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Mathematical Fallacies and Paradoxes

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Abstract: **Mathematical fallacies** are errors, typically committed with an intent to deceive, that occur in a mathematical proof or argument. A fallacy in an argument doesn't necessarily mean that the conclusion is necessarily incorrect, only that the argument itself is wrong. However, fallacious arguments can have surprising conclusions. Apart from a mathematical fallacy a **paradox** is a statement that goes against our intuition but may be true, or a statement that is or appears to be self-contradictory. Mathematical paradoxes result from either counter-intuitive properties of infinity, or self-reference.

FIRST FALLACY

A fallacy due to John Bernoulli, may be stated as follows. We have $(-1)^2 = 1$. Take logarithms, $2\log(-1) = \log 1 = 0$.

$$\begin{aligned} \log(-1) &= 0 \\ -1 &= e^0 \\ -1 &= 1 \end{aligned}$$

The same argument may be expressed thus. Let x be a quantity which satisfies the equation $e^x = -1$
Square both sides

$$e^{2x} = 1$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow x = 0$$

$$\Rightarrow e^x = e^0$$

But $e^x = -1$ and $e^0 = 1$,

$$-1 = 1$$

SECOND FALLACY

Suppose that $a = b$, then

$$\begin{aligned} ab &= a^2 \\ ab - b^2 &= a^2 - b^2 \\ b(a-b) &= (a+b)(a-b) \\ b &= a+b \\ b &= 2b \\ 1 &= 2 \end{aligned}$$

THIRD FALLACY

Let a and b be two unequal numbers, and let c be their arithmetic mean, hence

$$\begin{aligned} a+b &= 2c \\ (a+b)(a-b) &= 2c(a-b) \\ a^2 - 2ac &= b^2 - 2bc \\ a^2 - 2ac + c^2 &= b^2 - 2bc + c^2 \\ (a-c)^2 &= (b-c)^2 \\ a &= b \end{aligned}$$

FOURTH FALLACY

From Taylor's expansion, we know that

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots.$$

If $x = 1$, the resulting series is convergent; hence we have

$$\begin{aligned} \log 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \cdots. \\ 2\log 2 &= 2 - 1 + \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{2}{5} - \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{4} + \tfrac{2}{9} - \cdots. \end{aligned}$$

Taking those terms together which have a common denominator, we obtain

$$\begin{aligned} 2\log 2 &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} - \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \\ &= \log 2 \\ 2 &= 1 \end{aligned}$$

Fifth Fallacy

This fallacy is very similar to that last given. We have

$$\begin{aligned}
\log 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\
&= \left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots\right) - \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots\right) \\
&= \left\{\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots\right) + \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots\right)\right\} - 2\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots\right) \\
&= \left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots\right\} - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots\right) \\
&= 0
\end{aligned}$$

The error in each of the foregoing examples is obvious, but the fallacies in the next examples are concealed somewhat better.

Sixth Fallacy

We can write the identity $\sqrt{-1} = \sqrt{-1}$ in the form

$$\begin{gathered}
\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}} \\
\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}} \\
(\sqrt{-1})^2 = (\sqrt{1})^2 \\
-1 = 1
\end{gathered}$$

Seventh Fallacy

Again, we have

$$\begin{gathered}
\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \\
\sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} \\
(\sqrt{-1})^2 = \sqrt{1} \\
-1 = 1
\end{gathered}$$

Eighth Fallacy

The following demonstration depends on the fact that an algebraical identity is true whatever be the symbols used in it, and it will appeal only to those who are familiar with this fact.

We have, as an identity,

$$\sqrt{x-y} = i\sqrt{y-x} \tag{1}$$

where i stands either for $+\sqrt{-1}$ or for $-\sqrt{-1}$. Now an *identity* in x and y is necessarily true whatever numbers x and y may represent. First put $x = a$ and $y = b$,

$$\sqrt{a-b} = i\sqrt{b-a} \tag{2}$$

Next put $x = b$ and $y = a$,

$$\sqrt{b-a} = i\sqrt{a-b} \tag{3}$$

Also since (1) is an identity, it follows that in (2) and (3) the symbol i must be the same, that is, it represents $+\sqrt{-1}$ or $-\sqrt{-1}$ in both cases. Hence, from (2) and (3), we have

$$\begin{aligned}\sqrt{a-b}\,\sqrt{b-a} &= i^2\sqrt{b-a}\,\sqrt{a-b},\\ 1 &= i^2,\\ 1 &= -1.\end{aligned}$$

Ninth Fallacy

This paradox is due to my teacher Elias[1]
Consider the following limit
Case I

$$\begin{aligned}\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2} &= \lim_{n\to\infty}\frac{1}{n^2}+\lim_{n\to\infty}\frac{2}{n^2}+\cdots+\lim_{n\to\infty}\frac{n}{n^2}\\ &= \lim_{n\to\infty}\frac{1}{n^2}+\lim_{n\to\infty}\frac{2}{n^2}+\cdots+\lim_{n\to\infty}\frac{1}{n}\\ &= 0+0+\cdots+0\\ &= 0\end{aligned}$$

Case II

$$\begin{aligned}\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2} &= \lim_{n\to\infty}\frac{n(n+1)}{2n^2} \qquad (Gauss\ Sum)\\ &= \lim_{n\to\infty}\frac{n^2+n}{2n^2} = \lim_{n\to\infty}\left[\frac{n^2}{2n^2}+\frac{n}{2n^2}\right]\\ &= \lim_{n\to\infty}\frac{n^2}{2n^2}+\lim_{n\to\infty}\frac{n}{2n^2}\\ &= \frac{1}{2}+0\\ &= \frac{1}{2}\end{aligned}$$

Hence from Case I and Case II we can conclude that

$$0 = \frac{1}{2}$$

[1] Elias Bogale is one of a few teacher which I have been taught by.

References

[1] [Clifford A. Pickover] A Passion for Mathematics, 2005.

[2] [E. A. Maxwell] Fallacies in mathematics, 1963.

[3] [Bryan Bunch] Mathematical fallacies and paradoxes, 1982.

[4] [Wiki] Mathematical fallacy.