Interest Rate Swaptions - A Review & Derivation of Swaption Pricing Formulae

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Abstract

In this paper we outline the European interest rate swaption pricing formula from first principles using the Martingale Representation Theorem and the annuity measure. This leads to an expression that allows us to apply the generalized Black-Scholes result. We show that a swaption pricing formula is nothing more than the Black-76 formula scaled by the underlying swap annuity factor.

Firstly we review the Martingale Representation Theorem for pricing options, which allows us to price options under a numeraire of our choice. We also highlight and consider European call and put option pricing payoffs. Next we discuss how to evaluate and price an interest swap, which is the swaption underlying instrument. We proceed to examine how to price interest rate swaptions using the martingale representation theorem with the annuity measure to simplify the calculation. Finally applying the Radon-Nikodym derivative to change measure from the annuity measure to the savings account measure we arrive at the swaption pricing formula expressed in terms of the Black-76 formula. We also provide a full derivation of the generalized Black-Scholes formula for completeness.

Notation

The notation in table (1) will be used for pricing formulae.

A_N^{Fixed}	The swap fixed leg annuity scaled by the swap notional
A_N^{Float}	The swap float leg annuity scaled by the swap notional
b	The cost of carry, $b = r - q$
C	Value of a European call option
K	The strike of the European option
l	The Libor floating rate in % of an interest rate swap floating cashflow
m	The total number of floating leg coupons in an interest rate swap
M_t	A tradeable asset or numeraire M evaluated at time t
n	The total number of fixed leg coupons in an interest rate swap
N_t	A tradeable asset or numeraire N evaluated at time t
N	The notional of an interest rate swap
$N(z)$	The value of the Cumulative Standard Normal Distribution
P	Value of a European put option
p^{Market}	The market par rate in % for a swap. This is the fixed rate that makes the swap fixed leg price match the price of the floating leg
$P(t,T)$	The discount factor for a cashflow paid and time T and evaluated at time t, where $t < T$
ϕ	A call or put indicator function, 1 represents a call and -1 a put option. In the case of swap 1 represents a swap to receive and -1 to pay the fixed leg coupons
q	The continuous dividend yield or convenience yield
r	The risk-free interest rate (zero rate)
r^{Fixed}	The fixed rate in % of an interest rate swap fixed cashflow
s	The Libor floating spread in basis points of an interest rate swap floating cashflow
S	For options the underlying spot value
σ	The volatility of the underlying asset
T	The time to expiry of the option in years
t_E	The option pricing or valuation date with $t_E < T$ for newly transacted swaptions the valuation date will be the trade or effective date
τ	The year fraction of a swap coupon or cashflow
V	Value of a European call or put option
X_T	The option payoff evaluated at time T

Table 1: Notation

Introduction

A swaption is an option contract that provides the holder with the right, but not the obligation, to enter an interest rate swap starting in the future at a fixed rate set today. Swaptions are quoted as N x M, where N indicates the option expiry in years and M refers to the underlying swap tenor in years. Hence a 1 x 5 Swaption would refer to 1 year option to enter a 5 year swap[1].

Swaptions can be cash or physically settled meaning that on option expiry if exercised we can specify to enter into the underlying swap or receive the cash equivalent on expiry. Swaptions are specified as $payer$ or $receiver$ meaning that one has the option to enter a swap to pay or receive the fixed leg of the swap respectively. Furthermore swaptions have an associated option style with the main flavours being European, American and Bermudan, which refer to the option exercise date(s), giving the holder the right to exercise at option expiry only, at any date up to and on discrete intervals up to and including option expiry respectively.

In what follows we consider how European Swaptions on interest rate swaps with physical settlement are priced. Firstly we outline the necessary preliminaries namely the Martingale Representation Theorem and the pricing of underlying interest rate swaps. We then proceed to outline how swaptions are priced using the Martingale Representation Theorem, selecting the annuity measure. This leads to an Black-Scholes type expression that allows us to use the generalized Black-Scholes result to arrive at an analytical expression for the swaption price.

1 Martingale Representation Theorem

Using the Martingale Representation Theorem outlined in our Martingale paper [6] we can evaluate the price of an option as below, whereby the price V_t at time t of an option with payoff X_T at time T is evaluated with respect to a tradeable asset or numeraire N with corresponding probability measure $\mathbb{Q}_N$.

$$\frac{V_t}{N_t} = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{X_T}{N_T}\Big|\mathcal{F}_t\right] \tag{1}$$

or equivalently as

$$V_t = N_t\mathbb{E}^{\mathbb{Q}_N}\left[\frac{X_T}{N_T}\Big|\mathcal{F}_t\right] \tag{2}$$

A European Option with payoff X_T at time T takes the below form for a European Call

$$\begin{aligned} X_T &= max(S_T - K, 0) \\ &= (S_T - K)^+ \end{aligned} \tag{3}$$

and likewise for a European Put Option

$$\begin{aligned} X_T &= max(K - S_T, 0) \\ &= (K - S_T)^+ \end{aligned} \tag{4}$$

[1]Note the underlying 5 year swap in this case would be a forward starting swap, starting in 1 year with a tenor of 5 years and ending in 6 years from the contract spot date.

2 Swap Present Value

The net present value PV or price of an interest rate swap can be evaluated as outlined in our swap pricing paper [7] as follows

$$\begin{aligned} PV^{Swap} &= \phi\left(PV^{\text{Fixed Leg}} - PV^{\text{Float Leg}}\right) \\ &= \phi\left[\sum_{i=1}^{n} Nr^{Fixed}\tau_i P(t,t_i) - \sum_{j=1}^{m} N(l_{j-1}+s)\tau_j P(t,t_j)\right] \end{aligned} \tag{5}$$

Remark: Swap Float Leg Representation as a Fixed Leg

Interest rate swaps are generally quoted and traded in the financial markets as par rates. The par rate is the fixed rate required to make the present value of the fixed leg PV match that of the float leg PV. Such a swap is called a par swap and has a net PV of zero.

$$PV^{ParSwap} = \phi\left[\sum_{i=1}^{n} Nr^{Fixed}\tau_i P(t,t_i) - \sum_{j=1}^{m} N(l_{j-1}+s)\tau_j P(t,t_j)\right] = 0 \tag{6}$$

Since par swaps have zero PV we have

$$\sum_{i=1}^{n} Nr^{Fixed}\tau_i P(t,t_i) = \sum_{j=1}^{m} N(l_{j-1}+s)\tau_j P(t,t_j) \tag{7}$$

Furthermore par swaps have a fixed rate equal to the par rate i.e. $r^{Fixed} = p^{Market}$

$$\underbrace{\sum_{i=1}^{n} Np^{Market}\tau_i P(t,t_i)}_{\textit{Fixed Leg}} = \underbrace{\sum_{j=1}^{m} N(l_{j-1}+s)\tau_j P(t,t_j)}_{\textit{Float Leg}} \tag{8}$$

As outlined above and in our swap pricing paper [7] we can represent the float leg as a fixed leg traded at the market par rate p^{Market} becomes

$$\begin{aligned} PV^{Swap} &= \phi\left[\sum_{i=1}^{n} Nr^{Fixed}\tau_i P(t,t_i) - \sum_{j=1}^{m} N(l_{j-1}+s)\tau_j P(t,t_j)\right] \\ &= \phi\left[\sum_{i=1}^{n} N(r^{Fixed} - p^{Market})\tau_i P(t,t_i) - \sum_{j=1}^{m} Ns\tau_j P(t,t_j)\right] \\ &= \phi\left[\left(r^{Fixed} - p^{Market}\right) A_N^{Fixed} - sA_N^{Float}\right] \end{aligned} \tag{9}$$

In the case when there is no Libor spread s on the floating leg this simplifies to

$$\begin{aligned}
PV^{Swap} &= \phi\left(PV^{\text{Fixed Leg}} - PV^{\text{Float Leg}}\right)\\
&= \phi\left[\sum_{i=1}^{n} Nr^{Fixed}\tau_i P(t,t_i) - \sum_{j=1}^{m} Nl_{j-1}\tau_j P(t,t_j)\right]\\
&= \phi\left[A_N^{Fixed}\left(r^{Fixed} - p^{Market}\right)\right]
\end{aligned} \tag{10}$$

3 Swaption Price

In a receiver swaption the holder has the right to receive the fixed leg cashflows in the underlying swap at a strike rate agreed today and pay the float leg cashflows. A rational option holder will only exercise the option if the fixed leg cashflows to be received are larger than the float leg cashflows to be paid. The corresponding option payoff X_T can be represented as

$$\begin{aligned}
X_T &= max\left(\sum_{i=1}^{n} NK\tau_i P(t,t_i) - \sum_{j=1}^{m} Nl_{j-1}\tau_j P(t,t_j), 0\right)\\
&= max\left(A_N^{Fixed}K - A_N^{Fixed}p^{Market}, 0\right)\\
&= A_N^{Fixed} max\left(K - p^{Market}, 0\right)\\
&= A_N^{Fixed}\left(K - p^{Market}\right)^+
\end{aligned} \tag{11}$$

As can be seen by comparing (11) and (4) a receiver swaption payoff replicates the payoff of a put option scaled by the swap fixed leg annuity A_N^{Fixed}.

Likewise a payer swaption extends the holder the right to receive the fixed cashflows from the underlying swap and has payoff X_T.

$$\begin{aligned}
X_T &= max\left(\sum_{j=1}^{m} Nl_{j-1}\tau_j P(t,t_j) - \sum_{i=1}^{n} NK\tau_i P(t,t_i), 0\right)\\
&= max\left(A_N^{Fixed}p^{Market} - A_N^{Fixed}K, 0\right)\\
&= A_N^{Fixed} max\left(p^{Market} - K, 0\right)\\
&= A_N^{Fixed}\left(p^{Market} - K\right)^+
\end{aligned} \tag{12}$$

Again by comparing (12) and (46) a payer swaption payoff replicates the payoff of a call option scaled by the swap fixed leg annuity A_N^{Fixed}.

Remark: Payer / Receiver Swaptions

It can be easily seen from the swaption payoff that a *payer* swaption represents a call option payoff and a *receiver* swaption a put option payoff.

Both options give the right but not the obligation to enter into a swap contract in the future to pay or receive fixed cashflows respectively in exchange for floating cashflows with the fixed rate set today at the strike rate K.

In the general case we can represent a swaption payoff as

$$X_T = A_N^{Fixed}\left[\phi\left(p^{Market} - K\right)\right]^+ \tag{13}$$

where ϕ = 1 for a payer swaption and -1 for a receiver swaption.

Applying the martingale representation theorem from section (1) we can price the swaption using equation (2) using the swaption payoff from (13) giving

$$\begin{aligned} V_t &= N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{X_T}{N_T}\middle|\mathcal{F}_t\right] \\ &= N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{A_N^{Fixed}(T)\left[\phi\left(p^{Market} - K\right)\right]^+}{N_T}\middle|\mathcal{F}_t\right] \end{aligned} \tag{14}$$

As outlined in our Martingale paper [6] we may select a convenient numeraire to simplify the expectation term in (14). In this case we select the Annuity measure A_N^{Fixed} with corresponding probability measure $\mathbb{Q}_A$ which leads to

$$\begin{aligned} V_t &= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}_A}\left[\frac{A_N^{Fixed}(T)\left[\phi\left(p^{Market} - K\right)\right]^+}{A_N^{Fixed}(T)}\middle|\mathcal{F}_t\right] \\ &= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}_A}\left[\left[\phi\left(p^{Market} - K\right)\right]^+\middle|\mathcal{F}_t\right] \end{aligned} \tag{15}$$

We could at this stage see that expectation term in (15) can be evaluated using the generalized Black-Scholes formula as shown in (21) below. However for completeness we change the measure from the annuity measure $\mathbb{Q}_A$ to the more familiar and native Black-Scholes measure, namely the risk-neutral or savings account measure $\mathbb{Q}$. This is merely to help readers identify the Black-Scholes expectation and is not an actual requirement.

Remark: Radon-Nikodym Derivative

The Radon-Nikodym derivative allows us to change the numeraire and associated probability measure of an expectation and is often used in conjunction with the Martingale Representation Theorem.

$$V_t = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}X_T\middle|\mathcal{F}_t\right] \tag{16}$$

The Radon-Nikodym derivative $\left(\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\right)$ is defined as

$$\left(\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\right) = \frac{\left(\frac{M_t}{M_T}\right)}{\left(\frac{N_t}{N_T}\right)} = \left(\frac{N_T}{N_t}\right)\left(\frac{M_t}{M_T}\right) \tag{17}$$

To change numeraire from $\mathbb{Q}_N$ to $\mathbb{Q}_M$ we can multiply V_t by Radon-Nikodym derivative $\left(\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\right)$ giving

$$\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}X_T\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\left(\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\right)X_T\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\left(\frac{N_T}{N_t}\right)\left(\frac{M_t}{M_T}\right)X_T\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}_M}\left[\frac{M_t}{M_T}X_T\middle|\mathcal{F}_t\right]
\end{aligned} \tag{18}$$

In general we can denote the Radon-Nikodym derivative as $\left(\frac{d\mathbb{Q}_{New}}{d\mathbb{Q}_{Old}}\right)$ and apply as follows

$$E^{\mathbb{Q}_{Old}}\left[X_T\right] = E^{\mathbb{Q}_{New}}\left[\frac{d\mathbb{Q}_{New}}{d\mathbb{Q}_{Old}}X_T\right] \tag{19}$$

Applying the Radon-Nikodym derivative as outlined in our Martingale paper [6] to change the measure from the annuity measure $\mathbb{Q}_A$ to the risk-neutral savings account measure $\mathbb{Q}$ in (15) leads to a generalized Black-Scholes formula type expression as shown below.

$$
\begin{aligned}
V_t &= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Big[\frac{d\mathbb{Q}}{d\mathbb{Q}_A}\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Big] \\
&= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\frac{\left(\frac{e^{rt}}{e^{rT}}\right)}{\left(\frac{A(t)_N^{Fixed}}{A(T)_N^{Fixed}}\right)}\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Bigg] \\
&= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\left(\frac{e^{rt}}{e^{rT}}\right)\left(\frac{A_N^{Fixed}(T)}{A_N^{Fixed}(t)}\right)\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Bigg] \\
&= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[e^{-r(T-t)}\left(\frac{A_N^{Fixed}(T)}{A_N^{Fixed}(t)}\right)\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Bigg] \\
&= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\left(\frac{A_N^{Fixed}(T).e^{-r(T-t)}}{A_N^{Fixed}(t)}\right)\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Bigg]
\end{aligned} \tag{20}
$$

Noting that $e^{-r(T-t)}$ is the discount factor operator from time T to t under the savings account measure. If we discount the spot annuity $A_N^{Fixed}(T)$ back to time t by applying the discount factor operator we have that $A_N^{Fixed}(T).e^{-r(T-t)} = A_N^{Fixed}(t)$ giving

$$
\begin{aligned}
V_t &= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\left(\frac{A_N^{Fixed}(t)}{A_N^{Fixed}(t)}\right)\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Bigg] \\
&= A_N^{Fixed}(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\Big[\left[\phi\left(p^{Market}-K\right)\right]^+\Big|\mathcal{F}_t\Big]}_{\text{Black-Scholes Formula}}
\end{aligned} \tag{21}
$$

In the case where our underlying swap has a Libor spread on the floating leg using (9) gives

$$
\begin{aligned}
V_t &= A_N^{Fixed}(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\left(\phi\left(p^{Market}+s\left(\frac{A_N^{Float}(T)}{A_N^{Fixed}(t)}\right)-K\right)\right)^+\Big|\mathcal{F}_t\Bigg] \\
&= A_N^{Fixed}(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\Big[\left(\phi\left(p^{Market}-K'\right)\right)^+\Big|\mathcal{F}_t\Big]}_{\text{Black-Scholes Formula}}
\end{aligned} \tag{22}
$$

where

$$K' = K - s\left(\frac{A_N^{Float}(T)}{A_N^{Fixed}(t)}\right)$$

Remark: Generalized Black-Scholes and Black-76 Formulae

The generalized Black-Scholes formula evaluates the price $V(t)$ at time t of a European option with expiry at time T as

$$V(t)^{BS} = \phi e^{-r(T-t)}\Big[S_t e^{b(T-t)}N(\phi d_1) - KN(\phi d_2)\Big] \tag{23}$$

where

$$d_1 = \frac{ln(S_t/K) + (b + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

Furthermore as outlined in our Black-Scholes paper [8] setting the carry term $b = 0$ leads to the Black-76 formula for pricing interest rate options namely

$$V(t)^{B76} = \phi e^{-r(T-t)}\Big[S_t N(\phi d_1) - KN(\phi d_2)\Big] \tag{24}$$

where

$$d_1 = \frac{ln(S_t/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

As outlined in the appendix we should now recognise that the swaption pricing formula from (21) is nothing more than the generalized Black-Scholes (1973) formula scaled by the annuity factor $A_N^{Fixed}(t)$. In this particular case the underlying asset is an interest rate, therefore we customize the generalized Black-Scholes formula as outlined in our Black-Scholes paper [8] to price interest rate options by setting the carry term b to zero, which leads to the Black-76 formula.

Note that when using the Black-76 formula from (24) we have an unnecessary discounting term $e^{-r(T-t)}$, which we eliminate by setting the zero rate $r = 0$ to make this discount factor multiplier equal to unity.

Therefore applying the generalized Black-Scholes result to (21) with the carry term $b = 0$ and zero rate $r = 0$ leads to following result.

Result: Swaption Pricing Formula

European swaptions can be priced using the Black-76 analytical formula scaled by the interest rate swap fixed leg annuity term $A_N^{Fixed}(t)$

$$V_t = \phi A_N^{Fixed}(t)\text{Black-76}(p^{Market}, \textit{K, (T - t), }\sigma\textit{(K,T), r=0)} \tag{25}$$

quoting this explicitly we have

$$V_t = \phi A_N^{Fixed}(t)\left(p^{Market} N(\phi d_1) - K N(\phi d_2)\right) \tag{26}$$

where

$$d_1 = \frac{ln(p^{Market}/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}$$

and $\phi = 1$ denotes a payer swaption and $\phi = -1$ a receiver swaption.

Underlying Swaps with a Libor Floating Spread s
In the case where our underlying swap has a Libor floating spread we adjust the strike as outlined in (22) replacing K with K' where $K' = K - s\left(\frac{A_N^{Float}(T)}{A_N^{Fixed}(t)}\right)$.

4 Conclusion

In conclusion we reviewed the martingale representation theorem for pricing options, which allows us to price options under a numeraire of our choice. We also considered European call and put option pricing payoffs, discussed how to evaluate and price an interest swap and examined how to price interest rate swaptions using the Martingale Representation Theorem. We chose the annuity measure to simplify the swaption pricing calculation and finally applied the Radon-Nikodym derivative to change probability measure from the annuity measure to the savings account measure to arrive at a swaption pricing formula expressed in terms of the Black-76 formula. We showed that the swaption pricing formula is nothing more than the Black-76 formula scaled by the underlying swap annuity factor. In the appendix we also provide a full derivation of the generalized Black-Scholes formula for completeness.

Appendix: Derivation of the Generalized Black-Scholes Model

We first assume that the underlying asset S follows a Geometric Brownian Motion process with constant volatility σ namely

$$dS_t = rS_t dt + \sigma S_t dB_t \tag{27}$$

and more generally for assets paying continuous dividends q

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t \tag{28}$$

For a log-normal process we define $Y_t = ln(S_t)$ or $S_t = e^{Y_t}$ and apply Itô's Lemma to Y_t giving

$$dY_t = \frac{dY_t}{dS_t} dS_t + \frac{1}{2}\frac{d^2 Y_t}{dS_t^2} dS_t^2 \tag{29}$$

now $\frac{dY}{dS_t} = \left(\frac{1}{S_t}\right)$, $\frac{d^2Y_t}{dS_t^2} = \left(-\frac{1}{S_t^2}\right)$ and $dS_t^2 = \sigma^2 S_t^2 dt$ therefore we have

$$dY_t = \left(\frac{1}{S_t}\right)\left[\left(r - q\right) Sdt + \sigma S_t dB_t\right] + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left[\sigma_t^2 S_t^2 dt\right] \tag{30}$$

giving

$$dY_t = \left(r - q - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{31}$$

which leads to

$$dlnS_t = \left(r - q - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{32}$$

expressing this in integral form

$$\int_t^T lnS(u)du = \int_t^T \left(r - q - \frac{1}{2}\sigma^2\right) du + \int_t^T \sigma B(u)du \tag{33}$$

which implies[2]

$$lnS(T) - lnS(t) = \left(r - q - \frac{1}{2}\sigma^2\right)(T - t) + \sigma B(T) \tag{34}$$

$$ln\left(\frac{S(T)}{S(t)}\right) = \left(r - q - \frac{1}{2}\sigma^2\right)(T - t) + \sigma B(T) \tag{35}$$

knowing the dynamics of our Normal Brownian process namely $B(T) \sim N(0, T - t)$ and apply the Central Limit Theorem with mean μ and variance σ^2

$$z = \left(\frac{x - \mu}{\sigma}\right) = \left(\frac{B(T)}{\sqrt{(T - t)}}\right) \tag{36}$$

which we express as

$$B(T) = \sqrt{(T - t)}z \tag{37}$$

[2]Note when evaluating the stochastic integrand B(t) = 0

where z represents a standard normal variate. Applying (37) to our Brownian expression (35) and rearranging gives

$$\boxed{S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}} \tag{38}$$

Knowing (38) we could choose to use Monte Carlo simulation with random number standard normal variates z or proceed in search of an analytical solution.

For vanilla European option pricing we can evaluate the price as the discounted expected value of the option payoff namely as follows for call options

$$C(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\,(S(T)-K,0)\Big] \tag{39}$$

and likewise for put options

$$P(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\,(K-S(T),0)\Big] \tag{40}$$

for a call option we have

$$\text{Max}\,(S(T)-K,0) = \begin{cases} S(T)-K, & \text{if } S(T) \geq K \\ 0, & \text{otherwise} \end{cases} \tag{41}$$

from (38) we have

$$z = \left(\frac{ln\left(\frac{S(T)}{S(t)}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{42}$$

we can evaluate the call payoff from (41) using and evaluating (42) for $S_T \geq K$ giving

$$S(T) \geq K \iff z \geq \left(\frac{ln\left(\frac{K}{S(t)}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{43}$$

next we define the RHS of (43) as follows

$$-d_2 = \left(\frac{ln\left(\frac{K}{S(t)}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{44}$$

multiplying both sides by minus one gives

$$d_2 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r-q-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{45}$$

Substituting our definition of $S(T)$ from (38) and d_2 from (45) into our call option payoff (41) we arrive at

$$\text{Max}\,(S(T)-K,0) = \begin{cases} S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}, & \text{if } Z \geq -d_2 \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

from the definition of standard normal probability density function PDF for Z

$$P(Z=z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \tag{47}$$

we proceed to evaluate the risk neutral price of the discounted call option payoff from (39). Note we eliminate the max operator using (46) by evaluating the integrand from the lower bound $-d_2$ which guarantees a positive payoff.

$$\begin{aligned}
C(t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\left(S(T)-K,0\right)\Big] \\
&= e^{-r(T-t)}\int_{-d_2}^{\infty}\underbrace{\left(S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}-K\right)}_{\text{Payoff}}\underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}}_{\text{PDF}}dz \\
&= \frac{S(t)e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\left(e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}-K\right)e^{-\frac{1}{2}z^2}dz \\
&= \frac{S(t)e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\left(e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}\right)e^{-\frac{1}{2}z^2}dz-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}z^2}dz
\end{aligned} \tag{48}$$

factorizing the exponential r and q terms give

$$\begin{aligned}
C(t) &= \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\left(e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{(T-t)}z}\right)e^{-\frac{1}{2}z^2}dz-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}z^2}dz \\
&= \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\underbrace{e^{\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{(T-t)}z-\frac{1}{2}z^2\right)}}_{\text{Term 1}}dz-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}z^2}dz
\end{aligned} \tag{49}$$

we now complete the square of term 1 in (49) to get

$$C(t)=\frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\underbrace{e^{\left(-\frac{1}{2}\left(z-\sigma\sqrt{(T-t)}\right)^2\right)}}_{\text{Term 2}}dz-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}z^2}dz \tag{50}$$

now we make a substitution namely $y\triangleq z-\sigma\sqrt{(T-t)}$ such that term 2 in (50) becomes a standard normal function in y. When making this substitution our integration limits change; from a lower bound of $z=-d_2$ to $y=-d_2-\sigma\sqrt{(T-t)}\triangleq -d_1$ and from an upper bound of $z=\infty$ to $y=\infty$ leading to

$$C(t)=\frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}}\int_{-d_1}^{\infty}e^{-\frac{1}{2}y^2}dy-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}e^{-\frac{1}{2}z^2}dz \tag{51}$$

from the definition of the standard normal cumulative density function we know that

$$P(Z=z)=\frac{1}{\sqrt{2\pi}}\int_{z}^{\infty}e^{-\frac{1}{2}z^2}dz=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-z}e^{-\frac{1}{2}z^2}dz \tag{52}$$

since the standard normal distribution is symmetrical we can invert the bounds to give

$$C(t)=\frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d_1}e^{-\frac{1}{2}y^2}dy-\frac{Ke^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{d_2}e^{-\frac{1}{2}z^2}dz \tag{53}$$

applying the standard normal CDF expression (52) into (53)

$$C(t) = S(t)e^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{54}$$

finally applying put-call super-symmetry and with minor rearrangement we arrive at the generalized Black-Scholes result namely

$$V(t) = \phi e^{-r(T-t)}\left[S(t)e^{b(T-t)}N(\phi d_1) - KN(\phi d_2)\right] \tag{55}$$

where ϕ is our call-put indicator function and $d_1 = d_2 + \sigma\sqrt{(T-t)}$ giving

$$d_1 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{56}$$

and

$$d_2 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{57}$$

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