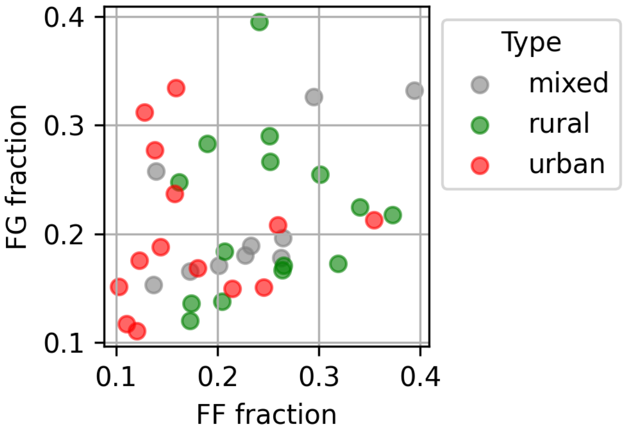
A linear classifier.

This section won't actually do anything that is different from the logistic regression we described in the previous section; instead it presents the same mathematics from a slightly different perspective. In the logistic regression example we started with a one-dimensional example, here we will look at another example, but this time a higher dimensional one, based on data from Irish elections.

As described previously, elections in Ireland use an elegant voting system called Single Transferable Vote that allows for constituency-based proportional voting. There are multi-seat constituencies and votes transfer as candidates are elected or eliminated, so it is a simplification to reduce the votes to the proportion for each party. However, for our purposes here this is what we have done. Based on the 2020 election[1] this figure show the fraction of 'first preference' vote for two of the main parties, Fianna Fáil and Fine Gael, for all 39 constituencies.



The constituencies have been labelled 'urban', 'mixed' and 'rural' and this has been marked by colour. Don't take these labels too seriously; I guess it based on my own knowledge.

It is clear there is a relationship between the voting pattern and settlement type with rural area more likely to vote for these two parties. There are exceptions, in voting local issues often change the result; the rural con-

[1] data.gov.ie/dataset/candidate-details-for-general-election-2020

stituency Galway-Roscommon, for example, has three TDs[2], one Sinn Féin, one who was in Fine Gael who left in protest following the closure of a local hospital and another who is an activist opposed to restrictions on the cutting of turf, a significant issue in an area with large amounts of bog. Similarly, the urban constituency with the largest Fianna Fáil vote is Cork South-Central; this is the constituency of Micheál Martin, who is leader of Fianna Fáil, in the event he went on to become Taoiseach[3] from 2020 to 2022.

Lets just concentrate on the urban and rural consistituencies and imagine we want to guess the settlement type of an unknown constituency based on the Fiánna Fail and Fine Gael votes. The obvious approach to that is to mark areas on the vote-fraction plane corresponding to rural and urban constituencies and, in a linear model that means dividing the plane using a straight line. In fact, this is the same as the logistic regression tasks. In logistic regression we modelled the probability using a logistic function:

$$p(\text{rural}) = \sigma(\beta_0 + \beta_1 x_1 + \beta_2 x_2) \tag{1}$$

where, in our current context, x_1 is the fraction voting Fianna Fáil and x_2 the fraction voting Fine Gael. We found values for the βs by optimising a loss function, the log-likelihood. With these parameters we can then estimate a probability that any point is 'rural'; for a binary classified we can dump this nicety and just predict that a constituency is rural if $p(\text{rural}) \geq 0.5$. In fact, $\sigma(0) = 0.5$ so the decision boundary where $\hat{p}(\text{rural}) = 0.5$ corresponds to

$$\beta_0 + \beta_1 x_1 + \beta_2 x_2 = 0 \tag{2}$$

or, put another way, the line

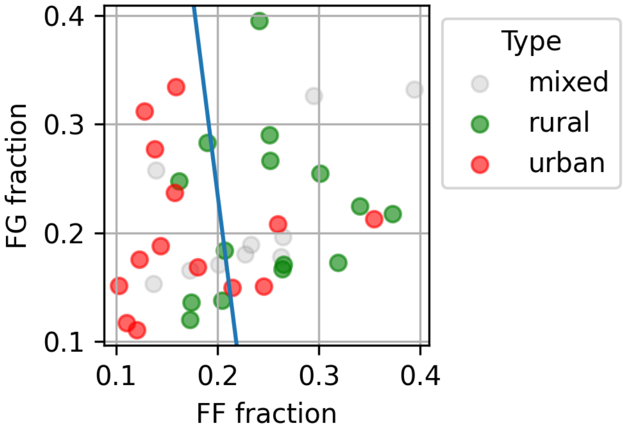
$$x_2 = -\frac{\beta_1}{\beta_2} x_1 - \frac{\beta_0}{\beta_2} \tag{3}$$

Points to the above and to the right of this line have a greater than 0.5 estimated chance of being rural, those below and to the right, less than 0.5.

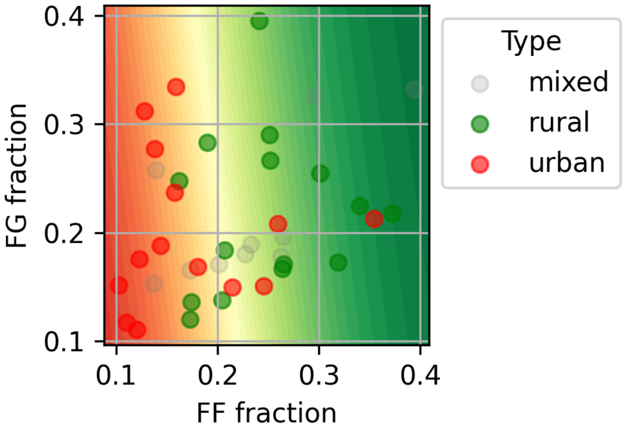
We can plot that line

[2] Teachta Dálá, the title given to a representative in Ireland

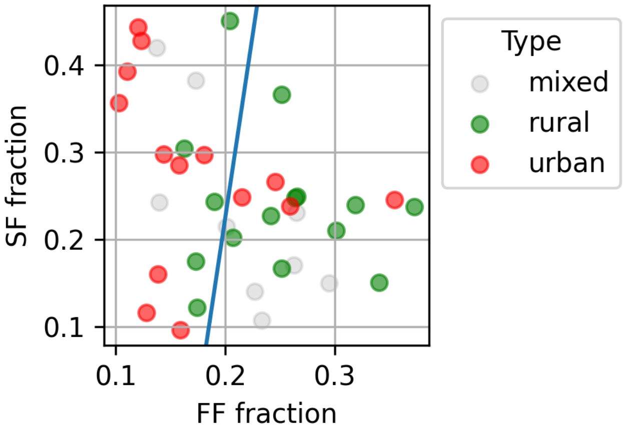
[3] the equivalent of Premier or Prime Minister



We can see it has done ok, it isn't perfect, but then no line is going to be. We can look at the assigned probability using a heatmap



Clearly, it is has fitted the points as best as it can using a linear classifier and this gives a useful model of settlement type, but a linear model, perhaps unsurprisingly, has failed to account for Cork South-Central, assigning it a high probability of being rural. We can also see that the line is nearly vertical; the Fine Gael vote does not seem to be playing a significant role in distinguishing rural and urban. There is a complex history in this, Fianna Fáil is a broad party which lost a lot of support in the tumult surrounding the Great Recession it lost support, particularly to Sinn Féin in urban areas; however, Sinn Féin also has rural support so including it isn't much better



There are lots of other parties we could consider, the left-wing and social democratic parties like The Labour Party, The Green Party, People Before Profits and The Social Democrats tend to do better in cities. However, I don't want to get too distracted so lets stop. The data are on the github though if you want to look, for example, if you want to do a classifier in higher dimensions where the line will be replaced by a plane or hyperplane.

Another topic we won't consider here is the objective function, we took the log-likelihood loss from logistic regression, but there are others used for this problem. These approaches, some of which also apply to regression, aim to *regularize*, that is, come up with a fit that is less influenced by outliers and more likely to work on unseen data; these leads to topics like *ridge regression*, *lasso regularizers* and *support vector machines*. Regularization is, in general, something we will consider more in the future. In the next section though, we will look at the perceptron, which at first seems like a weird way to reformulate linear classification but turns out to be useful.

Summary

This is a straight-forward section in that it doesn't really introduce any new mathematics, all it does is replace the logistic regressor

$$\hat{p} = \sigma(\beta_0 + \boldsymbol{\beta} \cdot \mathbf{x}) \tag{4}$$

with the classification boundary corresponding to $\hat{p} = 0$:

$$\beta_0 + \boldsymbol{\beta} \cdot \mathbf{x} = 0 \tag{5}$$