Eigenvalues and Diagonalization

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Eigenvalues and Eigenvectors

Definition 1. Let A be an $n \times n$ matrix. The scalar λ is called an eigenvalue[1] of A if there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

The vector $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Remark. Note that an eigenvector cannot be zero. Allowing $\mathbf{x}$ to be the zero vector would render the definition meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ is true for all real values of λ. An eigenvalue of $\lambda = 0$, however, is possible.

Definition

- Let $A = (a_{ij})$ be a square matrix of order n over F($M_n(F)$). A vector $x \in F^n$ is called an **eigenvector** of A if $\exists \lambda \in F$ such that

  $$Ax = \lambda x$$

  Where λ-**eigenvalue** of A corresponding to the **eigenvector** x.

  Geometric multiplicity

  Algebraic multiplicity

- $|A - \lambda I_n| = \det(A - \lambda I_n)$ is a polynomial of degree n known as characteristics polynomial of A denoted by $P(\lambda) = |A - \lambda I_n|$
- The equation $|\lambda I_n - A| = 0$ is called **characteristic equation** of a matrix A.
- ***Similar Matrix***: The matrices A and $B \in M_n(F)$ are Similar if there exists an invertible matrix P such that

  $$B = P^{-1}AP$$

- ***Diagonalizable matrix***: Let $A \in M_n(F)$. A is said to be diagonalizable if A is similar to a diagonal matrix D.

  $$A = PDP^{-1}$$

[1] The terms eigenvalue and eigenvector are derived from the German word *Eigenwert*(Proper value).

Example 1. Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -4 \\ 3 & -5 \end{bmatrix}$$

Solution. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & 4 \\ -3 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) - (-12) \\ &= \lambda^2 + 3\lambda - 10 + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \end{aligned}$$

Now, we have $\lambda_1 = -1$ and $\lambda_2 = -2$ as eigenvalue. To find the corresponding eigenvector $\square$

Remark (Formula for eigenvalue)**.** For a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The possible value for the eigenvalues are given by

$$\frac{1}{2}\left(a + d - \sqrt{a^2 + 4bc - 2ad + d^2}\right), \qquad \frac{1}{2}\left(a + d + \sqrt{a^2 + 4bc - 2ad + d^2}\right)$$

References

[1] [Demissu Gemeda] Topics in Linear Algebra, Addis Ababa University 2005.

[2] [Sheldon Axler] Linear Algebra Done Right, Springer Publishing 1997.

[3] [Kolman §Hill] Introduction to Linear Algebra with Applications 2000.