Basel problem using Fourier Series

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1 Fourier series

For any function $f(x)$ its fourier representation is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \qquad for \ -L \leq x \leq L \tag{1}$$

Where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

But if $f(x)$ even function, then its fourier representation is as follows

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad for \ -L \leq x \leq L \tag{2}$$

Where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

2 Basel problem

Theorem 2.1. *The Fourier series representation of x^2 on $-1 \leq x \leq 1$ is*

$$\frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2\pi^2}\cos(n\pi x)$$

Proof. Since x^2 is even function from (2) we have the following

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \qquad for \ -1 \le x \le 1 \tag{3}$$

Where

$$\begin{aligned} a_0 &= \int_{-1}^{1} f(x)dx \\ &= 2\int_{0}^{1} x^2 dx \\ &= 2\left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{2}{3} \end{aligned}$$

And

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx \\ &= \int_{-1}^{1} x^2 \cos(n\pi x)dx \\ &= 2\int_{0}^{1} x^2 \cos(n\pi x)dx \end{aligned}$$

Now, use integration by parts to evaluate the integral. Finally you will get

$$a_n = (-1)^n \frac{4}{n^2\pi^2}$$

Substituting a_0 and a_n in (3) gives the desired results.

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

□

Corollary 2.2 (Basel Problem)**.**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. Substitute $x = 1$ in Theorem 2.1 □

References

[1] [Ronald Bracewell] *The Fourier Transform and Its Applications.* Third Edition, 1999.

[2] [E. M. Stein and G. Weiss] *Introduction to Fourier Analysis on Euclidean Spaces.*

[3] [G. B. Folland] *Fourier Analysis and Its Applications.* 1992.