A Review of the Generalized Black-Scholes Formula & It's Application to Different Underlying Assets

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Abstract

The Black-Scholes (1973) formula is well used for pricing vanilla European options. There are several different variations used by market practitioners dependent on the underlying asset being modelled. In this brief paper we present the generalized Black-Scholes representation, outline it's derivation and review how to configure the model appropriately for different asset classes. In particular varying the cost of carry term incorporated in the model allows us to express the generalized Black-Scholes as the classical Black-Scholes (1973) formula or the canonical Black (1976) representation.

Keywords: Generalized Black-Scholes, Derivation, Black Model, Cost of Carry, European Option Pricing, Put-Call Parity, Put-Call Super-Symmetry

Notation

The notation in table (1) will be used for pricing formulae.

b	The cost of carry, $b = r - q$
C	Value of a European call option
K	The strike of the European option
$N(z)$	The value of the Cumulative Standard Normal Distribution
P	Value of a European put option
ϕ	A call or put indicator function, 1 represents a call and -1 a put option
q	The continuous dividend yield or convenience yield
r	The risk-free interest rate (zero rate)
S	The underlying spot value
σ	The volatility of the underlying asset
T	The time to expiry of the option in years
V	Value of a European call or put option

Table 1: Notation

Introduction

The Black-Scholes (1973) model from [3] can be ”generalized” by incorporating a cost of carry rate b. For an appropriate carry value the model can be used to price vanilla European options on stocks, stock indices paying a continuous dividend yield q, options on futures, commodity options incorporating the storage costs via a convenience yield q and currency options. The model's popularity amongst traders and market practitioners heavily relies on dynamic delta hedging, see [6] for details.

1 The Generalized Black-Scholes Formula

The generalized Black-Scholes (1973) formula is derived in the appendix and presented here as

$$V = \phi e^{-rT}\left[Se^{bT}N(\phi d_1) - KN(\phi d_2)\right] \tag{1}$$

where

$$d_1 = \frac{ln(S/K) + (b + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

for a call option setting $\phi = 1$ this becomes

$$C = e^{-rT}\left[Se^{bT}N(d_1) - KN(d_2)\right] \tag{2}$$

and likewise setting $\phi = -1$ for a put option we have

$$P = e^{-rT}\left[KN(-d_2) - Se^{bT}N(-d_1)\right] \tag{3}$$

Put-Call Super-Symmetry

The generalized formula (1) relies on Put-Call Super-Symmetry see [1] and [10], which also discuss negative volatility. Put-Call Super-Symmetry is a no-arbitrage condition whereby the price of a Call option C equals the negative price of an equivalent Put option P with negative volatility ceteris paribus namely

$$C(S, K, \sigma, T, r, b) = -P(S, K, -\sigma, T, r, b) \tag{4}$$

and equivalently

$$P(S, K, \sigma, T, r, b) = -C(S, K, -\sigma, T, r, b) \tag{5}$$

Cost of Carry

The cost of carry term b conveniently provides a mechanism to switch between the classical Black-Scholes (1973) and the Black (1976) formulas. Setting $b = r$ gives the Black-Scholes formula and likewise $b = 0$ the Black-76 formula as shown below.

2 The Black-Scholes (1973) Formula

The Black-Scholes (1973) formula from [3] is used for European stock options paying no dividends and can be deduced by setting $b = r$ in the generalized Black-Scholes formula (1) to give

$$V = \phi \left[SN(\phi d_1) - Ke^{-rT} N(\phi d_2) \right] \tag{6}$$

where

$$d_1 = \frac{ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

3 The Black (1976) Formula

The Black (1976) formula from [4] is used for European options on forwards and futures. It is useful for interest rate cap and floor pricing for example. Setting $b = 0$ in the generalized Black-Scholes formula (1) leads to the Black (1976) pricing formula as shown below

$$V = \phi \left[Se^{-rT} N(\phi d_1) - Ke^{-rT} N(\phi d_2) \right] \tag{7}$$

where

$$d_1 = \frac{ln(S/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

4 Put-Call Parity

Put-Call Parity is well known no-arbitrage relationship between call and put options see [8] and [14]. A trading arbitrage opportunity exists if the parity relationship does not hold, subject to several dynamic hedging and trade replication assumptions namely

- No bid-ask trading spreads
- The market is frictionless i.e. no transaction costs
- The underlying asset can be traded long or short
- There are no liquidity restrictions on trade size

Mathematically we can express put-call parity as follows

$$C - P = Se^{(b-r)T} - Ke^{-rT} \tag{8}$$

or equivalently in terms of dividend yield q

$$C - P = Se^{-qT} - Ke^{-rT} \tag{9}$$

5 Applications of the Generalized Formula

The generalized Black-Scholes model is applied to different underlying assets by configuring the cost of carry term b as follows.

5.1 Black-Scholes Model (1973) - Stock Options

To price European stock options with no dividends we set $b = r$ whereby the generalized model represents the Black-Scholes (1973) model namely

$$V = \phi\left[SN(\phi d_1) - Ke^{-rT}N(\phi d_2)\right] \tag{10}$$

where

$$d_1 = \frac{ln(S/K) + (r + \sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

5.2 Merton Model (1973) - Commodity Options, Stock Indices & Stocks with Dividends

To price European options with continuous dividends we set $b = r - q$ which leads to an expression for the Merton (1973) model see [12]. This is particularly useful for Commodity option pricing, whereby the q parameter incorporates storage and convenience yield costs. Likewise options on stock indices and stocks modelled having continuous dividends can be priced.

$$V = \phi \left[S e^{-qT} N(\phi d_1) - K e^{-rT} N(\phi d_2) \right] \tag{11}$$

where

$$d_1 = \frac{ln(S/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

5.3 Black Model (1976) - Options on Futures

To price European options on futures we set $b = 0$ and define the futures' price $F = Se^{bT}$. The generalized Black-Scholes formula becomes

$$V = \phi \left[F e^{-rT} N(\phi d_1) - K e^{-rT} N(\phi d_2) \right] \tag{12}$$

where

$$d_1 = \frac{ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

5.4 Asay (1982) - Options on Margined Futures

To price European options on futures, where the premium is paid into a margin account, we set both $b = 0$ and $r = 0$ and define the futures' price $F = Se^{bT}$, see [2]. The generalized Black-Scholes formula becomes

$$V = \phi \left[F N(\phi d_1) - K N(\phi d_2) \right] \tag{13}$$

where

$$d_1 = \frac{ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

5.5 Garman & Kohlhagen (1983) - Currency Options

To price Currency options see [7] we set $b = r_d - r_f$, $r = r_f$ and $q = r_d$ where r_d denotes the domestic (asset) interest rate and r_f the foreign (money) interest rate

$$V = \phi \left[Se^{-r_f T} N(\phi d_1) - Ke^{-r_d T} N(\phi d_2) \right] \tag{14}$$

where

$$d_1 = \frac{ln(S/K) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

6 Conclusion

In conclusion we have presented the generalized Black-Scholes formula and reviewed how by appropriately setting the cost of carry and interest terms we can apply the generalized result to price European options from different asset classes such as stock, commodity, futures and currency options. In the appendix 1 we present chart and tableau results. Finally in appendix 2 we present a derivation of the generalized Black-Scholes model for completeness.

Appendix 1 : Black-Scholes Price Results

Below we indicate Black-Scholes price results for call options. Note we calculate put option prices from put-call parity as described above in section (4)

Parameters	
CallOrPut	**CALL**
Spot	**100.00**
Strike	**100.00**
Vol	**10.00%**
Time	**1.00**
Rate	**1.00%**
Carry	**1.00%**

Price Breakdown	
d1	0.1500
d2	0.0500
Nd1	0.5596
Nd2	0.5199
Nminusd1	0.4404
Nminusd2	0.4801
exp((b-r)t)	1.0000
exp(-rt)	0.9900
Call Price	4.4852
Put Price	3.4902

ShiftSize	
SpotShift	**5.0%**
TimeShift	**5.0%**

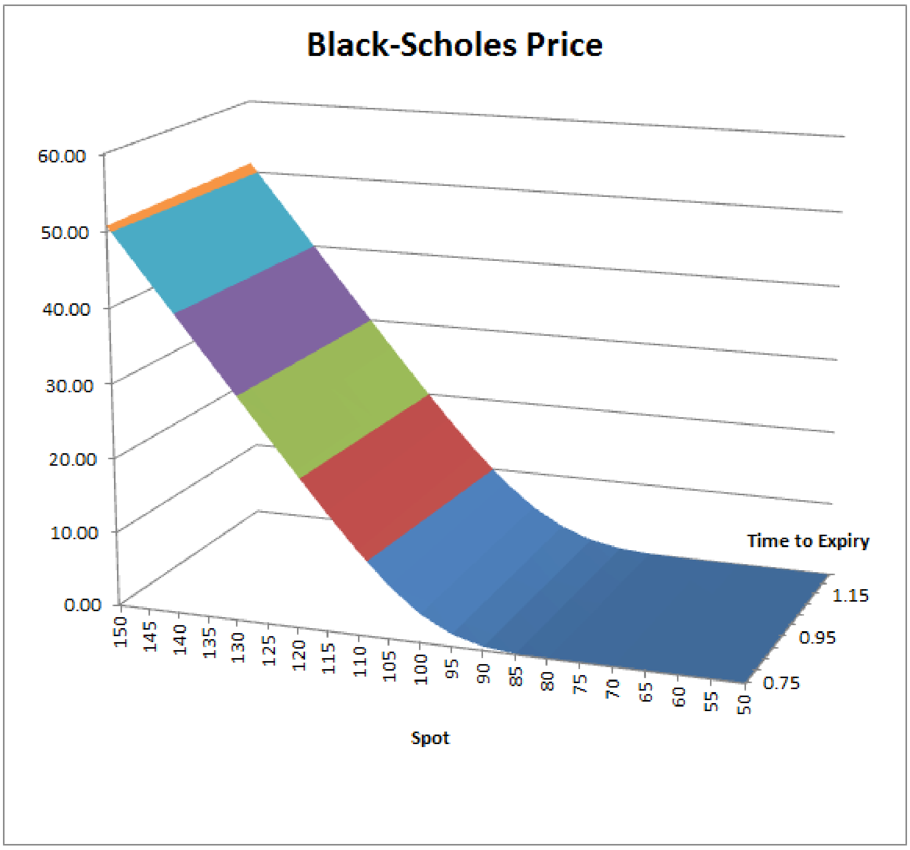

Figure 1: Black-Scholes Price Chart

Black-Scholes Price		75%	80%	85%	90%	95%	**100%**	105%	110%	115%	120%	125%
	Spot \ Expiry	0.75	0.80	0.85	0.90	0.95	**1.00**	1.05	1.10	1.15	1.20	1.25
150%	150.00	50.747196	50.796813	50.846406	50.895977	50.945528	**50.995060**	51.044575	51.094076	51.143565	51.193045	51.242520
145%	145.00	45.747208	45.796834	45.846443	45.896039	45.945626	**45.995207**	46.044789	46.094376	46.143974	46.193590	46.243229
140%	140.00	40.747277	40.796953	40.846634	40.896330	40.946052	**40.995808**	41.045609	41.095465	41.145387	41.195385	41.245467
135%	135.00	35.747670	35.797561	35.847530	35.897596	35.947777	**35.998091**	36.048554	36.099181	36.149984	36.200975	36.252164
130%	130.00	30.749672	30.800386	30.851365	30.902636	30.954219	**31.006134**	31.058394	31.111009	31.163986	31.217328	31.271037
125%	125.00	25.758711	25.812124	25.866166	25.920847	25.976165	**26.032114**	26.088682	26.145853	26.203609	26.261928	26.320788
120%	120.00	20.794427	20.855192	20.917029	20.979875	21.043661	**21.108320**	21.173787	21.239999	21.306893	21.374413	21.442505
115%	115.00	15.916101	15.992859	16.070689	16.149436	16.228962	**16.309146**	16.389881	16.471072	16.552637	16.634502	16.716603
110%	110.00	11.268061	11.370815	11.473428	11.575789	11.677809	**11.779422**	11.880576	11.981229	12.081353	12.180925	12.279929
105%	105.00	7.120155	7.250138	7.377749	7.503127	7.626400	**7.747684**	7.867087	7.984709	8.100639	8.214962	8.327756
100%	**100.00**	**3.827443**	**3.965444**	**4.099941**	**4.231236**	**4.359592**	**4.485236**	**4.608370**	**4.729169**	**4.847790**	**4.964372**	**5.079041**
95%	95.00	1.647862	1.759742	1.869876	1.978344	2.085225	**2.190593**	2.294518	2.397066	2.498300	2.598278	2.697055
90%	90.00	0.528814	0.592825	0.657906	0.723861	0.790527	**0.857766**	0.925463	0.993519	1.061852	1.130392	1.199078
85%	85.00	0.116360	0.140293	0.166107	0.193670	0.222858	**0.253552**	0.285640	0.319017	0.353587	0.389260	0.425951
80%	80.00	0.015938	0.021310	0.027655	0.034995	0.043343	**0.052704**	0.063074	0.074440	0.086789	0.100100	0.114351
75%	75.00	0.001213	0.001868	0.002747	0.003886	0.005320	**0.007080**	0.009196	0.011693	0.014597	0.017926	0.021699
70%	70.00	0.000045	0.000083	0.000144	0.000237	0.000370	**0.000555**	0.000803	0.001127	0.001539	0.002053	0.002682
65%	65.00	0.000001	0.000002	0.000003	0.000007	0.000013	**0.000022**	0.000037	0.000059	0.000091	0.000135	0.000195
60%	60.00	0.000000	0.000000	0.000000	0.000000	0.000000	**0.000000**	0.000001	0.000001	0.000003	0.000004	0.000007
55%	55.00	0.000000	0.000000	0.000000	0.000000	0.000000	**0.000000**	0.000000	0.000000	0.000000	0.000000	0.000000
50%	50.00	0.000000	0.000000	0.000000	0.000000	0.000000	**0.000000**	0.000000	0.000000	0.000000	0.000000	0.000000

Figure 2: Black-Scholes Price Tableau

Appendix 2: Derivation of the Generalized Black-Scholes Model

We first assume that the underlying asset S follows a Geometric Brownian Motion process with constant volatility σ namely

$$dS_t = rS_t dt + \sigma S_t dB_t \tag{15}$$

and more generally for assets paying continuous dividends q

$$dS_t = (r-q)S_t dt + \sigma S_t dB_t \tag{16}$$

For a log-normal process we define $Y_t = ln(S_t)$ or $S_t = e^{Y_t}$ and apply Itô's Lemma to Y_t giving

$$dY_t = \frac{dY_t}{dS_t} dS_t + \frac{1}{2} \frac{d^2 Y_t}{dS_t^2} dS_t^2 \tag{17}$$

now $\frac{dY}{dS_t} = \left(\frac{1}{S_t}\right)$, $\frac{d^2 Y_t}{dS_t^2} = \left(-\frac{1}{S_t^2}\right)$ and $dS_t^2 = \sigma^2 S_t^2 dt$ therefore we have

$$dY_t = \left(\frac{1}{S_t}\right) \left[\left(r-q\right) Sdt + \sigma S_t dB_t\right] + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left[\sigma_t^2 S_t^2 dt\right] \tag{18}$$

giving

$$dY_t = \left(r - q - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{19}$$

which leads to

$$dlnS_t = \left(r - q - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{20}$$

expressing this in integral form

$$\int_t^T lnS(u)du = \int_t^T \left(r - q - \frac{1}{2}\sigma^2\right) du + \int_t^T \sigma B(u)du \tag{21}$$

which implies[1]

$$lnS(T) - lnS(t) = \left(r - q - \frac{1}{2}\sigma^2\right)(T - t) + \sigma B(T) \tag{22}$$

$$ln\left(\frac{S(T)}{S(t)}\right) = \left(r - q - \frac{1}{2}\sigma^2\right)(T - t) + \sigma B(T) \tag{23}$$

knowing the dynamics of our Normal Brownian process namely $B(T) \sim N(0, T-t)$ and apply the Central Limit Theorem with mean μ and variance σ^2

$$z = \left(\frac{x - \mu}{\sigma}\right) = \left(\frac{B(T)}{\sqrt{(T-t)}}\right) \tag{24}$$

which we express as

$$B(T) = \sqrt{(T-t)}z \tag{25}$$

where z represents a standard normal variate. Applying (25) to our Brownian expression (23) and rearranging gives

$$\boxed{S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}} \tag{26}$$

Knowing (26) we could choose to use Monte Carlo simulation with random number standard normal variates z or proceed in search of an analytical solution.

For vanilla European option pricng we can evaluate the price as the discounted expected value of the option payoff namely as follows for call options

$$C(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\,(S(T) - K, 0)\Big] \tag{27}$$

and likewise for put options

$$P(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\,(K - S(T), 0)\Big] \tag{28}$$

for a call option we have

$$\text{Max}\,(S(T) - K, 0) = \begin{cases} S(T) - K, & \text{if } S(T) \geq K \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

[1]Note when evaluating the stochastic integrand B(t) = 0

from (26) we have

$$z = \left(\frac{ln\left(\frac{S(T)}{S(t)}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right) \tag{30}$$

we can evaluate the call payoff from (29) using and evaluating (30) for $S_T \geq K$ giving

$$S(T) \geq K \iff z \geq \left(\frac{ln\left(\frac{K}{S(t)}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right) \tag{31}$$

next we define the RHS of (31) as follows

$$-d_2 = \left(\frac{ln\left(\frac{K}{S(t)}\right) - \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right) \tag{32}$$

multiplying both sides by minus one gives

$$d_2 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \right) \tag{33}$$

Substituting our definition of $S(T)$ from (26) and d_2 from (33) into our call option payoff (29) we arrive at

$$\text{Max}\,(S(T) - K, 0) = \begin{cases} S(T) = S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}, & \text{if } Z \geq -d_2 \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

from the definition of standard normal probability density function PDF for Z

$$P(Z = z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{35}$$

we proceed to evaluate the risk neutral price of the discounted call option payoff from (27). Note we eliminate the max operator using (34) by evaluating the integrand from the lower bound $-d_2$ which guarantees a positive payoff.

$$\begin{aligned} C(t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\text{Max}\,(S(T) - K, 0)\Big] \\ &= e^{-r(T-t)} \int_{-d_2}^{\infty} \underbrace{\left(S(t)e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z} - K\right)}_{\text{Payoff}} \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}}_{\text{PDF}} dz \\ &= \frac{S(t)e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left(e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z} - K\right) e^{-\frac{1}{2}z^2} dz \\ &= \frac{S(t)e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left(e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}\right) e^{-\frac{1}{2}z^2} dz - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2} dz \end{aligned} \tag{36}$$

factorizing the exponential r and q terms give

$$\begin{aligned} C(t) &= \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left(e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{(T-t)}z}\right) e^{-\frac{1}{2}z^2} dz - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2} dz \\ &= \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \underbrace{e^{\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{(T-t)}z-\frac{1}{2}z^2\right)}}_{\text{Term 1}} dz - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2} dz \end{aligned} \tag{37}$$

we now complete the square of term 1 in (37) to get

$$C(t) = \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \underbrace{e^{\left(-\frac{1}{2}\left(z-\sigma\sqrt{(T-t)}\right)^2\right)}}_{\text{Term 2}} dz - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2} dz \tag{38}$$

now we make a substitution namely $y \triangleq z - \sigma\sqrt{(T-t)}$ such that term 2 in (38) becomes a standard normal function in y. When making this substitution our integration limits change; from a lower bound of $z = -d_2$ to $y = -d_2 - \sigma\sqrt{(T-t)} \triangleq -d_1$ and from an upper bound of $z = \infty$ to $y = \infty$ leading to

$$C(t) = \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-\frac{1}{2}y^2} dy - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2} dz \tag{39}$$

from the definition of the standard normal cumulative density function we know that

$$P(Z = z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-\frac{1}{2}z^2} dz \tag{40}$$

since the standard normal distribution is symmetrical we can invert the bounds to give

$$C(t) = \frac{S(t)e^{-q(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2} dy - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}z^2} dz \tag{41}$$

applying the standard normal CDF expression (40) into (41)

$$C(t) = S(t)e^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{42}$$

finally applying put-call super-symmetry and with minor rearrangement we arrive at the generalized Black-Scholes result namely

$$V(t) = \phi e^{-r(T-t)}\left[S(t)e^{b(T-t)}N(\phi d_1) - KN(\phi d_2)\right] \tag{43}$$

where ϕ is our call-put indicator function and $d_1 = d_2 + \sigma\sqrt{(T-t)}$ giving

$$d_1 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{44}$$

and

$$d_2 = \left(\frac{ln\left(\frac{S(t)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) \tag{45}$$

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