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Triangle Inequality

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November 13, 2015

Abstract: **Triangle Inequality** is one of the most important inequalities in mathematics.There are four proofs presented here and each of them are proved by using different areas of mathematics.

Definition 0.1 (Triangular inequality)**.** The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

1 Geometry

Proof:
Let ABC be a triangle. We need to show that $\mathsf{BA}+\mathsf{AC} > \mathsf{BC}$, $\mathsf{BA}+\mathsf{BC} > \mathsf{AC}$ and $\mathsf{BC}+\mathsf{AC} > \mathsf{BA}$.
We show only $\mathsf{BA}+\mathsf{AC} > \mathsf{BC}$. The others can be shown in similar manner. See the Figure on the right
Extend $\overline{\mathsf{BA}}$ to some point X on $\overleftrightarrow{\mathsf{AB}}$ such that $\mathsf{B}-\mathsf{A}-\mathsf{X}$ and $\overline{\mathsf{AX}} \equiv \overline{\mathsf{AC}}$. This is possible by Axiom of segment construction. Join C and X. Since $\overline{\mathsf{AX}} \equiv \overline{\mathsf{AC}}$ and $\mathsf{A\hat{X}C} \equiv \mathsf{X\hat{C}A}$. $\mathsf{XCA} < \mathsf{XCB}$ by definition of angle comparison. Thus, $\mathsf{A\hat{X}C} < \mathsf{X\hat{C}B}$(i.e. $\mathsf{B\hat{X}C} < \mathsf{X\hat{C}B}$). Now in $\triangle\mathsf{BCX}$ we have $\mathsf{BC} < \mathsf{BX}$. But $\mathsf{BX} = \mathsf{BA}+\mathsf{AX}$ (as B, A, X are collinear and $\mathsf{B}-\mathsf{A}-\mathsf{X}$,$\mathsf{BX} = \mathsf{BA}+\mathsf{AC}$ as $\overline{\mathsf{AC}} \equiv \overline{\mathsf{AX}}$. Therefore,

$$\mathsf{BC} < \mathsf{BA}+\mathsf{AC}$$

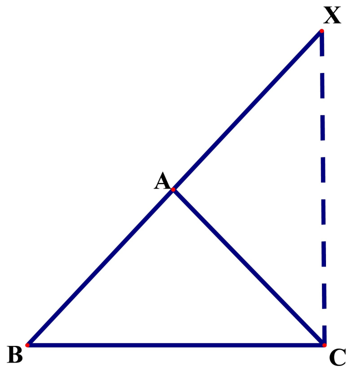

Figure 1.1: 1234

2 Algebra

Theorem 2.1 (Triangle inequality). $|x + y| \leq |x| + |y|$

Proof. From the definition of absolute value we have

$$
\begin{aligned}
x \leq |x| \qquad &\text{and} \qquad -x \leq |x| \\
y \leq |y| \qquad &\text{and} \qquad -y \leq |y|
\end{aligned}
$$

$$x + y \leq |x| + |y| \tag{2.1}$$

And

$$-(x + y) \leq |x| + |y| \tag{2.2}$$

From 2.1 and 2.2 we can conclude that

$$-(|x| + |y|) \leq x + y \leq (|x| + |y|)$$

$$\Rightarrow |x + y| \leq |x| + |y|$$

□

3 Complex Analysis

Definition 3.1. Let $z = x + iy$ be complex number with real part x ($\mathrm{Re}(z) = x$) and imaginary part y ($\mathrm{Im}(z) = y$), then the conjugate of z is defined by $\bar{z} = x - iy$.
Norm of z is given by $|z| = \sqrt{x^2 + y^2}$.

Theorem 3.2 (Triangle inequality).

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{3.1}$$

Proof.

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$$

$$
\begin{aligned}
(z_1 + z_2)\overline{(z_1 + z_2)} &= z_1\overline{(z_1 + z_2)} + z_2\overline{(z_1 + z_2)} \\
&= z_1\overline{(z_1)} + z_1\overline{(z_2)} + z_2\overline{(z_1)} + z_2\overline{(z_2)} \\
&= |z_1|^2 + \overline{\overline{z_1} z_2} + z_1\overline{z_2} + |z_2|^2 \\
&= |z_1|^2 + 2\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2, \qquad \text{since} \quad \mathrm{Re}(z) = \frac{z + \overline{z}}{2}. \\
&\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2
\end{aligned}
$$

$$\Rightarrow |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$$

□

4 Vector Analysis

Lemma 4.1 (Cauchy-Schwartz inequality)**.**

$$\|\vec{A}\cdot\vec{B}\| \leq \|\vec{A}\|\|\vec{B}\| \tag{4.1}$$

Proof.

$$|\cos(\theta)| \leq 1$$
$$-|\vec{A}||\vec{B}| \leq |\vec{A}||\vec{B}|\cos(\theta) \leq |\vec{A}||\vec{B}|$$
$$-|\vec{A}||\vec{B}| \leq |\vec{A}\cdot\vec{B}| \leq |\vec{A}||\vec{B}|$$
$$\|\vec{A}\cdot\vec{B}\| \leq \|\vec{A}\|\|\vec{B}\|$$

□

Theorem 4.2 (Triangular inequality)**.**

$$\|\vec{A}+\vec{B}\| \leq \|\vec{A}\| + \|\vec{B}\| \tag{4.2}$$

Proof.

$$\|\vec{A}+\vec{B}\|^2 = (A+B)(A+B)$$

$$\begin{aligned}
(\vec{A}+\vec{B})(\vec{A}+\vec{B}) &= \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B} \\
&= \|\vec{A}\|^2 + 2\vec{A}\cdot\vec{B} + \|\vec{B}\|^2 \\
&\leq \|\vec{A}\|^2 + \|\vec{B}\|^2 + \|\vec{A}\cdot\vec{B}\|
\end{aligned}$$

But from Cauchy-Schwartz inequality we have $\|\vec{A}\cdot\vec{B}\| \leq \|\vec{A}\|\|\vec{B}\|$
Thus,

$$\begin{aligned}
\|\vec{A}+\vec{B}\|^2 &\leq \|\vec{A}\|^2 + \|\vec{B}\|^2 + \|\vec{A}\|\|\vec{B}\| \\
&\leq (\|\vec{A}\| + \|\vec{B}\|)^2
\end{aligned}$$

$$\Rightarrow |\vec{A}+\vec{B}\| \leq \|\vec{A}\| + \|\vec{B}\|$$

□

References

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