Eigenvalues and Eigenvectors

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1 Introduction

Definition 1.1. Assume that $L : \mathbb{R}^k \to \mathbb{R}^k$ is a linear operator. If the vector $v \in \mathbb{R}^k$ and the scalar $\lambda \in \mathbb{R}$ satisfy $Lv = \lambda v$, then v is called an eigenvector of L. The scalar λ is called an eigenvalue of L.

Clearly, zero vector is always an eigen-vector. Also, if u is an eigenvector, then κu is also an eigenvector for every $\kappa \in \mathbb{R}$. Indeed, assuming that λ is the eigenvalue corresponding to u we have $A(\kappa u) = \kappa A(u) = \kappa \lambda u = \lambda \kappa u$.

Example 1.2. Find the eigenvalues and the eigenvectors of the operator with the matrix

$$A = \begin{bmatrix} 5 & 4 \\ -4 & -5 \end{bmatrix}.$$

Hide solution

We are looking for a vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$ and a scalar $\lambda \in \mathbb{R}$ such that $Au = \lambda u$. The last equation is equivalent to the system:

$$\begin{aligned} 5x + 4y &= \lambda x \\ -4x - 5y &= \lambda y. \end{aligned}$$

The previous system is equivalent to:

$$\begin{aligned} (5-\lambda)x + 4y &= 0 \\ -4x + (-5-\lambda)y &= 0. \end{aligned}$$

If we multiply the first equation by $5+\lambda$, the second equation by 4 and add the obtained two equations we deduce: $(25 - \lambda^2 - 16)x = 0$. If $9 - \lambda^2 \neq 0$, then we must have $x = 0$. This would imply that $y = 0$ which would lead to a trivial eigenvector. Assume that $9 - \lambda^2 = 0$. Then we have $\lambda \in \{-3, 3\}$, and these are the two eigenvalues.

The eigenvalue $\lambda_1 = 3$ corresponds to the eigenvector whose coordinates (x, y) satisfy the system:

$$\begin{aligned} 2x + 4y &= 0 \\ -4x - 8y &= 0. \end{aligned}$$

A non-trivial solution $(-2, 1)$ corresponds to the eigenvector $u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Similarly, the eigenvalue $\lambda_2 = -3$ gives us the system:

$$\begin{aligned} 8x + 4y &= 0 \\ -4x - 2y &= 0. \end{aligned}$$

This gives us another eigenvector $u_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

In the previous example, we found eigenvalues as the zeroes of the polynomial $\varphi_A(\lambda) = \lambda^2 - 9$. This is called the characteristic polynomial of the matrix A. More precisely,

Definition 1.3. Let A be an $n \times n$ matrix. The polynomial $\varphi_A(\lambda) = \det\ (A - \lambda I)$ is called the characteristic polynomial of the matrix A.

The proof of the following theorem is obvious once we have seen the solution of Example 1.

Theorem 1.4. *Assume that A is an $n \times n$ matrix. A real number η is an eigenvalue of A if and only if $\varphi_A(\eta) = 0$.*

Hide proof The real number η is an eigenvalue of A if and only if the equation $Av = \eta v$ has at least one non-trivial solution $v \in \mathbb{R}^n$. The last vector equation is equivalent to $(A - \eta I)v = 0$, which can be understood as a system of n equation with n variables. This system has a non-trivial solution if and only if the matrix $A - \eta I$ is non-invertible.

Polynomials with matrices

We will use the eigenvectors and eigenvalues to find closed formulas for n-th powers of matrices. We will illustrate the method by considering the following example.

Example 1.5. Let $A = \begin{bmatrix} 5 & 4 \\ -4 & -5 \end{bmatrix}$. Let us denote by a_n, b_n, c_n, and d_n the numbers such that $A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$. Find the formulas for a_n, b_n, c_n, and d_n.

Hide solution

The operator A^n is uniquely determined by the vectors $A^n v$ and $A^n w$ for any two vectors $v, w \in \mathbb{R}^2$ that form a basis for $\mathbb{R}^2$.

Assume that there are two eigenvectors of A that form a basis for $\mathbb{R}^2$. Assume that $v_1, v_2 \in \mathbb{R}^2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Then we have

$$A^n v_1 = A^{n-1}(Av_1) = A^{n-1}(\lambda_1 v_1) = \lambda_1 A^{n-1} v_1 = \lambda_1 A^{n-2}(Av_1)) = \lambda_1^2 A^{n-2} v_1 = \cdots = \lambda_1^n v_1,$$

and similarly $A^n v_2 = \lambda_2^n v_2$.

From Example 1 we know that $u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are the eigenvectors of A and that their corresponding eigenvalues are 3 and -3. Therefore $A^n \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3^n \\ 3^n \end{bmatrix}$ and $A^n \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \cdot (-3)^n \\ 2 \cdot (-3)^n \end{bmatrix}$.

It remains to find the matrix of A^n. In order to do that we need to find $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Calculating $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We need to find scalars α_1 and α_2 such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_1 u_1 + \alpha_2 u_2$. This gives us the system of equations:

$$\begin{aligned} -2\alpha_1 - \alpha_2 &= 1 \\ \alpha_1 + 2\alpha_2 &= 0. \end{aligned}$$

Multiplying the second equation by 2 and adding it to the first implies $\alpha_2 = \frac{1}{3}$. Substituting this value into the second equation yields $\alpha_1 = -\frac{2}{3}$. Thus

$$A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{2}{3}A^n u_1 + \frac{1}{3}A^n u_2 = \begin{bmatrix} \frac{2}{3} \cdot 2 \cdot 3^n + \frac{1}{3} \cdot (-(-3)^n) \\ -\frac{2}{3} \cdot 3^n + \frac{1}{3} \cdot 2 \cdot (-3)^n \end{bmatrix} = \begin{bmatrix} 4 \cdot 3^{n-1} + (-3)^{n-1} \\ -2 \cdot 3^{n-1} - 2 \cdot (-3)^{n-1} \end{bmatrix}.$$

Calculating $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We need to find scalars β_1 and β_2 for which $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \beta_1 u_1 + \beta_2 u_2$. The last equation is equivalent to the following system:

$$\begin{aligned} -2\beta_1 - \beta_2 &= 0 \\ \beta_1 + 2\beta_2 &= 1. \end{aligned}$$

If we multiply the second equation by 2 and add it to the first we obtain $\beta_2 = \frac{2}{3}$. Substituting this value in the first equation yields $\beta_1 = -\frac{1}{3}$. Therefore

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{3}A^n u_1 + \frac{2}{3}A^n u_2 = \begin{bmatrix} \frac{1}{3} \cdot 2 \cdot 3^n + \frac{2}{3} \cdot (-(-3)^n) \\ -\frac{1}{3} \cdot 3^n + \frac{4}{3} \cdot (-3)^n \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^{n-1} + 2 \cdot (-3)^{n-1} \\ -3^{n-1} - 4 \cdot (-3)^{n-1} \end{bmatrix}.$$

Thus

$$A^n = \begin{bmatrix} 4 \cdot 3^{n-1} + (-3)^{n-1} & 2 \cdot 3^{n-1} + 2 \cdot (-3)^{n-1} \\ -2 \cdot 3^{n-1} - 2 \cdot (-3)^{n-1} & -3^{n-1} - 4 \cdot (-3)^{n-1} \end{bmatrix},$$

or, equivalently:

$$\begin{aligned} a_n &= 4 \cdot 3^{n-1} + (-3)^{n-1} \\ b_n &= 2 \cdot 3^{n-1} + 2 \cdot (-3)^{n-1} \\ c_n &= -2 \cdot 3^{n-1} - 2 \cdot (-3)^{n-1} \\ d_n &= -3^{n-1} - 4 \cdot (-3)^{n-1}. \end{aligned}$$

Theorem 1.6. *Assume that A is an $n \times n$ matrix that has n linearly independent eigenvectors v_1, $\dots$, v_n. Assume that λ_1, $\dots$, λ_n are eigenvalues corresponding to v_1, $\dots$, v_n. Then there exists an invertible $n \times n$ matrix P such that*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

Proof. Assume that $v_i = \begin{bmatrix} v_{1i} \\ v_{2i} \\ \vdots \\ v_{ni} \end{bmatrix}$, and let $P = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n} \\ & & & \vdots & \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nn} \end{bmatrix}$.

Let e_1, $\dots$, e_n be the standard basis of $\mathbb{R}^n$. In other words, the vector e_i has 1 at the i-th position and 0 at every other position. Then we have $Pe_i = v_i$ for each $i \in \{1, 2, \dots, n\}$. Thus

$APe_i = Av_i = \lambda_i v_i$ and $P^{-1}APe_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1} v_i = \lambda_i e_i$. Thus

$$P^{-1}AP(e_i) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The i-th row of the matrix of the operator $P^{-1}AP$ must be equal to $P^{-1}AP(e_i)$, hence the matrix of the operator $P^{-1}AP$ must be $\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$. $\square$

Theorem 1.7 (Cayley-Hamilton)**.** *Assume that A is an $n \times n$ matrix and φ_A its characteristic polynomial. Then $\varphi_A(A) = 0$.*

Proof. We will prove this theorem under the assumption that A has n linearly independent eigenvectors. The proof of the general case follows the same idea but requires the theory of Jordan canonical forms. In Example 5 at the end of this article we will see some techniques involving Jordan canonical forms.

According to Theorem 2 there is a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. Then we have $A = PDP^{-1}$ and $A^k = PD^kP^{-1}$ for each $k \in \mathbb{N}$. Thus $\varphi_A(A) = P\varphi_A(D)P^{-1}$. However, $\varphi_A(D) = 0$, since the entries of D are the eigenvalues of A, and each of them is a zero of the characteristic polynomial according to Theorem 1. This completes the proof. $\square$

2 Recursive systems of equations

Our next goal is to use the techniques of eigenvalues and eigenvectors to solve the recursive systems of equations.

Example 2.1. Assume that $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are two sequence of real numbers defined in the following way: $x_0 = 3$, $y_0 = 2$, and

$$\begin{aligned} x_{n+1} &= 5x_n + 4y_n \\ y_{n+1} &= -4x_n - 5y_n, \end{aligned}$$

for $n \geq 0$. Determine the formulas for x_n and y_n.

Hide solution

Let $A = \begin{bmatrix} 5 & 4 \\ -4 & -5 \end{bmatrix}$. Then we have for $n \geq 0$: $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$. Therefore:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A^2 \begin{bmatrix} x_{n-2} \\ y_{n-2} \end{bmatrix} = \cdots = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

In Example 2 we showed that $A^n = \begin{bmatrix} 4 \cdot 3^{n-1} + (-3)^{n-1} & 2 \cdot 3^{n-1} + 2 \cdot (-3)^{n-1} \\ -2 \cdot 3^{n-1} - 2 \cdot (-3)^{n-1} & -3^{n-1} - 4 \cdot (-3)^{n-1} \end{bmatrix}$, hence:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 4 \cdot 3^{n-1} + (-3)^{n-1} & 2 \cdot 3^{n-1} + 2 \cdot (-3)^{n-1} \\ -2 \cdot 3^{n-1} - 2 \cdot (-3)^{n-1} & -3^{n-1} - 4 \cdot (-3)^{n-1} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3^n - (-3)^n + 4 \cdot 3^{n-1} + 4 \cdot (-3)^{n-1} \\ -2 \cdot 3^n + 2 \cdot (-3)^n - 2 \cdot 3^{n-1} - 8 \cdot (-3)^{n-} \end{bmatrix}$$

Thus:

$$\begin{aligned} x_n &= 4 \cdot 3^n - (-3)^n + 4 \cdot 3^{n-1} + 4 \cdot (-3)^{n-1} = 16 \cdot 3^{n-1} + 7 \cdot (-3)^{n-1} \\ y_n &= -2 \cdot 3^n + 2 \cdot (-3)^n - 2 \cdot 3^{n-1} - 8 \cdot (-3)^{n-1} = -8 \cdot 3^{n-1} - 14 \cdot (-3)^{n-1}. \end{aligned}$$

Using the technique described above we can solve the recursive equations. The following example provides the formula for Fibonacci numbers.

Example 2.2 (Fibonacci numbers)**.** Assume that $(F_n)_{n=0}^{\infty}$ is the sequence defined as $F_0 = 0$, $F_1 = 1$ and for $n \geq 0$ the following equation holds:

$$F_{n+2} = F_{n+1} + F_n.$$

Prove that

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

Hide solution

Let us denote $G_n = F_{n+1}$. Then we have $G_0 = 1$, $F_0 = 0$ and for $n \geq 0$ the following holds:

$$\begin{aligned} F_{n+1} &= \qquad G_n \\ G_{n+1} &= F_n + G_n, \end{aligned}$$

or, equivalently $\begin{bmatrix} F_{n+1} \\ G_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ G_n \end{bmatrix}$. Let us denote $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Similarly to the argument presented in Example 3 we get $\begin{bmatrix} F_n \\ G_n \end{bmatrix} = A^n \begin{bmatrix} F_0 \\ G_0 \end{bmatrix}$.

Therefore, our goal is to find A^n. In order to do so, we will first find eigenvalues and eigenvectors of the matrix A. The eigenvalues are the solutions of the equation $\varphi_A(x) = 0$, where $\varphi_A(x) = \det(A - xI) = -x(1-x) - 1 = x^2 - x - 1$. Since $\varphi_A(x) = \left(x - \frac{1-\sqrt{5}}{2}\right) \cdot \left(x - \frac{1+\sqrt{5}}{2}\right)$, we get that the eigenvalues are $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

We will now find an eigenvector corresponding to the eigenvalue λ_1. Let us denote by $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ the required eigenvector. Then we have $-\lambda_1 u_1 + u_2 = 0$ and $u_1 + (1 - \lambda_1) u_2 = 0$ and we may take $u = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$. Similarly, we find $v = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$. We have $A^n u = \lambda_1^n u$ and $A^n v = \lambda_2^n v$. In order to determine the matrix A^n it remains to find the vectors $A^n e_1$ and $A^n e_2$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the elements of the standard basis of $\mathbb{R}^2$. We will first express e_1 in terms of u and v. In order to do so we need to find scalars α_1 and α_2 such that $e_1 = \alpha_1 u + \alpha_2 v$. This gives us the system:

$$\begin{aligned} 1 &= \alpha_1 + \alpha_2 \\ 0 &= \alpha_1 \cdot \frac{1 - \sqrt{5}}{2} + \alpha_2 \cdot \frac{1 + \sqrt{5}}{2}. \end{aligned}$$

Solving the system gives us $(\alpha_1,\alpha_2)=\left(\frac{1+\sqrt{5}}{2\sqrt{5}},\frac{\sqrt{5}-1}{2\sqrt{5}}\right)$. We now obtain:

$$\begin{aligned} A^n e_1 &= A^n\left(\alpha_1 u+\alpha_2 v\right)=\frac{1+\sqrt{5}}{2\sqrt{5}}\lambda_1^n u+\frac{\sqrt{5}-1}{2\sqrt{5}}\lambda_2^n v \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{(1+\sqrt{5})\lambda_1^n+(\sqrt{5}-1)\lambda_2^n}{2} \\ \frac{(1+\sqrt{5})\lambda_1^{n+1}+(\sqrt{5}-1)\lambda_2^{n+1}}{2}\end{bmatrix}. \end{aligned}$$

In order to find $A^n e_2$ we first need to find the scalars β_1 and β_2 such that $e_2=\beta_1 u+\beta_2 v$. The last equation becomes:

$$\begin{aligned} 0 &= \beta_1+\beta_2 \\ 1 &= \beta_1\cdot\frac{1-\sqrt{5}}{2}+\beta_2\cdot\frac{1+\sqrt{5}}{2}. \end{aligned}$$

After solving the system we obtain $(\beta_1,\beta_2)=\left(-\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$. Then we have

$$\begin{aligned} A^n e_2 &= A^n\left(\beta_1 u+\beta_2 v\right)=-\frac{1}{\sqrt{5}}\lambda_1^n u+\frac{1}{\sqrt{5}}\lambda_2^n v \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_2^n-\lambda_1^n \\ \lambda_2^{n+2}-\lambda_1^{n+1}\end{bmatrix}. \end{aligned}$$

Therefore

$$A^n = \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{(1+\sqrt{5})\lambda_1^n+(\sqrt{5}-1)\lambda_2^n}{2} & \lambda_2^n-\lambda_1^n \\ \frac{(1+\sqrt{5})\lambda_1^{n+1}+(\sqrt{5}-1)\lambda_2^{n+1}}{2} & \lambda_2^{n+2}-\lambda_1^{n+1}\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} F_n \\ G_n\end{bmatrix}=A^n\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_2^n-\lambda_1^n \\ \lambda_2^{n+1}-\lambda_1^{n+1}\end{bmatrix}.$$

Thus

$$F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

In the next example we treat the recursive system of equations whose matrix does not have a basis of eigenvectors. This is an introductory example to Jordan forms of matrices.

Example 2.3. Consider the matrix $A=\begin{bmatrix}4 & 1\\ -1 & 2\end{bmatrix}$ and the following system of equations:

$$\begin{aligned} x_{n+1} &= 4x_n+y_n \\ y_{n+1} &= -x_n+2y_n, \end{aligned}$$

with the initial conditions $x_0=2$, $y_0=5$.

(a) Prove that A has only one eigenvalue λ and determine λ.

(b) Find an eigenvector u corresponding to λ.

(c) Does there exist an eigenvector w of A such that u and w are not scalar multiples of each other?

(d) Find a vector v such that $Av=\lambda v+u$. Here u and λ are the eigenvector and the eigenvalue from the previous parts of the problem.

(e) Determine the matrix A^n.

(f) Find the closed formulas for x_n and y_n.

Hide solution

(a) The characteristic polynomial of A is $\varphi_A(x) = (4-x)(2-x)+1 = x^2-6x+9 = (x-3)^2$, hence $\lambda = 3$ is the only eigenvalue of A.

(b) We need to find $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ such that $Au = 3u$. This gives us the following system of equations:

$$\begin{aligned} u_1 + u_2 &= 0 \\ -u_1 - u_2 &= 0, \end{aligned}$$

and each eigenvector of A must satisfy $u = \begin{bmatrix} t \\ -t \end{bmatrix}$ for $t \in \mathbb{R}$. We can take $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Such w does not exist because each eigenvector w of A is of the form $w = t \begin{bmatrix} 1 \\ -1 \end{bmatrix} = tu$.

(d) Let us denote $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. We will find v_1 and v_2. The numbers v_1 and v_2 must satisfy the following system of equations:

$$\begin{aligned} v_1 + v_2 &= 1 \\ -v_1 - v_u &= -1. \end{aligned}$$

We can take $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(e) Since the vectors u and v are linearly independent it suffices to find $A^n u$ and $A^n v$. Clearly, $A^n u = \lambda^n u = 3^n u$, and

$$\begin{aligned} A^n v &= A^{n-1}(Av) = A^{n-1}(\lambda v + u) = \lambda A^{n-1} v + A^{n-1} u = \lambda A^{n-1} v + \lambda^{n-1} u \\ &= \lambda A^{n-2}(Av) + \lambda^{n-1} u = \lambda A^{n-2}(\lambda v + u) + \lambda^{n-1} u = \lambda^2 A^{n-2} v + 2\lambda^{n-1} u \\ &\vdots \\ &= \lambda^n v + n\lambda^{n-1} u = \begin{bmatrix} 3^n + n3^{n-1} \\ -n3^{n-1} \end{bmatrix}. \end{aligned}$$

We now need to find $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have already found that $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3^n + n3^{n-1} \\ -n3^{n-1} \end{bmatrix}$. In order to find the other vector we first express $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in terms of u and v: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = v - u$, hence:

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A^n v - A^n u = \begin{bmatrix} 3^n + n3^{n-1} \\ -n3^{n-1} \end{bmatrix} - \begin{bmatrix} 3^n \\ -3^n \end{bmatrix} = \begin{bmatrix} n3^{n-1} \\ 3^n - n3^{n-1} \end{bmatrix}.$$

Thus:

$$A^n = \begin{bmatrix} 3^n + n3^{n-1} & n3^{n-1} \\ -n3^{n-1} & 3^n - n3^{n-1} \end{bmatrix}.$$

(f) We solve the recursive system of equations by first expressing it in the form $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$. In the same way as in Example 3 we conclude that $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, thus:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 3^n + n3^{n-1} & n3^{n-1} \\ -n3^{n-1} & 3^n - n3^{n-1} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3^n + 7n \cdot 3^{n-1} \\ 5 \cdot 3^n - 7n \cdot 3^{n-1} \end{bmatrix},$$

which is equivalent to:

$$
\begin{aligned}
x_n &= 2 \cdot 3^n + 7n \cdot 3^{n-1} \\
y_n &= 5 \cdot 3^n - 7n \cdot 3^{n-1}.
\end{aligned}
$$

Remark. The vectors u and v from previous example form a basis called Jordan basis for the matrix A.