Zero Determinants

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Abstract

The determinant of a matrix has a versatile application. For instance if the determinant of a coefficient matrix of a given system is zero, then it follows that the equations(which form the system) are not linearly independent. The determinant is also helpful in determining whether a given matrix is invertible or not. If the matrix contain zero row(column) or if one row(column) of the matrix is a multiple of the other, then the determinant of that matrix becomes zero. A matrix with zero determinant is known as a **singular matrix** and it is known that singular matrices are not invertible. In this paper, we will try to study some simple property which characterize singular matrices.

1 Definitions

- **Matrix**: is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns.
  e.g. Matrix A of order n is given by

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

- **Square Matrix**: is a matrix with equal number of rows and columns.

- **Determinant**: is a useful value that can be computed from a square matrix.
  Determinant for[1] 2×2 matrix: Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc.$$

- **Minor and Cofactor:** If A is a square matrix, then the **minor** A_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A. Then the cofactor Δ_{ij} is given by $\Delta_{ij} = (-1)^{i+j}\det(A_{ij})$.

[1]For 3×3 matrix one can use *Sarrus' rule* to determine the determinant.

- **Laplace Expansion(for[2] $n \times n$ matrix)**
  Let $A = (a_{ij})$ be a square matrix of order n where $n \geq 3$. Then, $\det(A)$ can be expressed as a cofactor expansion by using any row of A.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \tag{1}$$

2 Facts

Lemma 2.1

The determinant of any 3×3 matrix with **row(column) of common difference** is zero.

Proof. Let's take a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Since the matrix is with **row of common difference**[3] the rows satisfy some arithmetic progression. Say,

$$\begin{aligned} b &= a + x, \qquad & c &= b + x \\ e &= d + y, \qquad & f &= e + y \\ h &= g + z, \qquad & i &= h + z \end{aligned}$$

Then the matrix become

$$A = \begin{bmatrix} a & a+x & b+x \\ d & d+y & e+y \\ g & g+z & h+z \end{bmatrix} = \begin{bmatrix} a & a+x & a+2x \\ d & d+y & d+2y \\ g & g+z & g+2z \end{bmatrix}$$

Computing the determinant

$$\begin{aligned} \det(A) = \begin{vmatrix} a & a+x & a+2x \\ d & d+y & d+2y \\ g & g+z & g+2z \end{vmatrix} &= a \cdot \begin{vmatrix} d+y & d+2y \\ g+z & g+2z \end{vmatrix} - d \cdot \begin{vmatrix} a+x & a+2x \\ g+z & g+2z \end{vmatrix} + g \cdot \begin{vmatrix} a+x & a+2x \\ d+y & d+2y \end{vmatrix} \\ &= a \cdot (dz - gy) - d \cdot (az - gx) + g \cdot (ay - dx) \\ &= adz - agy - adz + dgx + agy - dgx \\ &= (adz - adz) + (agy - agy) + (dgx - dgx) = 0 \end{aligned}$$

□

[2] Leibniz formula can be also used to calculate the determinant of $n \times n$ matrix,

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

[3] In each row the entries are differ by a constant. This constant may **vary** from one row to the other.

Theorem 2.1 (Zero determinant Theorem)

Given any square matrix A of order $n \geq 3$. If A is a matrix with **row(column) of common difference**, then the determinant of A is zero.

Proof. Consider an $n \times n$ matrix A with row of common difference

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

From Laplace expansion we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

$$= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + \cdots + (-1)^{n+1} a_{1n} \det(A_{1n})$$

If each matrices $A_{11}, A_{12}, \cdots, A_{1n}$ are 3×3 matrix then we are done. But if not we will repeat this process until we get a 3×3 matrix.

Then this 3×3 matrix is a matrix with **row of common difference** since the bigger matrix A(from which it constructed) itself is a matrix with **row of common difference**.

Thus, the determinant of each matrix $A_{11}, A_{12}, \cdots, A_{1n}$ is zero by lemma (2.1). Then

$$\begin{aligned} \det(A) &= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + \cdots + (-1)^{n+1} a_{1n} \det(A_{1n}) \\ &= (-1)^{1+1} a_{11}(0) + (-1)^{1+2} a_{12}(0) + \cdots + (-1)^{n+1} a_{1n}(0) \\ &= 0 \end{aligned}$$

Therefore, $\det(A) = 0$. □

Corollary 2.1

If A is a square matrix with consecutive entries, then the determinant of A is zero.

Conclusion

This is just a nice way of saying; if a matrix is with row(column) of common difference, then somehow we can express one row(column) of that matrix as a linear combination of the other.

A trivial matrix with zero determinant is; a matrix with zero row(column) or a matrix containing a row(column) which is a scalar multiple of some other row(column). But by using our result, we can easily generate a non trivial matrix of determinant zero.

References

[1] [Serg Lang] Linear Algebra, Addison-Wesley Publishing, 1972.

[2] [Kolman § Hill] Introduction to Linear Algebra with Applications, 2000.

[3] [Demissu Gemeda] An Introduction to Linear Algebra, AAU Press, 2000.