Trace of a Matrix

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Definition

- ***Trace:*** is the sum of the diagonals of the matrix. Let A be a matrix of order n

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

  then the trace of A is given by

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

  Generally,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

  Basic properties of trace

  1. tr(A+B)=tr(A)+tr(B)
  2. tr(A+B)
  3. tr(AB)

- ***Similar Matrix***: The matrices A and B $\in M_n(F)$ are Similar if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

Lemma 0.1. *Trace of a matrix is commutative*

$$tr(AB) = tr(BA)$$

Proof.

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{ki} \tag{1}$$

$$\operatorname{tr}(BA) = \sum_{i=1}^{n}\sum_{k=1}^{n} B_{ik}A_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ki}B_{ik} \tag{2}$$

Changing the index in (2) from k to i completes the proof.

□

Theorem 0.2 (Similar trace Theorem). *Similar matrices have the same trace.*

Proof. Let A and B be Similar matrices by definition,

$$B = P^{-1}AP$$

$$\begin{aligned}
\operatorname{tr}(B) &= \operatorname{tr}(P^{-1}AP) \\
&= \operatorname{tr}(P^{-1}PA) \qquad \textit{from the lemma above} \\
&= \operatorname{tr}(A) \qquad \because\; P^{-1}P = I_n = 1 \\
\therefore \operatorname{tr}(B) &= \operatorname{tr}(A)
\end{aligned}$$

□

Corollary 0.3. *If A and B are similar matrices, then $A_iB^i = B_iA^i$. Where $A^i = i^{th}$ column of A and $A_i = i^{th}$ row of A.*

References

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