Optimization Techniques – SWE1002

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Evaluation Components

- CAT-1 – 15 marks
- CAT – 2 – 15 Marks
- DA1 – 10 Marks - Applications of OT
- DA2 – 10 Marks - Literature Survey
- QUIZ - 10 Marks

Optimization Everywhere

- In Management, resources need to be optimized.
- In Business, investment need to be optimized.
- In life, time need to be optimized.
- In software, code need to be optimized.
- In electronics, size need to be optimized.
- In electrical, energy need to be optimized.
- In mechanical, weight need to be optimized.
- In civil, space need to be optimized.
- In Physics, energy need to be optimized.

Applications

- Information System.
- Industrial Engineering and Manufacturing Systems.
- Engineering Design.
- Operations and Supply Chain Management
- Scientific fields…etc.

1. Introduction

- Optimization is the act of obtaining the best result under given circumstances.

- Optimization can be defined as the process of finding the conditions that give the maximum or minimum of a function.

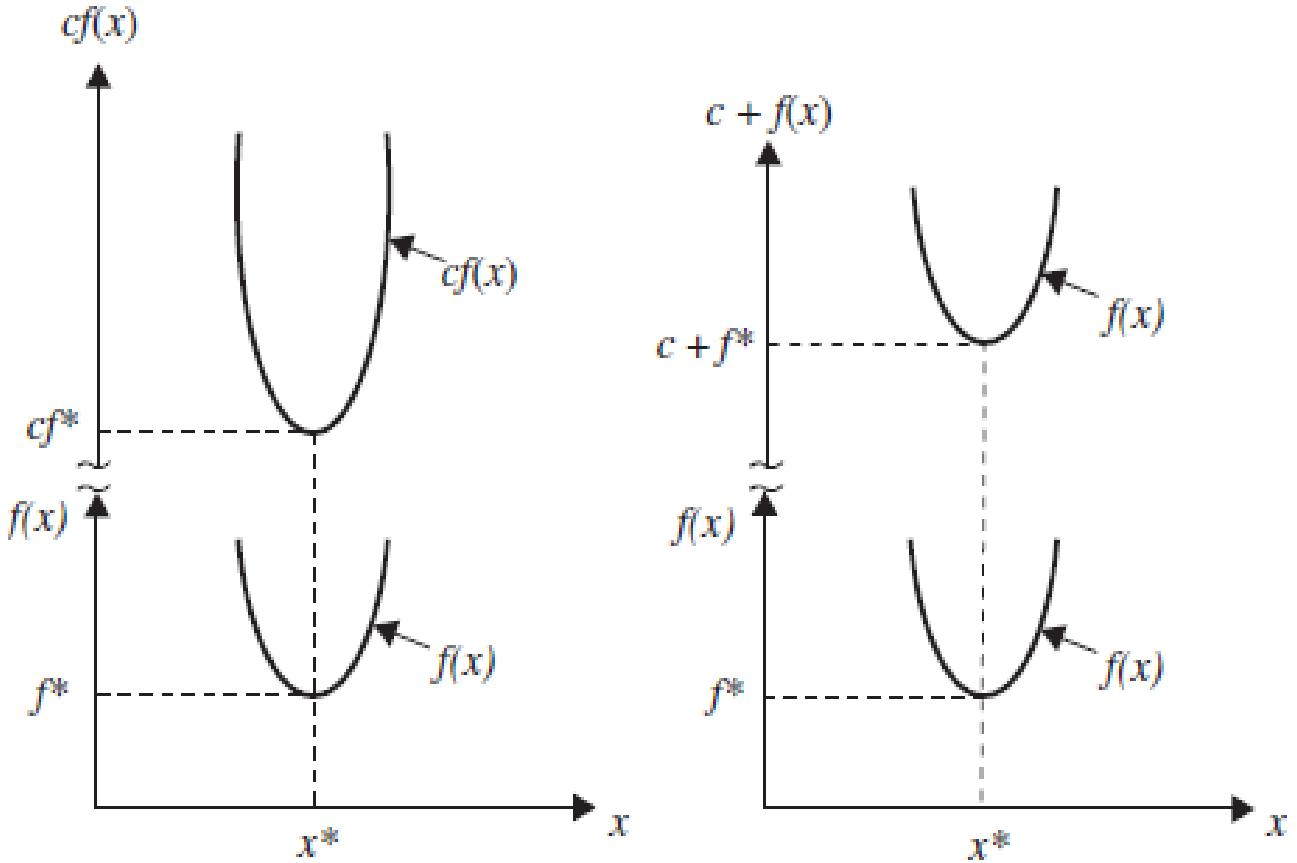
- The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research.

- *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions.

1. Introduction

- If a point x^* corresponds to the minimum value of the function $f(x)$, the same point also corresponds to the maximum value of the negative of the function, $-f(x)$. Thus optimization can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function.

Optimum solution of cf(x) or c+f(x) same as that of f(x)



1. Introduction

DESIGN VECTOR

- The conventional design procedures aim at finding an acceptable or adequate design which merely satisfies the functional and other requirements of the problem.
- In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available.
- Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one.

1. Introduction

DESIGN CONSTRAINT

. In practice, the design variables cannot be selected arbitrarily, but have to satisfy certain requirements. These restrictions are called design constraints.

- **Behaviour constraints:** Constraints that represent limitations on the behaviour or performance of the system are termed *behaviour* or *functional constraints.*

- **Side constraints:** Constraints that represent physical limitations on design variables such as manufacturing limitations.

Cont..

OBJECTIVE FUNCTION

- The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the **objective function**.

1. Introduction

- In civil engineering, the objective is usually taken as the minimization of the cost.

- In mechanical engineering, the maximization of the mechanical efficiency is the obvious choice of an objective function.

- In aerospace structural design problems, the objective function for minimization is generally taken as weight.

- In some situations, there may be more than one criterion to be satisfied simultaneously. An optimization problem involving multiple objective functions is known as a **multiobjective programming problem.**

1. Introduction

- With multiple objectives there arises a possibility of conflict, and one simple way to handle the problem is to construct an overall objective function as a linear combination of the conflicting multiple objective functions.

- Thus, if f_1 **(X)** and f_2 **(X)** denote two objective functions, construct a new (overall) objective function for optimization as:

$$f(\mathbf{X}) = \alpha_1 f_1(\mathbf{X}) + \alpha_2 f_2(\mathbf{X})$$

where α_1 and α_2 are constants whose values indicate the relative importance of one objective function to the other.

Mathematical programming or optimization techniques	Stochastic process techniques	Statistical methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	Design of experiments
Geometric programming	Renewal theory	Discriminate analysis (factor analysis)
Quadratic programming	Simulation methods	
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Modern or nontraditional optimization techniques		
Genetic algorithms		
Simulated annealing		
Ant colony optimization		
Particle swarm optimization		
Neural networks		
Fuzzy optimization		

CLASSIFICATIONS OF OPTIMIZATION PROBLEMS

- Table lists various mathematical programming techniques of operations research.
- Mathematical programming techniques are useful in finding the minimum of a function of several variables under a prescribed set of constraints.
- Stochastic process techniques can be used to analyze problems described by a set of random variables having known probability distributions.
- Statistical methods enable one to analyze the experimental data and build empirical models to obtain the most accurate representation of the physical situation.

- The modern optimization methods, also called as non traditional optimization methods, have emerged for solving complex engineering optimization problems in recent years.
- These methods include genetic algorithms, simulated annealing, particle swarm optimization, ant colony optimization, neural network-based optimization, and fuzzy optimization.
- The genetic algorithms are computerized search and optimization algorithms based on the mechanics of natural genetics and natural selection.
- The simulated annealing method is based on the mechanics of the cooling process of molten metals through annealing.

- The particle swarm optimization algorithm mimics the behavior of social organisms such as a colony or swarm of insects (for example, ants, termites, bees, and wasps), a flock of birds, and a school of fish.
- The ant colony optimization is based on the cooperative behavior of ant colonies, which are able to find the shortest path from their nest to a food source.
- The neural network methods are based on the computational power of the nervous system in the presence of massive amount of sensory data through its parallel processing capability.
- The fuzzy optimization methods were developed to solve optimization problems involving design data, objective function, and constraints stated in imprecise form involving vague and linguistic descriptions.

Engineering applications of optimization

1. Design of aircraft and aerospace structures for minimum weight

2. Finding the optimal trajectories of space vehicles

3. Design of civil engineering structures such as bridges, towers, chimneys, and dams for minimum cost

4. Minimum-weight design of structures for earthquake, wind, and other types of random loading

5. Design of water resources systems for maximum benefit

6. Optimal plastic design of structures

7. Optimum design of linkages, cams, gears, machine tools, and other mechanical components

8. Selection of machining conditions in metal-cutting processes for minimum production cost

9. Design of material handling equipment, such as conveyors, trucks, and cranes, for minimum cost

10. Design of pumps, turbines, and heat transfer equipment for maximum efficiency

11. Optimum design of electrical machinery such as motors, generators, and transformers

12. Optimum design of electrical networks

13. Shortest route taken by a salesperson visiting various cities during one tour

14. Optimal production planning, controlling, and scheduling

15. Analysis of statistical data and building empirical models from experimental results to obtain the most accurate representation of the physical phenomenon

16. Optimum design of chemical processing equipment and plants

17. Design of optimum pipeline networks for process industries

18. Selection of a site for an industry

19. Planning of maintenance and replacement of equipment to reduce operating costs

20. Inventory control

21. Allocation of resources or services among several activities to maximize the benefit

22. Controlling the waiting and idle times and queueing in production lines to reduce the costs

23. Planning the best strategy to obtain maximum profit in the presence of a competitor

24. Optimum design of control systems

RECAP

Tutorial - 1 Assignment

- What are the elements of an optimization problem. Identify any optimization problem. list and discuss the components of Optimization.
- Example.

Determine how many products to ship from each factory to each wareshouse, or from each warehouse to factory and direct to each end customer, to minimize shipping cost while meeting warehouse demands and not exceeding factory supplies

Statement of an optimization problem

An optimization, or a mathematical programming problem can be stated as follows.

Find

X $x = (x^1, x^2, ..., x^n)$

which minimizes

$$f(x)$$

subject to the constraints

$$g_j(x) \leq 0$$

for $j = 1, \ldots, m$, and

$$l_j(x) = 0$$

for $j = 1, \ldots, p$.

- The variable x is called the design vector, $f(x)$ is the objective function, $gj(x)$ are the inequality constraints and $lj(x)$ are the equality constraints. The number of variables n and the number of constraints $p + m$ need not be related. If $p + m = 0$ the problem is called an unconstrained optimization problem.

Constraint Surface

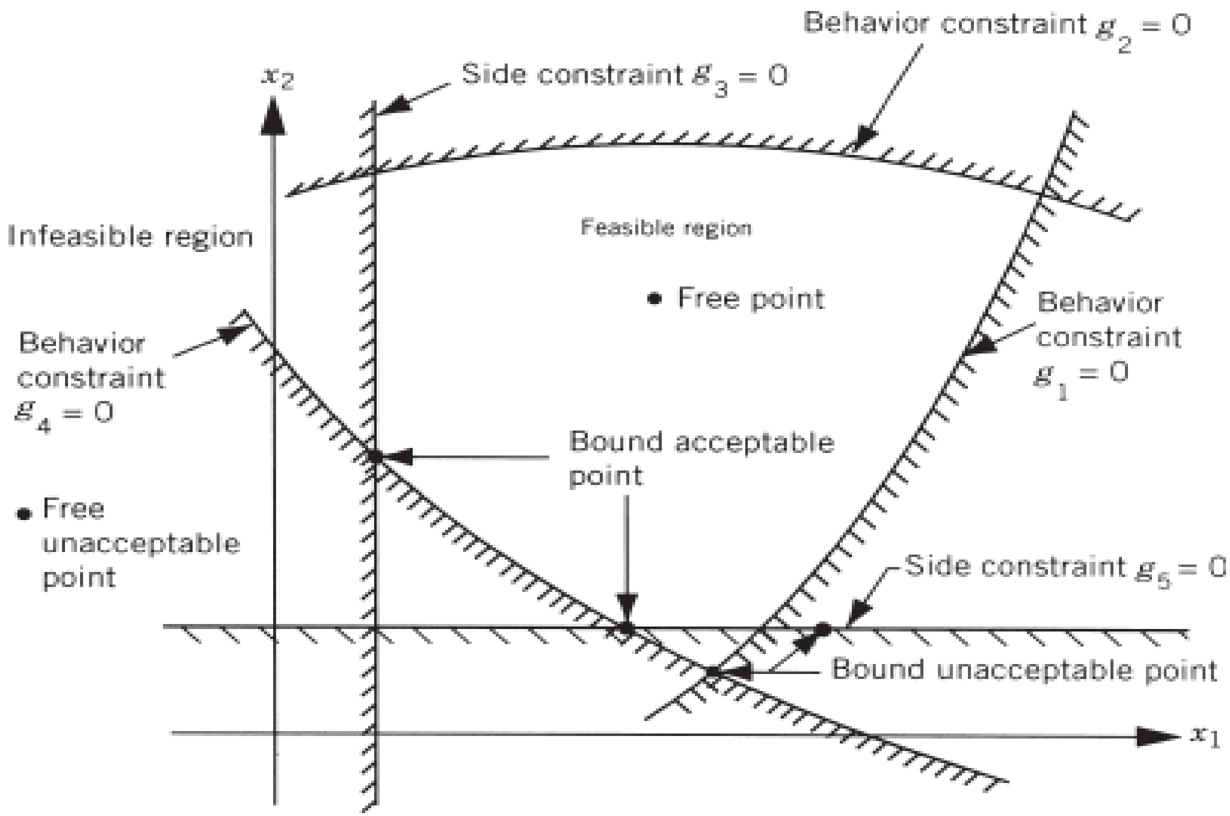


Figure 1.4 Constraint surfaces in a hypothetical two-dimensional design space.

Constraint Surface

Figure 1.4 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines.

- A design point that lies on one or more than one constraint surface is called a bound point, and the associated constraint is called an active constraint.
- Design points that do not lie on any constraint surface are known as free points.
- Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be identified as one of the following four types:

1. Free and acceptable point
2. Free and unacceptable point
3. Bound and acceptable point
4. Bound and unacceptable point

Steps to Solve Optimization Problems

- Visualize the problem
- Define the problem
- Write an equation for it
- Find the minimum or maximum for the problem (usually the derivatives or end-points)

A construction company specialized in transport has two offices at different locations, O_1 and O_2, with 8 trucks available at O_1 and 6 trucks available at O_2. The company is currently servicing two construction sites C_1 and C_2. The site C_1 needs 4 trucks to be operational, and the site C_2 needs 7 trucks. The distances from office O_1 to construction site C_1 is 8 kilometers, and to construction site C_2 it is 9 kilometers. The distances from office O_2 to construction site C_1 is 3 kilometers, and to construction site C_2 it is 5 kilometers, as illustrated in Figure 4.6. The construction company would like to minimize fuel cost and hence send trucks to the construction sites such that the overall distance the trucks have to drive is minimized. If we denote by x_1 the number of trucks sent from O_1 to C_1, by x_2 the number of trucks sent from O_1 to C_2, by x_3 the number of trucks sent from O_2 to C_1, by x_4 the number of trucks sent from O_2 to C_2, the company needs to make a choice satisfying the following conditions:

$$\begin{aligned} x_1 + x_2 &\leq 8 && \text{no more than 8 trucks at } O_1, \\ x_3 + x_4 &\leq 6 && \text{no more than 6 trucks at } O_2, \\ x_1 + x_3 &= 4 && \text{need 4 trucks at } C_1, \\ x_2 + x_4 &= 7 && \text{need 7 trucks at } C_2, \\ x_i &\geq 0 && \text{number of trucks need to be non-negative.} \end{aligned}$$

In addition, the choice of x_i, $i = 1, 2, 3, 4$ needs to be such that the fuel use is minimized, i.e. the objective function

$$8x_1 + 9x_2 + 3x_3 + 5x_4 \longrightarrow \min .$$

To simplify this problem, one can first eliminate the variables x_3 and x_4 using the equalities in (4.11),

$$x_3 = 4 - x_1, \qquad x_4 = 7 - x_2,$$

$$\begin{aligned} 5x_1 + 4x_2 + 47 &\longrightarrow \min \\ x_1 + x_2 &\leq 8 \\ -x_1 - x_2 &\leq -5 \\ x_1 &\leq 4 \\ x_2 &\leq 7 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

Objective Function Surfaces

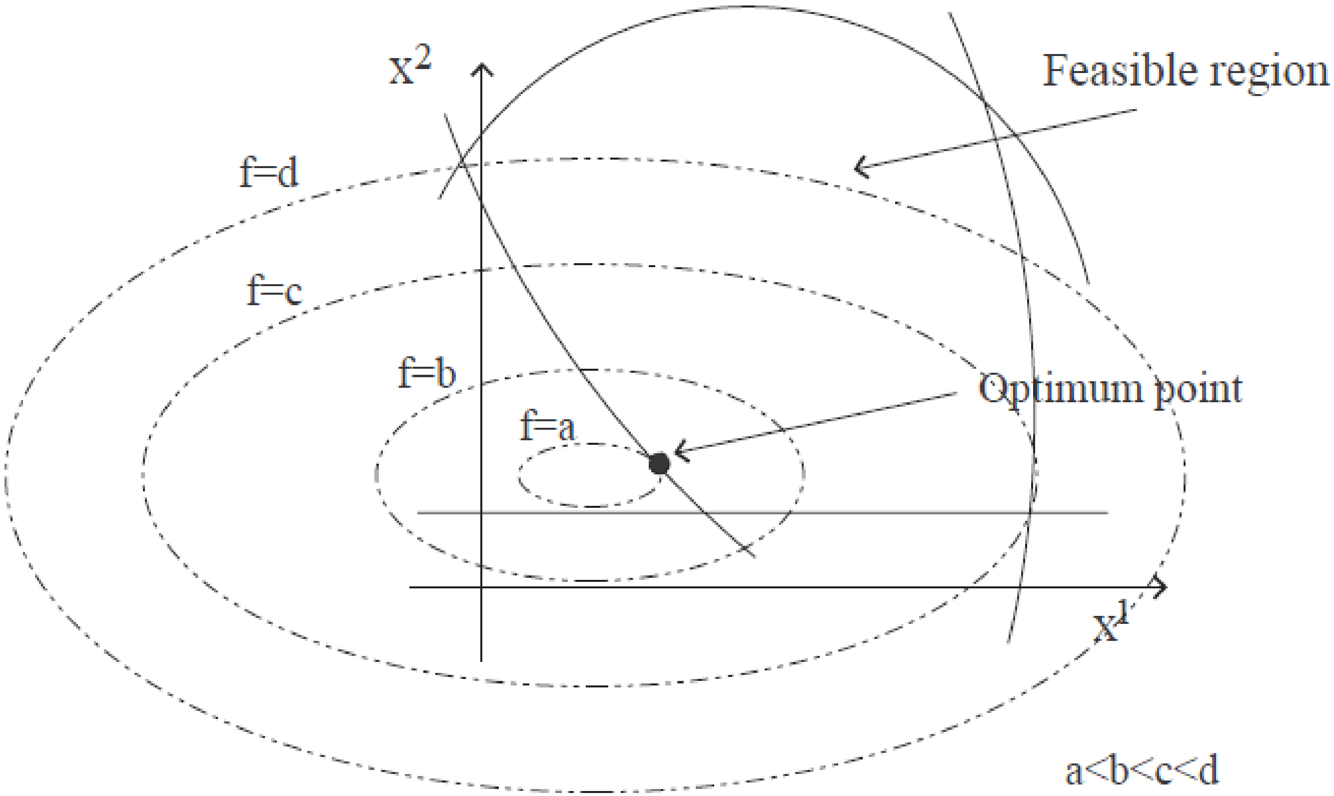


Figure 1.2: Design space, objective functions surfaces, and optimum point.

Objective Function Surfaces

- The locus of all points satisfying f (X) = C = constant forms a hypersurface in the design space, and each value of C corresponds to a different member of a family of surfaces.
- These surfaces, called objective function surfaces, are shown in a hypothetical two-dimensional design space in Fig. 1.5.
- Once the objective function surfaces are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty.
- But the main problem is that as the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem has to be solved purely as a mathematical problem.

Classification of Optimization Problems

- Optimization problem can be classified as constrained or unconstrained, depending on whether constraints exist or not.
- Based on the nature of design variables encountered, optimization problems can be classified into two broad categories.
- In the first category, the problem is to find values to a set of design parameters (constants) that make some prescribed function of parameters minimum subject to certain constraints.
- They are called *parameter* or *static optimization problems*.
- In the second category of problems, the objective is to find a set of design parameters, which are all continuous functions of some other parameter, that minimizes an objective function subject to a set of constraints.
- This type of problem, where each design variable is a function of one or more parameters, is known as a *trajectory* or *dynamic optimization problem.*

- Depending on the physical structure of the problem, optimization problems can be classified as optimal control and non-optimal control problems.
- An *optimal control* (OC) *problem* is a mathematical programming problem involving a number of stages, where each stage evolves from the preceding stage in a prescribed manner.
- It is usually described by two types of variables: the control (design) and the state variables.
- The *control variables* define the system and govern the evolution of the system from one stage to the next, and the *state variables* describe the behaviour or status of the system in any stage.
- The problem is to find a set of control or design variables such that the total objective function (also known as the *performance index*, PI) over all the stages is minimized subject to a set of constraints on the control and state variables.

- An OC problem can be stated as follows:
- Find **X** which minimizes $f(\mathbf{X}) = \sum_{i=1}^{l} f_i(x_i, y_i)$
- subject to the constraints

$$
\begin{aligned}
q_i(x_i, y_i) + y_i &= y_{i+1}, & i &= 1, 2, \ldots, l \\
\mathbf{g}_j(x_j) &\le 0, & j &= 1, 2, \ldots, l \\
\mathbf{h}_k(y_k) &\le 0, & k &= 1, 2, \ldots, l
\end{aligned}
$$

- Here, xi is the ith control variable, yi the ith state variable, and fi the contribution of the ith stage to the total objective function;
- **g**j , **h**k, and qi are functions of xj , yk, and xi and yi , respectively, and l is the total number of stages.

- Another important classification of optimization problems is based on the nature of expressions for the objective function and the constraints.
- According to this classification, optimization problems can be classified as linear, nonlinear, geometric, and quadratic programming problems.
- This classification is extremely useful from the computational point of view since there are many special methods available for the efficient solution of a particular class of problems.

Classical Optimization Techniques

- The classical methods of optimization are useful in finding the optimum solution of continuous and differentiable functions.
- These methods are analytical and make use of the techniques of differential calculus in locating the optimum points.
- Since some of the practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited scope in practical applications.

SINGLE-VARIABLE OPTIMIZATION

- A function of one variable $f(x)$ is said to have a *relative* or *local minimum* at $x = x*$

  if $f(x*) \leq f(x* + h)$ for all sufficiently small positive and negative values of h.

- Similarly, a point $x*$ is called a *relative* or *local maximum* if $f(x*) \geq f(x* + h)$ for all values of h sufficiently close to zero.

- A function $f(x)$ is said to have a *global* or *absolute minimum* at $x*$

if $f(x*) \leq f(x)$ for all x, and not just for all x close to $x*$, in the domain over which $f(x)$ is defined.

- Similarly, a point $x*$ will be a global maximum of $f(x)$

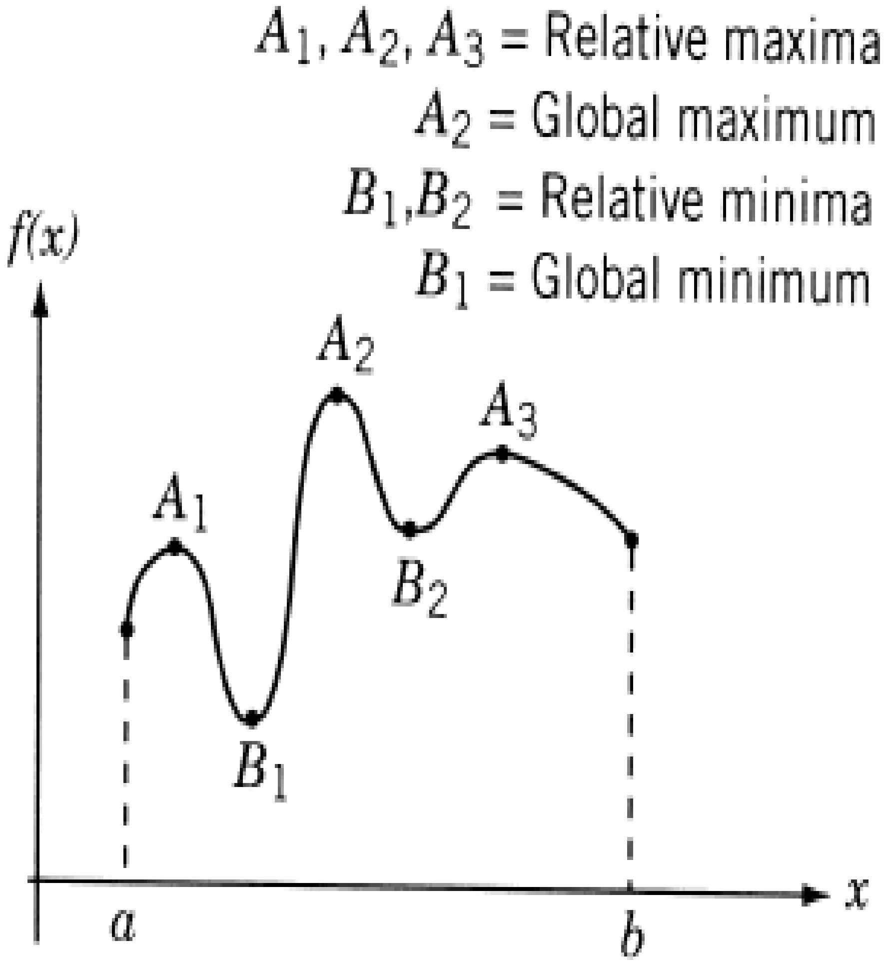
if $f(x*) \geq f(x)$ for all x in the domain.

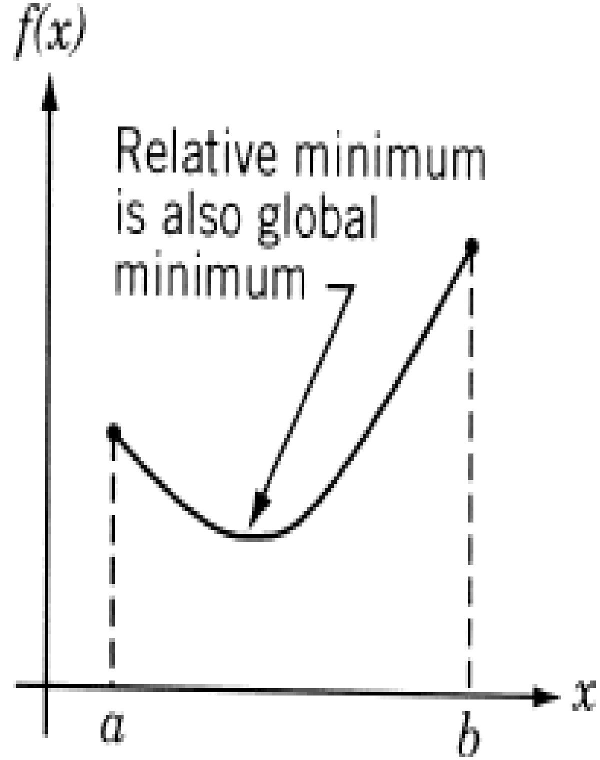
- A *single-variable optimization problem* is one in which the value of $x = x*$ is to be found in the interval $[a, b]$ such that $x*$ minimizes $f(x)$.

Relative and Global Optimum

- A function is said to have a *relative* or *local* minimum at $x = x^*$ if $f(x^*) \le f(x+h)$ for all sufficiently small positive and negative values of h, i.e. in the near vicinity of the point x.
- Similarly, a point x^* is called a *relative* or *local* maximum if $f(x^*) \ge f(x+h)$ for all values of h sufficiently close to zero.
- A function is said to have a *global* or *absolute* minimum at $x = x^*$ if $f(x^*) \le f(x)$ for all x in the domain over which $f(x)$ is defined.
- Similarly, a function is said to have a *global* or *absolute* maximum at $x = x^*$ if $f(x^*) \ge f(x)$ for all x in the domain over which $f(x)$ is defined.

Relative and global minima





- The following two theorems provide the necessary and sufficient conditions for the relative minimum of a function of a single variable.
- **Theorem : Necessary Condition** If a function $f(x)$ is defined in the interval $a \leq x \leq b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$, and if the derivative $df(x)/dx = f'(x)$ exists as a finite number at $x = x^*$, then $f'(x^*) = 0$.

- **Theorem Sufficient Condition** Let $f'(x*) = f''(x*) = \cdots = f^{(n-1)}(x*) = 0$, but $f^{(n)}(x*) = 0$.
- Then $f(x*)$ is (i) a minimum value of $f(x)$ if $f^{(n)}(x*)>0$ and n is even
- (ii) a maximum value of $f(x)$ if $f^{(n)}(x*) < 0$ and n is even
- (iii) neither a maximum nor a minimum if n is odd.(inflection point)

- ***Example*** Determine the maximum and minimum values of the function f (x) = 12x5 − 45x4 + 40x3 + 5
- SOLUTION
- Since f ′(x) = 60(x4 − 3x3 + 2x2) = 60x2(x − 1)(x − 2), f ′(x) = 0 at x = 0, x = 1, and x = 2.
- The second derivative is f ′′(x) = 60(4x3 − 9x2 + 4x)
- At x = 1, f ′′(x) = −60 and hence x = 1 is a relative maximum. Therefore, fmax = f (x = 1) = 12
- At x = 2, f ′′(x) = 240 and hence x = 2 is a relative minimum. Therefore, fmin = f (x = 2) = −11
- At x = 0, f ′′(x) = 0 and hence we must investigate the next derivative:
- f ′′′(x) = 60(12x2 − 18x + 4) = 240 at x = 0
- x = 0 is neither a maximum nor a minimum, and it is an inflection point.

Functions of a single variable

- Consider the function $f(x)$ defined for $a \leq x \leq b$
- To find the value of $x^* \in [a, b]$ such that x^* maximizes $f(x)$ we need to solve a *single-variable optimization* problem.
- We have the following theorems to understand the necessary and sufficient conditions for the relative maximum of a function of a single variable.

Functions of a single variable …contd.

- **Necessary condition :** For a single variable function $f(x)$ defined for $x \in [a, b]$ which has a relative maximum at $x = x^*$, $x^* \in [a, b]$ if the derivative $f'(x) = df(x)/dx$ exists as a finite number at $x = x^*$ then $f'(x^*) = 0$.
- We need to keep in mind that the above theorem holds good for relative minimum as well.
- The theorem only considers a domain where the function is continuous and derivative.
- It does not indicate the outcome if a maxima or minima exists at a point where the derivative fails to exist. This scenario is shown in the figure below, where the slopes m_1 and m_2 at the point of a maxima are unequal, hence cannot be found as depicted by the theorem.

First order derivative of $f(x)$

$f'(x) \to \frac{df}{dx}$, $f''(x) \to \frac{d^2f}{dx^2}$

$f'''(x) \to \frac{d^3f}{dx^3}$ $f^4(x) \to \frac{d^4f}{dx^4}$

$f^5(x) \to \frac{d^5f}{dx^5}$ - - - - - -

ex

$f(x) \to 4x^2 + 3x + 2 \Rightarrow 4(2)x^{2-1} + 3(1)x^{1-1} + 0$

$f'(x) = 8x + 3$

$f''(x) = 8(1)x^{1-1} + 0 = 8$

$f'''(x) = 0$

$f''(x) \to$ Higher order partial derivative and it is a even order since order of the function is $\underline{\underline{2}}$

Identification of Single-Variable Optima

Necessary Condition

- For finding local minima (maxima)

$$\left.\frac{df}{dx}\right|_{x=x^*} = 0 \qquad \text{AND} \qquad \left.\frac{d^2f}{dx^2}\right|_{x=x^*} \geqslant 0 \ (\leqslant 0)$$

- Proof follows…

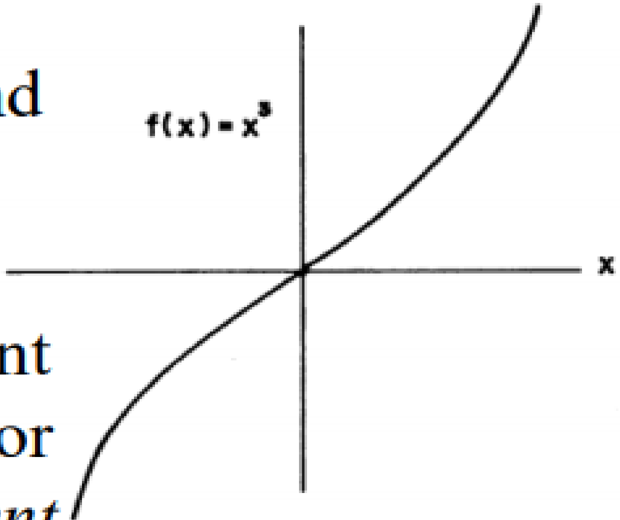
- These are necessary conditions, i.e., if they are not satisfied, x^* is not a local minimum (maximum).
- If they are satisfied, we still have no guarantee that x^* is a local minimum (maximum).

Stationary Point and Inflection Point

- A stationary point is a point x^* at which

$$\left.\frac{df}{dx}\right|_{x=x^*} = 0$$

- An *inflection point* or *saddle-point* is a stationary point that does not correspond to a local optimum (minimum or maximum).
- To distinguish whether a stationary point is a local minimum, a local maximum, or an inflection point, we need the *sufficient* conditions of optimality.



n th order condition for local maxima and minima

n th derivative test for relative extrema or inflection point.

If $\frac{d^2y}{dx^2}=0$ at a stationary point, continue differentiating until a non-zero higher-order derivative is obtained.

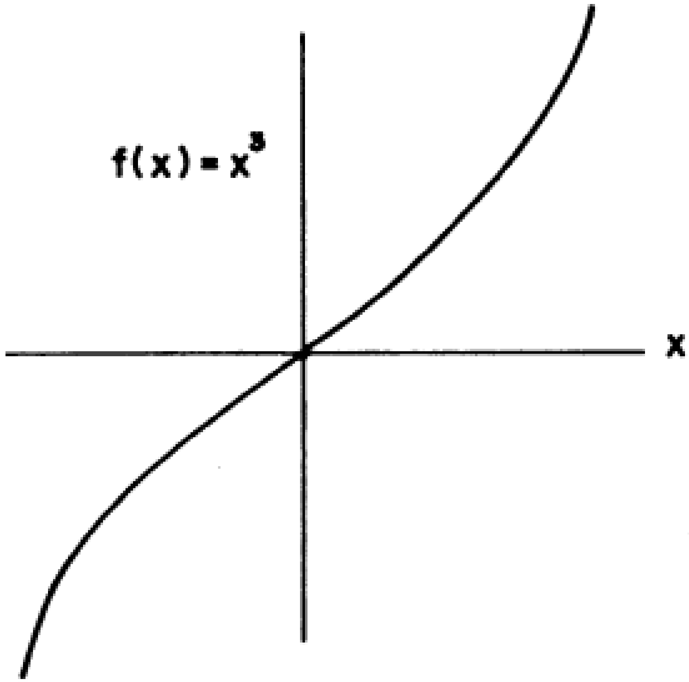
If $\frac{d^ny}{dx^n}\neq 0$ when evaluated at the stationary point then

- if n is an **odd** number , we have a **point of inflection**
- if n is an **even** number AND $\frac{d^ny}{dx^n}<0$ $\rightarrow$ a **local maximum**
- if n is an **even** number AND $\frac{d^ny}{dx^n}>0$ $\rightarrow$ a **local minimum**

Example : 1

$$f(x) = x^3$$

$$\left.\frac{df}{dx}\right|_{x=0} = 0 \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} = 0 \qquad \left.\frac{d^3f}{dx^3}\right|_{x=0} = 6$$



- Thus the first non-vanishing derivative is 3 (odd), and $x = 0$ is an inflection point.

$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$ → First Non Zero Value of the derivative

$\cancel{f^4(x) = 0}$

According to theorem the first non vanishing derivative is 3 (odd) and $x=0$ is an inflection point

inflection point

$f'(x) = 0$ find the value of 'x'

$3x^2 = 0$

$\boxed{x = 0}$

neither maximum nor minimum

So it is called as inflection Point

ex2

$\Rightarrow f'(x) = 0$

$3x^4 - 4x^3 - 24x^2 + 48x + 15$

$f'(x) = 12x^3 - 12x^2 + 48x + 48 = 0$

$12(x^3 - x^2 - 4x + 4) = 0$

$12((x^2 - 4)(x - 1) = 0$

$x - 1 = 0$

$\boxed{x = 1}$

$x^2 - 4 = 0$

$x^2 = 4$

$x = \sqrt{4}$

$\boxed{x = \pm 2}$

$x = 1, \quad x = 2, \quad x = -2$

$f''(x) = 36x^2 - 24x - 48$

at $x = 1$ $= 36(1)^2 - 24(1) - 48$

$= -36 < 0$

derivative is even, and derivative value is negative and hence $x = 1$ is the relative maximum.

F(x=1) = 38

at $x = -2$

$$f''(x) = 36(-2)^2 - 24(-2) - 48$$

$$= 36(4) + 24 - 48$$

$$= 144 > 0$$

hence $x = -2$ is relative minimum

at $x = +2$

$$f''(x) = 36(2)^2 - 24(2) - 48$$

$$= 36(4) - 48 - 48$$

$$= 48 > 0$$

$x = +2$ also a relative minimum.

Example : 2

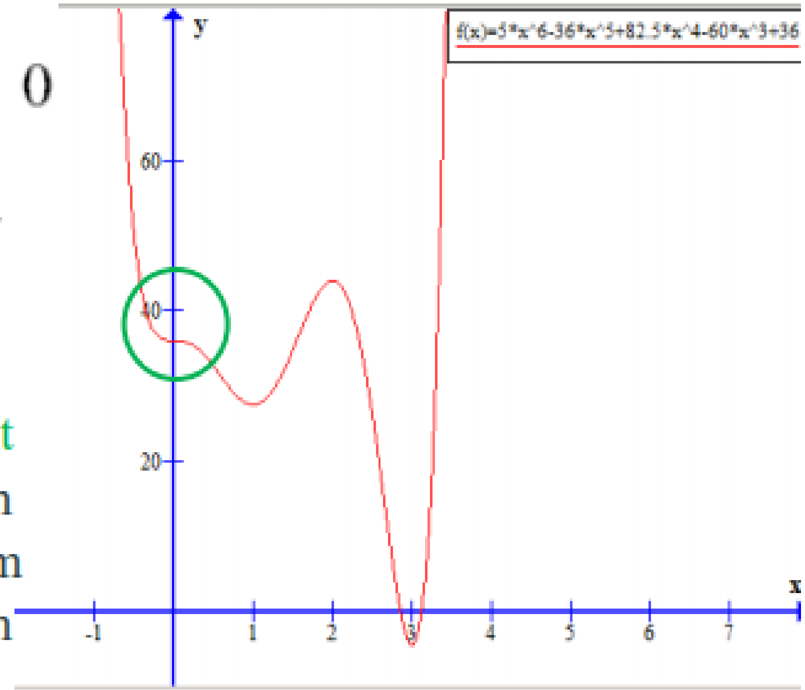
$$f(x) = 5x^6 - 36x^5 + \tfrac{165}{2}x^4 - 60x^3 + 36$$

$$\frac{df}{dx} = 30x^5 - 180x^4 + 330x^3 - 180x^2 = 30x^2(x-1)(x-2)(x-3)$$

Stationary points $x = 0, 1, 2, 3$ $\left.\dfrac{df}{dx}\right|_{x=x^*} = 0$

$$\frac{d^2f}{dx^2} = 150x^4 - 720x^3 + 990x^2 - 360x$$

x	$f(x)$	d^2f/dx^2	d^3f/dx^3	
0	36	0	-360	- Inflection point
1	27.5	60	60	-Local minimum
2	44	-120	-240	-Local maximum
3	5.5	540	2340	-Local minimum



$$f(x) = 5x^6 - 36x^5 + \frac{165}{2}x^4 - 60x^3 + 36$$

$$f'(x) = 30x^5 - 180x^4 + \frac{\cancel{4}^{2}(165)}{\cancel{2}}x^3 - 180x^2 + 0$$

$\frac{df}{dx}$

$$= 30x^5 - 180x^4 + 330x^3 - 180x^2$$

$$f'(x) = 0$$

$$30x^2(x^3 - 6x^2 + 11x - 6) = 0$$

$$30x^2(x-1)(x-2)(x-3) = 0$$

$$x^2 = 0 \quad x-1 = 0 \quad x-2 = 0 \quad x-3 = 0$$

Stationary Pts. $\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = 2}$ $\boxed{x = 3}$

for $x = 1$

$$f^2(x) = 150x^4 - 720x^3 + 990x^2 - 360x$$

$$f^2(1) = 150(1)^4 - 720(1)^3 + 990(1)^2 - 360(1)$$

$$= 150 - 720 + 990 - 360$$

$$= 60$$

even and +ve so local minimum

at $x=2$ 2

$f^2(2) = 150(2)^4 - 720(2)^3 + 990(2)^2 - 360(2)$

$= 150(16) - 720(8) - 990(4) - 360(2)$

$= -120$

'n' is even and value is -ve
So $x=2$ is a local maximum

at $x=0$

$f^2(0) = 150(0) + \cdots$

$f^2(0) = 0$

$f^3(x) = 600x^3 - 2160x^2 + 1980x - 360$

$f^3(0) = -360$

n is odd $\boxed{x=0}$ point of inflection

at $x=3$

$f^2(3) = 540$ so 3 is the local minimum

Classwork

- Determine the maximum and minimum values of the function $f\ (x) = 12x5 - 45x4 + 40x3 + 5$

Example 2.1 Determine the maximum and minimum values of the function

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5$$

SOLUTION Since $f'(x) = 60(x^4 - 3x^3 + 2x^2) = 60x^2(x-1)(x-2)$, $f'(x) = 0$ at $x = 0$, $x = 1$, and $x = 2$. The second derivative is

$$f''(x) = 60(4x^3 - 9x^2 + 4x)$$

At $x = 1$, $f''(x) = -60$ and hence $x = 1$ is a relative maximum. Therefore,

$$f_{\text{max}} = f(x = 1) = 12$$

At $x = 2$, $f''(x) = 240$ and hence $x = 2$ is a relative minimum. Therefore,

$$f_{\text{min}} = f(x = 2) = -11$$

At $x = 0$, $f''(x) = 0$ and hence we must investigate the next derivative:

$$f'''(x) = 60(12x^2 - 18x + 4) = 240 \quad \text{at} \quad x = 0$$

Since $f'''(x) \neq 0$ at $x = 0$, $x = 0$ is neither a maximum nor a minimum, and it is an inflection point.

Determine maximum and minimum values of the function.

C.W. $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

$f'(x) = 60x^4 - 180x^3 + 120x^2 + 0$

$f'(x) = 0$

$\cancel{60x^4 - 180x^3 + 120x^2}$

$60x^2(x^2 - 3x + 2) = 0$

$60x^2(x-1)(x-2) = 0$

$x^2 = 0 \rightarrow x = 0$ are the

$x - 1 = 0 \rightarrow x = 1$ Stationary

$x - 2 = 0 \rightarrow x = 2$ points

at $x=1$ $f''(x) = -60$ relative maximum

$x=2$ $f''(x) = 240$ rel. minimum.

.... F(1) = 12 and
F(2)= -11

Algorithm for finding Global Optima

$$\text{Maximize} \quad f(x)$$
$$\text{Subject to} \quad a \leqslant x \leqslant b$$

Step 1. Set $df/dx = 0$ and compute all stationary points.

Step 2. Select all stationary points that belong to the interval $[a, b]$. Call them $x_1, x_2, \ldots, x_N$. These points, along with a and b, are the only points that can qualify for a local optimum.

Step 3. Find the largest value of $f(x)$ out of $f(a), f(b), f(x_1), \ldots, f(x_N)$. This value becomes the global maximum point.

Example : 1

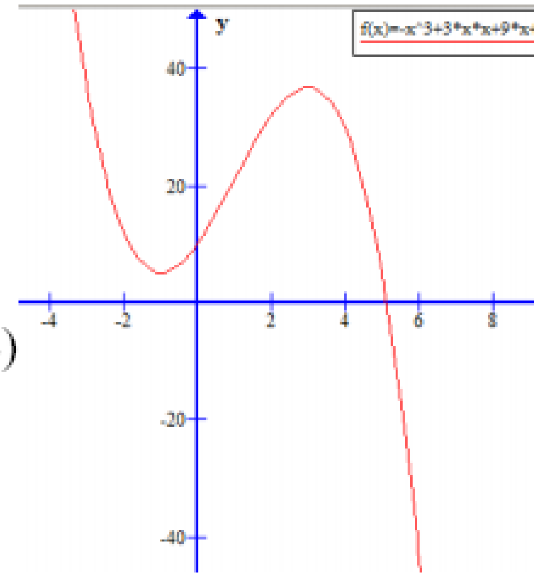
Find Global Maximum

Maximize $f(x) = -x^3 + 3x^2 + 9x + 10$ in the interval $-2 \leqslant x \leqslant 4$

$$\frac{df}{dx} = -3x^2 + 6x + 9 = 0$$

Stationary points x = -1, 3

To find the global maximum, evaluate $f(x)$ at $x = 3, -1, -2,$ and 4:



$$f(3) = 37 \qquad f(-1) = 5$$

$$f(-2) = 12 \qquad f(4) = 30$$

Hence $x = 3$ maximizes f over the interval $(-2, 4)$

Global maximum (4)

$f(x) = -x^3 + 3x^2 + 9x + 10$ in the interval $-2 \le x \le 4$

$\frac{df}{dx} = -3x^2 + 6x + 9$

$\div$ by -3

$= x^2 - 2x - 3$

$f'(x) = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

roots $x = 3$ $x = -1$

(7) Evaluate $f(x)$ at $x = 3, -1, -2,$ and 4

$f(3) = -(3)^3 + 3(3)^2 + 9(3) + 9$

$= -27 + 27 + 27 + 9$

$= 37$

$f(-1) = -(-1)^3 + 3(-1)^2 + 9(-1) + 10$

$= 1 + 3 - 9 + 10 = 5$

IIIrly $f(-2) = 12$, $f(4) = 30$

at $x = 3$ maximizes $f(x) = \boxed{x = 3}$

Optimal maximum

MULTIVARIABLE OPTIMIZATION WITH NO CONSTRAINTS

- Consider necessary, sufficient conditions for minimum or maximum of unconstrained function of several variables.
- **Theorem Necessary Condition** If f (**X**) has an extreme point (maximum or minimum) at **X** = **X**∗ and if the first partial derivatives of f (**X**) exist at **X**∗, then

$$\frac{\partial f}{\partial x_1}(\mathbf{X}^*) = \frac{\partial f}{\partial x_2}(\mathbf{X}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{X}^*) = 0$$

- **Theorem Sufficient Condition**   sufficient condition for a stationary point **X**∗ to be an extreme point is that the matrix of second partial derivatives (Hessian matrix) of f (**X**) evaluated at **X**∗ is (i) positive definite when **X**∗ is a relative minimum point, and (ii) negative definite when **X**∗ is a relative maximum point.

multivariable Optimization

$$z = x^2 - xy + y^2 + 3x$$

$$\frac{\partial z}{\partial x} = 2x - y + 3$$

$$\frac{\partial z}{\partial y} = -x + 2y$$

$$z \to f(x, y)$$

$$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial z}{\partial y} = 0$$

$$2x - y + 3 = 0 \to (1)$$

$$-x + 2y = 0 \qquad x = 2y \to (2)$$

Sub. (2) in (1)

$$2(2y) - y + 3 = 0$$

$$4y - y - 3 = 0$$

$$3y + 3 = 0$$

$$\boxed{y = -1}$$

$$x = 2y$$

$$x = 2(-1) = -2 \qquad \boxed{x = -2}$$

critical points check in eqn. ② ① ②.

$x^* = -2$

$y^* = -1$

To find these points are max/min need second order condition.

Before that find Extreme value

$$z = (-2)^2 - (-2)(-1) + (-1)^2 + 3(-2)$$
$$= 4 - (2) + (1) - 6 = 5 - 8 = -3$$

extreme value of z is -3.

How to calculate second order conditions

calculate Hessian matrix

- matrics of S.O.D.

$\frac{\partial z}{\partial x}$ $\quad \frac{\partial^2 z}{\partial x^2} = f_{11}$ | $\frac{\partial^2 z}{\partial y \partial x} = f_{21}$

$\frac{\partial^2 z}{\partial x \partial y} = f_{12}$ | $\frac{\partial^2 z}{\partial y^2} = f_{22}$

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & & & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \Rightarrow$$

2×2 matrix

Calculation of Principle Minus

for. 3×3 matrix., second partial derivatives. (3)

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$H_1 = |f_{11}|$$

$$H_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$H_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

Start with negative H_1 for finding maxima

$H_1 < 0$

$H_2 > 0$

$H_3 < 0$

minima $H_1 > 0$

$H_2 > 0$

$H_3 > 0$

$$\frac{\partial z}{\partial x} = 2x - y + 3 = 0$$

$$\frac{\partial z}{\partial y} = -x + 2y = 0$$

$$\frac{\partial^2 z}{\partial x^2} = f_{11} = 2 \qquad \frac{\partial z}{\partial x \partial y} = -1 = f_{12}$$

$$\frac{\partial^2 z}{\partial y^2} = 2 = f_{22} \qquad \frac{\partial^2 z}{\partial y \partial x} = -1 = f_{21}$$

$$H = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

principle – minors

$$H_1 = |2| = 2 > 0$$

$$H_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - (1) = 4 - 1 = 3 > 0$$

$H_1 > 0$, $H_2 > 0$ minima conds.

$x = -2$ $y = -1$ pt. of Minima

$$f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$ ①

Ex-2

$$\frac{\partial f}{\partial x_1} = 1 + 4x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4$$

$$\frac{\partial f}{\partial x_2} = -1 + 2x_1 + 2x_2$$

$$\frac{\partial f}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2 \qquad H = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Updownarrow$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2$$

$$H_1 = |4| = 4 > 0$$

$$H_2 = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

point of minima

$$1+4x_1+2x_2=0 \rightarrow (1) \quad (2)$$

$$-1+2x_1+2x_2=0 \rightarrow (2)$$

(–) (1) – (2)

$$\begin{array}{rl} & 1+4x_1+\cancel{2x_2}=0 \\ (-) & -1+2x_1+\cancel{2x_2}=0 \\ \hline & 2+2x_1 \quad =0 \end{array}$$

$$2x_1=-2$$

$$\boxed{x_1=-1}$$

Sub. in eqn (1)

$$1+4(-1)+2x_2=0$$

$$1-4+2x_2=0$$

$$-3+2x_2=0$$

$$2x_2=3$$

$$\boxed{x_2=3/2}$$

$x_1=-1$, $x_2=3/2$ are the Stationary points.

Substitute in $f(x)$ $(-5/4)$ extreme point.

Multivariable optimization with equality constraints

- **Problem statement:**

Minimize $f = f(\mathbf{X})$ subject to $g_j(\mathbf{X})=0$, $j=1,2,.....,m$ where

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

Here m is less than or equal to n, otherwise the problem becomes overdefined and, in general, there will be no solution.

- **Solution:**
  - **Solution by direct substitution**
  - **Solution by the method of constrained variation**
  - **Solution by the method of Lagrange multipliers**

The working rule (Algorithm)

Step 1: Ensure that the problem is of multivariable optimization with equality constraints, such as x_i in $f(x)$ subject to $g_j(x_i) = 0$; $j = 1, 2, ..., m$; $i = 1, 2, ..., n$.

Step 2: Express the equations in $(n - m)$ variables by suitable substitutions of one equation in the other.

Step 3: Substitute these expressions in the objective function $f(x)$ to make it unconstrained.

Step 4: Now if $f(x)$ has only one variable, use the maxima–minima rules (i.e. $\frac{df}{dx} = 0$ and check whether $\frac{d^2 f}{dx^2}$ is positive for minimum and negative for maximum).

Step 5: If $f(x)$ has two or more variables, find the partial derivatives and apply the necessary and sufficient conditions of multivariable constrained optimization techniques (i.e. set the equality partial derivatives to zero to find x_i values and find the Hessian matrix to evaluate definiteness and thus maxima/minima of $f(x)$).

Step 6: Write the solution.

(*Note:* If any of the above steps is not compatible practically, then discard this method.)

②

Multivariable with equality Constraints

ex) Find the minimum value of $x^2+y^2+z^2$

Subject to $x+y+2z=12$

Soln $f(x)$ has 3 variables (x, y, z)

direct method substitution method

$$2z = -x-y+12$$

$$z = \frac{12-x-y}{2}$$

to change 3 var. into 2 var.

$$f(x) = x^2+y^2+\left(\frac{12-x-y}{2}\right)^2$$

$$= x^2+y^2+\frac{1}{4}(12-x-y)^2$$

Now it has 2 variables.

necessary condn.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 2x + \tfrac{1}{2}(12 - x - y)(-1) \rightarrow 1$$

$$= 2x - \tfrac{1}{2}(12 - x - y)$$

$$\frac{\partial f}{\partial y} = 2y + \tfrac{1}{2}(12 - x - y)(-1) \rightarrow 2$$

$$= 2y - \tfrac{1}{2}(12 - x - y)$$

1 − 2

$$2x - 2y = 0$$

$$2x = 2y$$

$$\boxed{x = y}$$

Sub. in eqn 1

$$2x = \tfrac{1}{2}(12 - x - x)$$

$$2x = \tfrac{1}{2}(12 - 2x)$$

$$2x = 6 - x$$

$$2x + x = 6$$

$$3x = 6$$

$$\boxed{x = 2}$$ since $x = y$

so $\boxed{y = 2}$

To find z value

$$z = \frac{12 - x - y}{2}$$

$$= \frac{12 - 2 - 2}{2} = 8/2 = 4$$

$$\boxed{z = 4}$$

extreme pts. $(2, 2, 4)$

Sufficient cond. $\longrightarrow$ Hessian matrix.

$$\frac{\partial^2 f}{\partial x^2} = 2 + 1/2\,(0 - 1)$$ [redacted]

$$= 2 + 1/2 = 5/2$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 1/2 = 5/2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1/2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1/2$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix}$$

④

$$H = \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix}$$

$H_1 = 5/2 > 0$

$H_2 = \left[(5/2)(5/2) - (1/2)(1/2) \right]$

$= \left(\frac{25}{4} - 1/4 \right)$

$= \cancel{24/4} = 6 > 0$

positive definite $(2,2,4)$ is minimum pt.

$f(x) = 2^2 + 2^2 + 4^2 = 4 + 4 + 16 = 24$

So 24 is the minimum value.

ex 2

$$f(x_1, x_2) = x_1^2 + (x_2 - 1)^2$$

Subject to $-2x_1^2 + x_2 = 4$

Soln

$$f(x_1, x_2) = x_1^2 + (x_2-1)^2 \longrightarrow 1$$

$$-2x_1^2 + x_2 = 4$$

$$x_2 = 4 + 2x_1^2 \longrightarrow \text{Sub in } 1$$

$$= x_1^2 + (4 + 2x_1^2 - 1)^2$$

$$= x_1^2 + (3 + 2x_1^2)^2$$

$$= x_1^2 + 3^2 + 4x_1^4 + 2(3)2x_1^2$$

$$= x_1^2 + 4x_1^4 + 12x_1^2 + 9$$

$$f(x) = 4x_1^4 + 13x_1^2 + 9 \longrightarrow 2$$

Now eqn. 2 becomes single variable

$$\frac{\partial f}{dx} = 0$$

$4(4)x_1^3 + 2(13)x_1 + 0$

✓ $16x_1^3 + 26x_1 = 0$

$x_1(16x_1^2 + 26) = 0$

$\boxed{x_1 = 0}$ $\quad 16x_1^2 + 26 = 0$

Sub. $x_1 = 0$ in

$x_2 = 4 + 2x_1^2$

$= 4 + 2(0) = 4$

$(0, 4)$

ORDER IS even

$\frac{\partial^2 f}{dx^2} = 48x_1^2 + 26$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=0} = 48(0) + 26 = 26$

ie +ve

So f is rel. minimum at $x_1 = 0$

at $(x_1 = 0, x_2 = 4)$

$f(x_1, x_2) = x_1^2 + (x_2 - 1)^2$

$= 0 + (4 - 1)^2 = \underline{9}$

Global Minimum is 9

$\boxed{f_{min} = 9}$

Merits of the direct substitution method

1. Direct substitution method is easy to understand and use.
2. It is simple to calculate.
3. The problem is reduced in such a form that the principle of maxima–minima can be employed, which is straightforward and uncomplicated.

Limitations of the direct substitution method

Though the direct substitution method appears to be simple in theory, it is not that convenient from the practical point of view.

1. It is often impossible to solve when the constraint equations are nonlinear, particularly of a higher order.
2. It is often inconvenient or impossible to express m variables in the remaining $(n - m)$ variables when the equations are implicit.
3. This method is convenient to solve simpler problems but fails to solve complex problems.
4. This method fails if the objective function is not continuous (or not differentiable).

Lagrange Multiplier Technique

- The substitution method for solving constrained optimization problem cannot be used easily when the constraint equation is very complex and therefore cannot be solved for one of the decision variable. In such cases of constrained optimization we employ the Lagrangian Multiplier technique.
- In this Lagrangian technique of solving constrained optimisation problem, a combined equation called Lagrangian function is formed which incorporates both the original objective function and constraint equation.

Method of Lagrange Multipliers

The ***Method of Lagrange Multipliers*** is a useful way to determine the minimum or maximum of a surface subject to a constraint. It is an alternative to the method of substitution and works particularly well for non-linear constraints. For the following examples, all surfaces will be denoted as $f(x, y)$ and all constraints as $g(x, y) = c$.

The process follows an algorithm of steps which reveal another relationship between the input variables that was neither given nor evident beforehand. We can use this relationship in conjunction with the constraint to determine critical points of the surface on the constraint.

The *Method of Lagrange Multipliers* follows these steps:

1) Given a multivariable function $f(x,y)$ and a constraint $g(x,y)=c$, define the Lagrange function to be $L(x,y)=f(x,y)-\lambda(g(x,y)-c)$, where λ (lambda) is multiplied (distributed) through the constraint portion.
2) Determine the partial derivatives L_x and L_y.
3) Set the partial derivatives L_x and L_y equal to zero.
4) Make note of any immediate solutions for x and y that satisfy $L_x=0$ and $L_y=0$. Often there are none, so proceed to step 5.
5) Isolate the λ in each equation.
6) Equate the two λ equations, dropping out the λ altogether.
7) Reduce the equation from step 6 as far as possible. This is the relationship between x and y that shall be substituted into the constraint.
8) Substitute this equation into the constraint and algebraically solve for the variable that remains. This is where the critical points start to be determined.
9) Solve for the other variable by re-evaluating your results in step 8 back into the constraint.

Cont…

If just one critical point results, you cannot necessarily determine if it is a minimum or maximum just by itself, so you must study the shape of the surface and make inferences based on its shape and the relative location of the constraint (this is where a contour map helps). If two or more points result, then minima and maxima are easily determined by comparing their z-values. Some points may be neither minima nor maxima and are then ignored.

Compare the Z value

If Z value is less than zero – Minimum Point

Greater than Zero – Maximum Point

Lagrange Multipliers.

(ex) Determine the minimum and maximum values on $f(x,y) = x^2+y^2-2x-2y$ subject to the constraints $\cdot\ x+2y=4$

$$L(x,y) = f(x,y) - \lambda\, g(x,y) - c$$

$$L(x,y) = x^2+y^2-2x-2y-\lambda(x+2y-4)$$

$$L(x,y) = x^2+y^2-2x-2y-\lambda x-2\lambda y + 4\lambda \rightarrow 1$$

$$L_x = 2x-2-\lambda \rightarrow 2$$

$$L_y = 2y-2-2\lambda \rightarrow 3$$

$$L_x = L_y = 0$$

$$2x-2-\lambda = 0 \rightarrow 4$$

$$2y-2-2\lambda = 0 \rightarrow 5$$

Isolate λ in 4 and 5.
from 4

$$\boxed{2x-2 = \lambda} \rightarrow 6$$

$$\cancel{\div 2} \quad \cancel{\boxed{x-1 = \lambda}} \quad \cancel{\rightarrow 6}$$

2

$2y - 2 - 2\lambda = 0$

$y - 1 - \lambda = 0$

$\boxed{y - 1 = \lambda} \rightarrow 7$

Since by transitivity property λ & λ equal in 6 & 7

λ

$2x - 2 = y - 1$

$y = 2x - 2 + 1$

$y = 2x - 1$.

Sub. 'y' value in constraint

$x + 2y = 4$

$x + 2(2x - 1) = 4$

$x + 4x - 2 = 4$

$5x = 4 + 2$

$\boxed{x = 6/5}$

$6/5 + 2y = 4$

$2y = 4 - (6/5)$

③

$$2y = \frac{14}{5}$$

$$y = \frac{14}{5} / 2$$

$$= \frac{14}{5} \times 1/2 = \frac{14}{10} = 7/5$$

Substitute x, y values in f(x, y) to find minimum / max. value

$$f(x,y) = x^2 + y^2 - 2x - 2y$$

$$= (6/5)^2 + (7/5)^2 - 2(6/5) - 2(7/5)$$

$$= \frac{36}{25} + \frac{49}{25} - 12/5 - \frac{14}{5}$$

$$= \frac{36+49}{25} - \left(\frac{12+14}{5}\right)$$

$$= \frac{85}{25} - 26/5 = \frac{85 - 26(5)}{5}$$

Z

$$f(x,y) = -9/5$$

minimum pt.

(CI - 17ML80100

(ex) Find Minimum value of

$f(x,y) = x^2+4y^2-2x+8y$ subject

to constraints $x+2y=7$

ex3 Find minimum and maximum pts (6) on the surface $f(x,y) = x^2+y^2-2x-2y$ subject to constraint $x^2+y^2=4$

$$L(x,y) = x^2+y^2-2x-2y-\lambda(x^2+y^2-4)$$
$$= x^2+y^2-2x-2y-\lambda x^2-\lambda y^2+4\lambda \rightarrow (1)$$

$$L_x = 2x-2-2\lambda x = 0 \rightarrow (2)$$
$$L_y = 2y-2-2\lambda y = 0 \rightarrow (3)$$

from (2)

$$2x-2-2\lambda x = 0$$
$$2\lambda x = 2x-2$$
$$\lambda = \frac{2x-2}{2x} = \frac{x-1}{x}$$

from (3)

$$2y-2-2\lambda y = 0$$
$$2\lambda y = 2y-2$$
$$\lambda = \frac{2y-2}{2y} = \frac{y-1}{y}$$

$$\therefore \frac{x-1}{x} = \frac{y-1}{y} \Rightarrow y(x-1) = x(y-1)$$

(7)

$xy - y = xy - x \Rightarrow -y = xy - xy - x$

$-y = -x$

$\therefore \boxed{y = x}$

sub $y = x$ in constraint

$x^2 + y^2 = 4 \Rightarrow x^2 + (x)^2 = 4$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

since $y = x$ $\quad y = \pm\sqrt{2}$

$(\sqrt{2}, \sqrt{2}, z), (-\sqrt{2}, -\sqrt{2}, z)$

determine z

$f(x,y) = x^2 + y^2 - 2x - 2y$

$= (\sqrt{2})^2 + (\sqrt{2})^2 - 2(\sqrt{2}) - 2(\sqrt{2})$

$= 2 + 2 - 2\sqrt{2} - 2\sqrt{2}$

$= 4 - 4\sqrt{2}$

$f(x,y) = (-\sqrt{2})^2 + (\sqrt{-2})^2 - 2(-\sqrt{2}) - 2(-\sqrt{2})$

$= 4 + 2\sqrt{2} + 2\sqrt{2}$

$= 4 + 4\sqrt{2}$

(8)

the points are

$(\sqrt{2}, \sqrt{2}, 4-4\sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2}, 4+4\sqrt{2})$

minimum pt. maximum pt.

-1.648 9.6

---x---x---x---

ex4 Determine the minimum & maximum points of $f(x,y) = 3xy + 2$ subject to the constraint $x^2+y^2=1$

Soln

$$L(x,y) = 3xy + 2 - \lambda(x^2+y^2-1)$$
$$= 3xy+2 - \lambda x^2 - \lambda y^2 + \lambda$$

$$L_x = 3y - 2\lambda x = 0$$
$$L_y = 3x - 2\lambda y = 0$$

$$3y - 2\lambda x = 0$$
$$-2\lambda x = -3y$$
$$\lambda = \frac{3y}{2x}$$

$$3x - 2\lambda y = 0$$
$$2\lambda y = 3x$$
$$\lambda = \frac{3x}{2y}$$

$$\therefore \frac{3y}{2x} = \frac{3x}{2y}$$

$$3y(2y) = 3x(2x)$$
$$6y^2 = 6x^2$$
$$y^2 = x^2 \qquad y = \sqrt{x^2}$$
$$y = \pm x$$

$y = x$, $y = -x$ sub. in Constraint

$$y = x \Rightarrow x^2 + (x)^2 = 1$$
$$2x^2 = 1$$
$$x^2 = 1/2$$
$$x = \pm\sqrt{1/2}$$

$(\sqrt{1/2}, \sqrt{1/2}), (-\sqrt{1/2}, -\sqrt{1/2})$

$$y = -x \Rightarrow x^2 + (-x)^2 = 1$$
$$2x^2 = 1$$
$$x^2 = 1/2$$
$$x = \pm\sqrt{1/2}$$

$(\sqrt{1/2}, -\sqrt{1/2}), (\sqrt{1/2}, \sqrt{1/2})$

(10)

(10) find 'z' value for x, y values.
i.e for 4 pts. [∵ substitute in $f(x,y)$]

we get

$\left(\sqrt{1/2}, \sqrt{1/2}, \underline{\underline{7/2}}\right) \rightarrow$ Pt1

$\left(-\sqrt{1/2}, -\sqrt{1/2}, \underline{\underline{7/2}}\right) \rightarrow$ Pt2

$\left(-\sqrt{1/2}, \sqrt{1/2}, \underline{\underline{1/2}}\right) \rightarrow$ Pt3

$\left(\sqrt{1/2}, -\sqrt{1/2}, \underline{\underline{1/2}}\right) \rightarrow$ Pt3

first 2 points has same 'z' value.
So both tie for maximum
the other 2 points has same 'z' value
So both tie for minimum.

—x—

tutorial Assignment.

(4)

$f(x,y) = x^2+y^2+4x-2y$ subject

to - elliptical constraint $2x^2+y^2=4$

1

Kuh - Tucker (K.T) — Condition

— inequality constraints

Given $f(x)$

minimization $f(x)$ concentrates

subject to $g(x) \rightarrow$ many functions (many eqns).

$$g_1(x) \leq 0 \qquad \text{S.T.} \quad \lambda \geq 0$$
$$g_2(x) \leq 0 \qquad \lambda_1 \geq 0$$
$$g_3(x) \leq 0 \qquad \lambda_2 \geq 0$$
$$\vdots \qquad \qquad \vdots$$

$f(x) \rightarrow n$ variables

$g(x) \rightarrow n$ variables

$$x^* = (x_1, x_2, x_3 \ldots x_n)$$

(x^* values of $x_1, x_2 \ldots$)

$\hookrightarrow$ local minimum.

step 1

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} + \lambda_3 \frac{\partial g_3}{\partial x_1} = 0$$

(2) $\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} + \lambda_3 \frac{\partial g_3}{\partial x_2} = 0$

for every variable " "

Step2 $\lambda_i g_i = 0 \Rightarrow$

$$\lambda_1 g_1 = 0$$
$$\lambda_2 g_2 = 0$$
$$\lambda_3 g_3 = 0$$

step3. $g_i \leq 0$ $\quad i = 1, 2, 3 \ldots$

$$g_1 \leq 0$$
$$g_2 \leq 0$$
$$g_3 \leq 0$$

Step 4 $\lambda_i \geq 0$ $\quad i = 1, 2, 3 \ldots n$

$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$\lambda_3 \geq 0$$
$$\vdots$$
$$\lambda_n \geq 0$$

non negative always.

③

If minimization objective S.T.

constraint $g(x) \geq 0$

then $\lambda_i \leq 0$ non positive.

If maximization objective

$g(x) \geq 0$

then $\lambda_i \geq 0$ non negative

---

ex1 Minimize $f = x_1^2 + 2x_2^2 + 3x_3^2$ subject to constraints

$g_1 = x_1 - x_2 - 2x_3 \leq 12$

$g_2 = x_1 + 2x_2 - 3x_3 \leq 8$

using Kuhn Tucker conditions

1. $\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$

$$2x_1 + \lambda_1(1) + \lambda_2(1) = 0$$

$$2x_1 + \lambda_1 + \lambda_2 = 0 \rightarrow 1$$

2. $$\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$4x_2 - \lambda_1(-1) + \lambda_2(2) = 0$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0 \rightarrow 2$$

3. $$\frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial g_1}{\partial x_3} + \lambda_2 \frac{\partial g_2}{\partial x_3} = 0$$

$$6x_3 + \lambda_1(-2) + \lambda_2(-3) = 0$$

$$6x_3 + 2\lambda_1 - 3\lambda_2 = 0 \rightarrow 3$$

step 1

$\Rightarrow$

$$2x_1 + \lambda_1 + \lambda_2 = 0$$

$$4x_2 - \lambda_1 + 2\lambda_2 = 0$$

$$6x_3 - 2\lambda_1 - 3\lambda_2 = 0$$

Step 2 $\quad \lambda_i g_i = 0 \quad i = 1, 2, 3 \ldots$

✓ $\lambda_1 (x_1 - x_2 - 2x_3 - 12) = 0 \rightarrow (4)$

✓ $\lambda_2 (x_1 + 2x_2 - 3x_3 - 8) = 0 \rightarrow (5)$

Step 3. $\quad g_i \leq 0 \quad i = 1, 2$

$x_1 - x_2 - 2x_3 - 12 \leq 0 \rightarrow (6)$

$x_1 + 2x_2 - 3x_3 - 8 \leq 0 \rightarrow (7)$

Step 4 : $\quad \lambda_i \geq 0 \quad i = 0 \rightarrow (8)$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

from step 2

$\lambda_1 = 0$ (or) $x_1 - x_2 - 2x_3 - 12 = 0$

($\because$ prod. of 2 = 0)

Case 1

$\lambda_1 = 0$

Substitute $\lambda_1 = 0$ in Step 1 eqns.

(6)

$$2x_1 + 0 + \lambda_2 = 0$$
$$4x_2 - 0 + 2\lambda_2 = 0$$
$$6x_3 - 2(0) - 3\lambda_2 = 0$$

$$x_1 = \frac{-\lambda_2}{2}$$
$$-x_2 = \frac{2\lambda_2 - \lambda_2}{4} = \frac{\lambda_2}{2}$$
$$x_2 = \frac{-\lambda_2}{2}$$

$$x_3 = \frac{3\lambda_2}{6}$$
$$x_3 = \frac{\lambda_2}{2}$$

$$x_1 = x_2 = \frac{-\lambda_2}{2} \quad \& \quad x_3 = \frac{\lambda_2}{2}$$

Sub. x_1 & x_2 and x_3 in eqn. (5)

$$\lambda_2\left[\left(\frac{-\lambda_2}{2} + 2\left(\frac{-\lambda_2}{2}\right) - 3\left(\frac{\lambda_2}{2}\right) - 8\right] = 0$$

$$2\,\frac{\lambda_2}{\cancel{2}}\left(-\lambda_2 - 2\lambda_2 - 3\lambda_2 - 16\right) = 0$$

$$\frac{\lambda_2}{2}\left(-6\lambda_2 - 16\right) = 0$$

$$-3\lambda_2 - 8$$

$$\cancel{\lambda_2\left(-\frac{\lambda_2}{2} - \lambda_2 - \frac{3\lambda_2}{2} - 8\right)} = 0$$

$$= -2\lambda_2^2 - 4\lambda_2^2 - 6\lambda_2^2 - 32\lambda_2 = 0$$

$$-6\lambda_2^2 - 6\lambda_2^2 - 32\lambda_2 = 0$$

$$-12\lambda_2^2 - 32\lambda_2 = 0$$

$\div 4$

$$-3\lambda_2^2 - 8\lambda_2 = 0$$

Cancel (−)

$$3\lambda_2^2 + 8\lambda_2 = 0$$

$$\lambda_2(3\lambda_2 + 8) = 0$$

$$\boxed{\lambda_2 = 0} \quad ✓ \qquad 3\lambda_2 = -8 \qquad \boxed{\lambda_2 = -8/3} \quad ✗$$

According to our condn.

$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$\lambda_3 \geq 0.$$

$\lambda_2 = 0$ and $\lambda_1 = 0$ in step 1 eqns. (8)

we get,

$2x_1 + 0 + 0 = 0 \Rightarrow x_1 = 0$

$4x_2 - 0 + 0 = 0 \quad x_2 = 0$

$6x_3 - 2(0) - 3(0) = 0 \quad x_3 = 0$

Soln 1 $(0, 0, 0)$. which satisfies the given constraint

$$-12 \le 0$$

$$-8 \le 0$$

Case 2:

$$x_1 - x_2 - 2x_3 - 12 = 0$$

step 1

$$x_1 = \frac{-\lambda_1 - \lambda_2}{2}$$

$$x_2 = \frac{\lambda_1 - 2\lambda_2}{4}$$

$$x_3 = \frac{2\lambda_1 + 3\lambda_2}{6}$$

(9)

$17\lambda_1 + 12\lambda_2 = -144$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0$

$$\left(\frac{-\lambda_1 - \lambda_2}{2}\right) - \left(\frac{\lambda_1 - 2\lambda_2}{4}\right) - 2\left(\frac{2\lambda_1 + 3\lambda_2}{6}\right) - 12 = 0$$

$$\frac{-\lambda_1 - \lambda_2}{2} - \frac{\lambda_1 - 2\lambda_2}{4} - \left(\frac{4\lambda_1 + 6\lambda_2}{6}\right) - 12 = 0$$

$$\frac{-6\lambda_1 - 6\lambda_2 - 3\lambda_1 + 6\lambda_2 - 8\lambda_1 - 12\lambda_2 - 144}{12} = 0$$

$$-17\lambda_1 - 12\lambda_2 - 144 = 0$$

Cancel (−) $\quad 17\lambda_1 + 12\lambda_2 + 144 = 0$

$$17\lambda_1 + 12\lambda_2 = -144$$

λ_1 & $\lambda_2 \leq 0$

λ_1 & $\lambda_2 \leq 0$ from the above equations.

If, check the possibility.

(or) $\lambda_1 = 0$ $\lambda_2 = -12$ ×

$\lambda_1 = -8$ & $\lambda_2 = 0$ ×

$\lambda_1 < 0$ $\lambda_1 < 0$. ×

[λ_1 & λ_2 is not is not possible for +ve]

Thus case 2 is not possible for given objective function since λ values are negative. So case 2 is ignored.

and hence $x_1 = 0, x_2 = 0$ $x_3 = 0$ are the minimum pts. for the given objective function.

1. minimize $f(x)$ — condn $\leq \Rightarrow \lambda \geq 0$

2. minimize $f(x)$ — cond $\geq \Rightarrow \lambda \leq 0$

3. maximize
   $f(x)$ $\geq \Rightarrow \lambda \geq 0$
   $f(x)$ $\leq \Rightarrow \lambda \leq 0$

(5)

ex3 minimize $f = x_1^2 + x_2^2 + 60x_1$
subject to the constraints

$$g_1 = x_1 - 80 \geq 0$$
$$g_2 = x_1 + x_2 - 120 \geq 0$$

$$2x_1 + 60 + \lambda_1 + \lambda_2 = 0$$
step1
$$2x_2 + \lambda_2 = 0$$

step2

$$\lambda_i g_i = 0$$
$$\lambda_1 (x_1 - 80) = 0$$
$$\lambda_2 (x_1 + x_2 - 120) = 0$$

step3 $g_i \leq 0$
$$x_1 - 80 \geq 0$$
$$x_1 + x_2 + 120 \geq 0$$

Step 4 : $\lambda j \le 0$

$\lambda_1 \le 0$

$\lambda_2 \le 0$

Case 1

$\lambda_1 = 0$

Case 2

$x_1 - 80 = 0$

Sub $\lambda_1 = 0$ in eqn 1

$$1 \Rightarrow 2x_1 + 60 + \lambda_1 + \lambda_2 = 0$$

$$2x_1 + 60 + 0 + \lambda_2 = 0$$

$$2x_1 + \lambda_2 = -60$$

$$2x_1 = -\lambda_2 - 60$$

$$x_1 = \frac{-\lambda_2}{2} - 30$$

$$2 \Rightarrow 2x_2 = -\lambda_2$$

$$x_2 = \frac{-\lambda_2}{2}$$

Sub x_1 & x_2 in 4

(1)

$$\lambda_2 (x_1 + x_2 - 120) = 0$$

$$\lambda_2 \left(\left(\frac{-\lambda_2}{2} - 30 \right) + \left(\frac{-\lambda_2}{2} \right) - 120 \right) = 0$$

$$\lambda_2 \cancel{(-\lambda_2 - 60) + (-\lambda_2 - 240)} = 0$$

$$\cancel{\lambda_2 (-\lambda_2 - 60 - \lambda_2 - 240)}$$

$$\lambda_2 \left(\frac{-\lambda_2 - 60}{2} \right) + \left(\frac{-\lambda_2 - 240}{2} \right) = 0$$

$$\lambda_2 \left(\frac{-\lambda_2}{2} - 30 - \frac{\lambda_2}{2} - 120 \right)$$

$$\lambda_2 \left(\frac{-\lambda_2}{2} - 150 - \frac{\lambda_2}{2} \right)$$

$$\lambda_2 (-\lambda_2 - 150) = 0$$

$\lambda_2 0$ $\qquad$ $\lambda_2 = +150.$

$\boxed{\lambda_2 = 0}$ or $\boxed{\lambda_2 = -150}$

(8)

$\lambda_1 = 0, \lambda_2 = 0$

$2x_1 = \cdot$ $\quad x_1 = -\frac{\lambda_2}{2} - 30$

$x_2 = -\frac{\lambda_2}{2}$

$x_1 = -30.$

$x_2 = 0$

$[-30, 0]$

$\lambda_1 = 0, \lambda_2 = -150$

$x_1 = \frac{150}{2} - 30$

$= 75 - 30 = 45$

$x_2 = \frac{-\lambda_2}{2} = \frac{150}{2} = -75$

$[45, 75] > 0$

But solution violates (7)

<u>Case 2</u>: $(x_1 - 80) = 0$

$x_1 = 80$

$x_1 = 80$ in eqn ① & ② ⑨

$2x_1 + 60 + \lambda_1 + \lambda_2 = 0$

$2(80) + 60 + \lambda_1 + \lambda_2 = 0$

$160 + 60 - 2x_2 + \lambda_1 = 0$

$220 - 2x_2 + \lambda_1 = 0$

$\boxed{\lambda_1 = 2x_2 - 220}$

$2x_2 + \lambda_2 = 0$

$\boxed{\lambda_2 = -2x_2}$

sub x_2 & λ_1 in ④

$(-2x_2)(x_1 + x_2 - 120) = 0$

$-2x_2(80 + x_2 - 120) = 0$

$-2x_2(x_2 - 40) = 0$

possible values

either $x_2 = 0$

(or) ~~$x_2 = 40$~~

$\boxed{x_2 - 40 = 0}$

For $x_2 = 0$

$\lambda_1 = -220$ violates !

for $x_2 - 40 = 0 \Rightarrow x_2 = 40$

$\lambda_1 = -140$

$\lambda_2 = -80$

$\therefore$ this solution satisfies all eqns. from KKT condns. $\therefore$ this solution set is the optimization problem $x^* = [80, 40]$

Q. Minimize $f(x,y) = x^2+y^2$ subjected to

$g(x,y) = 2x+y \leq 2$.

Sol:

Step-1: $\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$.

$2x + 2\lambda = 0$ — ①

$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$

$2y + \lambda = 0$ — ②

Step-2: $\lambda_i g_i = 0$

$\lambda(2x+y-2) = 0$ — ③

Step-3: $g_i \leq 0$.

$2x+y-2 \leq 0$ — ④

Step-4: $\lambda_i \geq 0$.

$\lambda \geq 0$ — ⑤.

From step-2:

Case-1:

$\lambda = 0 \Rightarrow$ Substitute in eq ①, ②.

$2x = 0 \Rightarrow x = 0$

$2y = 0 \Rightarrow y = 0$.

Substitute $(x, y) = (0,0)$ in constraint ④.

$-2 \leq 0$. (satisfies)

Case-2:

$$2x+y-2=0.$$

From (1) and (2)

$$2x+2\lambda=0 \Rightarrow x=-\lambda \left(=\frac{4}{5}\right)$$

$$2y+\lambda=0 \Rightarrow y=\frac{-\lambda}{2}. \left(=\frac{2}{5}\right)$$

$$\Rightarrow 2x+y-2=0$$

$$-2\lambda-\frac{\lambda}{2}-2=0.$$

$$\frac{-4\lambda-\lambda-4}{2}=0$$

$$-5\lambda-4=0$$

$$\lambda=\frac{-4}{5}\ngeq 0 \text{ (not possible)}$$

From (5), $\lambda \geq 0$

case-2 is not possible for given objective function since λ value is negative. So, case-2 is ignored.

And hence $x=0$, $y=0$ are the minimum points for the given objective function.