A Note on Differential Equations

Miliyon T.
Addis Ababa University
Department of Mathematics
http://www.albohessab.weebly.com

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1 Introduction

Definition 1.1 (Lipschitz). A map $f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in x if the inequality

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|, \tag{1}$$

holds wherever $(t, x_1), (t, x_2) \in \Omega$.

As a consequence of Definition (1.1) a function f is Lipschitz iff there exists a constant $K > 0$ such that

$$\frac{|f(t, x_1) - f(t, x_2)|}{|x_1 - x_2|} \leq K, \qquad x_1 \neq x_2,$$

whenever $(t, x_1), (t, x_2) \in \Omega$.

Theorem 1.2. *Define rectangle R by*

$$R = \{(t, x) : |t - t_0| \leq a, \ |x - x_0| \leq b\}$$

where $a, b \in \mathbb{R}$. If the following two conditions are satisfied

1. *$f : \mathbb{R} \to \mathbb{R}$ be a real valued continuous function.*
2. *$\frac{\partial f}{\partial x}$ be defined and continuous on R.*

Then f is Lipschitz on x.

Proof. □

Lemma 1.3 (Gronwall). *Assume that $f, g : [t_0, \infty) \to \mathbb{R}^+$ are non negative continuous functions. let $k > 0$ be a constant. Then the inequality*

$$f(t) \leq k + \int_{t_0}^{t} g(s)f(s)ds, \qquad t \geq t_0 \tag{2}$$

implies the inequality

$$f(t) \leq k \exp\left(\int_{t_0}^{t} g(s)ds\right), \qquad t \geq t_0. \tag{3}$$

Proof. By hypothesis we have

$$\frac{f(t)g(t)}{k + \int_{t_0}^{t} g(s)f(s)ds} \leq g(t), \qquad t \geq t_0 \tag{4}$$

Since

$$\frac{d}{dt}\left(k + \int_{t_0}^{t} g(s)f(s)ds\right) = g(t)f(t)$$

By integrating (4) between the limits t_0 and t, we have

$$\ln\left(k + \int_{t_0}^{t} g(s)f(s)ds\right) - \ln(k) \leq \int_{t_0}^{t} g(s)ds$$

In other words,

$$k + \int_{t_0}^{t} g(s)f(s)ds \leq k \exp\left(\int_{t_0}^{t} g(s)ds\right)$$

Hence,

$$f(t) \leq k \exp\left(\int_{t_0}^{t} g(s)ds\right)$$

□

Theorem 1.4 (**Gronwall**[1])**.** *Let x, ψ and χ be real continuous functions defined in $I = [a, b]$, $\psi(t) \geq 0$ for $t \in I$. Suppose on I the following inequality holds*

$$x(t) \leq \psi(t) + \int_a^t \chi(s)x(s)ds \tag{5}$$

Then

$$x(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s) \exp\left[\int_s^t \chi(u)du\right] ds \tag{6}$$

in I.

Proof. □

Corollary 1.5. *Let f and k be as in Lemma (1.3) If the inequality*

$$f(t) \leq k \int_{t_0}^t g(s)ds, \qquad t \geq t_0 \tag{7}$$

holds then,

$$f(t) \equiv 0, \text{ for } t \geq t_0. \tag{8}$$

Proof. □

1.1 Picard's Successive Approximations

Definition 1.6 (**IVP**)**.** An initial value problem is given by

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{9}$$

Consider the IVP in (9), the Picard's approximation to (9) is given by

$$x_{n+1}(t) = x_0 + \int_{t_0}^t f(s, x_n(s))ds \tag{10}$$

Example 1.7. Find the fourth Picard's approximation to the differential equation

$$\frac{dx}{dt} = -x, \qquad x(0) = 1, \ t \geq 0.$$

Solution. □

Example 1.8. Find the third Picard's approximation to the differential equation

$$\frac{d^2y}{dx^2} = xy + 1, \qquad y_0 = 1, \frac{dy}{dx} = 0.$$

Solution. Transform the second order equation $y'' = f(x, y, y')$ into first order system,

$$\begin{aligned} y' &= z \\ z' &= f(x, y, z) \\ y_0 &= 1 \\ z_0 &= 0 \end{aligned}$$

[1] T.H. Gronwall proved the inequality in 1919.

Then the Picard iteration is given by

$$y_{n+1} = y_0 + \int_0^t z_n(s)ds$$
$$z_{n+1} = z_0 + \int_0^t f(s, y_n(s), z_n(s))ds$$

Hence,

$$\begin{aligned} y_1 &= 1 + \int_0^t 0ds \\ &= 1 \end{aligned}$$

$$\begin{aligned} z_1 &= 0 + \int_0^t f(s, y_0(s), z_0(s))ds \\ &= \int_0^t (s+1)ds \end{aligned}$$

□

Example 1.9. Find the third Picard's approximation to the differential equation

$$\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right), \qquad y_0 = 1,\ \frac{dy}{dx} = \frac{1}{2}.$$

Solution. □

2 Existence and Uniqueness of solution

Definition 2.1 (**ε-approximate**)**.** Let $f \in C(\mathbb{R})$ on D. An ε-approximate solution to the IVP $x' = f(t, x)$ is a function $\Phi \in C$ on a t interval I such that

1. $(t, \Phi(t)) \in D$,for $t \in I$.
2. $\Phi(t) \in C^1$, except for a finite set S on which Φ has discontinuity.
3. $\|\Phi'(t) - f(t, \Phi(t))\| \leq \varepsilon$, for $t \in I \backslash S$.

Theorem 2.2 (**Picard-Lindelof**)**.** *Assume that f is a continuous and Lipschitz w.r.t x on the rectangle*

$$R = \{(t, x) : |t - t_0| \leq a,\ |x - x_0| \leq b\}.$$

Let

$$M := \max_{(t,x)\in R}\{\|f(t, x)\|\}$$

and

$$\alpha := \min\left\{a, \frac{b}{M}\right\}$$

Then the initial value problem (9) has a unique solution x on $[t_0, t_0 + \alpha]$. Furthermore,

$$|x(t) - x_0| < b, \qquad \text{for } t \in [t_0, t_0 + \alpha].$$

Proof. □

2.1 Cauchy-Peano Theorem

Let $I = [a, b] \subset \mathbb{R}$ be an interval and let $F(I, \mathbb{R})$ denote the set of all real valued functions defined on I.

Definition 2.3 (**Equicontinuous**). A set of functions $F = \{f\}$ defined on a real interval I is said to be equicontinuous on I if, given any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$, independent of $f \in F$ and also $t, \widetilde{t} \in I$ such that

$$|f(t) - f(\widetilde{t})| < \epsilon, \text{ whenever } |t - \widetilde{t}| < \delta.$$

Theorem 2.4 (**Cauchy-Peano**). *If $f \in C(\mathbb{R})$, then there exists a solution $\phi \in C'$ of the IVP $x' = f(t, x)$ on $|t - \tau| < \alpha$ for which $\phi(\tau) = \xi$.*

Proof. □

Theorem 2.5 (**Variation of constant formula**). *If $p : (a, b) \to \mathbb{R}$ and $q : (a, b) \to \mathbb{R}$ are continuous functions, where $-\infty \leq a < b \leq \infty$. Then the unique solution of the IVP:*

$$x' = p(t)x + q(t), \qquad x(t_0) = x_0$$

where $t_0 \in (a, b)$, $x_0 \in \mathbb{R}$ is given by

$$x(t) = e^{\int_{t_0}^t p(r)dr} x_0 + e^{\int_{t_0}^t p(r)dr} \int_{t_0}^t e^{-\int_{t_0}^t p(r)dr} q(s)ds, \qquad t \in (a, b). \tag{11}$$

Proof. □

3 Linear systems

We consider systems of differential equations of the form

$$\begin{aligned} x_1' &= ax_1 + bx_2, \\ x_2' &= cx_1 + dx_2, \end{aligned} \tag{12}$$

where $a, b, c,$ and d are real numbers. At first glance the system may seem to be first-order; however, it is coupled, and this two-dimensional system is more closely related to a second-order differential equation. Before finding the solution, some notation is introduced to simplify the form of the system. The new notation implicitly removes most of the $=$ and $+$ operations. The form of the system suggests arranging the unknown functions x_1, x_2 vertically and enclosed in brackets (*i.e.* matrix form)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{13}$$

This new object is similar to a number or function. Indeed, operations may be performed on (13). Let $a \in I\!R$. We define

$$\begin{pmatrix} u \\ v \end{pmatrix}' := \begin{pmatrix} u' \\ v' \end{pmatrix} \qquad a \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} u \\ v \end{pmatrix} a := \begin{pmatrix} au \\ av \end{pmatrix}.$$

The new object can be part of mathematical statements. We say

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

if and only if $a = c$ and $b = d$. Moreover, we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

Then (12) is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

often written more simply as

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}.$$

In the calculations and expressions that follow both the matrix form (above) and the component form will be given and separated by "or." They are equivalent. Based on our discussion in class, we expect a solution of (12) to have the form

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt} \qquad \text{or} \qquad \begin{array}{c} x_1 = \xi_1 e^{rt} \\ x_2 = \xi_2 e^{rt} \end{array} \tag{14}$$

where as of now, ξ_1, ξ_2, and r are unknown. To be more concrete, we consider a specific system.
Example 1. Consider the system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} \qquad \text{or} \qquad \begin{array}{c} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{array} . \tag{15}$$

Our guess, (14), is substituted in (15), and we find

$$re^{rt} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{rt} \qquad \text{or} \qquad \begin{array}{c} r\xi_1 = \xi_1 + \xi_2 \\ r\xi_2 = 4\xi_1 + \xi_2 \end{array} .$$

Rearranging gives

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} (1-r)\xi_1 + \xi_2 = 0 \\ 4\xi_1 + (1-r)\xi_2 = 0 \end{array} . \tag{16}$$

Geometrically the last system of equations describe two lines passing through the origin. We do not want a unique solution (zero in this case), for that would say $\xi_1 = \xi_2 = 0$, and our guess would not produce anything interesting. We want the lines in (16) to be the same, so we require the slopes to be equal. That is,

$$r^2 - 2r - 3 = 0.$$

The roots, called *eigenvalues*, are $r = 3, -1$.
Left to find are ξ_1 and ξ_2. Inserting $r = 3$ back in (16), we find

$$\begin{array}{c} -2\xi_1 + \xi_2 = 0 \\ 4\xi_1 - 2\xi_2 = 0 \end{array} .$$

As expected this system has infinitely many solutions, one of them is

$$\xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} \xi_1 = 1 \\ \xi_2 = 2 \end{array} ,$$

and, a solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \qquad \text{or} \qquad \begin{array}{c} x_1 = e^{3t} \\ x_2 = 2e^{3t} \end{array} .$$

A similar calculation for $r = -1$ shows

$$\xi = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} \xi_1 = 1 \\ \xi_2 = -2 \end{array} ,$$

and, a solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t} \qquad \text{or} \qquad \begin{array}{c} x_1 = e^{3t} \\ x_2 = -2e^{3t} \end{array} .$$

The general solution is therefore,

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \qquad \text{or} \qquad \begin{array}{c} x_1 = c_1 e^{3t} + c_2 e^{-t} \\ x_2 = c_1 2e^{3t} - c_2 2e^{-t} \end{array} .$$

Example 2. Find the general solution of the ODE

$$y'' - 2y' - 3y = 0, \qquad y(0) = 2, \;\; y'(0) = 2.$$

using the approach in Example 1.

Solution. We may turn the ODE into a system of ODEs by setting $x_1 = y$ and $x_2 = y'$. Then the second-order ODE is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A comparison of (15) and (16) in Example 1 shows that we can immediately find the equations for ξ in our guess. Here

$$\begin{pmatrix} 0-r & 1 \\ 3 & 2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} -r\xi_1 + \xi_2 = 0 \\ 3\xi_1 + (2-r)\xi_2 = 0 \end{array}.$$

The polynomial for r is just the determinate or the upper-left diagonal element times the lower-right minus the product of the remaining two diagonal elements. Here, we find $-r(2-r) - 3 = 0$ or

$$r^2 - 2r - 3 = 0.$$

Note that this is the characteristic polynomial for the original second-order ODE! The roots are $r = 3, -1$. Inserting $r = 3$ in the equations for ξ, we find

$$\begin{array}{c} -3\xi_1 + \xi_2 = 0 \\ 3\xi_1 - \xi_2 = 0 \end{array}.$$

As expected this system has infinitely many solutions, one of them is

$$\xi = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} \xi_1 = 1 \\ \xi_2 = 3 \end{array},$$

The other root, $r = -1$ gives

$$\begin{array}{c} \xi_1 + \xi_2 = 0 \\ 3\xi_1 + 3\xi_1 = 0 \end{array},$$

and

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{or} \qquad \begin{array}{c} \xi_1 = 1 \\ \xi_2 = -1 \end{array}.$$

The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \qquad \text{or} \qquad \begin{array}{c} x_1 = c_1 e^{3t} + c_2 e^{-t} \\ x_2 = c_1 3 e^{3t} - c_2 e^{-t} \end{array}.$$

Since we set $y = x_1$ in the original transformation, the top line is the general solution to the second-order ODE, and the bottom line is the derivative of the top ($x_2 = y' = x_1'$).
To find c_1 and c_2 we use the initial data -

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solving, we find $c_1 = 1 = c_2$, and the solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \qquad \text{or} \qquad \begin{array}{c} y = e^{3t} + e^{-t} \\ y' = 3e^{3t} - e^{-t} \end{array}.$$

In this section we mostly drop the component form of the equations (get used to it). As in previous problems we use Euler's formula to change exponents with complex numbers to oscillatory functions.
If $z = a + ib$ is a complex number, the complex conjugate is $\bar{z} = a - ib$. That is, i is replaced with $-i$. In addition, the real and imaginary parts are denoted

$$\mathrm{Re}(z) = a \qquad \mathrm{Im}(z) = b.$$

Both are real numbers.

Example 3 Solve the initial-value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution. As in the previous example, the system for ξ is

$$\begin{pmatrix} 2-r & 8 \\ -1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The nontrivial solution is obtained by requiring the determinate to be zero. That is,

$$(2-r)(-2-r) - (-1)(8) = r^2 + 4 = 0.$$

The eigenvalues (roots) are $r = \pm 2i$. The procedure is the same as in Example 2. Suppose $r = 2i$. The equations for ξ are

$$\begin{array}{c} (2-2i)\xi_1 + 8\xi_2 = 0 \\ -1\xi_1 + (-2-2i)\xi_2 = 0 \end{array}.$$

It is not obvious the two equations are multiples of one another. However, you can verify that $(-2+2i)$ times the second equation gives the first equation. A non-trivial solution is required. If we opportunistically set $\xi_2 = 1$ in the second equation, then $\xi_1 = -2 - 2i$. So

$$\xi = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix}$$

will work, and one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} e^{2i\,t}.$$

Since the other root (eigenvalue) is just the complex conjugate of the first, a second solution is found simply by changing i in the first solution to $-i$ (its complex conjugate). So the general solutions is

$$\mathbf{x} = C_1 \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} e^{2i\,t} + C_2 \begin{pmatrix} 2-2i \\ -1 \end{pmatrix} e^{-2i\,t}. \tag{17}$$

The complex numbers are undesirable. We hide them by using Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. The exponential functions in (17) are replaced, and the result expressed as the real plus imaginary part -

$$\mathbf{x} = C_1 \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} (\cos 2t + i \sin 2t) + C_2 \begin{pmatrix} 2-2i \\ -1 \end{pmatrix} (\cos 2t - i \sin 2t).$$

Read across the first line and pick out all the terms without an i. Then do the same for the bottom line (that is, find the real part). The result is

$$\mathrm{Re}(\mathbf{x}) = (C_1 + C_2) \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix}.$$

The imaginary part consists of all the terms with an i in front. In particular,

$$i\mathrm{Im}(\mathbf{x}) = i(C_1 - C_2)\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}.$$

Finally, set $c_1 = C_1 + C_2$ and $c_2 = i(C_1 - C_2)$. Then

$$\mathbf{x} = c_1\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}$$

is the general solution.

There is a slight short-cut to this procedure. Since there is so much symmetry in the solutions for the two roots (they differ by replacing i with -i), one might expect that all the necessary information is contained in one of the solutions. This is the case. The solution for $r = 2i$ (first part of Equation (17)) is

$$\mathbf{x}^{(\mathbf{1})} = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} e^{2i\,t} = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix}(\cos 2t + i\sin 2t).$$

To find the general solution, only the real and imaginary parts of this solution need to be found. Again rewriting as real plus imaginary,

$$\mathbf{x}^{(\mathbf{1})} = \begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + i\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}.$$

The general solution is $\mathbf{x}^{(\mathbf{1})} = c_1\mathrm{Re}(\mathbf{x}^{(\mathbf{1})}) + c_2\mathrm{Im}(\mathbf{x}^{(\mathbf{1})})$ or

$$\mathbf{x} = c_1\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}$$

as before.

Repeated Eigenvalues

Of course the eigenvalues need not all be distinct. Consider for example

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}. \tag{18}$$

Proceeding as before, the system for ξ is

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

The nontrivial solution is obtained by requiring the determinate to be zero. That is,

$$(1-r)(3-r) - (-1) = r^2 - 4r + 4 = 0.$$

The eigenvalues (roots) are $r = 2, 2$. The equations for ξ are

$$\begin{array}{c} -\xi_1 - \xi_2 = 0 \\ \xi_1 + \xi_2 = 0 \end{array}.$$

A nontrivial solution is

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

In analogy with second-order, linear, constant coefficient, homogeneous ODEs, we might expect the second solution to be

$$\mathbf{x}^{(2)} = t\mathbf{x}^{(1)} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}. \tag{20}$$

Unfortunately, this is NOT a solution. What to do?
Somehow a candidate for the second solution has to be constructed from the only solution we have. Based on previous experiences, (20) has to be close to the correct solution. We alter it slightly and try again. Set

$$\mathbf{x}^{(2)} = t\mathbf{x}^{(1)} + \eta e^{2t} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{2t}. \tag{21}$$

We have to see if a choice for η exists so that $\mathbf{x}^{(2)}$ solves (18). The left side of (18) (the time derivative of (21)) is

$$\mathbf{x}^{(2)\prime} = \mathbf{x}^{(1)} + t\mathbf{x}^{(1)\prime} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} 2e^{2t}. \tag{22}$$

The right side is

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{2t}. \tag{23}$$

However,

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} = 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

Equating (22) and (23), canceling the common e^{2t} and the other common terms, we find

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} 2 = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

After rearranging ($r = 2$ here)

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

or more generally the system for η is

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{24}$$

Notice (24) is an iteration of (19). For the current example, (24) in component form is

$$\begin{aligned} -\eta_1 - \eta_2 &= 1 \\ \eta_1 + \eta_2 &= -1. \end{aligned}$$

As for the system for ξ, the two equations for η are multiples of one another. Any solution will suffice. The simplest way to find a solution is to set either η_1 or η_2 to zero. Hence, $\eta_1 = -1$ and $\eta_2 = 0$ will work, and the second solution is

$$\mathbf{x}^{(2)} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t},$$

and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} \right].$$

Example 4. Find the solution of the following system of differential equations.

$$\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Solution. The system for ξ is

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The nontrivial solution is obtained by requiring the determinate to be zero. That is,

$$(3-r)(-3-r) - (-1)(9) = r^2 = 0.$$

The eigenvalues (roots) are $r = 0, 0$. The equations for ξ are

$$\begin{aligned} 3\xi_1 + 9\xi_2 &= 0 \\ 1\xi_1 + 3\xi_2 &= 0 \end{aligned}.$$

A nontrivial solution is $\xi_1 = 3$, $\xi_2 = -1$, and

$$\mathbf{x}^{(\mathbf{1})} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

The system for η is

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Any non-trivial solution be sufficient. The choice $\eta_2 = 0$ and $\eta_1 = 1$ is a solution. Therefore, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 \left[t \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].$$

To find the solution, the initial data must be applied. At $t = 0$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We find $c_1 = -4$ and $c_2 = 14$, and the solution is

$$\mathbf{x}(t) = \begin{pmatrix} 2 + 42t \\ -4 - 14t \end{pmatrix}.$$

3.1 Equilibrium Points

Consider the autonomous system of ordinary differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{25}$$

Definition 3.1 (Equilibrium point)**.** An equilibrium(a critical) point of the system (25) is a point (x_*, y_*) such that

$$f(x_*, y_*) = g(x_*, y_*) = 0$$

Let $\mathbf{x} = (x(t), y(t)), \mathbf{x}_0 = (x_0, y_0)$, and $\mathbf{x}_* = (x_*, y_*)$. A critical point x_* is **stable** provided that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\mathbf{x}_0 - \mathbf{x}_*\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{x}_*\| < \varepsilon \text{ for all } t > 0.$$

The critical point (x_*, y_*) is called **unstable** if it is not stable.
A critical point x_* is **asymptotically stable** if there exists $\delta > 0$ such that

$$\|\mathbf{x}_0 - \mathbf{x}_*\| < \delta \Rightarrow \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_*$$

Let λ_1 and λ_2 be the eigenvalues of

$$J(\mathbf{x}_*) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{\mathbf{x}_*}$$

Then the stability of the equilibrium points of the system (25) is summarized in table 1.

3.2 Stability

Table 1: Stability properties of linear systems

Eigenvalue	Type of critical point	Stability
$\lambda_1, \lambda_2 > 0$	Improper node	Unstable
$\lambda_1, \lambda_2 < 0$	Improper node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1 > 0$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = \lambda \pm i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$\lambda_1, \lambda_2 = \pm i\mu$	Centre	Stable

Table 1 provides information on the type of stability that could result based on the nature of the eigenvalues of a linear system.

3.3 The Routh-Hurwitz Criteria

Given a polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$$

where the coefficients a_i are real constants,$i = 1, 2, \ldots, n$ define the n Hurwitz matrices using the coefficients of the characteristic polynomial:

$$H_1 = \begin{pmatrix} a_1 \end{pmatrix}, H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \cdots$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \cdot & \cdot & 0 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & a_n \end{pmatrix}$$

where $a_j = 0$ for $j > n$. All the roots of polynomial $P(\lambda)$ are negative or have negative real parts iff the determinants of all Hurwitz matrices are positive:

$$\det H_j > 0, \quad j = 1, 2, \ldots, n$$

For $n = 2$, $\det H_1 = a_1 > 0$ and $H_2 = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$ or $a_1 > 0$ and $a_2 > 0$. For polynomials of degree $2, 3, 4$ and 5, the Routh Hurwitz Criteria are summarized as follows:

$$n = 2 : a_1 > 0 \text{ and } a_2 > 0 \tag{26}$$

$$n = 3 : a_1 > 0, a_3 > 0 \text{ and } a_1 a_2 - a_3 > 0 \tag{27}$$

$$n = 4 : a_1 > 0, a_3 > 0, a_4 > 0 \text{ and } a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0 \tag{28}$$

$$n = 5 : a_i > 0, i = 1, 2, 3, 4 \text{ and } 5, a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0, \tag{29}$$

and

$$(a_1 a_4 - a_5)(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) - a_5(a_1 a_2 - a_3)^2 - a_1 a_5^2 > 0 \tag{30}$$

References

[1] [W.G. Kelley and A.C. Peterson] The Theory of Differential Equations: Classical and Qualitative.

[2] [Coddington and Levinson] Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.