Rightarrow |\omega(G_2 - S)| = 3 \geq 2 = |S|$$

(c) The degree sequence of G_1 is; $1, 4, 4, 4, 4, 5$ and
The degree sequence of G_2 is; $5, 5, 3, 3, 2, 2$
(d) NO! It's impossible(Because of Chvatal's Theorem).

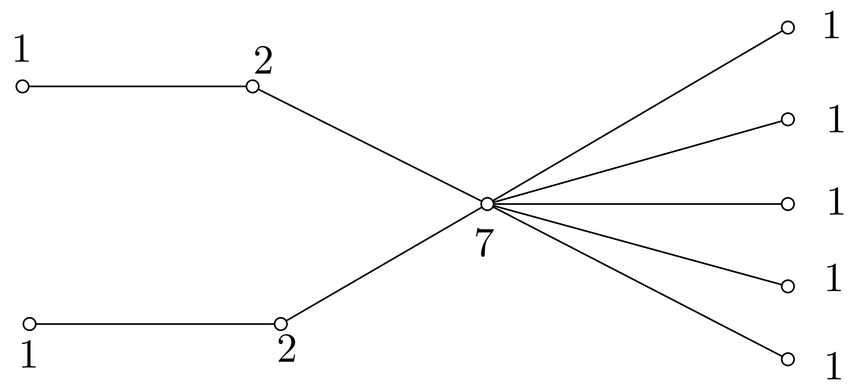
19. The minimum number of edges in a simple graph of order $n \geq 2$ that guarantee Hamiltonian is

$$\binom{n-1}{2} + 2$$

20. $\underbrace{n-2, n-2, \ldots, n-2}_{(n-2)\text{ times}}, n-1, 1$ is the degree of non-Hamiltonian simple graph with n vertices and $\binom{n-1}{2} + 1$ edges.

21. $4, 3, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1$ is the degree sequence of the given tree.

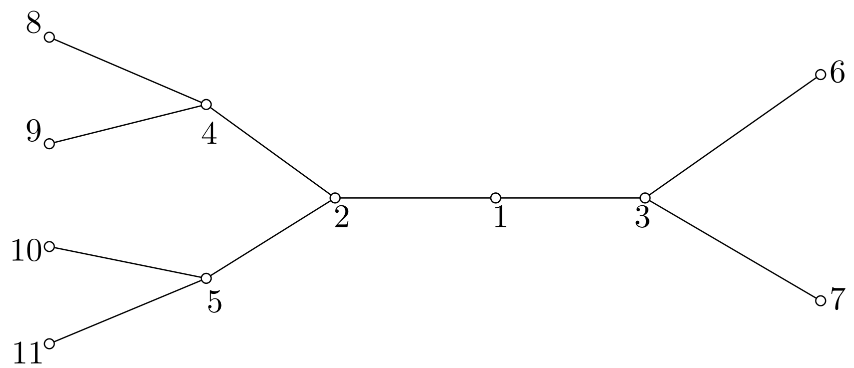
22. The sequence $7, 2, 2, 1, 1, 1, 1, 1, 1, 1$ is the degree sequence of tree because we have found one. Look at the tree below



23. From Cayley's Theorem, we have $\tau(K_n) = n^{n-2}$

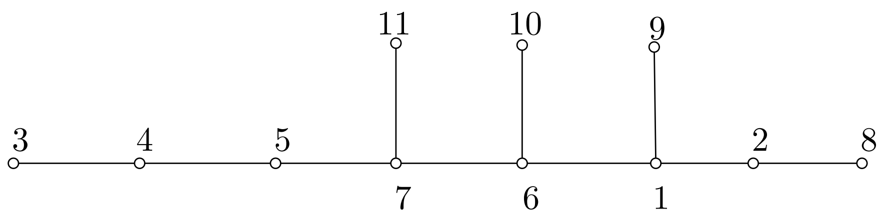
$$\tau(K_{10}) = 10^8$$

24. (a) $V = \{1, 2, 3, \dots, 11\}$, $E = \{12, 13, 24, 25, 36, 37, 48, 49, 510, 511\}$



$3, 3, 1, 2, 4, 4, 2, 5, 5$ is the corresponding prüfer sequence.

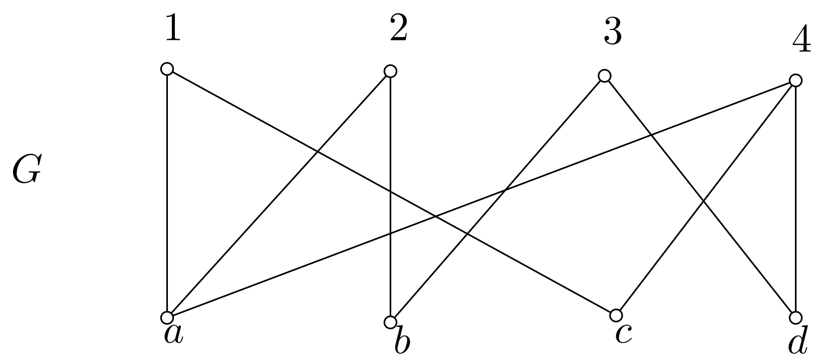
(b) We have assumed $V = \{1, 2, \dots, 11\}$
For the prüfer code $4, 5, 7, 2, 1, 1, 6, 6, 7$, the corresponding tree is



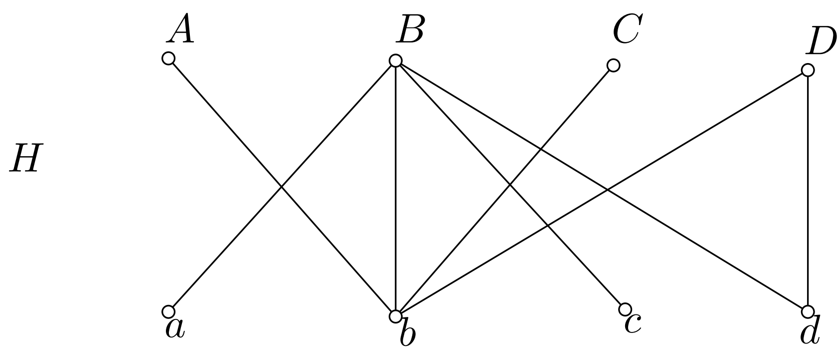
25. $m - 1$ edges.

26. (a) There has to be two vertices of degree one in a tree of order $n \geq 2$. (b) The maximum possible number of pendant vertices in a tree of order n is $n-1$.

27. (a) $M = \{1a, 2b, 3d, 4c\}$



M saturates G(i.e. M is a perfect matching).
(b) $M = \{Ba, Cb, Dd\}$



M is a maximum matching. Here, there is no perfect matching because Hall's condition is not satisfied. Take $S = \{a, c, d\}$, $N(S) = \{B, D\}$

$$\Rightarrow |N(s)| = 2 \geq 3 = |S|, \qquad \rightarrow\leftarrow$$

28. It is a matter of finding a perfect matching in G. But Hall's Theorem tells us it is impossible to find such matching. Take $S = \{A, B, D\}$. Thus $N(S) = \{c, d\}$

$$\Rightarrow |N(S)| = 2, |S| = 3 \Rightarrow |N(S)| < |S|$$

Therefore, there is no job for each of the applicants.

29. Given G is 4-regular planar graph of order n and $f = 10$.
From Euler formula we have

$$V + f - e = 2 \tag{3}$$

But since G is 4-regular,

$$\sum_{v \in V(G)} \deg(v) = 4V = 2e$$

This implies

$$e = 2V \tag{4}$$

Now, substitute (4) in (3)

$$V + f - 2V = 2$$

After simplification

$$V = f - 2$$

But $f = 10$ is given. Hence

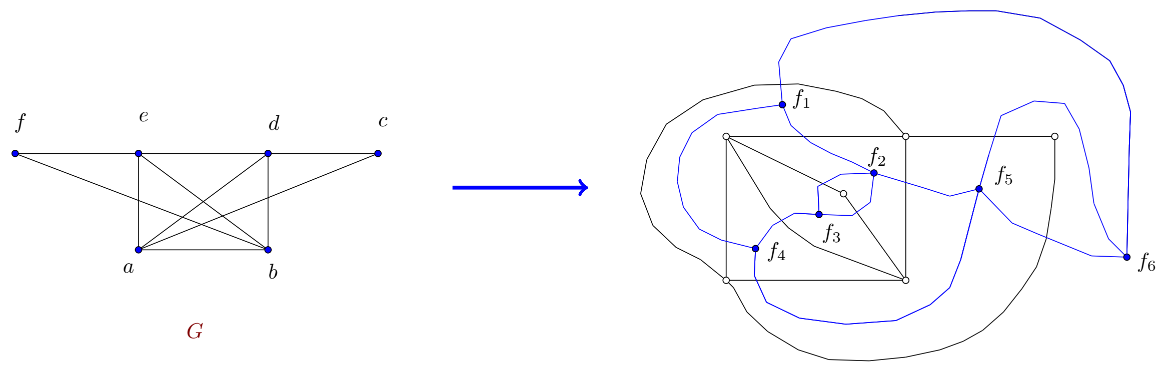
$$V = 8$$

Therefore, the order of G is equal to 8.

30. K_n is planar for $n \leq 4$.

31. For $m = 1, 2$ and $\forall n \in \mathbb{N}$, $K_{m,n}$ is planar.

32. The graph in blue is the dual of G.



The number of edges in G, $|E| = 10$. Now

$$\begin{aligned}\sum_{i=1}^{6} \deg(f_i) &= 3 + 4 + 3 + 3 + 4 + 3 \\ &= 20 \\ &= 2(10) = 2|E|\end{aligned}$$

Therefore,

$$\sum_{\text{all faces}} \deg(\text{face}) = 2|E|$$