Applications of matrices

Miliyon T.

October 7, 2013

1 Area using matrices

Theorem 1.1. *Given a triangle with vertices* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3), *then the area of the given triangle is given by*

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Proof. Consider the following figure

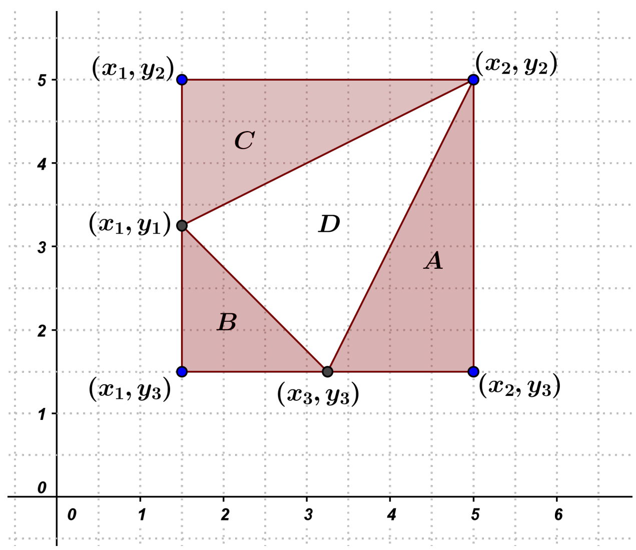

Figure 1: A triangle on a cartesian coordinate

Our goal is to find the area of $\triangle D(A_{\triangle D})$. But we can accomplish this just by subtracting the area of $\triangle A + \triangle B + \triangle C$ from the area of the $\square$.
Now, the area of $\square$ is

$$\begin{aligned} A_{\square} &= (x_2 - x_1)(y_2 - y_3) \\ &= x_2y_2 - x_2y_3 - x_1y_2 + x_1y_3 \end{aligned}$$

The area of $\triangle A$ is given by

$$\begin{aligned} A_{\triangle A} &= \frac{1}{2}(x_2 - x_3)(y_2 - y_3) \\ &= \frac{1}{2}(x_2y_2 - x_2y_3 - x_3y_2 + x_3y_3) \end{aligned}$$

And area of $\triangle B$ is given by

$$\begin{aligned} A_{\triangle B} &= \frac{1}{2}(x_3 - x_1)(y_1 - y_3) \\ &= \frac{1}{2}(x_3y_1 - x_3y_3 - x_1y_1 + x_1y_3) \end{aligned}$$

Finally, the area of $\triangle C$ is given by

$$\begin{aligned} A_{\triangle C} &= \frac{1}{2}(x_2 - x_1)(y_2 - y_1) \\ &= \frac{1}{2}(x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1) \end{aligned}$$

Now,

$$\begin{aligned} A_{\triangle A} + A_{\triangle B} + A_{\triangle C} &= \begin{cases} \frac{1}{2}(x_2y_2 - x_2y_3 - x_3y_2 + \cancel{x_3y_3}) \\ +\frac{1}{2}(x_3y_1 - \cancel{x_3y_3} - \cancel{x_1y_1} + x_1y_3) \\ +\frac{1}{2}(x_2y_2 - x_2y_1 - x_1y_2 + \cancel{x_1y_1}) \end{cases} \\ &= x_2y_2 + \frac{1}{2}(-x_2y_3 - x_3y_2 + x_3y_1 + x_1y_3 - x_2y_1 - x_1y_2) \end{aligned}$$

But we know that

$$A_{\triangle D} = A_{\square} - (A_{\triangle A} + A_{\triangle B} + A_{\triangle C})$$

Thus

$$\begin{aligned} A_{\triangle D} &= (\cancel{x_2y_2} - x_2y_3 - x_1y_2 + x_1y_3) - \Big(\cancel{x_2y_2} + \frac{1}{2}(-x_2y_3 - x_3y_2 + x_3y_1 + x_1y_3 - x_2y_1 - x_1y_2)\Big) \\ &\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(some algebraic simplification)} \\ &= \frac{1}{2}\Big((x_3y_2 - x_2y_3) - (x_3y_1 - x_1y_3) + (x_2y_1 - x_1y_2)\Big) \\ &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \end{aligned}$$

□

Corollary 1.2. *If* $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, *then the points* $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) *lies on the same line that means they are collinear.*