The Logarithmic Constant e is Irrational

Miliyon T.
Addis Ababa University
Department of Mathematics

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1 Characterization of the number e.

1.1 Using limit

Proposition 1.1. *The limit*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \tag{1}$$

converges to e.

Proof. The Limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

is indeterminate form of the type 1^{∞}. So let's change it into the standard indeterminate form i.e. ($\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$), so that we could use L'Hopital's rule to evaluate the limit.
Now,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to\infty}e^{\ln(1+\frac{1}{x})^{x}}=\lim_{x\to\infty}e^{x\ln(1+\frac{1}{x})}=\lim_{x\to\infty}e^{\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}=e^{\lim\limits_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}}$$

Now the limit is in $\frac{0}{0}$ form. Thus we can apply L'Hopital's rule

$$\begin{aligned}
e^{\lim\limits_{x\to\infty}\frac{(\ln(1+\frac{1}{x}))'}{(\frac{1}{x})'}} &= e^{\lim\limits_{x\to\infty}\frac{\frac{1}{(1+\frac{1}{x})}\cdot(\frac{1}{x})'}{(\frac{1}{x})'}}\\
&= e^{\lim\limits_{x\to\infty}\frac{1}{(1+\frac{1}{x})}}\\
&= e^{\frac{1}{\left(1+\lim\limits_{x\to\infty}\frac{1}{x}\right)}}\\
&= e
\end{aligned}$$

$\square$

1.2 Using power series

Proposition 1.2. *The infinite series*

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \tag{2}$$

converges to the number e.

Proof. The power series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{3}$$

At $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{4}$$

$\square$

2 Irrationality of e

Theorem 2.1. *The number* e *is irrational.*

Proof. We now that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Denote the partial sum

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Clearly, the following inequality is true for any $n = 1, 2, 3, 4, \ldots$

$$\begin{aligned} 0 < e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots && (5) \\ &= \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+2)} + \frac{1}{((n+3)(n+2)} + \cdots\right] && (6) \end{aligned}$$

Now replace every $n + k$ in the denominator by $n + 1$ since $k > 1$ we've $n + k > n + 1$. This implies us

$$\frac{1}{(n+k)} < \frac{1}{(n+1)}$$

Thus,

$$\begin{aligned}
\frac{1}{(n+1)!}\left[1+\frac{1}{(n+2)}+\frac{1}{(n+3)(n+2)}+\cdots\right] &< \frac{1}{(n+1)!}\left[1+\frac{1}{(n+1)}+\frac{1}{(n+1)(n+1)}+\cdots\right] \\
&= \frac{1}{(n+1)!}\left[1+\frac{1}{(n+1)}+\frac{1}{(n+1)^2}+\cdots\right] \\
&= \frac{1}{(n+1)!}\left[\frac{n+1}{n}\right] \qquad \text{(Geometric sum)} \\
&= \frac{1}{n!n}
\end{aligned}$$

Thus we have

$$\frac{1}{(n+1)!}\left[1+\frac{1}{(n+2)}+\frac{1}{(n+3)(n+2)}+\cdots\right] < \frac{1}{n!n} \tag{7}$$

Now use (7) and (6) to obtain

$$0 < n!(e-s_n) < \frac{1}{n}, \qquad \text{for } n = 1, 2, 3, \ldots \tag{8}$$

Assume e is rational i.e. $e = \frac{p}{q}$. Where p and q are relatively prime (the fraction is in lowest term).

Now choose any $n > q$. Which implies $n > 1$ so is $\frac{1}{n} < 1$. Thus, (8) becomes

$$0 < n!(e-s_n) < \frac{1}{n} < 1$$

This shows that $n!(e-s_n)$ is not an integer. But

$$\begin{aligned}
n!(e-s_n) &= n!\left(\frac{p}{q}-s_n\right) \\
&= n!\left(\frac{p}{q}\right) - n!(s_n) \\
&= \underbrace{\left(\frac{n!}{q}\right)}_{\text{integer}} p - \underbrace{n!\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)}_{\text{integer}} \qquad (q|n! \because n > q.) \\
&= \text{integer} - \text{integer} = \text{integer}.
\end{aligned}$$

Meaning $n!(e-s_n)$ is an integer which is a contradiction! Hence our assumption e is rational is wrong.

Therefore, e is irrational. $\square$

References

[1] [Robert Ellis] Calculus with Analytic Geometry.

[2] [Tom Apostol] Mathematical Analysis, 2nd ed.

[3] [Walter Rudin] Principles of Mathematical Analysis.