$\text{sinc}(x)$

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1 Introductions

2 Limit

Find

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

To determine a limit a 0 it is adequate to consider $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\theta \neq 0$. We shall exhibit a diagram and utilize trigonometric relationships and an intuitive geometrical argument to establish the needed equations and inequalities.

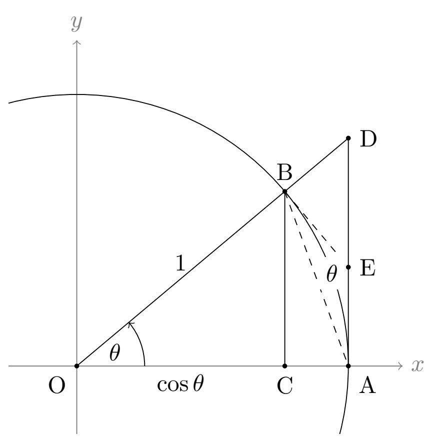

Figure 1

Let A and B be points of a unit circle with center at O such that $0 < \theta < \frac{\pi}{2}$ where θ is the *radian measure* of angle AOB. As in Figure 1 draw BC perpendicular to OA where C is on OA, and draw AD perpendicular to OA where D is on OB. In radian measure the angle θ is defined as the corresponding arc over the radius. Since the radius is 1 we have

$$\theta = \frac{\widehat{AB}}{1}, \quad \Rightarrow \quad \widehat{AB} = \theta$$

We let "AB" stand for the segment AB and also the length of AB, and we let "$\widehat{AB}$" stand for the arc AB and also the length of $\widehat{AB}$.

Consider these statements for $0 < \theta < \frac{\pi}{2}$.

1. $0 < BC$ since B and C are distinct points.
2. $BC = \sin\theta$ by the definition of $\sin\theta$.
3. $BC < \widehat{AB}$ because BC is the perpendicular from B to line OA and $\widehat{AB}$ is not.

Therefore,

$$0 < \sin\theta < \theta \qquad \text{if} \qquad 0 < \theta < \frac{\pi}{2} \tag{1}$$

Next draw segment AB to form acute triangle AOB. Then $0 < AB < \theta$ and by the law of cosines,

$$\overline{AB}^2 = 1^2 + 1^2 - 2\cos\theta.$$

This implies

$$\theta^2 > 2 - 2\cos\theta$$

Then

$$1 - \frac{\theta^2}{2} < \cos\theta$$

Therefore, $\cos\theta < 1$ whenever $0 < \theta < \frac{\pi}{2}$ and

$$\theta^2 > 2 - 2\cos\theta < 1 \qquad \text{if} \qquad 0 < \theta < \frac{\pi}{2} \tag{2}$$

Finally, draw BE tangent to circle O at B with E on AD. Since two tangents BE and EA could be sides of a polygon circumscribed about circle O, then

$$\overline{AE} + \overline{BE} > \widehat{AB}$$

Since $\widehat{AB} = \theta$,

$$\theta < \overline{AE} + \overline{BE}.$$

Since triangle BED is a right triangle with right angle at B, then $BE < ED$ and

$$\overline{AE} + \overline{BE} < \overline{AE} + \overline{ED}$$

But $\overline{AE} + \overline{ED} = \overline{AD}$ and $AD = \tan\theta$. Therefore,

$$0 < \theta < \tan\theta \qquad \text{if} \qquad 0 < \theta < \frac{\pi}{2} \tag{3}$$

Now multiply the inequality in (3) by $\dfrac{\cos\theta}{\theta}$ we obtain

$$0 < \cos\theta < \frac{\sin\theta}{\theta} \qquad \text{if} \qquad 0 < \theta < \frac{\pi}{2}$$

Bus since $\sin\theta < \theta$ if $0 < \theta < \frac{\pi}{2}$ by (1), then $\dfrac{\sin\theta}{\theta} < 1$ and

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \tag{4}$$

If $-\frac{\pi}{2} < \theta < 0$, then $0 < -\theta < \frac{\pi}{2}$ and the statement (4) yields

$$\cos(-\theta) < \frac{\sin(-\theta)}{-\theta} < 1 \qquad \text{if} \qquad -\frac{\pi}{2} < \theta < 0 \tag{5}$$

But $\cos(-\theta) = \cos(\theta)$ and $\frac{\sin(-\theta)}{-\theta} = \frac{\sin(\theta)}{\theta}$, thus (4) and (5) combine to yield

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \qquad \text{if} \qquad 0 < |\theta| < \frac{\pi}{2}. \tag{6}$$

Thus by applying *the squeezing theorem*[1] on (6) we have

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

[1] what?