Canonical forms

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1 Smith canonical form

Let A be an $m \times n$ matrix over $F[x]$ and $k \leq \min\{m, k\}$. Choose arbitrary k column and k rows of A. There are $\binom{m}{k}\binom{n}{k}$ matrices of such type.

Definition 1.1. The determinant of any of the above matrices is called a k^{th} **order minor**.

Let's look at the following example

Example 1.2. Find a Smith canonical form of a matrix A.

$$A = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 2 & 3 & x-1 \end{pmatrix}$$

Let $d_k(x)$ be the gcd of the k^{th} order minor of A, for $k = 1, 2, 3$. Now,

$$d_1(x) = \gcd(x, 1, 0, 2, 3, x-1) = 1$$
$$d_2(x) = \gcd\left(\begin{vmatrix} x & 1 \\ 0 & x \end{vmatrix}, \begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix}, \begin{vmatrix} 0 & x \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 2 & x-1 \end{vmatrix}, \begin{vmatrix} x & 1 \\ 3 & x-1 \end{vmatrix}, \begin{vmatrix} x & 1 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} x & 0 \\ 2 & x-1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 3 & x-1 \end{vmatrix} \right) = 1$$
$$d_3(x) = |A| = x^3 - x^2 - 3x + 2$$

Thus the invariant factors f_1, f_2 and f_3 are given

$$f_1 = d_1(x) = 1, \qquad f_2 = \frac{d_2(x)}{d_1(x)} = d_2(x) = 1, \qquad f_3 = d_3(x) = x^3 - x^2 - 3x + 2$$

Hence B the Smith canonical form of A is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_3(x) \end{pmatrix}$$

Definition 1.3. Let A be an $n \times n$ matrix in F, then $xI_n - A$ is an $n \times n$ matrix in $F[x]$. The **invariant factors** of $xI_n - A$ are called the **similarity invariant factors** of A.

Remark 1.4. 1. Let A be a square matrix of order n in F, then d_n of $(xI - A)$ is $\chi_A(x)$.

2. The **minimal polynomial** $m_A(x) = f_n$, where f_n is the highest order invariants of A.

Example 1.5. Let

$$A = \begin{pmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

Then

$$xI - A = \begin{pmatrix} x-6 & -2 & 2 \\ 2 & x-2 & -2 \\ -2 & -2 & x-2 \end{pmatrix}$$

$$d_1(x) = 1, \qquad d_2(x) = x-4, \qquad d_3(x) = (x-2)(x-4)^2$$

Thus the invariant factors of A are

$$d_1(x) = 1 = f_1, \qquad \frac{d_2(x)}{d_1(x)} = x-4 = f_2, \qquad \frac{d_3(x)}{d_2(x)} = (x-2)(x-4) = f_3$$

2 Rational canonical form

2.1 Invariant factors

Consider the following monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

Definition 2.1. The companion matrix of f which is denoted by $C(f)$ is given as follows

$$C(f) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} \tag{2}$$

Example 2.2. Find the companion matrix of

1. $f(x) = x + a_0$
2. $f(x) = x^2 + a_1 x + a_0$
3. $f(x) = x^3 + a_2 x^2 + a_1 x + a_0$

Solution:

1. $C(f) = a_0$
2. $C(f) = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$
3. $C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$

Theorem 2.3. *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then the* ***characteristic polynomial*** *and the* ***minimal polynomial*** *of $C(f)$ are both equal to f.*

Proof.

$$xI - C(f) = \begin{pmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \xrightarrow[\substack{c_1 + x^5 c_6 \\ \vdots \\ c_1 + x^{n-1} c_n}]{\substack{c_1 + x c_2 \\ c_1 + x^2 c_3 \\ c_1 + x^3 c_4 \\ c_1 + x^4 c_5}} \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ f(x) & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

$\square$

Definition 2.4. Let A be a square matrix of order n and $f_1, f_2, ..., f_r$ be the **non trivial** similarity invariants of A. Let $c_i = C(f_i), \quad i = 1, 2, ..., r$.
Then $B = Diag(c_1, c_2, ..., c_r)$ is called the **Rational canonical form** of all matrices similar to A.

Example 2.5. Find the Rational canonical form (RCF) of the matrix A in example (1.5)
Solution: The non trivial invariant factors of A are: $f_1(x) = x - 4, f_2(x) = x^2 - 6x + 8$ then

$$C(f_1) = (4), \qquad C(f_2) = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}$$

Hence the RCF of A is a matrix B which is written

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & 6 \end{pmatrix}$$

2.2 Elementary divisors

Let A be a square matrix of order n in F and $\chi_A(x)$ characteristic polynomial of A.
Let $f_1, f_2, ..., f_k$ be the non trivial similarity invariants of A.
Suppose $\chi_A(x) = f_1 f_2 \cdots f_r = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct monic polynomials that are irreducible over F and each α_i is a positive integer.

$$\chi_A(x) = f_1 f_2 \cdots f_r = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \tag{3}$$

$\Rightarrow f_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots p_k^{\alpha_{ik}} \qquad (i = 1, 2, ..., r)$.
Since $f_i | f_{i+1}$, then $\alpha_{(i+1)j} \geq \alpha_{ij}$

Remark 2.6. α_{ij} can be zero but if a_{ij} is positive, then $\alpha_{(i+1)}$ is also positive.

Definition 2.7. The polynomials $p_i^{\alpha_{ij}}$ that appear in the similarity invariants of A, with $\alpha_{ij} > 0$ are called **elementary divisors** of A over F.

Remark 2.8. The list of elementary divisors may include duplications.

Example 2.9. Find the elementary divisors of

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

Solution:

$$d_1 = \gcd(1, -1, 2, -5, x, x-3, x-4) = 1$$

$$d_2 = \gcd\left(\begin{vmatrix} x-3 & -1 \\ 1 & x \end{vmatrix}, \begin{vmatrix} x-3 & 2 \\ 1 & -5 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ x & -5 \end{vmatrix}, \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -5 \\ 1 & x-4 \end{vmatrix}, \begin{vmatrix} x & -5 \\ 1 & x-4 \end{vmatrix}, \begin{vmatrix} x-3 & -1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} x-3 & 2 \\ 1 & x-4 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 1 & x-4 \end{vmatrix} \right) = 1$$

$$d_3 = |IX - A| = x^3 - 7x^2 + 16x - 12 = (x-3)(x-2)^2$$

The similarity invariants are

$$f_1 = d_1 = 1, \qquad f_2 = \frac{d_2}{d_1} = 1, \qquad f_3 = \frac{d_3}{d_2} = (x-3)(x-2)^2$$

The elementary divisors are

$$x - 3, (x-2)^2$$

3 Normal canonical form

Definition 3.1. Let A be a square matrix of order n over F and $g_1, g_2, ..., g_r$ be its elementary divisors. Let $c_i = C(g_i)$, then the matrix $Diag(c_1, c_2, ..., c_r)$ is called **Normal canonical form** of A.

4 Jordan canonical form

Let A be a square matrix of order n over F. Suppose the elementary divisors of A are of the form $(x-\lambda_i)^{\alpha_{ij}}, \forall i$. This is possible of F is algebraically closed field. For $(x-\lambda_j)^{\alpha_{ij}}$, we define the **Jordan Block** corresponding to the elementary divisor $(x-\lambda_j)^{\alpha_{ij}}$ to be the $\alpha_{ij} \times \alpha_{ij}$ matrix $J_{\alpha_{ij}}(\lambda_j)$ given by

$$J_{\alpha_{ij}}(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix} \tag{4}$$

Example 4.1. If $(x-3)^2$ is an elementary divisor of a square matrix A, then the Jordan block of $(x-3)^2$