Mathematical Proofs

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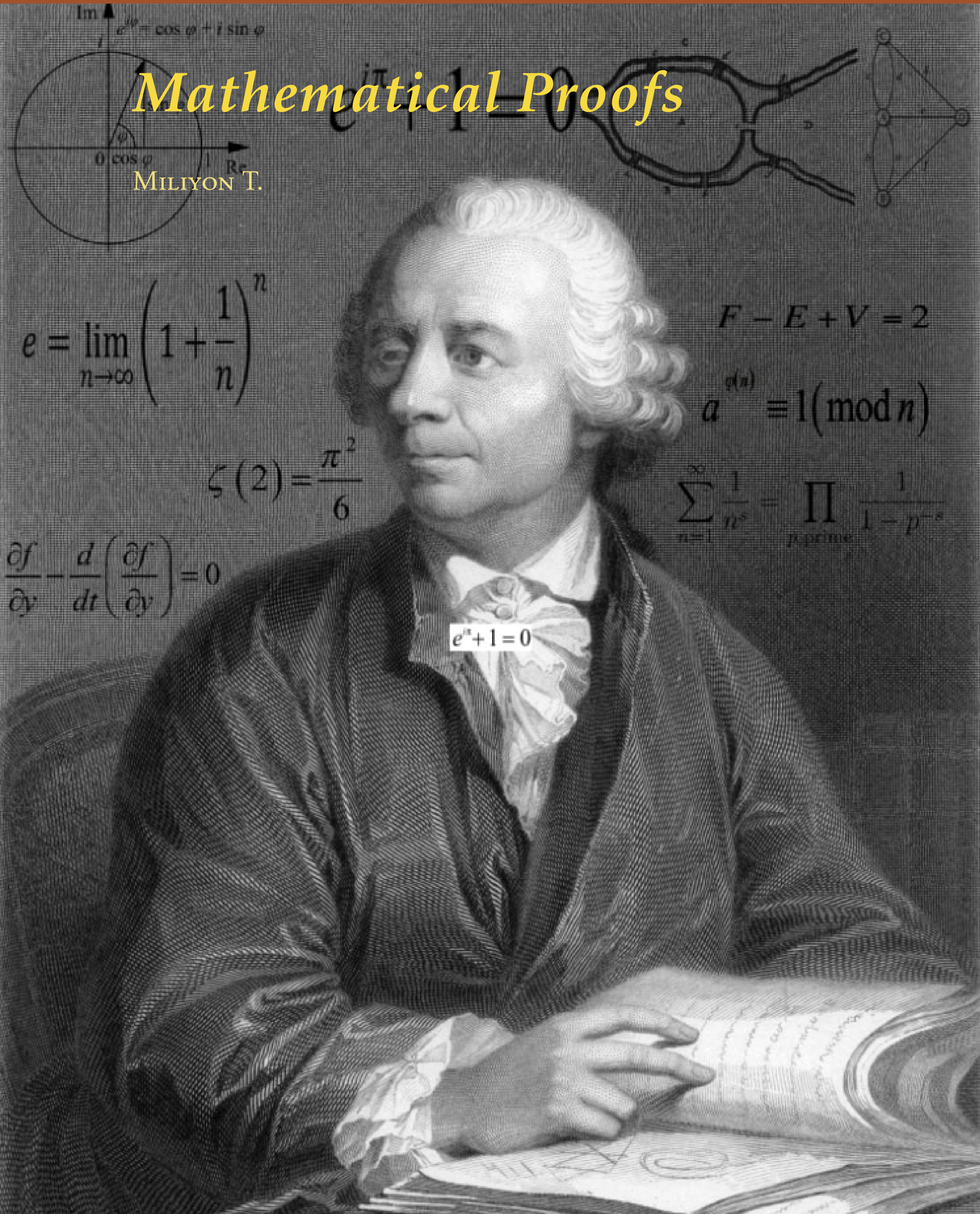

Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works do consist of proofs, just as poems do consist of characters.

-Vladimir Arnold

Contents

Preface

"Proof is the glue that holds mathematics together."
- Sir Michael Atiyah

Proofs in mathematics are indispensable. Mathematics without proof is just a life without oxygen. Proving something is just convincing something is true to someone or to ourselves. Despite all the other types of proof, mathematical proofs are irrefutable. Once something is proved to be true mathematically in a *clear manner*, it's hard (impossible) to disprove it.

There are different ways of proving in mathematics; proof by contradiction, proof by *induction*, *direct proof*, *indirect proof*, proof by *contra positive*, proof by *exhaustion* and so on. From all those methods(ways) of proving, a proof by contradiction is as the British mathematician G.H Hardy quote "mathematicians' finest weapon", the same method that people thought for centuries and yet that it was used by Euclid, to show that Primes goes forever(which isn't true[1]).

There are dozens of beautiful proofs in mathematics and most of them are the contribution of the great Leonhard Euler.

$$\zeta(2) = \frac{\pi^2}{6},$$
$$e^{i\pi} + 1 = 0,$$
$$\zeta(s) = \prod_{p\in\mathbb{P}}\left(\frac{1}{1-\frac{1}{p^s}}\right)$$

are some of Euler's beautiful result.

The Hungarian mathematician Paul Erdos spent his whole life in finding an *elementary proof*[2] what he call a "*proof from the book*". The proofs presented in this book are simple as they are given at high school and scarcely at undergraduate level.

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[1] Prime Simplicity by Michael Hardy and Catherine Woodgold
[2] An Elementary proof is a proof without the concept of complex analysis.

Acknowledgment

First of all I would like to thank the almighty God to whom I owe my breath.

To Zer
Abe
Fiker

ons

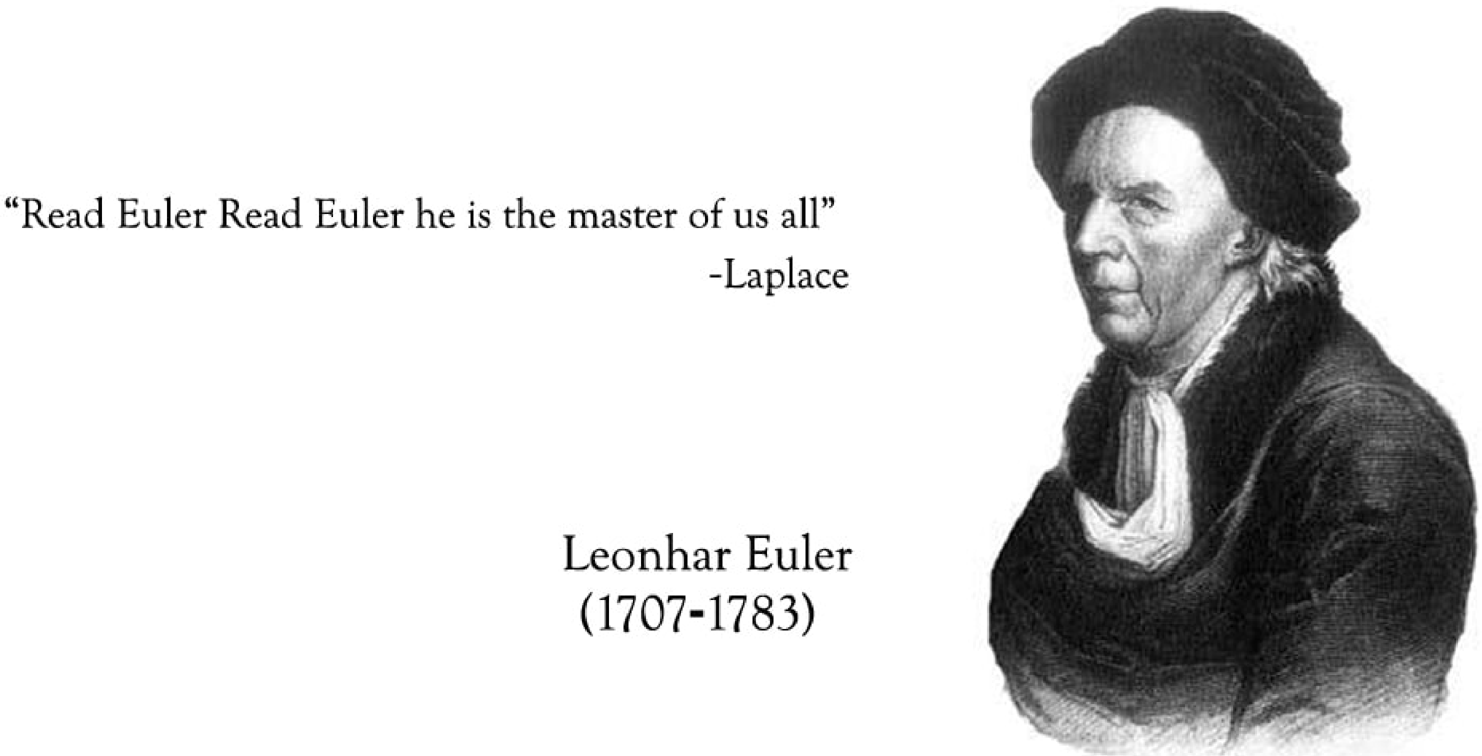
$\in$	Membership in a set
$A \times B$	Cartesian product
$\ni$	Such that
$\cap$	Intersection
$\cup$	Union
$\Rightarrow$	Implies
$\Leftrightarrow$	If an only if
$\subset$	Subset
$\emptyset$	Null set
$\exists$	There exists
$\forall$	For all
$\square$	End of a proof
aH	Left coset
Ha	Right coset
$(G,*)$	Group
e	Identity element
a^{-1}	The inverse of a
G/H	Quotient group
$\lhd$	Normal subgroup
$\|G\|$	Number of elements in G
$\mathbb{Z}$	The of Integers
$\mathbb{Q}$	The set of Rational numbers
$\mathbb{Q}^{\times}$	$\mathbb{Q} - \{0\}$
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers

"For example is not a proof."

-Jewish Proverb

Part I

Main



1.1 Euclid's Theorem

Theorem 1.1: (Euclid)

There are an infinite number of primes.

Proof. Write the primes $2, 3, 5, 7, 11, \ldots$ in ascending order. For any particular prime p, consider the number $N = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots p) + 1$. That is, form the product of all the primes from 2 to p, and increase this product by one. Because $N > 1$, we can use the fundamental theorem to conclude that N is divisible by some prime q. But none of the primes $2, 3, 5, \cdots, p$ divides N. For if q were one of these primes, then on combining the relation $q \mid 2 \cdot 3 \cdot 5 \cdots p$ with $q \mid N$, we would get $q \mid (N - 2 \cdot 3 \cdot 5 \cdots p)$, or what is the same thing, $q \mid 1$. The only positive divisor of the integer 1 is 1 itself, and since $q > 1$, the contradiction is obvious. Consequently, there exists a new prime q larger than p. $\square$

Lemma 1.1: (Euclid's Lemma)

Any composite number is divisible by a prime.

Proof. For a composite number n, there exists an integer d satisfying the conditions $d \mid n$ and $1 < d < n$. among all such integers d, choose p to be the smallest. Then p must be a prime number. Otherwise, it too would possess a divisor q with $1 < q < p$; but $q \mid p$ and $p \mid n$ implies that $q \mid n$, which contradicts our choice of p as the smallest divisor, not equal to 1, of n. Thus, there exists a prime p with $p \mid n$. $\square$

Theorem 1.1

If p is a prime and $p \mid ab$, then either $p \mid a$ or $p \mid b$.

Proof. If $p \mid a$, then we need go no further, so let us assume that $p \nmid a$. Since the only positive divisors of p (hence, the only candidates for the value of $\gcd(a, p)$.) are 1 and p itself, this implies that $\gcd(a, p) = 1$. Citing Euclid's lemma, it follows immediately that $p \mid b$. □

Corollary 1.1

For any prime p if $p|a^n$, then $p|a$.

1.2 Fundamental Theorem of Arithmetic

Theorem 1.2: (Fundamental Theorem of Arithmetic)

Every positive integer $n > 1$ is either a prime or can be expressed as a product of primes; this representation is unique, apart from the order in which the factors occur.

Proof. Either n is a prime or it is composite. In the first case there is nothing to prove. If n is composite, then there exists a prime divisor of n, as we have shown. Thus, n may be written as $n = p_1 n_1$, where p_1 is prime and $1 < n_1 < n$. If n_1 is prime, then we have our representation. In the contrary case, the argument is repeated to produce a second prime number p_2 such that $n_1 = p_2 n_2$; that is,

$$n = p_1 p_2 n_2; 1 < n_2 < n_1 :$$

If n_2 is a prime, then it is not necessary to go further. Otherwise, write $n_2 = p_3 n_3$, with p_3 a prime; hence,

$$N = p_1 \cdot p_2 \cdot p_3 \cdot n_3; 1 < n_3 < n_2 :$$

The decreasing sequence $n > n_1 > n_2 > \cdots > 1$ cannot continue indefinitely, so that after a finite number of steps n_k is a prime,say p_k. This leads to the prime factorization

$$n = p_1 p_2 \cdots p_k :$$

The second part of the proof the uniqueness of the prime factorization is more difficult. To this purpose let us suppose that the integer n can be represented as a product of primes in two ways; say,

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s; r \leq s;$$

Where the p_i and q_j are all primes, written in increasing order, so that $p_1 \leq p_2 \leq \cdots \leq p_r$ and $q_1 \leq q_2 \leq \cdots \leq q_s$: Because $p_1 \mid q_1 q_2 \cdots q_s$, we know that $p_1|q_k$ for some value of k. Being a prime, q_k has only two divisors, 1 and itself. Because p_1 is greater than 1, we must conclude that $p_1 = q_k$; but then it must be that $p_1 \geq q_1$. An entirely similar argument (starting with q_1 rather than p_1) yields $q_1 \geq p_1$, so that in fact $p_1 = q_1$. We can cancel this common factor and obtain

$$p_2 p_3 \cdots p_r = q_2 q_3 \cdots q_s :$$

Now repeat the process to get $p_2 = q_2$; cancel again, to see that

$$p_3 p_4 \cdots p_r = q_3 q_4 \cdots q_s :$$

Continue in this fashion. If the inequality $r < s$ held, we should eventually arrive at the equation

$$1 = q_{r+1} q_{r+2} \cdots q_s;$$

Which is absurd, since each $q_i > 1$. It follows that $r = s$ and that

$$p_1 = q_1; p_2 = q_2, \cdots, p_r = q_r;$$

This makes the two factorizations of n identical. □

1.3 The Division Algorithm

Definition 1.3: (Well Ordered)

A nonempty set S of real numbers is said to be well-ordered if every nonempty subset of S has a least element.

Remark 1.3

Every nonempty finite set of real numbers is well-ordered.

Definition 1.3: (The Well-Ordering Principle)

The set $\mathbb{N}$ of positive integers is well-ordered.

Theorem 1.3

For each integer m, the set

$$S = \{i \in \mathbb{Z} : i \geq m\}$$

is well-ordered.

Proof. We need only show that every nonempty subset of S has a least element. So let T be a nonempty subset of S. If T is a subset of $\mathbb{N}$, then, by **the Well-Ordering Principle**, T has a least element. Hence we may assume that T is not a subset of $\mathbb{N}$. Thus $T - \mathbb{N}$ is a finite nonempty set and so contains a least element t. Since $t \leq 0$, it follows that $t \leq x$ for all $x \in T$; so t is a least element of T. □

Theorem 1.3: (The Division Algorithm)

Let a be any integer and b a positive integer. Then there exist unique integers q and r such that

$$a = qb + r \qquad \text{where } 0 \leq r < b$$

Proof. The proof consists of two parts. First, we must establish the existence of the integers q and r, and then we must show they are indeed unique.

1. Existence

   Consider the set $S = \{a - bn | \; n \in \mathbb{Z} \text{ and } a - bn \geq 0\}$. Clearly, $S \subset \mathbb{W}$. We shall show that S contains a least element. To this end, first we will show that S is a non empty subset of $\mathbb{W}$:

   Case 1: Suppose $a \geq 0$. Then $a = a - b \cdot 0 \in S$, so S contains an element.

   Case 2: Suppose $a < 0$. Since $b \in \mathbb{Z}^+, b \geq 1$. Then $-ba \geq -a$; that is, $a - ba \geq 0$.

   Consequently, $a - ba \in S$. In both cases, S contains at least one element, so S is a nonempty subset of $\mathbb{W}$. Therefore, by theorem (1.3), S contains a least element r. Since $r \in S$, an integer q exists such that $r = a - bq$, where $r \geq 0$.

   To show that $r < b$: We will prove this by contradiction. Assume $r \geq b$. Then $r - b \geq 0$. But $r - b = (a - bq) - b = a - b(q + 1)$. Since $a - b(q + 1)$ is of the form $a - bn$ and is greater than 0, $a - b(q + 1) \in S$; that is, $r - b \in S$. Since $b > 0$, $r - b < r$. Thus, $r - b$ is smaller than r and is in S. This contradicts our choice of r, so $r < b$. Thus, there are integers q and r such that $a = bq + r$, where $0 \leq r < b$.

2. Uniqueness

   We would like to show that the integers q and r are unique. Assume there are integers q, q', r, and r' such that $a = bq + r$ and $a = bq' + r'$, where $0 \le r < b$ and $0 \le r' < b$.

   Assume, for convenience, that $q \ge q'$. Then $r - r' = b(q - q')$. Because $q \ge q'$, $q - q' \ge 0$ and hence $r - r' \ge 0$. But, because $r < b$ and $r' < b$, $r - r' < b$. Suppose $q > q'$; that is, $q - q' \ge 1$. Then $b(q - q') \ge b$; that is, $r - r' \ge b$. This is a contradiction because $r - r' < b$. Therefore, $q \not> q'$; thus, $q = q'$, and hence, $r = r'$. Thus, the integers q and r are unique, completing the uniqueness proof.

□

1.4 Curious Sum

Theorem 1.4: (Gauss Sum)

For any positive integer n we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{1.1}$$

This result has been known by mathematicians[1] far more from Gauss. But there is a beautiful mystical story about this problem and how the child prodigy **Carl Friedreich Gauss** approached the problem. The story goes like this... when Gauss was only ten his math teacher was bored of teaching and he asked his students to add the numbers from 1 up to 100. Eventually, Gauss came up with the correct answer less than in a minute.

This is how Gauss did it.

$$\begin{array}{ccccccccc} & 1 & +2 & +3 & +\cdots & +98 & +99 & +100 \\ + & 100 & +99 & +98 & +\cdots & +3 & +2 & +1 \\ \hline = & 101 & +101 & +101 & +\cdots & +101 & +101 & +101 \end{array}$$

Clearly, as we can see from the above there are hundred 101 summands. Thus

$$2(1 + 2 + 3 + \cdots + 100) = 100(101)$$
$$(1 + 2 + 3 + \cdots + 100) = \frac{100(101)}{2}$$

The proof of Theorem 1.4 is now trivial. Just put n in place of 100.

Gauss sum can be extended to non consecutive equally spaced numbers. i.e. Sequences of numbers with common difference. Suppose

$$A_1 + (A_1 + d) + (A_1 + 2d) + \cdots + (A_1 + (n-1)d)$$

be the sum of sequence of numbers with common difference d.

Now, let's apply Gauss sum

$$\begin{array}{cccccc} & A_1 & +(A_1+d) & +\cdots & +(A_1+(n-2)d) & +(A_1+(n-1)d) \\ + & (A_1+(n-1)d) & +(A_1+(n-2)d) & +\cdots & +(A_1+d) & +A_1 \\ \hline = & (2A_1+(n-1)d) & +(2A_1+(n-1)d) & +\cdots & +(2A_1+(n-1)d) & +(2A_1+(n-1)d) \end{array}$$

There are n number of $(2A_1 + (n-1)d)$.

$$2(A_1 + (A_1 + d) + (A_1 + 2d) + \cdots + (A_1 + (n-1)d)) = n(2A_1 + (n-1)d) \tag{1.2}$$
$$A_1 + (A_1 + d) + (A_1 + 2d) + \cdots + (A_1 + (n-1)d) = \frac{n}{2}(2A_1 + (n-1)d) \tag{1.3}$$

[1] In 499 AD Arybhata gave a formula for the sum of the first n integers. Brahamagupta extended Arybhta's result to squares and cubes of the first n integers.

Let's do a substitution $A_n = A_1 + (n-1)d$ just to simplify things. Now (1.3) becomes

$$A_1 + A_2 + A_3 + \cdots + A_n = \frac{n}{2}(2A_1 + (n-1)d), \qquad \text{where} \qquad d = A_i - A_{i-1} \tag{1.4}$$

Notation 1.4

$$S_n := A_1 + A_2 + A_3 + \cdots + A_n = \frac{n}{2}(2A_1 + (n-1)d) \tag{1.5}$$

where A_i is i^{th} term in the sequence $A_1, A_2, A_3, \cdots, A_n$.

$$D_n := A_1 - A_2 - A_3 - \cdots - A_n \tag{1.6}$$

where A_i is i^{th} term in the sequence $A_1, A_2, A_3, \cdots, A_n$.

Corollary 1.4: (Gauss Difference)

$$D_n = (2-n)A_1 - \frac{nd}{2}(n-1)$$

Proof. [2]Consider

$$\begin{aligned} D_n &= A_1 - A_2 - A_3 - \cdots - A_n \\ &= A_1 - (A_2 + A_3 + \cdots + A_n) \\ &= A_1 - (A_1 + A_2 + A_3 + \cdots + A_n - A_1) \\ &= A_1 - (S_n - A_1) \\ &= 2A_1 - S_n \end{aligned}$$

But from (1.5) we have $S_n = \frac{n}{2}(2A_1 + (n-1)d)$

$$D_n = 2A_1 - \frac{n}{2}(2A_1 + (n-1)d) \tag{1.7}$$

$$\therefore D_n = (2-n)A_1 - \frac{nd}{2}(n-1)$$

□

Theorem 1.4: (Arithmetic Sum)

$$S_n = \frac{n}{2}(A_1 + A_n) \tag{1.8}$$

Proof. We know

$$A_n = A_1 + (n-1)d$$

where A_1 is the first term of the arithmetic sequence, A_n is the n^{th} term and d is the common difference.

$$S_n = A_1 + A_2 + A_3 + \cdots + A_n$$

[2]Næl H/Mariam.

$$\begin{array}{cccccc}
S_n = & A_1 & +(A_1+d) & +\cdots & +(A_1+(n-2)d) & +(A_1+(n-1)d) \\
+\ S_n = & (A_1+(n-1)d) & +(A_1+(n-2)d) & +\cdots & +(A_1+d) & +A_1 \\
\hline
2S_n = & (2A_1+(n-1)d) & +(2A_1+(n-1)d) & +\cdots & +(2A_1+(n-1)d) & +(2A_1+(n-1)d)
\end{array}$$

$$\begin{aligned}
2S_n &= n(2A_1+(n-1)d) \\
S_n &= \frac{n}{2}(2A_1+(n-1)d) \\
&= \frac{n}{2}(A_1+A_1+(n-1)d) \\
&= \frac{n}{2}(A_1+A_n)
\end{aligned}$$

□

Theorem 1.4: (Geometric Sum)

$$G_1+rG_1+r^2G_1+\cdots+r^{n-1}G_1=G_1\left(\frac{1-r^n}{1-r}\right)$$

Proof. □

Theorem 1.4: (Basel Problem)

The sum of the reciprocal of the integers converges,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$$

Proof. Consider the Cardinal Sine function

$$\frac{\sin x}{x}, \quad \text{which has non zero roots at } \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \ldots$$

So we can write this function as infinite product of polynomials like this

$$\begin{aligned}\frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\left(1 - \frac{x}{4\pi}\right)\left(1 + \frac{x}{4\pi}\right)\cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots\end{aligned}$$

Expand this infinite product to get

$$\frac{\sin x}{x} = 1 + \left(-\frac{x^2}{\pi^2} - \frac{x^2}{4\pi^2} - \frac{x^2}{9\pi^2}\cdots\right) + \cdots$$

Since we are only interested on the coefficient of x^2

$$\begin{aligned}\frac{\sin x}{x} &= 1 - \frac{x^2}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right) + \cdots \\ &= 1 - \frac{\zeta(2)}{\pi^2}x^2 + \cdots \end{aligned} \tag{1.9}$$

But from Taylor expansion we know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7}! + \cdots$$

Divide both side by x then it becomes

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \tag{1.10}$$

Now equate the coefficients of x^2 in (1.9) and (1.10).

$$\begin{aligned}-\frac{\zeta(2)}{\pi^2} &= -\frac{1}{3!} \\ \zeta(2) &= \frac{\pi^2}{3!}.\end{aligned}$$

□

Euler solved this problem in 1735. Which is now known to be $\zeta(2)$. He also calculated the value of zeta function($\zeta(s)$) up to $s = 26$. which is

$$\zeta(26) = \sum_{n=1}^{\infty} \frac{1}{n^{26}} = \frac{1315862}{11094481976030578125}\pi^{26}$$

Furthermore, Euler generalized this for any even integer $2n$,

$$\zeta(2n) = \frac{(-1)^{n-1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where $n \in \mathbb{N}$ and B_{2n} is Bernoulli number.

Theorem 1.4: (Euler)

The sum of natural number converges.

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

Proof. From Geometric series we have

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \cdots \tag{1.11}$$

Evaluating at $x = -1$

$$\frac{1}{2} = 1 - +1 - 1 + \cdots$$

Take (1.11) again and differentiate both side

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Substitute $x = -1$

$$\frac{1}{4} = 1 - 2 + 3 - 4 + \cdots \tag{1.12}$$

Then, introduce

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \tag{1.13}$$

Multiplying by $\frac{2}{2^s}$

$$\frac{2}{2^s}\zeta(s) = \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} \cdots \tag{1.14}$$

Now subtract (1.14) from (1.13)

$$\zeta(s) - \frac{2}{2^s}\zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots\right) - \left(\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} \cdots\right)$$

$$(1 - \frac{2}{2^s})\zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots\right) - \left(\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} \cdots\right)$$

$$(1 - \frac{2}{2^s})\zeta(s) = (1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots)$$

Evaluate the last equation at $s = -1$

$$(1 - \frac{2}{2^{-1}})\zeta(-1) = (1 - \frac{1}{2^{-1}} + \frac{1}{3^{-1}} - \frac{1}{4^{-1}} + \cdots)$$

Which gives us

$$(1-4)\zeta(-1) = (1 - 2 + 3 - 4 + \cdots)$$

$$(-3)\zeta(-1) = (1 - 2 + 3 - 4 + \cdots)$$

But from (1.12) we know that $1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$.
And from (1.13) we could easily derive $\zeta(-1) = (1 + 2 + 3 + 4 + \cdots)$. Thus

$$(-3)\zeta(-1) = \frac{1}{4}$$

Finally we get

$$(1 + 2 + 3 + 4 + \cdots) = -\frac{1}{12}$$

□

1.5 Partitions

D(n) is the number of ways of writing n as the sum of distinct whole numbers.
O(n) is the number of ways of writing n as the sum of (not necessarily distinct)odd numbers.

Example 1.5

Find $D(7)$ and $O(7)$
The partition of 7 into odd parts

$$7, \quad 5+1+1, \quad 3+3+1, \quad 3+1+1+1+1, \quad 1+1+1+1+1+1+1$$

The partition of 7 into distinct parts

$$7, \quad 6+1, \quad 5+2, \quad 4+3, \quad 4+2+1$$

Thus $D(7) = O(7) = 5$.

Theorem 1.5: (Distinct sum is equal to Odd sum)

$$O(n) = D(n) \tag{1.15}$$

(Non-standard proof). Introduce

$$\begin{aligned} P(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= 1 + x + x^2 + (x^3 + x^{2+1}) + (x^4 + x^{3+1}) + (x^5 + x^{4+1} + x^{3+2}) + \cdots \end{aligned}$$

So

$$P(x) = 1 + \sum_{n=1}^{\infty} D(n)x^n \tag{1.16}$$

Introduce

$$\begin{aligned} Q(x) &= \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} \cdots \\ &= (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)\cdots \\ &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots) \cdot (1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots) \cdot \\ &\quad (1 + x^5 + x^{5+5+5} + x^{5+5+5} + \cdots)\cdots \end{aligned}$$

So

$$Q(x) = 1 + \sum_{n=1}^{\infty} O(n)x^n \tag{1.17}$$

What we have done so far is we introduce two function $P(x)$ and $Q(x)$. Additionally we have proved that they are actually equal to the following infinite sums.

$$P(x) = (1+x)(1+x^2)(1+x^3)\cdots = 1 + \sum_{n=1}^{\infty} D(n)x^n$$

$$Q(x) = \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} \cdots = 1 + \sum_{n=1}^{\infty} O(n)x^n$$

Our aim is to show $D(n) = O(n)$. WLOG, suppose our generating functions $P(x)$ and $Q(x)$ are equal.

$$P(x) = Q(x)$$

$$1 + \sum_{n=1}^{\infty} D(n)x^n = 1 + \sum_{n=1}^{\infty} O(n)x^n$$

$$\Rightarrow D(n) = O(n)$$

Now, we are only expected to show our assumption $P(x) = Q(x)$ is true.

Let's pick $P(x)$ and do some trick

$$\begin{aligned} P(x) &= (1+x)(1)(1+x^2)(1)(1+x^3)\cdots \\ &= (1+x)\left(\frac{1-x}{1-x}\right)(1+x^2)\left(\frac{1-x^2}{1-x^2}\right)(1+x^3)\cdots \\ &= \frac{\cancel{(1+x)(1-x)}\cancel{(1+x^2)(1-x^2)}(1+x^3)(1-x^3)(1+x^4)(1-x^4)}{(1-x) \quad \cdot \quad \cancel{(1-x^2)} \quad \cdot \quad (1-x^3) \quad \cdot \quad \cancel{(1-x^4)}}\cdots \end{aligned}$$

If we keep multiplying by this pattern the entire numerator will cancel out and becomes 1. All the expressions with even power will cancel out and the odds left in the de-numerator. Like this

$$= \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} \cdots$$

which is equal to $Q(x)$.
Hence we can conclude that

$$D(n) = O(n).$$

$\square$

(Standard proof). The generating function for $D(n)$ beautifully factored into[3]

$$\sum_{n=0}^{\infty} D(n)q^n = \prod_{n=1}^{\infty}(1+q^n) \tag{1.18}$$

The generating function for $O(n)$ beautifully factored into

$$\sum_{n=0}^{\infty} O(n)q^n = \prod_{n \text{ odd}} \frac{1}{(1-q^n)} \tag{1.19}$$

Now,

$$\begin{aligned} \prod_{n=1}^{\infty}(1+q^n) &= (1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)(1+q^6)(1+q^7)\cdots \\ &= \left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^4}{1-q^2}\right)\left(\frac{1-q^6}{1-q^3}\right)\left(\frac{1-q^8}{1-q^4}\right)\left(\frac{1-q^{10}}{1-q^5}\right)\left(\frac{1-q^{12}}{1-q^6}\right)\cdots \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)}\cdots \qquad \text{(by cancelling common factors)} \\ &= \prod_{n \text{ odd}} \frac{1}{(1-q^n)} \end{aligned}$$

Thus the generating functions are identical; hence for every $n \geq 0$, equation (1.15) is true. $\square$

[3] George Andrew's Integer partition.

1.6 Irrational Numbers

Theorem 1.6

The n^{th} root of a prime is irrational.

Proof. Suppose not. i.e suppose it is rational, thus we can write $\sqrt[n]{p} = \frac{a}{b}$ where $n \in \mathbb{Z} \geq 2$ and $a, b \in \mathbb{Z}$ and they are relatively prime. Taking a power n both side gives

$$p = \frac{a^n}{b^n} \tag{1.20}$$

$$pb^n = a^n$$

$$p \mid a^n \Rightarrow a \neq 1$$

From Fundamental theorem of Arithmetic

$$a = \prod_{i=1}^{k} p_i \tag{1.21}$$

$$a = p_1 \cdot p_2 \cdot p_3 \cdots p_k, k \geq 1$$

$$\Rightarrow p|(p_1 \cdot p_2 \cdot p_3 \cdots p_k)^n$$

This implies p divides p_i for some i between 1 and k.
Prime number divides prime number

$$\Rightarrow p = p_i$$

Thus, $p \mid a$ since $p_i \mid a$

$$\because p \mid a^n \Rightarrow p|a$$

Now we can write a as $a = pk$, where $k \in \mathbb{Z}$. Let's substitute this on (1.20).

$$p = \frac{(pk)^n}{b^n}$$

$$pb^n = p^n \cdot k^n$$

$$b^n = p^{n-1} \cdot k^n = p \cdot p^{n-2} k^n$$

$$b^n = p \cdot p^{n-2} k^n$$

Which implies $p \mid b^n$ then by similar argument as the above we can easily show that $p \mid b$. Now we have shown that $p \mid a$ and $p \mid b$ but this contradict the fact that a and b are relatively prime.
Hence our assumption that $\sqrt[n]{p}$ is rational is wrong.
$\therefore \sqrt[n]{p}$ is irrational. □

Corollary 1.6

$\sqrt{2}$ is Irrational.

Proof. Suppose not. i.e. $\sqrt{2}$ is rational. So it can be written as $\frac{a}{b}$ in which a and b are relatively prime($\gcd(a,b) = 1$).

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2 \tag{1.22}$$

This implies

$$2|a^2$$

But from (1.1)

$$2|a^2 \Rightarrow 2|a$$

Hence a is even there for we can write a as $a = 2k$. Let's substitute this fact into (1.22)

$$\Rightarrow (2k)^2 = 2b^2$$

$$\Rightarrow 4k^2 = 2b^2$$

$$\Rightarrow b^2 = 2k^2$$

This implies

$$2|b^2$$

By the same reason as the above $2|b$. In other word b is even. Thus, a contradiction! a and b are not relatively prime.

Hence our assumption $\sqrt{2}$ is rational is wrong. Therefore $\sqrt{2}$ is irrational. □

Chapter 2

Geometry

Euclid
(unkown)

2.1 Pythagoras Formula

Theorem 2.1: (Pythagoras)

Let a, b and c be the three sides of a right triangle, then

$$a^2 + b^2 = c^2 \tag{2.1}$$

Proof 1. Look at Figure 2.1

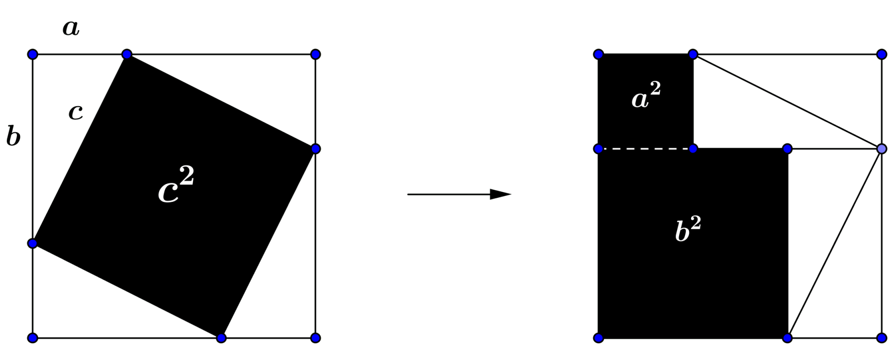


Figure 2.1: Proof of Pythagoras Formula #1

□

Proof 2. [1] Consider the trapezoid $ABCD$ shown below

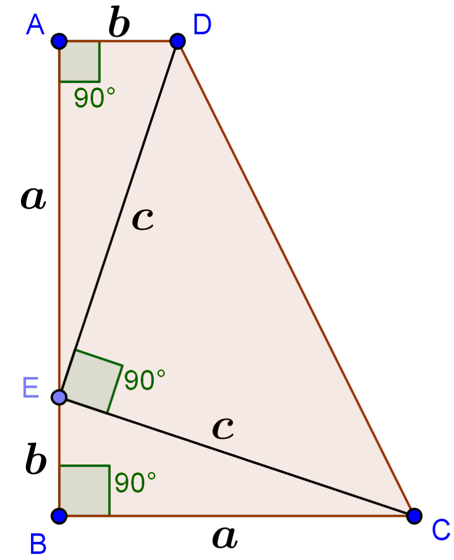


Figure 2.2: Proof of Pythagoras Formula #2

The area of the trapezoid $ABCD$ is the sum of the three triangles inside it.

$$A_{ABCD} = A_{\triangle ADE} + A_{\triangle DCE} + A_{\triangle BCE}$$

Thus

$$\begin{aligned}
\frac{1}{2}(a+b)(a+b) &= \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab \\
(a+b)^2 &= 2ab + c^2 \\
a^2 + 2ab + b^2 &= 2ab + c^2 \\
a^2 + b^2 &= c^2
\end{aligned}$$

□

Proof 3. Here is another proof without word (see Figure 2.3) □

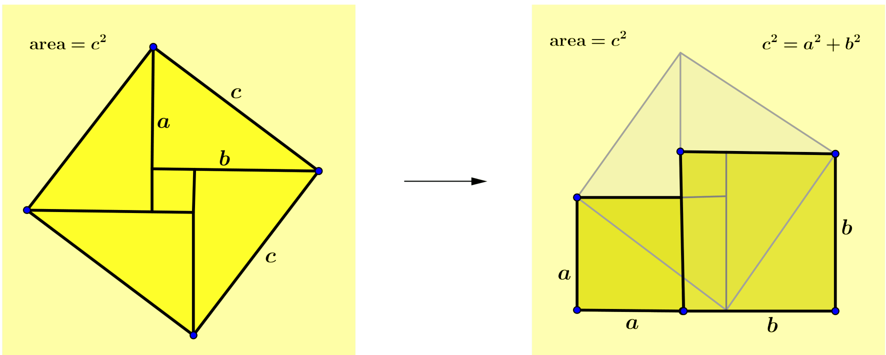


Figure 2.3: Proof of Pythagoras Formula #3

[1]This proof is due to James A. Garfield, the twentieth president of the United States, he developed the proof in 1876.

2.2 Trigonometric Addition Rule

Theorem 2.2: (Sine Addition)

Let α and β be two angles, then

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Proof. Let α and β be measures of positive angle such that $\alpha+\beta < 90°$ (consider figure 2.4).

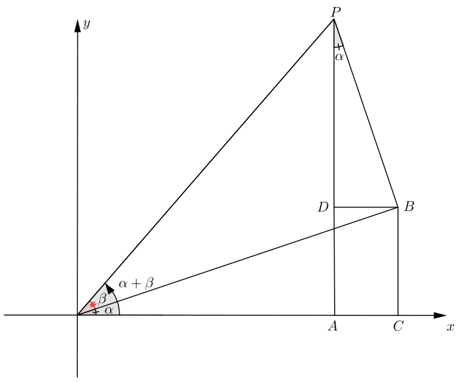


Figure 2.4: Sine Addition

Now $\measuredangle APB = \alpha$ since corresponding sides (OA and AP, OB and BP) are perpendicular. Then

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{AP}{OP} = \frac{AD+DP}{OP} = \frac{CB+DP}{OP} = \frac{CB}{OP} + \frac{DP}{OP} = \frac{CB}{OB}\frac{OB}{OP} + \frac{DP}{BP}\frac{BP}{OP}\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

□

Theorem 2.2: (Cosine Addition)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Proof. Consider figure 2.5

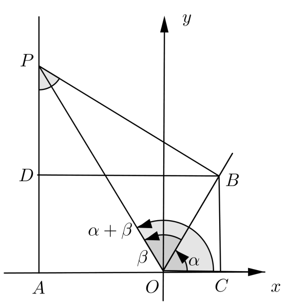


Figure 2.5: Cosine Addition

From which we get

$$\begin{aligned}\cos(\alpha+\beta) &= \frac{OA}{OP} = \frac{OC-AC}{OP} = \frac{OC-DB}{OP} = \frac{OC}{OP} - \frac{DB}{OP} = \frac{OC}{OB}\frac{OB}{OP} - \frac{DB}{BP}\frac{BP}{OP}\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

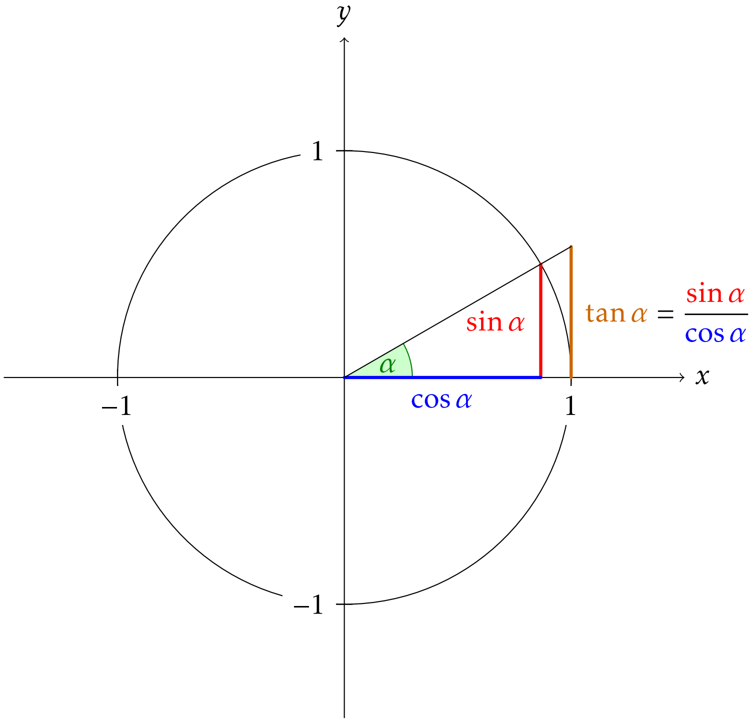
□

2.3 Trigonometric Identities

Theorem 2.3: (Pythagorean Identity)

$$\cos^2\alpha + \sin^2\alpha = 1 \tag{2.2}$$

Proof. Consider the right angle triangle inside the unit circle below



Using Pythagoras formula with $a = \sin\alpha, b = \cos\alpha, c = 1$. The identity follows

$$\cos^2\alpha + \sin^2\alpha = 1 \tag{2.3}$$

Using the above identity with simple algebraic manipulation, we can derive the following identity.

$$1 + \cot^2\alpha = \csc^2\alpha \tag{2.4}$$

which is obtained by dividing (2.3) by $\sin^2\alpha$.

$$\tan^2\alpha + 1 = \sec^2\alpha \tag{2.5}$$

which is obtained by dividing (2.3) by $\cos^2\alpha$. □

2.4 Ptolomy's Theorem

Theorem 2.4: (Ptolomy)

$$AD \cdot BC = AC \cdot BD \tag{2.6}$$

Chapter 3

Algebra

al-Khwārizmī
(780-850)

3.1 Quadratic formula

Theorem 3.1

The quadratic equation $ax^2 + bx + c = 0$ $(a, b, c \in \mathbb{R}, a \neq 0)$ *has solutions*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. We use the method of completing the square to rewrite $ax^2 + bx + c$.

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x+\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 +\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}. \end{aligned}$$

Therefore $ax^2 + bx + c = 0$ can be rewritten as

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = 0,$$

which can in turn be rearranged as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots gives

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

which implies

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as required. □

3.2 Cubic formula

Theorem 3.2: (Euler's Derivation)

A solution to the depressed equation $x^3 = mx + n$ *is given by*

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}} + \sqrt[3]{\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}}.$$

Proof. Assume $x = \sqrt[3]{p} + \sqrt[3]{q}$, cube both sides:

$$\begin{aligned} x^3 &= p + 3\sqrt[3]{p^2 q} + 3\sqrt[3]{pq^2} + q \\ &= 3\sqrt[3]{pq}(\sqrt[3]{p} + \sqrt[3]{q}) + (p+q) \\ &= 3\sqrt[3]{pq}x + (p+q). \end{aligned}$$

The resulting equation obviously has the same structure as the original depressed cubic equation. This suggests letting $3\sqrt[3]{pq} = m$ and $p + q = n$; and subsequently finding p and q and then also $x = \sqrt[3]{p} + \sqrt[3]{q}$. So, $4pq = 4m^3/27$ and from $(p+q)^2 = p^2 + 2pq + q^2 = n^2$. Combining these yields

$$(p^2 + 2pq + q^2) - 4pq = n^2 - \frac{4m^3}{27}$$

which simplifies to $(p-q)^2 = n^2 - \frac{4m^3}{27}$. Thus $p - q = \sqrt{n^2 - \frac{4m^3}{27}}$.
Adding and subtracting $p + q = n$, gives

$$2p = n + \sqrt{n^2 - \frac{m^3}{27}}$$

$$2q = n - \sqrt{n^2 - \frac{m^3}{27}}$$

So that the solution of the original cubic is $x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}} + \sqrt[3]{\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}}$. □

Theorem 3.2: (Tartagilia Derivation)

A solution for an equation in the form $x^3 + ax = b$, $a > 0$, $b > 0$ *is given by* $x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{-b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$.

Proof. Let $x = p - q$, $a = 3pq$ and $b = p^3 - q^3$.
This leads to a quadratic equation in q^3;

$$27(q^3)^2 + 27bq^3 - a^3 = 0$$

solving which gives

$$q^3 = \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

so that $q = \sqrt[3]{\frac{-b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$.
where the possibility of a negative q has been discarded as an unacceptable oddity. Quite similarly we

can find that $p = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$.

Thus $x = p - q = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} - \sqrt[3]{\frac{-b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$. $\square$

3.3 Heron's Formula

Theorem 3.3

The area of a triangle with sides a, b and c is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{a+b+c}{2} \tag{3.1}$$

3.4 Trigonometric Identities

3.5 Binet's Formula

Basic Introduction

Definition 3.5: (Golden Ratio)

The golden ratio (golden section) is defined as follows

$$\frac{a}{b} = \frac{a+b}{a}$$

or we can define the golden ratio using continued fraction

$$\phi := 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}} \tag{3.2}$$

The other way to define the golden ratio would be this

$$\phi := \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}} \tag{3.3}$$

Interesting Identities

The identity that we get from (3.2) directly

$$\phi = 1 + \frac{1}{\phi} \tag{3.4}$$

The following identity is derived from (3.3)

$$\phi^2 = 1 + \phi \tag{3.5}$$

Some Linear Algebra facts

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} \tag{3.6}$$

We are going to see the importance of (3.6) in proving the following lemma.

Lemma 3.5

For any $n \in \mathbb{N}$

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Proof. [Induction]
For $n = 1$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = I_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $n = k$. This step is called Induction hypothesis (IH).

$$\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $n = k+1$. But from (3.6) we have

$$\begin{aligned}
\begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (by\ IH) \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{k} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence by using principle of mathematical induction we can conclude that

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \forall n \in \mathbb{N}.$$

□

Generating Functions

Definition 3.5: (Generating Function)

The (ordinary) ***generating function*** *for the sequence* $a_0, a_1, a_2, \cdots$ *of real numbers is the formal power series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

or any equivalent closed form expression.

Note:

1. Generating functions[1] can be used to solve recurrence relations. As we will see soon.

2. Each sequence a_n defines a unique generating function $f(x)$, and conversely.

3. Generating functions are considered as algebraic forms and can be manipulated as such, without regard to actual convergence of the power series.

Example 3.5

The generating function for the sequence $1, 1, 1, 1, 1, \cdots$ *is*

$$f(x) = 1 + x + x^2 + x^3 + \cdots \tag{3.7}$$

As we know from geometric series the closed form of (3.7) is given by

$$f(x) = \frac{1}{1-x}$$

[1] They were invented in 1718 by French mathematician Abraham De Moivre (1667 – 1754).[Koshy, Catalan numbers with Application pp. 382]

Example 3.5

The generating function for the sequence $1, 1, 1, 1, 1, \ldots$ *is*

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Differentiating both sides of this expression produces

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

Thus, $\frac{1}{(1-x)^2}$ *is a closed form expression for the generating function of the sequence* $1, 2, 3, 4, \ldots$.

Recurrence Relation

Definition 3.5

A ***recurrence relation*** *for the sequence* $a_0, a_1, a_2, \ldots$ *is an equation relating the term* a_n *to certain of the preceding terms* a_i, $i \le n$, *for each* $n \ge n_0$.

Theorem 3.5: (Binet)

For any $n \in \mathbb{N}$ *the* n^{th} *term of the Fibonacci sequence is given by*

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right]$$

where ϕ *is a golden ratio.*

Combinatorial proof

Proof. The Fibonacci sequence is defined by the following recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

This can be rewritten as follows

$$a_n - a_{n-1} - a_{n-2} = 0 \tag{3.8}$$

Which is clearly a homogeneous equation. The characteristic[2](χ) equation of (3.8) is given by

$$\lambda^2 - \lambda - 1 = 0 \tag{3.9}$$

Thus using quadratic formula the solutions to (3.9) are

$$\lambda = \frac{1+\sqrt{5}}{2} \qquad \& \qquad \lambda = \frac{1-\sqrt{5}}{2}$$

Hence the general solution to the recurrence relation (3.8) is

$$a_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{3.10}$$

[2]Dr. Yirgalem

But from Fibonacci sequence we know that the values of a_0 and a_1 are 0 and 1 respectively. So by using this we are going to find the values of α and β
Substitute $n = 1$ in (3.10)

$$a_1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^1$$

But $a_1 = 1$, then we have

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1 \tag{3.11}$$

Substitute $n = 2$ in (3.10)

$$a_1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^2 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^2$$

But $a_2 = 1$, then we have

$$\alpha\left(\frac{3+\sqrt{5}}{2}\right) + \beta\left(\frac{3-\sqrt{5}}{2}\right) = 1 \tag{3.12}$$

From (3.11) and (3.12) we have two equations with two variable. Hence by using simultaneous equation or other method we will get the values of α and β to be $\frac{1}{\sqrt{5}}$ and $\frac{-1}{\sqrt{5}}$ respectively. Therefore we have

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

□

Proof by Induction

Proof. [3]For $n = 1$

$$\begin{aligned}
F_1 + F_2 &= \frac{1}{\sqrt{5}}\left[\phi - \frac{1}{\phi}\right] + \frac{1}{\sqrt{5}}\left[\phi^2 - \frac{1}{\phi^2}\right] \\
&= \frac{1}{\sqrt{5}}\left[2\phi - 1\right] + \frac{1}{\sqrt{5}}\left[\frac{\phi^4 - 1}{\phi^2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\sqrt{5}\right] + \frac{1}{\sqrt{5}}\left[\frac{(\phi^2-1)(\phi^2+1)}{\phi^2}\right] \\
&= 1 + \frac{1}{\sqrt{5}}\left[\frac{\phi(\phi^2-1)}{\phi^2}\right] \qquad (\phi^2 - 1 = \phi \; by \; (3.5)) \\
&= 1 + \frac{1}{\sqrt{5}}\left[\frac{\phi^2-1}{\phi}\right] = 1 + \frac{1}{\sqrt{5}}\left[\phi - \frac{1}{\phi}\right] \\
&= 1 + \frac{1}{\sqrt{5}}\left[\sqrt{5}\right] \\
&= 2 = F_3
\end{aligned}$$

For $n = k$

$$F_k + F_{k-1} = F_{k-2}$$

[3]Alfred and Ingmar

Now, for $n = k+1$

$$\begin{aligned}
F_{k+1}+F_{k+2} &= \frac{1}{\sqrt{5}}\left[\phi^{k+1}-\left(-\frac{1}{\phi}\right)^{k+1}\right]+\frac{1}{\sqrt{5}}\left[\phi^{k+2}-\left(-\frac{1}{\phi}\right)^{k+2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\phi^{k+2}+\phi^{k+1}-\left(-\frac{1}{\phi}\right)^{k+2}-\left(-\frac{1}{\phi}\right)^{k+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\phi^{k+1}(\phi+1)-\left(-\frac{1}{\phi}\right)^{k+1}\left(\left(-\frac{1}{\phi}\right)+1\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\phi^{k+1}(\phi^2)-\left(-\frac{1}{\phi}\right)^{k+1}\left(-\frac{1}{\phi}\right)^{2}\right] \qquad \left(\because \left(-\frac{1}{\phi}\right)^2=\left(-\frac{1}{\phi}\right)+1\right) \\
&= \frac{1}{\sqrt{5}}\left[\phi^{k+3}-\left(-\frac{1}{\phi}\right)^{k+3}\right] \\
&= F_{k+3}
\end{aligned}$$

Hence, by principles of mathematical induction it follows that

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^{n}-\left(-\frac{1}{\phi}\right)^{n}\right] \qquad \forall n \in \mathbb{N}.$$

□

Linear algebra proof

Theorem 3.5: (Binet)

A closed form of a Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}}\left[\phi_1^n - \phi_2^n\right] \tag{3.13}$$

Where $\phi_1 = \frac{1+\sqrt{5}}{2}$ *and* $\phi_2 = \frac{1-\sqrt{5}}{2}$.

Proof. From (3.5) we have

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

If some how we could diagonalize matrix A (i.e to write A in the form $A = PDP^{-1}$), taking any powers of A would be simple. Because we know that $A^n = PD^nP^{-1}$ and we would get such a simple formula for F_n. So let's start diagonalize A.
First let's find the eigenvalues. Which can be found as follows

$$\begin{aligned}
|A-\lambda I_2| &= \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= -\lambda(1-\lambda)-1 \\
&= \lambda^2-\lambda-1
\end{aligned}$$

Thus, by using quadratic formula we would get

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \phi_1 \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} = \phi_2$$

Now, let's find the corresponding eigenvectors
For $\lambda_1 = \phi_1$

$$(A - \lambda_1 I_2)V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$(A - \lambda_1 I_2)V = (A - \phi_1 I_2)V = \begin{bmatrix} -\phi_1 & 1 \\ 1 & 1-\phi_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

After multiplying the matrices in the left side and equating with the right side we will get the following system of equation

$$-\phi_1 v_1 + v_2 = 0 \tag{3.14}$$

$$v_1 + (1-\phi_1)v_2 = 0 \tag{3.15}$$

Multiplying (3.15) by ϕ_1

$$\phi_1 v_1 - v_2 = 0 \tag{3.16}$$

Using (3.14) and (3.16)

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ \phi_1 v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} v_1$$

Hence the corresponding eigenvector for λ_1 is $\begin{bmatrix} 1 \\ \phi_1 \end{bmatrix}$.

Similarly, one can find that the eigenvector for λ_2 is $\begin{bmatrix} 1 \\ \phi_2 \end{bmatrix}$. Now we are going to write matrix in a form $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$
Finally, let's compute

$$\begin{aligned}
\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}^{n-1} \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} \phi_2 & -1 \\ -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\phi_2 - \phi_1} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1} & 0 \\ 0 & \phi_2^{n-1} \end{bmatrix} \begin{bmatrix} \phi_2 - 1 \\ -\phi_1 + 1 \end{bmatrix} \\
&= \frac{1}{-\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1^{n-1}(\phi_2 - 1) \\ \phi_2^{n-1}(-\phi_1 + 1) \end{bmatrix} \\
&= -\frac{1}{\sqrt{5}} \begin{bmatrix} \phi_1^{n-1}(-\phi_1) + \phi_2^{n-1}(\phi_2) \\ \phi_1^{n}(-\phi_1) + \phi_2^{n}(\phi_2) \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \phi_1^n - \phi_2^n \\ \phi_1^{n+1} - \phi_2^{n+1} \end{bmatrix}
\end{aligned}$$

That was to be shown! □

Proof using generating function

Proof. [4]Consider a Fibonacci sequence which is given by the following recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2. \tag{3.17}$$

[4]The derivation of the solution by using a generating function is due to Abraham De Moivre (1718).

And F_0 and F_1 are defined to be 0 and 1 respectively. The generating function for Fibonacci sequence is

$$f(x) = F_0 + F_1 x + F_2 x^2 + \cdots = \sum_{i=0}^{\infty} F_i x^i$$

When we assign their respective numbers for each F_i's $f(x)$ becomes

$$f(x) = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots \tag{3.18}$$

The generating function (3.18) won't help us to solve the recurrence relation in (3.17). So, let's find the closed form. In order to find the closed form we are going to do some trick, which indeed based at how the Fibonacci sequence recursively defined. Here is what we are going to do, first write (3.18) as it is,

$$\begin{array}{ccccccccc}
f(x) & = & 0 & +1x & +1x^2 & +2x^3 & +3x^4 & +5x^5 & +\cdots \\
xf(x) & = & 0x & +1x^2 & +1x^3 & +2x^4 & +3x^5 & +5x^6 & +\cdots \\
x^2 f(x) & = & 0x^2 & +1x^3 & +1x^4 & +2x^5 & +3x^6 & +5x^7 & +\cdots \\
\hline
f(x) - xf(x) - x^2 f(x) & = & 0 & +x & +x^2\{1-1\} & +0\cdot x^3 & +0\cdot x^4 & +0\cdot x^5 & +\cdots
\end{array}$$

Clearly, on the right side we are left with x because the summands goes to 0. Hence

$$f(x) - xf(x) - x^2 f(x) = x$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2} \tag{3.19}$$

So far, we have found a generating function (3.19) for (3.17). Now, let's use method partial fraction

$$f(x) = \frac{-x}{x^2 + x - 1} = \frac{-x}{(x+\phi_1)(x+\phi_2)} = \frac{A}{(x+\phi_1)} + \frac{B}{(x+\phi_2)}$$

To find A and B we are going to solve the following system of equation

$$A + B = -1$$
$$\phi_2 A + \phi_1 B = 0$$

Which results $A = \frac{-\phi_1}{\sqrt{5}}$ and $B = \frac{\phi_2}{\sqrt{5}}$.Thus,

$$\begin{aligned}
f(x) &= -\frac{1}{\sqrt{5}} \frac{\phi_1}{(x+\phi_1)} + \frac{1}{\sqrt{5}} \frac{\phi_2}{(x+\phi_2)} \\
&= \frac{1}{\sqrt{5}} \left[\frac{\phi_2}{(x+\phi_2)} - \frac{\phi_1}{(x+\phi_1)} \right] \\
&= \frac{1}{\sqrt{5}} \left[\frac{1}{(\frac{x}{\phi_2}+1)} - \frac{1}{(\frac{x}{\phi_1}+1)} \right] = \frac{1}{\sqrt{5}} \left[\frac{1}{(-\phi_1 x + 1)} - \frac{1}{(-\phi_2 x + 1)} \right] \\
&= \frac{1}{\sqrt{5}} \left[\frac{1}{(1-\phi_1 x)} - \frac{1}{(1-\phi_2 x)} \right]
\end{aligned}$$

But we know that $\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$. Hence

$$\frac{1}{1-\phi_1 x} = 1 + \phi_1 x + \phi_1^2 x^2 + \phi_1^3 x^3 + \cdots$$

$$\frac{1}{1-\phi_2 x} = 1 + \phi_2 x + \phi_2^2 x^2 + \phi_2^3 x^3 + \cdots$$

Then, we will get

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{5}}\left[\frac{1}{(1-\phi_1 x)} - \frac{1}{(1-\phi_2 x)}\right] \\
&= \frac{1}{\sqrt{5}}\left[1+\phi_1 x+\phi_1^2x^2+\phi_1^3x^3+\cdots-(1+\phi_2 x+\phi_2^2x^2+\phi_2^3x^3+\cdots)\right] \\
&= \frac{1}{\sqrt{5}}\left[(\phi_1-\phi_2)x+(\phi_1^2-\phi_2^2)x^2+(\phi_1^3-\phi_2^3)x^3+\cdots\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}(\phi_1^n-\phi_2^n)x^n
\end{aligned}$$

Therefore,

$$F_n = \frac{1}{\sqrt{5}}\left[\phi_1^n - \phi_2^n\right]$$

□

3.6 Binomial Theorem

Theorem 3.6: (Newton)

For any $x, y \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Proof. Clearly, it's true for $n = 1$. Assume it is true for n. We are going to show that it is true for $n+1$

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k && \text{(by Induction Hypothesis)} \\
&= x\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k + y\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
&= x^{n+1}y^0 + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}
\end{aligned}$$

But

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} &= \binom{n}{0}x^ny^1 + \binom{n}{1}x^{n-1}y^2 + \cdots + \binom{n}{n-1}x^2y^n + \binom{n}{n}x^0y^{n+1} \\
&= \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^k + x^0y^{n+1}
\end{aligned}$$

Now

$$\begin{aligned}
(x+y)^{n+1} &= x^{n+1}y^0 + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^k + x^0y^{n+1} \\
&= x^{n+1}y^0 + x^0y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]x^{n+1-k}y^k
\end{aligned}$$

From Pascal's Identity(C.2) we have

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Thus

$$\begin{aligned}(x+y)^{n+1} &= x^{n+1}y^0 + x^0y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^k \\ &= x^0y^{n+1} + \sum_{k=0}^{n}\binom{n+1}{k}x^{n+1-k}y^k \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k\end{aligned}$$

Hence

$$(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k$$

That was to be shown. □

Augustin-Louis Cauchy
(1789-1857)

4.1 Limit and Continuity

Definition 4.1: Limit

Let f be a function defined at each point of some open interval containing a except possibly at a itself. The number L is the limit of $f(x)$ as x approaches a,

$$\lim_{x\to a} f(x) = L \quad \textit{iff} \quad (\forall \epsilon > 0)(\exists \delta = \delta(\epsilon) > 0) \ni |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Theorem 4.1: (Uniqueness Theorem)

If $f(x) \to p$ and $f(x) \to q$ as $x \to a$ then $p = q$.

Proof. Let $\frac{\epsilon}{2}$ be given. There exists δ_1 and δ_2 such that when

$$|x-a| < \delta_1 \Rightarrow |f(x) - p| < \frac{\epsilon}{2} \text{ and } |x-a| < \delta_2 \Rightarrow |f(x) - q| < \frac{\epsilon}{2}.$$

Set $\delta_m = \min\{\delta_1, \delta_2\}$. Then by the triangle inequality when

$$|x-a| < \delta_m \text{ we have } |p-q| \le |p - f(x)| + |f(x) - q| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and since ϵ was arbitrary $|p - q| \le 0 \Rightarrow p = q$. $\square$

Basic Properties of Limit

Theorem 4.1: (Sum and Difference Rule)

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a}[f(x) + g(x)] = L + M$$

Proof. WTS: $(\forall \epsilon > 0)(\exists \delta = \delta(\epsilon) > 0) \ni |x-a| < \delta \Rightarrow |f(x) + g(x) - (L+M)| < \epsilon$
Let $\epsilon > 0$ then because $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\text{If } 0 < |x-a| < \delta_1, \text{ then } |f(x) - L| < \epsilon/2.$$
$$\text{If } 0 < |x-a| < \delta_2, \text{ then } |g(x) - L| < \epsilon/2.$$

Now choose $\delta = \min(\delta_1, \delta_2)$, then we need to show that

$$|x-a| < \delta \Rightarrow |f(x) + g(x) - (L+M)| < \epsilon$$

Assume that $0 < |x-a| < \delta$ we then get

$$\begin{aligned} |f(x) + g(x) - (L+M)| &= |f(x) - L + g(x) - M| \\ &\le |f(x) - L| + |g(x) - M| && \text{(Triangular inequality)} \\ &\le \epsilon/2 + \epsilon/2 \le \epsilon \end{aligned}$$

Similarly, one can easily prove that $\lim_{x\to a}[f(x) - g(x)] = L - M$. □

Lemma 4.1

If $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that $|f(x)| < 1 + |L|$ whenever $0 < |x-a| < \delta$.

Proof. We know for any $\epsilon > 0$ there exist $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x-a| < \delta$.
Since this is true for any $\epsilon > 0$, it works for $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(x) - L| < 1$ whenever $0 < |x-a| < \delta$.
By the triangle inequality we get

$$\begin{aligned} |f(x)| &= |f(x) - L + L| \\ &\le |f(x) - L| + |L| \\ &< 1 + |L|, \end{aligned}$$

which gives the desired control on the values of f near a. □

Theorem 4.1: (Product Rule)

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We want to control $|f(x)g(x) - LM|$ by restricting x near a. By adding 0 in a clever way and using properties of the absolute value, we have that

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\le |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &= |f(x)|\,|g(x) - M| + |M|\,|f(x) - L|. \end{aligned}$$

Because $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we can control the terms $|f(x) - L|$ and $|g(x) - M|$. There is no problem controlling the constant term $|M|$. But what about controlling the term $|f(x)|$?

This is where Lemma (4.1) comes into play. Because $\lim_{x\to a} f(x) = L$, there is by Lemma (4.1) a $\delta_1 > 0$ such that $|f(x)| < 1 + |L|$ whenever $0 < |x - a| < \delta_1$.
We then have that

$$|f(x)|\,|g(x) - M| + |M|\,|f(x) - L| < (1 + |L|)|g(x) - M| + |M|\,|f(x) - L|.$$

We can control the term $(1 + |L|)|g(x) - M|$ because $\lim_{x\to a} g(x) = M$: there exists $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\epsilon}{2(1 + |L|)}$$

whenever $0 < |x - a| < \delta_2$.
We then have that

$$(1 + |L|)|g(x) - M| + |M|\,|f(x) - L| < \frac{\epsilon}{2} + |M|\,|f(x) - L|.$$

If $M = 0$, then by choosing $\delta < \min \delta_1, \delta_2$, for the restriction $0 < |x - a| < \delta$ we get the control

$$|f(x)g(x) - LM| < \frac{\epsilon}{2} + 0 < \epsilon.$$

If $M \neq 0$, then there is δ_3 such that $|f(x) - L| < \epsilon/|M|$ when $0 < |x - a| < \delta_3$.
In this case, the choice of $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ the restriction $0 < |x - a| < \delta$ gives the control

$$|f(x)g(x) - LM| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus we have shown that the limit of a product is the product of the limits. $\square$

4.2 Derivative

Definition 4.2: (Derivative)

If a function $f(x)$ is continuous and differentiable, then its derivative is given by

$$f^{'}(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{4.1}$$

or

$$f^{'}(c) = \lim_{h \to 0} \frac{f(x+h) - f(c)}{h} \tag{4.2}$$

Theorem 4.2: (Differentiability implies Continuity)

If $f(x)$ is differentiable at a point c, then its is continuous at c.

Proof. Let $f(x)$ be differentiable function. To prove that f is continuous at c, we have to show that $\lim_{x\to c} f(x) = f(c)$. We do this by showing that the difference $f(x) - f(c)$ approaches 0. We divide and multiply $f(x) - f(c)$ by $x - c$ for $(x \neq c)$:

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$$

Thus, using the Product Law of limits, we can write

$$\begin{aligned}
\lim_{x \to c}[f(x) - f(c)] &= \frac{f(x) - f(c)}{x - c}(x - c) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c) \\
&= f'(x) \cdot 0 = 0 && \text{(Differentiability of } f \text{ at } a\text{)}
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c}[f(x) - f(c) + f(c)] \\
&= \lim_{x \to c}[f(x) - f(c)] + \lim_{x \to c} f(c) \\
&= 0 + f(c) = f(c)
\end{aligned}$$

Therefore, f is continuous at c. ☐

Remark 4.2

The converse of Theorem (4.2) is false; that is, there are functions that are continuous but not differentiable. The most common example is the absolute valued function. In fact, there is a nowhere differentiable function which is continuous everywhere[a].

[a] Karl Weierstrass(1872)

Rules for derivative

1. **Power Rule**
   Let $f(x) = x^n$ then the derivative of f is given by

$$f'(x) = nx^{n-1}$$

Method 1. [1] In our first proof we use the following identity

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}) \tag{4.3}$$

Now $f(x) = x^n$, using definition (4.1)

$$\begin{aligned}
f'(x) &= \lim_{x\to a} \frac{f(x) - fa}{x-a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1})}{\cancel{(x-a)}}, \qquad \text{by (4.3)} \\
&= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + x^2a^{n-3} + xa^{n-2} + a^{n-1}), \qquad (x \neq a) \\
&= \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1} + a^{n-1}}_{n \text{ times}} \\
&= na^{n-1}
\end{aligned}$$

□

Method 2. To use definition (4.2), we need to use Binomial theorem to expand $(x+h)^n$.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n \tag{4.4}$$

Using definition (4.2) the derivative of $f(x) = x^n$ is

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} \left[nx^{n-1} + \underbrace{\frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}}_{\text{goes to } 0}\right] \\
&= nx^{n-1}
\end{aligned}$$

□

2. **Constant multiple rule**
   If c is a constant and is a f differentiable, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \tag{4.5}$$

[1] James Stewart Calculus 6th ed.

Proof. Let $g(x) = cf(x)$, then

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\
&= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

□

3. **Sum and Difference rule**
   If f and g are differentiable, then

$$(f+g)'(x) = f'(x) + g'(x) \tag{4.6}$$

Proof. By using definition (4.2)

$$\begin{aligned}
(f(x)+g(x))' &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h} + \frac{[g(x+h)-g(x)]}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}\right] - \lim_{h\to 0}\left[\frac{[g(x+h)-g(x)]}{h}\right] \\
&= f'(x) + g'(x)
\end{aligned}$$

Similarly, it is easy to prove that $(f-g)'(x) = f'(x) - g'(x)$. □

4. **Product rule**
   If f and g are differentiable, then

$$(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \tag{4.7}$$

Proof. By definition (4.1)

$$\begin{aligned}
(f\cdot g)'(x) &= \lim_{x_0\to x} \frac{f(x_0)g(x_0)-f(x)g(x)}{x_0-x} \\
&= \lim_{x_0\to x} \frac{f(x_0)g(x_0)-f(x_0)g(x)+f(x_0)g(x)-f(x)g(x)}{x_0-x} \\
&= \lim_{x_0\to x} \frac{f(x_0)(g(x_0)-g(x))+g(x)(f(x_0)-f(x))}{x_0-x} \\
&= \lim_{x_0\to x}\left[\frac{f(x_0)(g(x_0)-g(x))}{x_0-x} + \frac{g(x)(f(x_0)-f(x))}{x_0-x}\right] \\
&= \lim_{x_0\to x}\left[\frac{(f(x_0)-f(x))g(x)}{x_0-x} + \frac{f(x_0)(g(x_0)-g(x))}{x_0-x}\right] \\
&= \lim_{x_0\to x}\left[\frac{f(x_0)-f(x)}{x_0-x}\right]g(x) + \lim_{x_0\to x} f(x_0)\lim_{x_0\to x}\left[\frac{g(x_0)-g(x)}{x_0-x}\right] \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

□

5. **Quotient rule**
   If f and g are differentiable, then

$$\left(\frac{f}{g}\right)'(x) = \tag{4.8}$$

6. **Chain rule** If f and g are differentiable, then

$$(f(g(x)))' = \tag{4.9}$$

4.3 Integral

Theorem 4.3: Fundamental Theorem of Calculus Part 1

If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b \tag{4.10}$$

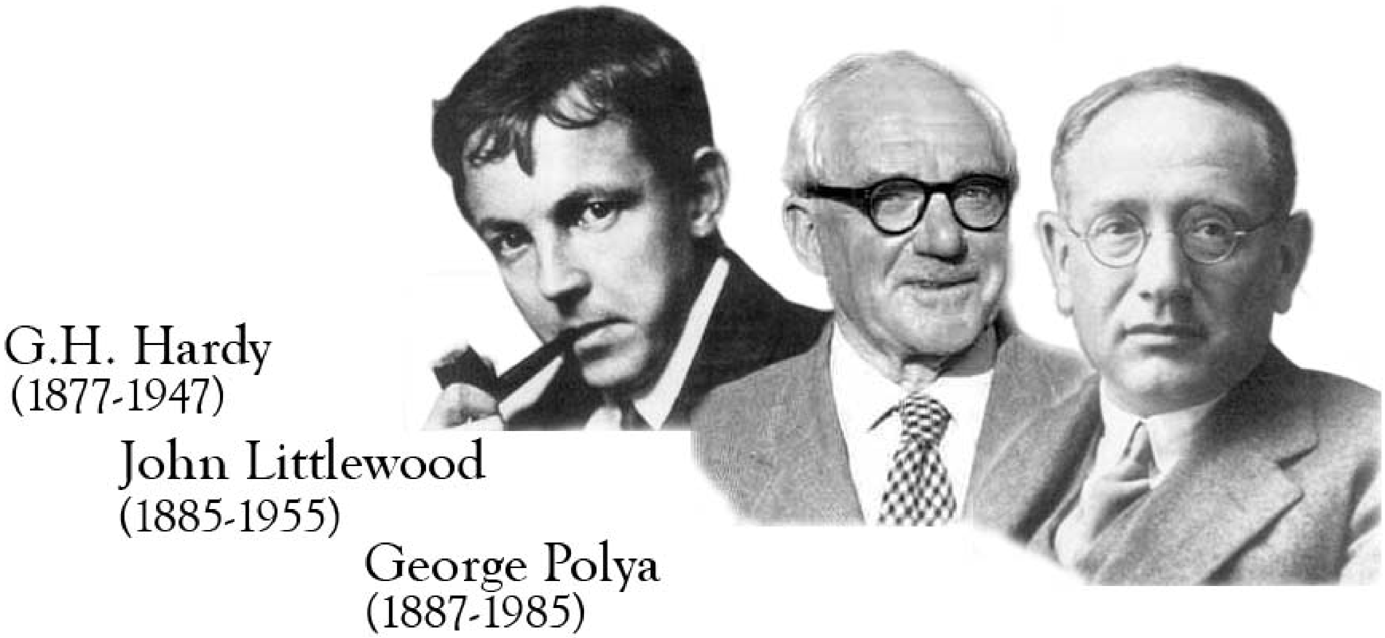
is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Theorem 4.3: Fundamental Theorem of Calculus Part 2

If f is continuous on $[a,b]$, then

$$\int_a^b f(x) = F(b) - F(a) \tag{4.11}$$

where F is any antiderivative of f, that is, a function such that $F' = f$.



5.1 Young's inequality

Theorem 5.1: (Young, 1912)

Let α and β be any positive real numbers. For any $p > 1$ define q such that $\frac{1}{p} + \frac{1}{q} = 1$ (p and q are said to be conjugate exponents), then we have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \tag{5.1}$$

with equality when $\alpha^p = \beta^q$.

Proof. Obviously, for $\alpha = 0$, $\beta = 0$ this inequality holds. Suppose $\alpha \neq 0$ and $\beta \neq 0$. Consider a function $u = t^{p-1}$

Then the area of the rectangle (see Figure 5.1)

$$\begin{aligned}
\alpha\beta &\leq \int_0^\alpha t^{p-1}\,dt + \int_0^\beta u^{q-1}\,du \\
&= \left(\frac{t^p}{p}\right)_0^\alpha + \left(\frac{u^q}{q}\right)_0^\beta \\
&= \frac{\alpha^p}{p} + \frac{\beta^q}{q}
\end{aligned}$$

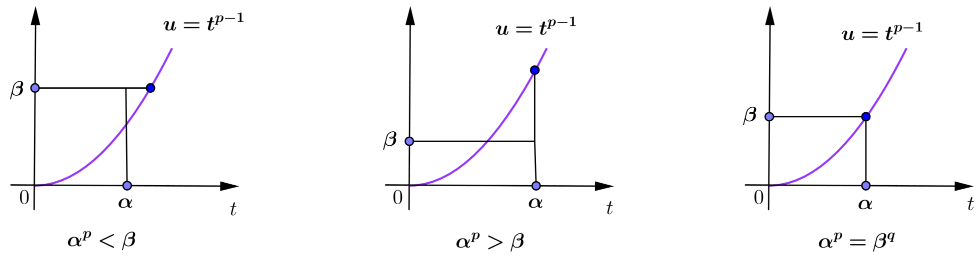


Figure 5.1: Young's Inequality

$\square$

Lemma 5.1

Let $(\bar{\xi}_j)$ and $(\bar{\eta}_j)$ be two sequences such that

$$\sum_{j=1}^{\infty} |\bar{\xi}_j|^p = 1, \qquad \sum_{j=1}^{\infty} |\bar{\eta}_j|^q = 1$$

Then

$$\sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le 1$$

where p and q are conjugate exponents.

Proof. Take $\alpha = |\bar{\xi}_j|$ and $\beta = |\bar{\eta}_j|$.
Use (5.1) so that

$$|\bar{\xi}_j \bar{\eta}_j| \le \frac{1}{p}|\bar{\xi}_j|^p + \frac{1}{q}|\bar{\eta}_j|^q$$

Take summation

$$\sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le \frac{1}{p} + \frac{1}{q} = 1$$

$\square$

5.2 Holder's inequality

Theorem 5.2: (Holder, 1889)

Let $x = (\xi_j) \in \ell^p$, $y = (\eta_j) \in \ell^q$ and $p > 1$. Then we have

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{\frac{1}{q}} \tag{5.2}$$

where p and q are conjugate exponents.

Proof. Take

$$\bar{\xi}_j = \frac{\xi_j}{\left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}}}, \qquad \bar{\eta}_j = \frac{\eta_j}{\left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{\frac{1}{q}}}$$

Notice that

$$\sum_{j=1}^{\infty} |\bar{\xi}_j|^p = 1 \quad \text{and} \quad \sum_{j=1}^{\infty} |\bar{\eta}_j|^q = 1$$

Now use (lemma 5.1) to get

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{\frac{1}{q}} \tag{5.3}$$

□

5.3 Minkowski's inequality

Theorem 5.3: (Minkowski, 1896)

Let $x = (\xi_j) \in \ell^p$, $y = (\eta_j) \in \ell^p$ and $p \geq 1$. Then we have

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty} |\eta_i|^p\right)^{\frac{1}{p}} \tag{5.4}$$

Proof. For $p = 1$, use triangle inequality and apply summation.
Let $p > 1$ and let $\xi_j + \eta_j = \omega_j$

$$|\omega_j|^p = |\xi_j + \eta_j|^p = |\xi_j + \eta_j||\omega_j|^{p-1} \tag{5.5}$$

$$\leq \underbrace{|\xi_j||\omega_j|^{p-1}}_{\mathrm{I}} + \underbrace{|\eta_j||\omega_j|^{p-1}}_{\mathrm{II}} \tag{5.6}$$

Hence we first prove the result by choosing $j = 1, \ldots, n$ (any fixed n)

$$\mathrm{I}: \sum_{j=1}^{n} |\xi_j||\omega_j|^{p-1}$$

where $x = (\xi_j) \in \ell^p$ and $(|\omega_j|^{p-1}) \in \ell^q$. Since

$$(|\omega_j|^{p-1})^q = |\omega_j|^{(p-1)q} = |\omega_j|^p \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{j=1}^{n} \left(|\omega_j|^{(p-1)q}\right) = \sum_{j=1}^{n} |\omega_j|^p < \infty \Rightarrow (|\omega_j|^{p-1}) \in \ell^q$$

Use Holder's inequality(5.2)

$$\mathrm{I}: \sum_{j=1}^{n} |\xi_j||\omega_j|^{p-1} \leq \left(\sum_{k=1}^{n} |\xi_j|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} (|\omega_m|^{p-1})^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^{n} |\xi_j|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\omega_m|^p\right)^{\frac{1}{q}} \tag{5.7}$$

Similarly,

$$\mathrm{II}: \sum_{j=1}^{n} |\eta_j||\omega_j|^{p-1} \leq \left(\sum_{k=1}^{n} |\eta_j|^p\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |\omega_m|^p\right)^{\frac{1}{q}} \tag{5.8}$$

Thus, using (5.6,5.7,5.8) we conclude that

$$\sum_{j=1}^{n}|\omega_j|^p \leq \sum_{j=1}^{n}(|\xi_j|+|\eta_j|)|\omega_j|^{p-1} \leq \left[\left(\sum_{k=1}^{n}|\xi_j|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|\eta_j|^p\right)^{\frac{1}{p}}\right]\cdot\left(\sum_{m=1}^{n}|\omega_m|^p\right)^{\frac{1}{q}}$$

Let $n \to \infty$

$$\left(\sum_{j=1}^{\infty}|\omega_j|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{\infty}|\xi_j|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{\infty}|\eta_j|^p\right)^{\frac{1}{p}}$$

Therefore

$$\left(\sum_{j=1}^{\infty}|\omega_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty}|\xi_j|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{\infty}|\eta_j|^p\right)^{\frac{1}{p}}$$

□

Part II

Additional

Appendix

Chapter A

A.1 $1 = 2$

Proof 1. [1]

$$
\begin{aligned}
x^2 &= \underbrace{x + x + \cdots + x}_{(x \text{ times})} \\
\frac{d}{dx} x^2 &= \frac{d}{dx} [\underbrace{x + x + \cdots + x}_{(x \text{ times})}] \\
2x &= 1 + 1 + \cdots + 1 = x \\
2 &= 1
\end{aligned}
$$

$\square$

Proof 2. Suppose that $a = b$, then

$$
\begin{aligned}
ab &= a^2 && \text{(Multiplying } a = b \text{ by } a.) \\
ab - b^2 &= a^2 - b^2 && \text{(Subtract } b^2) \\
b(a - b) &= (a + b)(a - b) && \text{(Factoring } a - b) \\
b &= a + b = 2b && \text{(Cancel } a - b \text{ and use } a = b) \\
1 &= 2
\end{aligned}
$$

$\square$

[1] Taking one x as a variable and the other x as constant.

A.2 $a = b$

Proof. Let a and b be two unequal numbers, and let c be their arithmetic mean, hence

$$
\begin{aligned}
a + b &= 2c \\
(a+b)(a-b) &= 2c(a-b) \\
a^2 - 2ac &= b^2 - 2bc \\
a^2 - 2ac + c^2 &= b^2 - 2bc + c^2 \\
(a-c)^2 &= (b-c)^2 \\
a &= b
\end{aligned}
$$

□

Brook Taylor
(1685-1731)

B.1 Introduction

Definition B.1: Taylor Expansion

Let f be a function with derivative of all orders throughout some interval containing a as an interior point. Then Taylor Series generated by f at $x = a$, is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

Remark B.1: Maclaurine

The Taylor series generated by $x = 0$, is called Maclaurine Series given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x)^n}{n!} = f(0) + f'(0)(x) + \cdots + \frac{f^{n}(0)(x)^n}{n!} + \cdots$$

Example B.1: e^x

Taylor expansion of e^x

$$\frac{d}{dx}e^x = \frac{d^2}{dx^2}e^x = \frac{d^n}{dx^n}e^x = e^x$$

At $x = 0$, $e^x = 1$. *Therefore*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x)^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$

B.2 Some special series

Trigonometric Functions

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \text{(for all } x\text{)}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad \text{(for all } x\text{)}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)x^{2n-1}}{(2n)!}, \qquad \text{(for } |x| < \tfrac{\pi}{2}\text{)}$$

Inverse Trigonometric Functions

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{4^n(2n+1)(n!)^2}, \qquad \text{(for } |x| < 1\text{)}$$

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{4^n(2n+1)(n!)^2}, \qquad \text{(for all } x\text{)}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2n+1}, \qquad \text{(for } |x| < 1\text{)}$$

Logarithmic Functions

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^n}{(n)}, \qquad \text{(for } |x| < 1\text{)}$$

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n, \qquad \text{(for } |x| < 1\text{)}$$

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n} \tag{B.1}$$

Jacob Bernoulli
(1655-1705)

Definition C.0: Bernoulli

Setting $B_0 = 1$ Bernoulli numbers[a] *are defined*

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0$$

[a] By convention if $B_1 = \frac{-1}{2}$, the given Bernoulli sequence is called **first Bernoulli numbers** and **second Bernoulli numbers** if $B_1 = \frac{1}{2}$.

One can also define the Bernoulli numbers[1] using Pascal's triangle

Definition C.0

Bernoulli numbers can be defined using the recurrence relation

$$\begin{aligned} B_0 &= 1 \\ B_2 + 2B_1 + 1 &= B_2 \\ B_3 + 3B_2 + 3B_1 + 1 &= B_3 \\ B_4 + 4B_3 + 6B_2 + 4B_1 + 1 &= B_4 \\ &\vdots \end{aligned}$$

[1] Discovered by Jacob Bernoulli(1654 – 1705) and discussed by him in a posthumous work *Ars Conjectandi*(1713).

Definition C.0: Pascal

Pascal's triangle is a triangular array of the binomial coefficients.

Pascal's triangle determines the coefficients which arise in binomial expansions. For an example, consider the expansion

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Notice the coefficients are the numbers in row two of Pascal's triangle: 1, 7, 21, 35, 35, 21, 7, 1. In general, when a binomial like $x+y$ is raised to a positive integer power we have:

$$(x+y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n \tag{C.1}$$

where the coefficients a_i in this expansion are precisely the numbers on row n of Pascal's triangle.

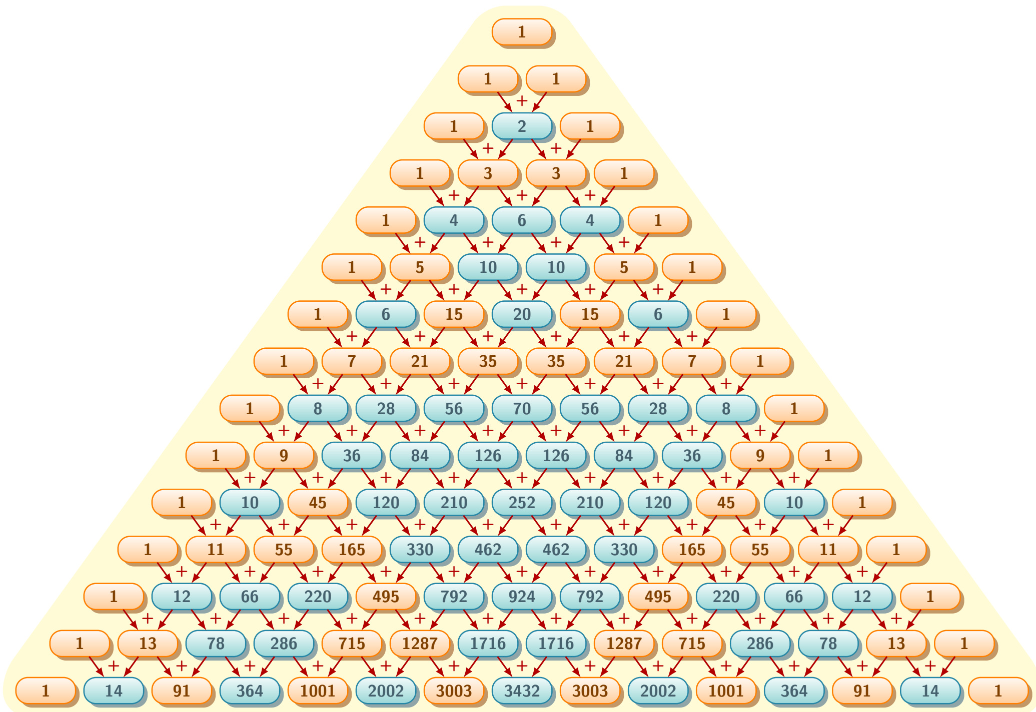


Figure C.1: Pascal Tree

Pascal Identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{C.2}$$

Proof.

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{(n-k)!k!} + \frac{n!}{(n-(k-1))(n-(k-1))!(k-1)k!} \\
&= \frac{(n-(k-1))(k-1)n! + n!}{(n-(k-1))(n-(k-1))!(k-1)k!} \\
&= \frac{((n-(k-1))(k-1)+1)n!}{(n-(k-1))(n-(k-1))!(k-1)k!} = \frac{(n+1)!}{(n+1-k)!(k!)} = \binom{n+1}{k}
\end{aligned}$$

□

C.1 How to Compute B_n's

Using definition (C) it is easy to compute that

$$B_0 = 1, \quad B_1 = \frac{-1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{-1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \cdots \tag{C.3}$$

C.2 Some Facts

Lemma C.2: Binomial Convolution

Let $f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n, g(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n$ and $h(z) = f(z) * g(z)$. Then there exists d_n such that

$$h(z) = \sum_{n\geq 0} \frac{d_n}{n!} z^n, \qquad \text{with } d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \tag{C.4}$$

Proof. Multiplying $f(z)$ and $g(z)$ gives us

$$\left(\frac{a_0}{0!}z^0 + \frac{a_1}{1!}z^1 + \frac{a_2}{2!}z^2 + \cdots\right)\left(\frac{b_0}{0!}z^0 + \frac{b_1}{1!}z^1 + \frac{b_2}{2!}z^2 + \cdots\right)$$

After expanding and regrouping we would get

$$\frac{a_0 b_0}{0!0!}z^0 + \left(\frac{a_0 b_1}{0!1!} + \frac{a_1 b_0}{1!0!}\right)z^1 + \left(\frac{a_0 b_2}{0!2!} + \cdots + \frac{a_2 b_0}{2!0!}\right)z^2 + \cdots \tag{C.5}$$

If we let c_n to be the coefficient of z^n. For example $c_0 = \frac{a_0 b_0}{0!0!}$. Then we have the following formula

$$c_n = \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \tag{C.6}$$

Using (C.5) and (C.6) we write $h(z)$ as the following sum

$$h(z) = \sum_{n\geq 0} c_n z^n \tag{C.7}$$

Now let's define a value d_n such that

$$\begin{aligned} d_n = n! c_n = n! \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \\ = \sum_{k=0}^{n} \frac{n! a_k b_{n-k}}{k!(n-k)!} \\ = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \end{aligned}$$

This gives us that

$$\frac{d_n}{n!} = c_n \tag{C.8}$$

Substituting (C.8) on (C.7) completes the proof.

$$h(z) = \sum_{n\geq 0} \frac{d_n}{n!} z^n, \qquad \text{with } d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

□

C.3 Generating function

Theorem C.3: Generating function for Bernoulli Numbers

The generating function for Bernoulli numbers is

$$G(z) = \frac{z}{e^z - 1}$$

Proof. Let

$$G(z) = \sum_{n\geq 0} \frac{B_n}{n!} z^n, \qquad \text{where } B_n \text{ stands for } n^{th} \text{ Bernoulli number.} \tag{C.9}$$

Multiply both sides by e^z

$$e^z G(z) = \sum_{n\geq 0} \frac{z^n}{n!} \cdot \sum_{n\geq 0} \frac{B_n}{n!} z^n \tag{C.10}$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} B_k \right) \frac{z^n}{n!} \qquad \text{by lemma (C.2).} \tag{C.11}$$

Now, by our definition of the Bernoulli number in (C)

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0 \qquad \text{with } B_0 = 1.$$

If we set $n = m+1$ and add B_n to both sides, then we have

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j + B_n = \sum_{j=0}^{n} \binom{n}{j} B_j = 0 + B_n = B_n \tag{C.12}$$

or $B_n + 1$ in the case where $n = m + 1 = 1$.
This enables us to simplify the result in (C.11) to get

$$e^z G(z) = z + \sum_{n\geq 0} B_n \frac{z^n}{n!} = z + G(z) \tag{C.13}$$

N.B. The z at the bottom comes from the fact that at $n = 1$, our result is

$$(B_1 + 1)\frac{z^1}{1!} = \frac{B_1}{1!} z^1 + z$$

If we subtract $G(z)$ from both sides, we get

$$e^z G(z) - G(z) = z$$
$$G(z)(e^z - 1) = z$$

Dividing both sides by $e^z - 1$ gives us

$$G(z) = \frac{z}{e^z - 1} \tag{C.14}$$

□

C.4 Why the odds vanish?

Corollary C.4

$B_{2i+1} = 0$ if $i \geq 1$.

Proof. From theorem (C.3) we have

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!}$$

Then using the fact that $B_1 = -1/2$, we have the following

$$\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1} - \frac{z}{2} \tag{C.15}$$

Now, look closely at the following algebraic simplification

$$\begin{aligned} \frac{z}{e^z - 1} - \frac{z}{2} &= \frac{2z + z(e^z - 1)}{2(e^z - 1)} \\ &= \left(\frac{z}{2}\right)\frac{(e^z + 1)}{(e^z - 1)} \\ &= \left(\frac{z}{2}\right)\frac{(e^z + 1)}{(e^z - 1)} * \left(\frac{e^{-z/2}}{e^{-z/2}}\right) \\ &= \left(\frac{z}{2}\right)\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right) \end{aligned}$$

If we replace z by $-z$ in our final result, the identity remain unchanged($\frac{z}{e^z-1} - \frac{z}{2}$ is even function).

$$\left(\frac{-z}{2}\right)\left(\frac{e^{-z/2} + e^{-(-z/2)}}{e^{-z/2} - e^{-(-z/2)}}\right) = \left(\frac{-z}{2}\right)\left(\frac{e^{-z/2} + e^{z/2}}{e^{-z/2} - e^{z/2}}\right) = \left(\frac{z}{2}\right)\left(\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}\right)$$

But this means that the same must hold true for $\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!}$. Since in (C.15), we showed that they are equal functions. So that we have

$$\sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{z^n}{n!} = \sum_{\substack{n \geq 1 \\ n \neq 1}} B_n \frac{(-z)^n}{n!} = \sum_{\substack{n \geq 1 \\ n \neq 1}} (-1)^n B_n \frac{z^n}{n!}$$

Matching up each of the terms according to powers of z^n, this gives us

$$B_n = (-1)^n B_n \tag{C.16}$$

Thus, $B_n = 0$, whenever n is odd. Since we have excluded the case $n = 1$, we conclude that

$$B_{2i+1} = 0 \text{ if } i \geq 1.$$

□

C.5 Simple but Elegant Application

Faulhaber's Formula

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k m^{n+1-k} \tag{C.17}$$

Bernhard Riemann
(1826-1866)

D.1 Introduction

Definition D.1: (Zeta Function)

The Riemann zeta function $\zeta(s)$ is defined for $\Re\mathfrak{e}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Note that the series converges absolutely in the given region: for $s = x + iy$ with $x > 1$, we have

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty}\left|\frac{1}{n^x n^{iy}}\right| = \sum_{n=1}^{\infty}\left|\frac{e^{-iy\log(n)}}{n^x}\right| = \sum_{n=1}^{\infty}\frac{1}{n^x} \leq \int_1^{\infty}\frac{1}{t^x}dt = \left[\frac{t^{1-x}}{1-x}\right]_1^{\infty} = \frac{1}{1-x}$$

Thus $\zeta(s)$ is indeed defined for $\Re\mathfrak{e}(s) > 1$.

D.2 Basic Results

Lemma D.2

The power series(B.1) of $f(x)$ which is

$$f(x) = \ln\left(\frac{1}{1-x}\right)$$

is given by

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Proof. $f(x) = \ln\left(\frac{1}{1-x}\right)$, the Maclaurin series of $f(x)$ is by definition(B.1)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

Now,

$$\begin{aligned}
f^{(1)}(x) &= \frac{1}{1-x} && \Rightarrow f^{(1)}(0) = 1 \\
f^{(2)}(x) &= \frac{1}{(1-x)^2} && \Rightarrow f^{(2)}(0) = 1 \\
f^{(3)}(x) &= \frac{2}{(1-x)^3} && \Rightarrow f^{(3)}(0) = 2 \\
f^{(4)}(x) &= \frac{2\cdot 3}{(1-x)^4} && \Rightarrow f^{(4)}(0) = 2\cdot 3 \\
&\vdots && \vdots \\
f^{(n)}(x) &= \frac{(n-1)!}{(1-x)^n} && \Rightarrow f^{(n)}(0) = (n-1)!
\end{aligned}$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{(n-1)!x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n}$$

□

D.3 Euler Product Formula

Theorem D.3: (Euler Product)

The Riemann zeta function can be represent as a product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{(1-\frac{1}{p^s})}$$

where $\mathbb{P}$ is the set of prime numbers.

Proof. Start with

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{D.1}$$

Multiply it by $\frac{1}{2^s}$

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s}\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \tag{D.2}$$

Now subtract (D.2) from (D.1)

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

$$(1 - \frac{1}{2^s})\zeta(s) = \sum_{\substack{n=1 \\ n\neq 2k}}^{\infty} \frac{1}{n^s}, \qquad k \in \mathbb{N} \tag{D.3}$$

Multiply the last equation by $\frac{1}{3^s}$

$$\frac{1}{3^s}(1 - \frac{1}{2^s})\zeta(s) = \frac{1}{3^s}\sum_{\substack{n=1 \\ n\neq 2k}}^{\infty} \frac{1}{n^s} = \sum_{\substack{n=0 \\ n\neq 2k}}^{\infty} \frac{1}{(3n)^s} \tag{D.4}$$

Again subtract (D.4) from (D.3) to get

$$(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s) = \sum_{\substack{n=1 \\ n\neq 2k}}^{\infty} \frac{1}{n^s} - \sum_{\substack{n=0 \\ n\neq 2k}}^{\infty} \frac{1}{(3n)^s}$$

$$(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s) = \sum_{\substack{n=1 \\ n\neq 2k \\ n\neq 3k}}^{\infty} \frac{1}{n^s}$$

If we repeat this pattern over again and again we will arrive at

$$((1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\cdots)\zeta(s) = \sum_{\substack{n=1 \\ n\neq 2k \\ n\neq 3k \\ \vdots}}^{\infty} \frac{1}{n^s}$$

Since we have taken out all the multiple of primes on the right side Fundamental Theorem of Arithmetic tells us that we had left with 1.

$$((1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\cdots)\zeta(s) = 1$$

$$\zeta(s) = \frac{1}{((1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\cdots)}$$

Which is nothing but

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{(1 - \frac{1}{p^s})}$$

□

Corollary D.3: Euler

There are infinitely many primes.

Proof. If there were a finite number of primes, harmonic series would converge. □

Theorem D.3

The sum of reciprocal of primes

$$\sum_{p \in \mathbb{P}} 1/p \to \infty$$

diverges(converges to ∞).

Proof. Because of the fact that $\lim_{s \to 1^+} \zeta(s) = \infty$, it follows that

$$\lim_{s \to 1^+} \ln(\zeta(s)) = \infty \tag{D.5}$$

But,

$$\begin{aligned}
\ln(\zeta(s)) &= \ln\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \\
&= \ln\left(\prod_{p} \frac{1}{1-1/p^s}\right) = \sum_{p} \ln\left(\frac{1}{1-1/p^s}\right) \\
&= \sum_{p} \sum_{n=1}^{\infty} \frac{1}{np^{sn}} && \text{(lemma (D.2))} \\
&= \sum_{p} \left(\frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{np^{sn}}\right)
\end{aligned}$$

Hence,

$$\ln(\zeta(s)) = \sum_{p} \frac{1}{p^s} + \sum_{p} \sum_{n=2}^{\infty} \frac{1}{np^{sn}}$$

Thus,

$$\ln(\zeta(s)) < \sum_{p} \frac{1}{p^s} + \sum_{p} \sum_{n=2}^{\infty} \frac{1}{p^{sn}}$$

From geometric sum, we have

$$\sum_{n=2}^{\infty} \frac{1}{p^{sn}} = \frac{1}{p^{2s} - p^s}$$

Therefore, for $s > 1$, we obtain that

$$\begin{aligned}
\ln(\zeta(s)) &< \sum_{p} \frac{1}{p^s} + \sum_{p} \frac{1}{p^{2s} - p^s} \\
&= \sum_{p} \frac{1}{p^s} + \sum_{p} \frac{1}{p^s(p^s - 1)} \\
&< \sum_{p} \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{n^s(n^s - 1)} \\
&< \sum_{p} \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \\
&< \sum_{p} \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{(n-1)^2}
\end{aligned}$$

Thus, it yields

$$\lim_{s \to 1^+} \ln(\zeta(s)) \leq \lim_{s \to 1^+} \left(\sum_p \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \right)$$

and thus, by (D.5), it is evident that

$$\lim_{s \to 1^+} \left(\sum_p \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \right) = \infty$$

However, the series

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$$

converges to a real number. Hence,

$$\lim_{s \to 1^+} \sum_p \frac{1}{p^s} = \infty$$

Therefore,

$$\sum_p \frac{1}{p} = \infty$$

□

Karl Weierstrass
(1815-1897)

E.1 Elegant Identities

Complex raise complex

$$i^i = \frac{1}{e^{\frac{\pi}{2}}} \tag{E.1}$$

Proof: From Euler's formula

$$e^{i\phi} = \cos\phi + i \cdot \sin\phi$$

when $\phi = \frac{\pi}{2}$

$$e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i \cdot \sin\frac{\pi}{2} = 0 + i \cdot 1 = i$$

Hence

$$i^i = (e^{\frac{i\pi}{2}})^i = e^{i^2 \cdot \frac{\pi}{2}} = e^{-\frac{\pi}{2}} = \frac{1}{e^{\frac{\pi}{2}}}$$

Complex root of complex

$$\sqrt[i]{i} = e^{\frac{\pi}{2}} \tag{E.2}$$

Proof: The proof is similar to (E.1)

$$e^{\frac{i\pi}{2}} = i$$

from (E.1)

$$\Rightarrow i^{\frac{1}{i}} = (e^{\frac{i\pi}{2}})^{\frac{1}{i}}$$

$$\Rightarrow \sqrt[i]{i} = e^{\frac{\pi}{2}}$$

Remark E.1

Exponentiation of a complex number is multi-valued. So the results in (E.1) and (E.2) are NOT the only ones.

Formal Proof. We have $\log z = \ln r + i\theta$, where r is a modulus of z and θ is the argument of z. From polar representation of complex number; $z = r(\cos\theta + i\sin\theta)$ and from Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It follows that

$$z = re^{i\theta}$$

Hence,

$$\log z = \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta}) = \ln r + i(\theta + 2k\pi),\ k \in \mathbb{Z}.$$

Now

$$i^i = e^{\log i^i} = e^{i\log i}$$

and

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right) = 0 + i\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}.$$

Thus

$$i^i = e^{i^2\left(\frac{\pi}{2}+2k\pi\right)} = e^{-\frac{\pi}{2}-2k\pi},\ k \in \mathbb{Z}.$$

$\square$

Euler's Identity

$$e^{i\pi} = -1$$

Proof: Recall Euler's formula

$$e^{i\phi} = \cos\phi + i\cdot\sin\phi$$

Letting $\phi = \pi$, we get

$$e^{i\pi} = -1 + i\cdot 0 = -1$$

Hardy-Ramanujan number

Hardy thought it was a dull one but Ramanujan[a] did not.

1729

$$12^3 + 1^3 = 1729$$
$$9^3 + 10^3 = 1729$$

[a] "No," Ramanujan replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

Telescoping to One

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}+\cdots=1 \tag{E.3}$$

Proof: The series in (E.3) can be rewritten as

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n\cdot(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \lim_{n\to\infty}\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1\end{aligned}$$

♣

Leibniz Sum

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{36}+\frac{1}{45}+\cdots=2 \tag{E.4}$$

Proof 1: Here is how Leibniz[a] did it,

$$\begin{aligned}1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\cdots &= 2\left[\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\cdots\right]\\ &= 2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots\right]\end{aligned}$$

Everything inside the brace becomes 1(the fractions cancel out). Hence $2\cdot 1=2$.

Proof 2: If you see closely each of the denominator in (E.4) are triangular numbers, so using sigma notation the series can be written as

$$\sum_{n=1}^{\infty}\frac{1}{\frac{n(n+1)}{2}}=\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$$

But this is Telescoping series in the form

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{2}{n+1}\right)=\lim_{n\to\infty}\left[(2-1)+\left(1-\frac{2}{3}\right)+\cdots+\left(\frac{2}{n-1}-\frac{2}{n}\right)+\left(\frac{2}{n}-\frac{2}{n+1}\right)\right]\\ &= \lim_{n\to\infty}\left(2-\frac{2}{n+1}\right)=2\end{aligned}$$

[a] Gottfried Wilhelm Leibniz(1646 – 1716) was a German mathematician and philosopher.

Ramanujan nested sum

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}=3$$

Proof[a]: Consider

$$(x+n+a)^2=(x+(n+a))^2=x^2+2x(n+a)+(n+a)^2=ax+(n+a)^2+x(x+2n+a)$$

This implies

$$x+n+a=\sqrt{ax+(n+a)^2+x\underbrace{(x+2n+a)}} \tag{E.5}$$

Now, substituting $x+n$ in place of x in (E.5)

$$x+2n+a=(x+n)+(n+a)=\sqrt{a(x+n)+(n+a)^2+(x+n)(x+3n+a)}$$

Then keep replacing the last term by writing $(x+\beta n+a)$ in the form of $((x+(\beta-1)n)+n+a)$ to give

$$x+n+a=\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\ldots}}}$$

Now setting $x=2$, $n=1$ and $a=0$, gives the desired result.

[a]Srinivasa Ramanujan(1887 – 1920) was an Indian mathematician

Oresme Sum

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$$

Proof:[a] Consider the function

$$f(z) = \frac{1}{1-z}$$

The power series expansion of $f(z)$ is

$$f(z) = \sum_{n=1}^{\infty} z^n$$

Now take the derivative of f, which is

$$f'(z) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}$$

Multiplying by z

$$zf'(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n \tag{E.6}$$

Substituting $z = 1/2$ in (E.6)

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

[a] Ponnusamy S. and Silverman H., Complex variables

Lagrange Identity

For any $\{a,b,c,d\}$,

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$$

Just expand!

Alternate Harmonic Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \tag{E.7}$$

Proof: The proof is the direct result of Maclaurin expansion of $\ln(1+x)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{E.8}$$

Substitute $x = 1$ in (E.8)

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Sophie Germain's

For any $\{x, y\}$,

$$x^4 + 4y^4 = ((x+y)^2 + y^2)((x-y)^2 + y^2) = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

Catalan's Identity

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

Proof: Let

$$S_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

Now

$$\begin{aligned}
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} &= S_{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= S_{2n} - S_n \\
&= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}
\end{aligned}$$

E.2 Leibniz Formula

$$\sqrt{6} = \sqrt{1 - \sqrt{-3}} + \sqrt{1 + \sqrt{-3}}$$

E.3 Representation for π

Cardinal Sine Function

Consider the cardinal sine function

$$\frac{\sin x}{x}, \qquad \text{which has a non zero root at } \pm\pi, \pm 2\pi, \pm\pi, \ldots$$

Now, express the cardinal sine function by infinite product polynomials like this

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \tag{E.9}$$

Evaluate (E.9) at $x = \pi/2$

$$\frac{\sin \pi/2}{\pi/2} = \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4}\right)\left(1 - \frac{1}{6}\right)\left(1 + \frac{1}{6}\right)\cdots$$

After some algebraic simplification we get

$$\frac{2}{\pi} = \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}\cdots$$

Arctan Function

The Taylor expansion of $\arctan x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{E.10}$$

Evaluate (E.10) at $x = 1$ to get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Gaussian Integrals

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{E.11}$$

Proof. Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Now,

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}}_{\text{Fubini theorem}} dx dy \\
&= \int_0^{\infty}\int_0^{\infty} e^{-r^2} r dr d\theta \qquad \text{(Jacobian)} \\
&= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr \\
&= \theta\Big|_0^{2\pi}\left(-\frac{1}{2} e^{-r^2}\Big|_0^{\infty}\right) \\
&= 2\pi\left(\frac{1}{2}\right) \\
&= \pi
\end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$

□

E.4 Mathematical Dreams

Theorem E.4: Freshman's Dream

Let R be a commutative ring with identity of prime characteristic p. If $a, b \in R$ then

$$(a \pm b)^{p^n} = a^{p^n} \pm b^{p^n}$$

for all integers $n \geq 0$.

Proof. Since R is commutative with identity we have that

$$(a+b)^{p^n} = \sum_{k=0}^{p^n} \binom{p^n}{k} a^{p^n-k} b^k$$

Then since $p | \binom{p^n}{k}$ for all $a \leq k \leq p^{n-1}$ and R has characteristic p, we know that

$$\binom{p^n}{k} a^{p^n-k} b^k = 0 \quad \forall 1 \leq k \leq p^n - 1$$

Therefore

$$(a+b)^{p^n} = \binom{p^n}{0} a^{p^n-0} b^0 + \binom{p^n}{p^n} a^{p^n-p^n} b^{p^n}.$$

Thus $(a+b)^{p^n} = a^{p^n} + b^{p^n}$. Also $(a-b)^{p^n} = a^{p^n} + (-b)^{p^n}$.
When $p = 2$ we have that $(-b)^{p^n} = b^{p^n}$, otherwise $(-b)^{p^n} = -b^{p^n}$. This completes the proof. □

Theorem E.4: Sophomore's Dream

$$\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n} = -\sum_{n=1}^{\infty} (-n)^{-n} = 0.78343051071213440705926438652697546940 7\ldots$$

Proof. Expand x^x as

$$x^x = \exp(x \log x) = \sum_{n=0}^{\infty} \frac{x^n (\log x)^n}{n!}$$

Therefore, we have

$$\int_0^1 x^x dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n (\log x)^n}{n!} dx$$

By uniform convergence of the power series, we may interchange summation and integration

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n (\log x)^n}{n!} dx$$

To evaluate the above integrals we perform the change of variable in the integral $x = \exp(-\frac{u}{n+1})$, with $0 < u < \infty$, giving us

$$\int_0^1 x^n (\log x)^n dx = (-1)^n (n+1)^{-(n+1)} \int_0^{\infty} u^n e^{-u} du$$

By the well-known Euler's integral identity for the Gamma function

$$\int_0^{\infty} u^n e^{-u} du = n!$$

so that

$$\int_0^1 \frac{x^n(\log x)^n}{n!}dx = (-1)^n(n+1)^{-(n+1)}$$

Summing these (and changing indexing so it starts at n = 1 instead of n = 0) yields the formula. □

E.5 Perpendicular lines

If m_1 and m_2 are the slop of two perpendicular lines, then

$$m_1 \cdot m_2 = -1$$

Proof. By definition Two lines are perpendicular iff the angle between them is 90°

$$\tan 90^\circ = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

$$= \frac{m_1 - m_2}{\infty} = 1 + m_1 m_2$$

$$1 + m_1 m_2 = 0 \quad , for\ m_1 \neq m_2$$

$$\Rightarrow 1 + m_1 \cdot m_2 = 0$$

$$\Rightarrow m_1 \cdot m_2 = -1$$

□

E.6 Poisson Probability Distribution

$$\begin{aligned}
\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^k &= \lim_{n\to\infty}\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}, \qquad \lambda = np \\
&= \lim_{n\to\infty}\frac{n!}{(n-k)!k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n\to\infty}\frac{n!}{(n-k)!k!}\left(\frac{\lambda^k}{n^k}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&= \lim_{n\to\infty}\frac{n!}{(n-k)!n^k}\left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n!}{(n-k)!n^k}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!}\left(\lim_{n\to\infty}\frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k\ factors}}{(n^k)}\right)e^{-\lambda}\cdot 1 \\
&= \frac{\lambda^k e^{-\lambda}}{k!}
\end{aligned}$$

E.7 Goldbach on prime

Lemma E.7: Goldbach

The Fermat numbers $F_n = 2^{2^n} + 1$ are pairwise relatively prime.

Proof. It's easy to show by induction that the following recursion holds

$$\prod_{k=0}^{n-1} F_k = F_n - 2$$

This means that if d divides both $F_k \& F_n$ (with $k < n$), then d also divides $F_n - 2$; so d divides 2. But every Fermat number is odd. So d is 1.
To prove the recursion we use induction on n. For $n = 1$ we have $F_0 = 3$ and $F_1 - 2 = 3$. With induction we now conclude that

$$\begin{aligned} \prod_{k=0}^{n} F_k &= \left(\prod_{k=0}^{n-1} F_k\right) F_n \\ &= (F_n - 2) F_n && \text{(Induction hypothesis)} \\ &= (2^{2^n} - 1)(2^{2^n} + 1) = (2^{2^{n+1}} - 1) = F_{n+1} - 2 \end{aligned}$$

□

Theorem E.7: Goldbach

There are infinitely many primes.

Proof. Choose a prime divisor p_n of each Fermat number F_n. By (E.7) we know these primes are all distinct, showing that there are infinitely many primes. □

E.8 More on Irrationals

1. An irrational number to an irrational power may be rational.

   Proof. To prove this, we need only give an example a^b where a and b are irrational and a^b rational. If[1] $\sqrt{2}^{\sqrt{2}}$ is rational, then it is our example. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is our example. □

2. An irrational number to an irrational power may be irrational.

   Proof. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then it is our example. If $\sqrt{2}^{\sqrt{2}}$ is rational, then $\sqrt{2}^{\sqrt{2}+1} = \sqrt{2}^{\sqrt{2}} + \sqrt{2}$ is our example. □

[1] In fact, $\sqrt{2}^{\sqrt{2}}$ is irrational, since it is the square root of the **Gelfand-Schneider** number $2^{\sqrt{2}}$, which is known to be transcendental.[Claudi Charming Proofs pp. 35]

Karl Weierstrass
(1815-1897)

$$\frac{a+b^n}{n} = x, \qquad \text{God exists!}$$

Euler Hilarious Triumph

Once at the Catherine's court
The great Euler face Didrot
Didrot was a French atheist
Who claims "God doesn't exist"
He was preaching this doctrine
To the youth's of Lenin
Though the princes Catherine
Was very annoyed by this doctrine
Thus, she asked for Euler's help
To get rid of Didrot
Hence, Euler appear in the court
And said ...
A plus B raise n the whole over n is equal to x
That implies God existence!
Didrot was poooooor at Math's
On the next day he returned to Paris.

-Miliyon T.

ISBN 978-80-85955-35-4

9 788085 955354

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