FUNDAMENTAL GROUPS OF SPACES

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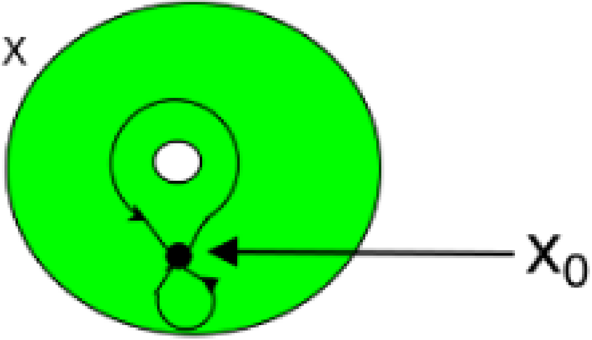
Introduction

- X is a topological space, $x_0 \in X$ fixed



Introduction

- X be a topological space, $x_0 \in X$ fixed. A path in X is a continuous map
$$f : [0,1] \longmapsto X \tag{1}$$
- Loops are paths such that
$$\{f : [0,1] \longmapsto X : f(0) = x_0 = f(1)\} \tag{2}$$
- Two loops α, β at a fixed point x_0 are equivalent if one can be deformed in to the other without breaking.
- The fundamental group will be defined in terms of loops and deformation of these loops.

Homotopy of paths

- Two paths α, β with common end points $(\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1))$ are equivalent or homotopic if there exists a continuous map

$$H : [0,1] \times [0,1] \longmapsto X \tag{3}$$

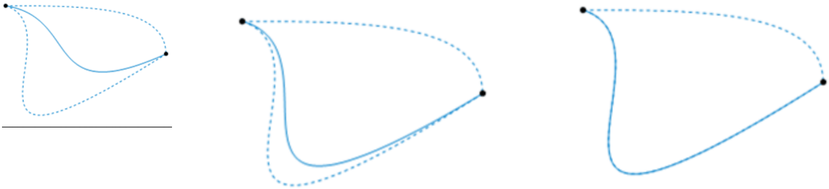
such that

$$H(0,t) = \alpha(t), \qquad H(1,t) = \beta(t), t \in [0,1] \tag{4}$$

and

$$H(s,0) = \alpha(0) = \beta(0), \quad H(s,1) = \alpha(1) = \beta(1), \forall s \in [0,1]$$

Family of homotopic paths



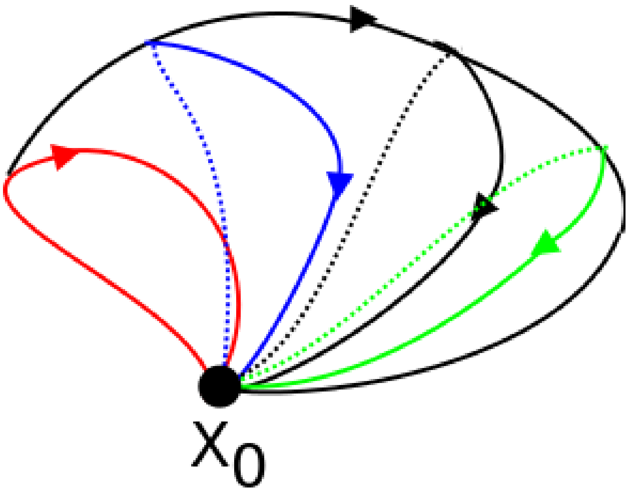
Homotopy of paths

- H gives a family of paths connecting α and β
- The map H is said to be a homotopy, and we write

$$\alpha \simeq \beta$$

  for 'α is homotopic to β' or
- We can write $H : \alpha \simeq \beta$ to specify the homotopy
- Homotopy is an equivalence relation and we denote the equivalence class by $[\alpha]$. If $\alpha \simeq \beta$ we write $[\alpha] = [\beta]$

Family of homotopic loops



Composition of loops

- If α and β are two loops with $\alpha(1) = \beta(0)$, then the composition or product loop $\alpha * \beta$ that traverses first α and then β defined by the formula

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

- The map $\alpha * \beta$ is continuous since $\alpha(1) = \beta(0)$, α and β are continuous by assumption
- The new path travels twice as fast along α part and twice as fast along β part in order $\alpha * \beta$ to be travelled in unit time
- multiplication is well defined on equivalence class. That is

$$[\alpha] * [\beta] = [\alpha * \beta]$$

Fundamental Groups of spaces

- The set all of all homotopy classes $[\alpha]$ of loops $\alpha : [0,1] \longmapsto X$ at the base point x_0 is denoted by $\pi_1(X, x_0)$

Proposition

$\pi_1(X, x_0)$ is a group with respect to the product

$$[\alpha] * [\beta] = [\alpha * \beta]$$

- This group is called the fundamental group of X at the base point x_0

Fundamental Groups of spaces

- The identity element is the constant map α_0 at the base point x_0
- The inverse of a loop α is the loop β defined by $\beta(t) = \alpha(1-t)$. That is, β going backwards along α
- Associativity:
$$[\alpha] * ([\beta] * [\gamma]) = ([\alpha] * [\beta]) * [\gamma]$$
- It is only in the level of classes. So the left path is not exactly the same as the right path.

- **Trivial fundamental group:** it is a group with one element.
- Example of trivial

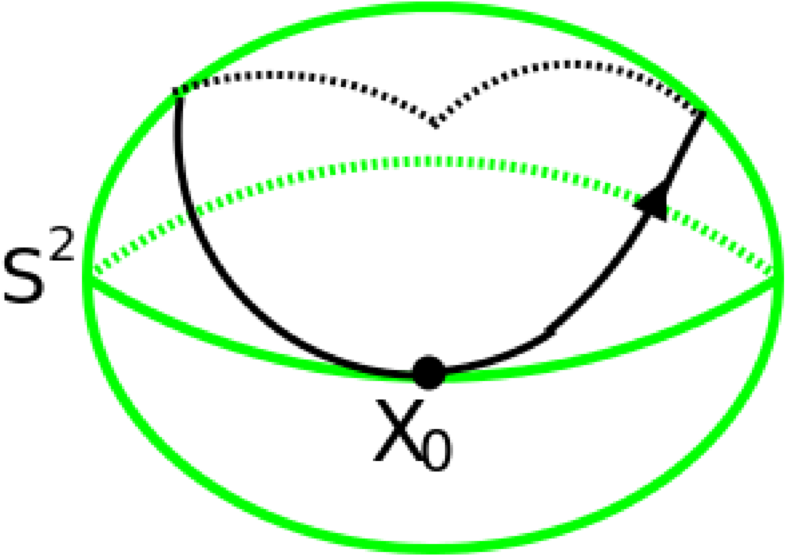
$$\pi_1(\mathbb{R}^n) = \pi_1(Disk) = \pi_1(Polygon) = \pi_1(\mathbb{S}^2) = \pi_1(\mathbb{S}^3) = [\alpha_0]$$

- **Infinite cyclic fundamental group:** each homotopy class consists of all loops which wind a round the circle.
- In this case the product is given by

$$[\alpha_n] * [\alpha_m] = [\alpha_{m+n}]$$

$$\begin{aligned} \pi_1(\mathbb{S}^1) = \pi_1(Annulus) &= \pi_1(PuncturedDisk) \\ &= \{\alpha_k : k \in \mathbb{Z}\} \simeq (\mathbb{Z}, +) \end{aligned}$$

$\pi_1(\mathbb{S}^2)$



The Poincaré Conjecture(1904)

The Poincaré Conjecture

If X is a closed connected 3-manifold and

$$\pi_1(X) \cong \pi_1(\mathbb{S}^3)$$

then X is homeomorphic to $\mathbb{S}^3$

- Grigori Perelman (Russian) solved this conjecture in a series of three papers in 2002-2003

Further Reading

1. M.A.Armstrong, *Basic Topology,* Springer
2. J.M Boardman, *Homotopy Invariant Algebraic Structures On Topological Spaces*