Interest Rate Modelling and Derivative Pricing

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WS, 2018/19

Part VI

Model Calibration

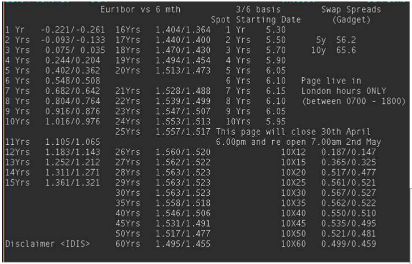
Outline

Yield Curve Calibration

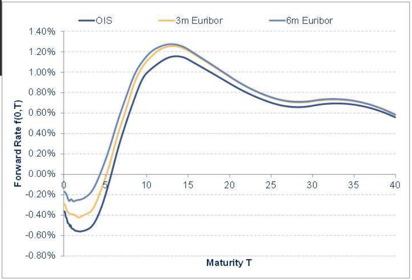
Calibration Methodologies for Hull White Model

Outline

What is the goal of yield curve calibration?



	Euribor vs 6 mth			3/6 basis Spot Starting Date		Swap Spreads (Gadget)	
1 Yr	-0.221/-0.261	16Yrs	1.404/1.364	1 Yr	5.30		
2 Yrs	-0.093/-0.133	17Yrs	1.440/1.400	2 Yrs	5.50	5y	56.2
3 Yrs	0.075/ 0.035	18Yrs	1.470/1.430	3 Yrs	5.70	10y	65.6
4 Yrs	0.244/0.204	19Yrs	1.494/1.454	4 Yrs	5.90		
5 Yrs	0.402/0.362	20Yrs	1.513/1.473	5 Yrs	6.05		
6 Yrs	0.548/0.508			6 Yrs	6.10	Page live in	
7 Yrs	0.682/0.642	21Yrs	1.528/1.488	7 Yrs	6.15	London hours ONLY	
8 Yrs	0.804/0.764	22Yrs	1.539/1.499	8 Yrs	6.10	(between 0700 - 1800)	
9 Yrs	0.916/0.876	23Yrs	1.547/1.507	9 Yrs	6.05		
10Yrs	1.016/0.976	24Yrs	1.553/1.513	10Yrs	5.95		
		25Yrs	1.557/1.517	This page will close 30th April			
11Yrs	1.105/1.065			6.00pm and re open 7.00am 2nd May			
12Yrs	1.183/1.143	26Yrs	1.560/1.520			10X12	0.187/0.147
13Yrs	1.252/1.212	27Yrs	1.562/1.522			10X15	0.365/0.325
14Yrs	1.311/1.271	28Yrs	1.563/1.523			10X20	0.517/0.477
15Yrs	1.361/1.321	29Yrs	1.563/1.523			10X25	0.561/0.521
		30Yrs	1.563/1.523			10X30	0.567/0.527
		35Yrs	1.558/1.518			10X35	0.562/0.522
		40Yrs	1.546/1.506			10X40	0.550/0.510
		45Yrs	1.531/1.491			10X45	0.535/0.495
		50Yrs	1.517/1.477			10X50	0.521/0.481
Disclaimer <IDIS>		60Yrs	1.495/1.455			10X60	0.499/0.459



We aim at finding a set of yield curves that allows re-pricing a set of market instruments

We start with a single-curve setting example to illustrate the general principle

Consider Vanilla swaps as market instruments with the pricing formula (single-curve setting, $t \leq T_0$)

$$\text{Swap}^k(t) = \underbrace{[P(t, T_0) - P(t, T_{n_k})]}_{\text{float leg}} - \underbrace{\sum_{i=1}^{n_k} R \cdot \tau_i \cdot P(t, T_i)}_{\text{fixed Leg}}$$

A market swap quote R_k for a T_{n_k}-maturing (and spot-starting) Vanilla swap is the fixed rate that prices the swap at par, i.e.

$$\underbrace{0}_{\text{Market}(R_k)} = \underbrace{\text{Swap}^k(0) = [P(0, T_0) - P(0, T_{n_k})] - \sum_{i=1}^{n_k} R_k \cdot \tau_i \cdot P(0, T_i)}_{\text{Model}[P](R_k)}$$

We associate a calibration helper operator $\mathcal{H}_k = \mathcal{H}_k[P]$ with each market instrument which takes as input a yield curve $P(0, T)$ and calculates (for a market quote)

$$\mathcal{H}_k[P](R_k) = \text{Model}[P](R_k) - \text{Market}(R_k)$$

Yield curve calibration is formulated as minimisation problem

(Single-Curve) Yield Curve Calibration Problem

For a given set of market quotes $\{R_k\}_{k=1,\ldots q}$ with corresponding instruments and calibration helpers $\mathcal{H}_k[P]$, the yield curve calibration problem is given by

$$\min_P \left\| \left[\mathcal{H}_1[P](R_1), \ldots, \mathcal{H}_q[P](R_q)\right]^\top \right\|.$$

- Effectively, we only need a finite set of $P(0, T_i)$
- Without further constraints there are multiple yield curves $P(0, T)$ that give optimal solution

$$\left[\mathcal{H}_1[P](R_1), \ldots, \mathcal{H}_q[P](R_q)\right]^\top = 0 \in \mathbb{R}^q$$

- We need to add sensible regularisation to
  - make calibration problem tractable, in particular finite dimensional domain
  - ensure unique, accurate and sensible solution
  - allow for efficient computation

Regularisation is achieved by discretisation and interpolation of the yield curve

Order market quotes R_k and calibration helpers $\mathcal{H}_k[P]$ by increasing final maturity T_{n_k} $(k = 1, \ldots, q)$ of underlying instruments.
Set

$$R = [R_1, \ldots, R_q] \quad \text{and} \quad \mathcal{H}[P] = [\mathcal{H}_1[P], \ldots, \mathcal{H}_q[P]]$$

Define a vector of yield curve parameters $z = [z_1, \ldots, z_q]^\top \in \mathbb{R}^q$ which specify the yield curve via

$$P = P[z]$$

- Typically, z_k are zero, forward rates or discount factors for maturities T_{n_k}
- Compare with interpolation traits in QuantLib

Specify $P[z](0, T)$ via interpolation/extrapolation based on curve parameters z.

- E.g. monoton cubic spline interpolation

We re-formulate the calibration problem in terms of model parameters

Finite Dimensional Yield Curve Calibration Problem

The yield curve calibration problem in terms of yield curve model parameters is given by

$$\min_z \left\| \mathcal{H}\left[P\left[z\right]\right]\left(R\right) \right\|$$

where

$$z = \left[z_1, \ldots, z_q\right]^\top, \quad R = \left[R_1, \ldots, R_q\right]^\top \quad \text{and} \quad \mathcal{H}\left[P\right] = \left[\mathcal{H}_1\left[P\right], \ldots, \mathcal{H}_q\left[P\right]\right].$$

- In general, parametrised calibration problem can be solved by general purpose optimisation methods
- This can be computationally expansive if number of inputs and parameters q is large
- We can also exploit the structure of the problem to reduce computational complexity

The multi-dimensional calibration problem can be reduced to a sequence of one-dimensional calibration problems I

Lemma

Consider our parametrised calibration problem setting. Assume a yield curve parametrisation $P[z]$ such that discount factors $P[z](0,T)$ are continuously differentiable w.r.t. z for all maturities T, and parametrised locally in the sense that

$$\frac{\partial}{\partial z_k} P[z](0,T) = 0 \quad \textit{for} \quad T \le T_{n_{k-1}}$$

then the Jacobi matrix $\frac{d}{dz}\mathcal{H}[P[z]](R)$ is of lower triangular form.

Proof:

Consider a component of the Jacobi matrix

$$\begin{aligned}
\frac{d}{dz_l}\mathcal{H}_k[P[z]](R) &= \frac{d}{dz_l}\text{Model}[P[z]](R_k) \\
&= \frac{d}{dz_l}\left[P(0,T_0) - P(0,T_{n_k}) - \sum_{i=1}^{n_k} R_k \cdot \tau_i \cdot P(0,T_i)\right] \\
&= \frac{d}{dz_l}P(0,T_0) - \frac{d}{dz_l}P(0,T_{n_k}) - \sum_{i=1}^{n_k} R_k \cdot \tau_i \cdot \frac{d}{dz_l}P(0,T_i)
\end{aligned}$$

The multi-dimensional calibration problem can be reduced to a sequence of one-dimensional calibration problems II

The largest maturity is T_{n_k}. Thus, due to local parametrisaion property, for $l > k$ $\frac{d}{dz_l} P(0, T_{n_k}) = 0$. Same holds for maturities $T_i \leq T_{n_k}$. Consequently,

$$\frac{d}{dz_l} \mathcal{H}_k \left[P \left[z \right] \right] \left(R_k \right) = 0 \quad \text{for} \quad l > k$$

and

$$\frac{d}{dz} \mathcal{H} \left[P \left[z \right] \right] \left(R \right) = \begin{bmatrix} \star & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \star & \dots & \dots & \star \end{bmatrix}$$

- solve calibration problem row-by-row
- for each row k leave z_l fixed for $l < k$ and only use z_k for optimisation

Sequential yield curve calibration is also called yield curve bootstrapping

- If there is an exact solution z such that $\mathcal{H}\left[P\left[z\right]\right](R) = 0$ then we can find it by solving sequence of one-dimensional equations

$$h_k(z_k) = \mathcal{H}_k\left[P\left[z_1, \ldots z_{k-1}, z_k, z_k, \ldots\right]\right](R_k) = 0 \quad \text{for} \quad k = 1, 2, \ldots, q$$

- If there is no exact solution, we can still exploit lower triangular form of Jacobi matrix in efficiently solving

$$\min_z \left\| \mathcal{H}\left[P\left[z\right]\right](R) \right\|$$

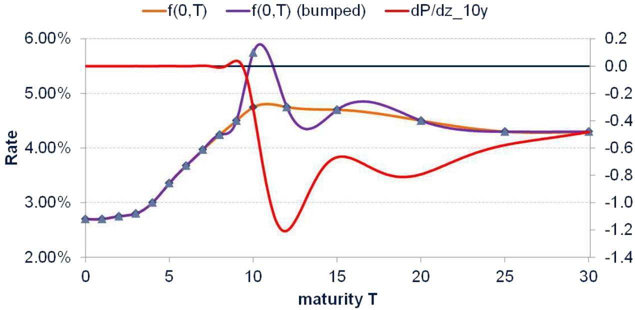
- Local parametrisation is achieved e.g. by spline interpolation methods that are fully specified by two neighboring points (e.g. linear interpolation)

- Note that local parametrisations typically yield less smooth forward rate curves than parametrisations where a change in a single parameter impacts a broader range of discount factors

Do we really need the restriction to local parametrisation?

- In many curve parametrisations/interpolations sensitivity $\frac{\partial}{\partial z_k} P[z](0, T)$ is small for $T \leq T_{n_{k-1}}$

Example: Interpolated forward rates $f(0, T)$ with cubic C^2-splines bumped by 1% at $10y$



- 10y rate bump does affect curve before $9y$ time point
- However, impact is small compared to impact around $10y$ maturity

We can extend the bootstrapping method to non-local parametrisations

Iterative Bootstrapping Method

Suppose we have a calibration problem set up via

$$\mathcal{H}\left[P\left[z\right]\right] = \left[\mathcal{H}_1\left[P\left[z\right]\right], \ldots, \mathcal{H}_q\left[P\left[z\right]\right]\right].$$

The iterative bootstrapping solves the calibration problem $\mathcal{H}\left[P\left[z\right]\right] = 0$ via the following steps

1. Set initial solution $z^0 = \left[z_1^0, \ldots z_q^0\right]$ via standard bootstrapping
2. If $\mathcal{H}\left[P\left[z^0\right]\right] \neq 0$ repeat the fixpoint iteration
   2.1 for $k = 1, 2, \ldots, q$ find z_k^i such that
   $$h_k(z_k^i) = \mathcal{H}_k\left[P\left[z_1^i, \ldots z_{k-1}^i, z_k^i, z_{k+1}^{i-1}, \ldots, z_q^{i-1}\right]\right](R_k) = 0.$$
   2.2 stop iteration if $\left\|z^i - z^{i-1}\right\| < \varepsilon$

- Iterative bootstrapping method usually converges in a few iterations

Outline

Single-curve calibration procedure is typically applied to discount curves from OIS swaps

Recall

$$\text{CompSwap}(t) = \underbrace{\sum_{j=1}^{m} L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j)}_{\text{compounding leg}} - \underbrace{\sum_{j=1}^{m} R \cdot \tau_j \cdot P(t, T_j)}_{\text{fixed leg}}$$

$$L(t, T_{j-1} T_j) = \left[\frac{P(t, T_{j-1})}{P(t, T_j)} - 1 \right] \frac{1}{\tau_j} \quad \text{(compounded OIS rate)}$$

Compounding swap rate helper can be defined solely in terms of discount curve P via

$$\mathcal{H}^{\text{CS}} [P] (K) = \text{CompSwap}(0) - 0$$

Single curve calibration procedure can be applied straight away

OIS discount curves can be derived from OIS swaps via single-curve calibration procedure

Forward rate agreements (FRA) can be used to specify short end of projection curves

Market quote of FRA with start date T_0 and tenor δ is the fixed rate K that prices the FRA at par as of today. Consider present value

$$\mathrm{FRA}(t) = \underbrace{P(t, T_0)}_{\text{discounting to } T_0} \cdot \underbrace{\left[L^{\delta}(t; T_0, T_0 + \delta) - K\right] \cdot \tau}_{\text{payoff}} \cdot \underbrace{\frac{1}{1 + \tau \cdot L^{\delta}(t; T_0, T_0 + \delta)}}_{\text{discounting from } T_0 \text{ to } T_0+\delta}$$

Condition $\mathrm{FRA}(t) = 0$ yields FRA calibration helper

$$\mathcal{H}^{\mathrm{FRA}}\left[P^{\delta}\right](K) = L^{\delta}(0; T_0, T_0 + \delta) - K = \left[\frac{P^{\delta}(0, T_0)}{P^{\delta}(0, T_0 + \delta)} - 1\right]\frac{1}{\tau} - K$$

- Typical tenors δ are 1m, 3m, 6m and 12m (corresponding to Libor rate indices)
- Typical expiries T_0 are up to 2y
- Both, available tenors and expiries, depend on the market (or currency)
- Note that FRA rate helper only depends on projection curve $P^{\delta}(0, T_0)$

Vanilla swaps are used to specify projection curves for longer maturities

Multi-curve swap price is given by

$$\mathrm{Swap}(t) = \underbrace{\sum_{j=1}^{m} L^{\delta}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j)}_{\text{float leg}} - \underbrace{\sum_{i=1}^{n} K \cdot \tau_i \cdot P(t, T_i)}_{\text{fixed Leg}}$$

Vanilla swap rate helper becomes

$$\mathcal{H}^{\mathrm{VS}}\left[P^{\delta}, (P)\right](K) = \sum_{j=1}^{m} L^{\delta}(0, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j) - \sum_{i=1}^{n} K \cdot \tau_i \cdot P(0, T_i)$$

- Rate helper depends on forward curve P^{δ} via forward Libor rates $L^{\delta}(0, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$
- Rate helper also depends on discount curve P via discount factors $P(t, \tilde{T}_j)$ and $P(0, T_i)$
  - this is reflected by notation $\mathcal{H}^{\mathrm{VS}}\left[\cdot, (P)\right]$
  - we put dependence in parentheses (P) because usually discount curve P is calibrated earlier already from OIS swaps

Projection curve calibration is analogous to single curve calibration I

- Specify projection curve parameters z^δ and projection curve $P^\delta = P^\delta\left[z^\delta\right]$
  - use methodologies/interpolations analogous to discount curves
- Set up calibration problem in terms of z^δ via

$$\mathcal{H}^\delta\left[P^\delta\left[z^\delta\right]\right] = \begin{bmatrix} \mathcal{H}_1^{\text{FRA}}\left[P^\delta\left[z^\delta\right]\right] \\ \vdots \\ \mathcal{H}_{q_{\text{FRA}}}^{\text{FRA}}\left[P^\delta\left[z^\delta\right]\right] \\ \mathcal{H}_1^{\text{VS}}\left[P^\delta\left[z^\delta\right],(P)\right] \\ \vdots \\ \mathcal{H}_{q_{\text{VS}}}^{\text{VS}}\left[P^\delta\left[z^\delta\right],(P)\right] \end{bmatrix}$$

  where calibration helpers are ordered by last cash flow date
- Obtain a set of market quotes

$$R^\delta = \left[R_1^{\text{FRA}},\ldots,R_{q_{\text{FRA}}}^{\text{FRA}},R_1^{\text{VS}},\ldots,R_{q_{\text{VS}}}^{\text{VS}}\right]^\top$$

Projection curve calibration is analogous to single curve calibration II

- Depending on curve parametrisation solve

$$\min_z \left\| \mathcal{H}^\delta \left[P^\delta \left[z^\delta \right], (P) \right] \left(R^\delta \right) \right\|$$

  via iterative bootstrapping or multi-dimensional optimisation method
- In principle, discount curve P and projection curve P^δ could also be solved simultanously by augmented optimisation problem

$$\min_{z, z^\delta} \left\| \hat{\mathcal{H}} \left[P[z], P^\delta \left[z^\delta \right] \right] \left(R, R^\delta \right) \right\|$$

  - keep in mind increased computational effort and complexity

Basis swaps are further instruments which are liquidely traded and also used for curve calibration I

Tenor Basis Swap

Floating rate payments of a longer Libor tenor are exchanged against floating rate payments of a shorter Libor tenor plus fixed spread.

$$
\begin{aligned}
\text{TenorSwap}(t) = & \sum_{j=1}^{m_1} L^{\delta_1}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta_1) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j) \\
& - \sum_{j=1}^{m_2} \left[L^{\delta_2}(t, \hat{T}_{j-1}, \hat{T}_{j-1} + \delta) + s \right] \cdot \hat{\tau}_j \cdot P(t, \hat{T}_j)
\end{aligned}
$$

- For example, $\delta_1 = 6m$ and $\delta_2 = 3m$
- Market quote is spread s (corresponding to maturity) which prices swap at par
- Typical use case: USD curves
  - in USD the following swap instruments are quoted:

Basis swaps are further instruments which are liquidely traded and also used for curve calibration II

  - OIS vs. fixed
  - 3m Libor vs. fixed
  - 6m Libor vs. 3m Libor plus spread

- first calibrate OIS discount curve P and 3m projection curve P^{3m}
- then use P and P^{3m} and calibrate P^{6m} from quoted tenor basis spreads

Cross currency basis swaps reference Libor rates in two currencies

Cross Currency Basis Swap

In a (constant notional) cross currency basis swap floating rate payments of a Libor rate in one currency are exchanged against floating rate payments of a Libor rate in another currency plus fixed spread.

$$\begin{aligned}\mathrm{XCcySwap}(t) = N_1 &\left\{ \sum_{j=1}^{m_1} L^{\delta_1}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta_1)\tilde{\tau}_j P^1(t, \tilde{T}_j) + P^1(t, \tilde{T}_{m_1}) \right\} \\ &- Fx(t) N_2 \left\{ \sum_{j=1}^{m_2} \left[L^{\delta_2}(t, \hat{T}_{j-1}, \hat{T}_{j-1} + \delta) + s \right] \hat{\tau}_j P^2(t, \hat{T}_j) + P^2(t, \hat{T}_{m_2}) \right\}\end{aligned}$$

- N_1 domestic currency notional, N_2 foreign currency notional
- $Fx(t)$ spot FX rate CCY2 / CCY1
- At trade date t_d notionals N_1 and N_2 are exchanged at time-t_d spot FX rate, i.e. $N_1 = Fx(t_d) N_2$

We have a look at the curves involved

projection curve from CCY-1 Libor swaps

discount curve from CCY-1 OIS swaps

$$\mathrm{XCcySwap}(t) = N_1 \left\{ \sum_{j=1}^{m_1} L^{\delta_1}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta_1) \tilde{\tau}_j P^1(t, \tilde{T}_j) + P^1(t, \tilde{T}_{m_1}) \right\}$$

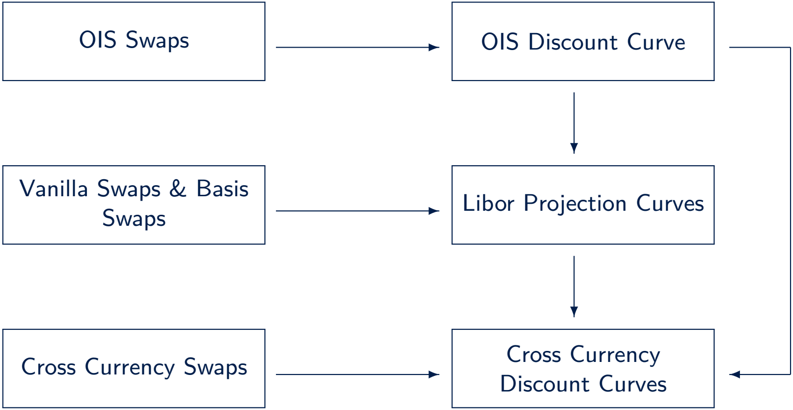
$$- Fx(t) N_2 \left\{ \sum_{j=1}^{m_2} \left[L^{\delta_2}(t, \hat{T}_{j-1}, \hat{T}_{j-1} + \delta) + s \right] \hat{\tau}_j P^2(t, \hat{T}_j) + P^2(t, \hat{T}_{m_2}) \right\}$$

projection curve from CCY-2 Libor swaps

discount curve specific to XCCY discounting in CCY-2

- Cross currency swaps require particular discount curves
- Cross currency discount curves (here P^2) are calibrated from quoted cross currency swap spreads (here s)
- Theoretical background is established via Collateralised Discounting
- For details, see e.g. M. Fujii and Y Shimada and A. Takahashi, Collateral Posting and Choice of Collateral Currency - Implications for Derivative Pricing and Risk Management. `https://ssrn.com/abstract=1601866`

In summary multi-curve calibration leads to a hierarchy of discount and projection curves



Outline

What are the parameters we need to calibrate in Hull White model?

forward rate from initial discount curve $P(0,t)$

short rate volatility from Vanilla options

$$r(t) = f(0,t) + x(t)$$
$$dx(t) = \left[\int_0^t \sigma(u)^2 \cdot e^{-2a(t-u)} du - a \cdot x(t)\right] \cdot dt + \sigma(t) \cdot dW(t)$$
$$x(0) = 0$$

mean reversion e.g. from other exotic option prices

- Short rate volatility $\sigma(t)$ mainly impacts overall variance of the rates
- Mean reversion a impacts forward volatility (and other related properties)

We first focus on volatility calibration (assuming mean reversion externally specified) and then look into mean reversion calibration

Outline

Market instruments for Volatility calibration are European swaptions

EUR ATM Swaption Straddles - BP Volatilities (Calendar day vols)
Please call +44 (0)20 7532 3080 for further details

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	25Y	30Y
1M Opt	45.3	45.0	48.0	53.8	55.3	61.6	70.1	78.6	85.2	88.7	90.0
2M Opt	38.8	40.9	44.8	48.3	51.4	58.6	67.0	76.3	82.5	84.5	85.5
3M Opt	35.6	37.3	41.7	46.8	50.9	58.3	66.7	75.0	80.5	82.5	84.1
6M Opt	34.9	37.7	42.1	46.9	51.0	59.3	66.3	74.1	78.7	80.1	81.3
9M Opt	35.4	38.0	43.1	47.3	51.5	59.1	66.9	73.8	77.5	78.7	79.0
1Y Opt	37.0	40.3	44.3	48.1	52.4	59.8	67.0	73.2	76.0	77.2	77.4
18M Opt	41.3	44.7	48.0	50.6	55.0	61.6	68.3	72.6	74.8	75.7	76.1
2Y Opt	46.5	49.4	52.6	55.0	58.2	63.9	69.8	73.0	74.2	75.1	75.5
3Y Opt	56.9	58.8	60.6	62.5	64.4	68.3	72.6	73.2	72.9	73.4	73.7
4Y Opt	64.1	65.5	66.0	67.4	68.6	71.1	73.9	72.4	71.5	71.1	71.0
5Y Opt	68.7	69.2	70.0	70.8	71.5	73.0	74.7	71.8	70.2	69.3	69.0
7Y Opt	73.0	73.3	73.6	73.8	74.1	74.5	74.8	70.1	67.6	66.4	66.0
10Y Opt	73.2	73.8	74								
15Y Opt	70.8	71.2	71								
20Y Opt	67.7	68.4	68								
25Y Opt	64.6	64.8	64								
30Y Opt	60.4	60.9	59								

EUR Vega - Normal Vol Skews

	Receivers						Payers				
	-200	-150	-100	-50	-25	ATM	+25	+50	+100	+150	+200
1y2y		22.29	14.02	5.40	1.84	40.72	0.91	4.20	13.83	24.45	
1y5y		0.20	-2.25	-2.44	-1.59	52.79	2.29	5.14	11.97	19.60	
1y10y		0.24	-1.69	-2.09	-1.40	67.86	2.10	4.80	11.53	19.32	
1y20y		13.45	7.57	2.33	0.63	76.97	0.67	2.64	9.62	18.89	
1y30y		7.75	4.34	1.43	0.46	79.14	0.16	1.00	4.59	10.15	
2y2y		11.95	6.40	1.54	0.14	49.98	1.32	3.90	11.32	19.98	
2y5y		-3.21	-3.26	-2.23	-1.28	58.62	1.61	3.52	8.09	13.38	
2y10y		-3.50	-2.97	-1.83	-1.01	70.41	1.21	2.63	6.04	10.10	
2y20y		1.10	0.20	-0.30	-0.28	75.04	0.57	1.44	4.09	7.81	
2y30y		4.86	2.51	0.58	0.07	76.50	0.46	1.48	5.08	10.29	
5y2y		-1.06	-1.41	-1.15	-0.70	69.84	0.95	2.14	5.18	8.91	
5y5y		-3.97	-2.93	-1.73	-0.94	72.02	1.11	2.39	5.42	9.00	
5y10y		-3.73	-2.55	-1.40	-0.74	75.23	0.84	1.80	4.04	6.69	
5y20y		-1.66	-1.12	-0.68	-0.38	70.67	0.49	1.10	2.72	4.85	
5y30y		-1.51	-0.99	-0.61	-0.35	69.54	0.47	1.07	2.69	4.86	
10y2y		-3.45	-2.56	-1.43	-0.75	74.34	0.83	1.74	3.79	6.11	
10y5y		-4.90	-3.28	-1.70	-0.87	74.37	0.92	1.89	4.00	6.33	
10y10y		-3.04	-1.95	-1.03	-0.54	73.36	0.60	1.29	2.91	4.89	
10y20y		-2.31	-1.32	-0.64	-0.33	65.38	0.36	0.78	1.79	3.08	
10y30y		-1.95	-1.17	-0.65	-0.36	63.77	0.46	1.02	2.53	4.54	

For Hull-White model calibration we assume that we can already price European swaptions at market level

- In practice, European swaption models depend on available market data (and business case)
- If only normal ATM volatilities are available (or should be used)[6]
  - interpolate ATM volatilities
  - assume normal model $dS = \sigma dW$
  - use Bachelier formula for Swaption pricing
- If Swaption smile data is available (in addition to ATM prices/volatilities)
  - calibrate e.g. Shifted SABR models per expiry/swap term to available data
  - interpolate models (e.g. via SABR model parameters β, ρ, ν)
  - make sure interpolated model fits (interplated) ATM swaption data (e.g. calibrate SABR α individually)
  - use interpolated model to price European swaption

[6] Same holds for (shifted) lognormal volatilities and corresponding basic models. But keep in mind implicit smile assumption!

How can we use European swaption prices to calibrate Hull-White volatility?

$$V^{\text{Swpt}}(T_E) = \left[\phi\left\{K \cdot \sum_{i=1}^{n} \tau_i \cdot P(T_E, T_i) - \sum_{j=1}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j)\right\}\right]^+$$

$$V^{\text{CBO}}(T_E) = \left[\phi\left\{\sum_{k=0}^{n+m+1} C_k \cdot P(T_E, \bar{T}_k)\right\}\right]^+$$

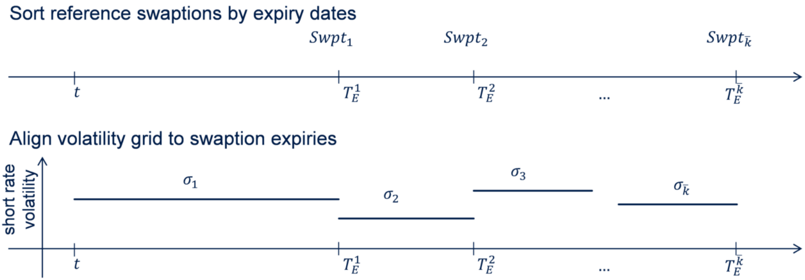
$$V^{\text{Swpt}}(t) = V^{\text{CBO}}(t) = \sum_{k=0}^{n+m+1} C_k \cdot V_k^{\text{ZBO}}(t)$$

$$V_k^{\text{ZBO}}(t) = P(t, T_E) \cdot \text{Black}\left(P(t, \bar{T}_k)/P(t, T_E), R_k, \nu_k, \phi\right)$$

$$\nu_k = G(T_E, \bar{T}_k)^2 \int_t^{T_E} \left[e^{-a(T_E - u)}\sigma(u)\right]^2 du$$

Price of a European swaption depends on short rate volatility $\sigma(t)$ from $t = 0$ to swaption expiry T_E.

We can calibrate a piece-wise constant volatility to a strip of reference European swaptions



We set up calibration helpers

$$\mathcal{H}_k[\sigma]\left(V_k^{\text{Swpt}}\right) = \underbrace{V_k^{\text{CBO}}(t)}_{\text{Model}[\sigma]} - \underbrace{V_k^{\text{Swpt}}}_{\text{Market}\left(\sigma_N^k\right)}$$

- $V_k^{\text{CBO}}(t)$ Hull-White model price of swaption represented as coupon bond option
- V_k^{Swpt} (quasi-)market price of swaption obtained from Vanilla model or implied (normal) volatility

Calibration problem is formulated in terms of short rate volatility values

Set

$$\sigma(t) = \sigma\left[\sigma_1, \ldots, \sigma_{\bar{k}}\right](t) = \sum_{k=1}^{\bar{k}} \mathbb{1}_{\{T_E^{k-1} \le t < T_E^k\}} \cdot \sigma_k$$

- Assume distinct expiry/grid dates T_E^k for reference swaptions
- Assume mean reversion is exogenously given

Hull-White Volatility Calibration Problem

For a given set of market quotes (or Vanilla model prices) $\left\{V_k^{\text{Swpt}}\right\}_{k=1,\ldots,\bar{k}}$ of reference European swaptions with corresponding calibration helpers $\mathcal{H}_k\left[\sigma\left[\sigma_1, \ldots, \sigma_{\bar{k}}\right]\right]$ the Hull-White volatility calibration problem is given as

$$\min_{\sigma_1,\ldots,\sigma_{\bar{k}}} \left\| \left[\mathcal{H}_1[\sigma]\left(V_1^{\text{Swpt}}\right), \ldots, \mathcal{H}_{\bar{k}}[\sigma]\left(V_{\bar{k}}^{\text{Swpt}}\right)\right]^\top \right\|.$$

We analyse the optimisation problem in more detail.

Multi-dimensional calibration problem can be decomposed into sequence of one-dimensional problems

Note that for $l > k$

$$\frac{d}{d\sigma_l}\mathcal{H}_k\left[\sigma\left[\sigma_1,\ldots,\sigma_{\bar{k}}\right]\right] = 0$$

Thus we could write

$$\begin{aligned}
\mathcal{H}_1\left[\sigma\left[\sigma_1\right]\right] &= 0\\
\mathcal{H}_2\left[\sigma\left[\sigma_1,\sigma_2\right]\right] &= 0\\
&\vdots\\
\mathcal{H}_{\bar{k}}\left[\sigma\left[\sigma_1,\sigma_2,\ldots,\sigma_{\bar{k}}\right]\right] &= 0
\end{aligned}$$

System of equations can be solved row-by-row (i.e. bootstrapping method) via one-dimensional root search method.

Sequentiel Hull-White volatility calibration is analogous to yield curve bootstrapping.

We can also formulate general optimisation problem if short rate volatilities and reference swaptions are not aligned

Suppose time grid $0 = t_0, t_1, \ldots, t_n$ and piece-wise constant volatility $\sigma(t)$ via $\bar{\sigma} = [\sigma_1, \ldots, \sigma_n]^\top$

$$\sigma(t) = \sigma[\bar{\sigma}](t) = \sum_{k=1}^{n} \mathbb{1}_{\{t_{k-1} \leq t < t_k\}} \cdot \sigma_k.$$

Denote $V^{\text{Swpt}} = \left[V_1^{\text{Swpt}}, \ldots, V_q^{\text{Swpt}}\right]$ a set of reference European swaption prices with calibration helper

$$\mathcal{H}[\sigma[\bar{\sigma}]]\left(V^{\text{Swpt}}\right) = \left[\mathcal{H}_1[\sigma[\bar{\sigma}]]\left(V_1^{\text{Swpt}}\right), \ldots, \mathcal{H}_q[\sigma[\bar{\sigma}]]\left(V_q^{\text{Swpt}}\right)\right].$$

Then calibration problem becomes

$$\min_{\bar{\sigma}} \left\| \mathcal{H}[\sigma[\bar{\sigma}]]\left(V^{\text{Swpt}}\right) \right\|.$$

The choice of reference European swaptions is critical for model calibration - What is the usage of your model?

Global calibration to available market data

General purpose calibration for curve simulation or pricing of a variety products with same model.

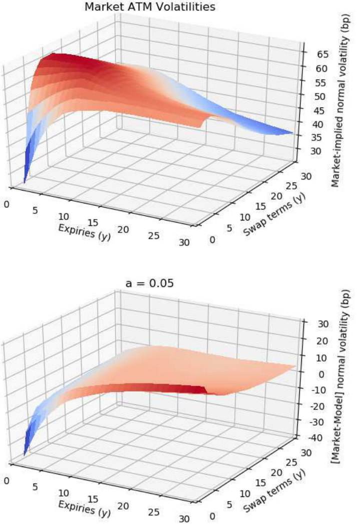
- Keep in mind model properties and limitations
- HW model cannot model smile - use ATM swaptions because most liquid
- Do not use too many reference swaptions per expiry - HW model has only one volatility parameter per expiry

Product-specific calibration

Pricing particular exotic product as most as possible consistent to underlying more simple products.

- Identify building blocks of exotic product - these are typically priced on simpler models if stand-alone product
- Calibrate HW model to prices of building blocks obtained from simpler model (or feeder model)

We illustrate market volatilities and global calibration fit



Lower mean reversion appears to yield slightly better global fit.

Building blocks of Bermudan swaption are co-terminal European swaptions

Recall decomposition

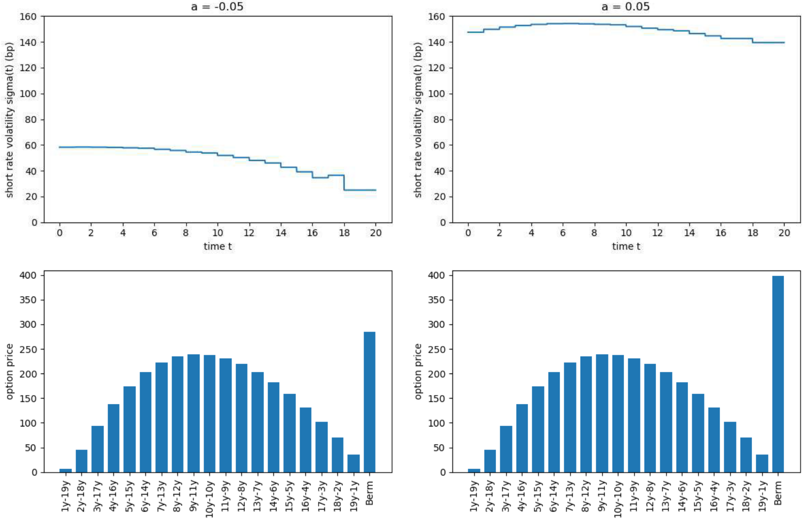
$$V^{\text{Berm}}(t) = \max_k \left\{ V_k^{\text{Swpt}}(t) \,|\, k = 1, \ldots, \bar{k} \right\} + \text{SwitchOption(t)}$$

where $V_k^{\text{Swpt}}(t)$ is price of European option to enter into swap at T_E^k (plus spot) with fixed maturity T_n.

- European swaption prices $V_k^{\text{Swpt}}(t)$ can be obtained from Vanilla model
- Consistent Hull-White model must produce max-European price $\max_k \left\{ V_k^{\text{Swpt}}(t) \,|\, k = 1, \ldots, \bar{k} \right\}$ consistent to Vanilla model

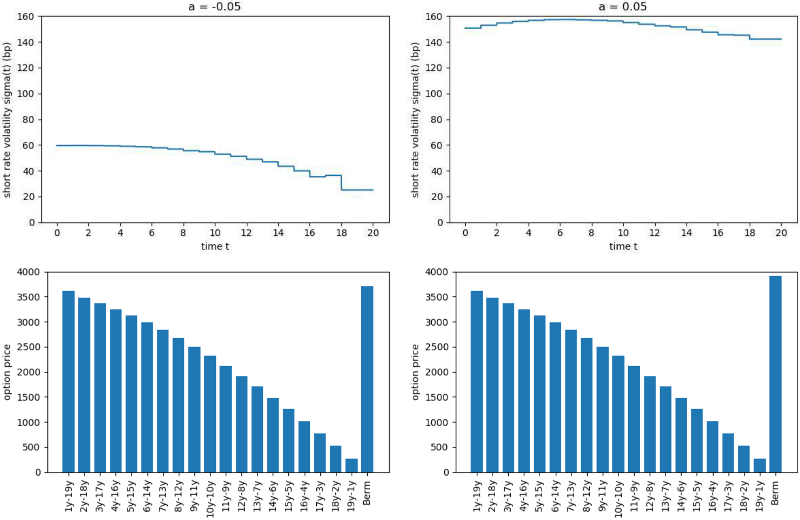
Hull-White model for Bermudan pricing is calibrated to corresponding co-terminal European swaptions.

20y-nc1y 3% Receiver Bermudan, (Fwd-)Rates at 5% (flat) and Implied Vols at 100bp (flat)



Out-of-the-money option shows concave co-terminal European swaption profile.

20y-nc1y 3% Receiver Bermudan, (Fwd-)Rates at 1% (flat) and Implied Vols at 100bp (flat)



In-the-money option shows decreasing co-terminal European swaption profile.

Outline

Mean reversion controls switch option value of Bermudan swaption I

Recall decomposition of Bermudan price into max-European price plus residual switch value

$$V^{\text{Berm}}(t) = \max_k \left\{ V_k^{\text{CBO}}(t) \,|\, k = 1, \ldots, \bar{k} \right\} + \text{SwitchOption(t)}.$$

- $V_k^{\text{CBO}}(t)$ is the Hull-White price of the co-terminal European swaptions reformulated as bond option
- SwitchOption(t) is the Hull-White price of the option to postpone exercise decision

We get

$$\frac{\partial}{\partial a} V^{\text{Berm}}(t) = \frac{\partial}{\partial a} \max_k \left\{ V_k^{\text{CBO}}(t) \,|\, k = 1, \ldots, \bar{k} \right\} + \frac{\partial}{\partial a} \text{SwitchOption(t)}.$$

If model is calibrated to match co-terminal swaptions from market prices V_k^{Swpt} then

$$V_k^{\text{CBO}}(t) = V_k^{\text{Swpt}} \quad \forall a.$$

Mean reversion controls switch option value of Bermudan swaption II

Thus

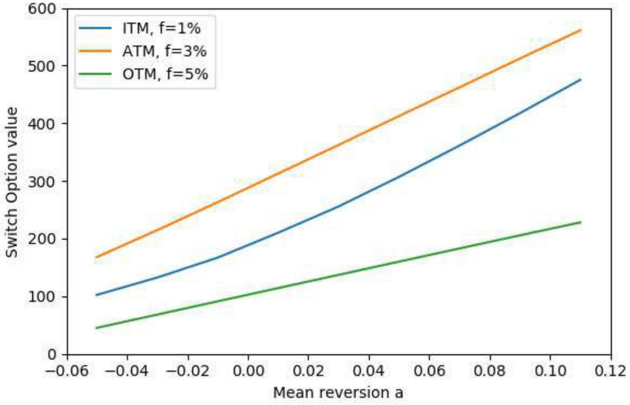
$$\frac{\partial}{\partial a} V_k^{\mathrm{CBO}}(t) = 0 \quad (\forall k) \quad \text{and} \quad \frac{\partial}{\partial a} \max_k \left\{ V_k^{\mathrm{CBO}}(t) \mid k = 1, \ldots, \bar{k} \right\} = 0.$$

Consequently,

$$\frac{\partial}{\partial a} V^{\mathrm{Berm}}(t) = \frac{\partial}{\partial a} \mathrm{SwitchOption(t)}$$

Switch option value (and Bermudan price) increase as mean reversion increases

- 20y-nc1y 3% Receiver Bermudan, (Fwd-)Rates $f \in \{1\%, 3\%, 5\%\}$ (flat) and Implied Vols at 100bp (flat)



If prices for reference Bermudan options are available we can use these prices to calibrate mean reversion.

If we don't have Bermudan prices available we can resort to alternative objectives to calibrate mean reversion

- Ratio of short-tenor and long-tenor option volatilities
- Auto-correlation (or inter-temporal correlation) of historical rates
- Payment-delay convexity adjustment

Mean reversion impacts the slope of short-tenor volatilities versus long-tenor volatilities

- For the analysis of short- vs. long-tenor volatilities we make several approximations
- Consider continuous forward yield

$$F(t, T_0, T_M) = \ln\left[\frac{P(t, T_0)}{P(t, T_M)}\right]\frac{1}{T_M - T_0}$$

- We will analyse standard deviation ratio for a $T_M - T_0$ forward yield and a $T_N - T_0$ forward yield

$$\lambda = \frac{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t\right]}}{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_N) \mid \mathcal{F}_t\right]}}$$

How are forward yields (and standard dev's) related to forward swap rates (and implied volatilities)?

We approximate swap rate by continuous forward yield I

Consider swap rate with start date T_0 and maturity T_M

$$S(t) = \frac{\sum_j L_j^\delta(t)\tilde{\tau}_j P(t, \tilde{T}_j)}{\sum_i \tau_i P(t, T_i)}$$

First we re-write swap rate in terms of single-curve rate plus basis spread)

$$S(t) = \frac{\sum_j L_j(t)\tilde{\tau}_j P(t, \tilde{T}_j)}{\sum_i \tau_i P(t, T_i)} + \underbrace{\frac{\sum_j \left[D_j^\delta - 1\right] \tilde{\tau}_j P(t, \tilde{T}_{j-1})}{\sum_i \tau_i P(t, T_i)}}_{b(t)}$$

Assume $b(t)$ is deterministic (similar to assuming D_j^δ are deterministic).
Simplifying single-curve swap rate yields

$$S(t) = \frac{P(t, T_0) - P(t, T_M)}{\sum_i \tau_i P(t, T_i)} + b(t)$$

We approximate swap rate by continuous forward yield II

Approximate annuity with only single long fixed-leg period T_0 to T_M with $\tau_1 = T_M - T_0$.
Then

$$S(t) \approx \frac{P(t, T_0) - P(t, T_M)}{(T_M - T_0)\, P(t, T_M)} + b(t) = \left[\frac{P(t, T_0)}{P(t, T_M)} - 1 \right] \frac{1}{T_M - T_0} + b(t)$$

First-order Taylor-approximation $\ln(x) \approx x - 1$ leads to

$$S(t) \approx \ln \left[\frac{P(t, T_0)}{P(t, T_M)} \right] \frac{1}{T_M - T_0} + b(t) = F(t, T_0, T_M) + b(t)$$

Deterministic basis spread assumption for $b(t)$ yields

$$\mathrm{Var}\left[S(T_0) \mid \mathcal{F}_t \right] \approx \mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t \right]$$

Also we approximate implied ATM volatility with standard deviation

Swap rate $S(t)$ is approximately normally distributed in Hull-White model. Thus

$$dS(t) \approx \sigma_S(t) dW^A(t)$$

for a deterministic volatility function $\sigma_S(t)$ depending on Hull-White model parameters.
Ito-isometry yields

$$\nu^2 = \mathrm{Var}\left[S(T_0) \mid \mathcal{F}_t\right] = \int_t^{T_0} \left[\sigma_S(t)\right]^2 dt$$

Vanilla options depend only on terminal distribution of swap rate. Thus an alternative swap rate with

$$d\tilde{S}(t) \approx \sigma_N dW^A(t) \quad \text{with} \quad \sigma_N^2 = \nu^2 / (T_0 - t)$$

yields same Vanilla option prices.
However, by construction σ_N is also the implied normal volatility of $\tilde{S}(T_0)$ and $S(T_0)$. This yields the relation

$$\mathrm{Var}\left[S(T_0) \mid \mathcal{F}_t\right] = \sigma_N^2 (T_0 - t)$$

We get the relation of the volatility ratio I

$$\lambda = \frac{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t\right]}}{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_N) \mid \mathcal{F}_t\right]}} \approx \frac{\sqrt{\left[\sigma_N^{T_0, T_M}\right]^2 (T_0 - t)}}{\sqrt{\left[\sigma_N^{T_0, T_N}\right]^2 (T_0 - t)}} = \frac{\sigma_N^{T_0, T_M}}{\sigma_N^{T_0, T_N}}$$

It remains to calculate $\mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t\right]$ with

$$F(T_0, T_0, T_M) = \ln\left[\frac{1}{P(T_0, T_M)}\right] \frac{1}{T_M - T_0} = -\frac{\ln\left[P(T_0, T_M)\right]}{T_M - T_0}$$

From $P(T_0, T_M) = \frac{P(t, T_M)}{P(t, T_0)} e^{-G(T_0, T_M)x(T_0) - \frac{1}{2} G(T_0, T_M)^2 y(T_0)}$ we get

$$\begin{aligned} F(T_0, T_0, T_M) &= -\left\{\ln\left[\frac{P(t, T_M)}{P(t, T_0)}\right] - G(T_0, T_M)x(T_0) - \frac{1}{2} G(T_0, T_M)^2 y(T_0)\right\} / (T_M - T_0) \\ &= F(t, T_0, T_M) + \frac{G(T_0, T_M)x(T_0) - \frac{1}{2} G(T_0, T_M)^2 y(T_0)}{T_M - T_0} \end{aligned}$$

This yields

$$\mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t\right] = \frac{G(T_0, T_M)^2}{(T_M - T_0)^2} \mathrm{Var}\left[x(T_0) \mid \mathcal{F}_t\right]$$

We get the relation of the volatility ratio II

and

$$\lambda = \frac{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_M) \mid \mathcal{F}_t\right]}}{\sqrt{\mathrm{Var}\left[F(T_0, T_0, T_N) \mid \mathcal{F}_t\right]}} = \frac{G(T_0, T_M)/(T_M - T_0)}{G(T_0, T_N)/(T_N - T_0)}$$

Substituting $G(T_0, T_1) = \left[1 - e^{-a(T_1 - T_0)}\right]/a$ yields

$$\lambda = \frac{\left[1 - e^{-a(T_M - T_0)}\right]/(T_M - T_0)}{\left[1 - e^{-a(T_N - T_0)}\right]/(T_N - T_0)}$$

Note that

- λ is independent of short rate volatility $\sigma(t)$
- λ only depends on mean reversion and time differences (i.e. swap terms) $T_M - T_0$ and $T_N - T_0$

Further simplification gives a relation only depending on $T_M - T_N$

Consider second order Taylor approximation

$$e^{-a(T_M - T_0)} \approx 1 - a\left(T_M - T_0\right) + \frac{1}{2}a^2\left(T_M - T_0\right)^2$$

This yields

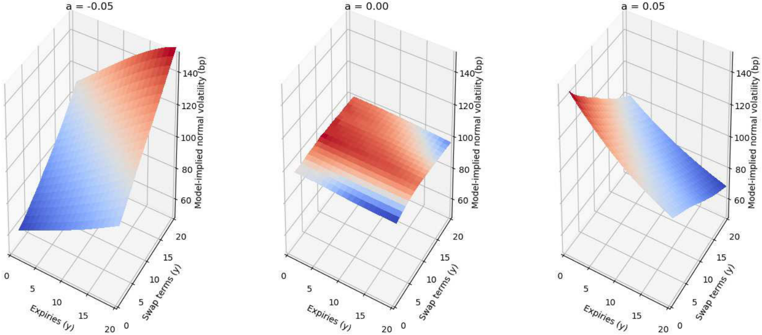
$$\begin{aligned}
\lambda &\approx \frac{\left[a\left(T_M - T_0\right) - \frac{1}{2}a^2\left(T_M - T_0\right)^2\right] / \left(T_M - T_0\right)}{\left[a\left(T_N - T_0\right) - \frac{1}{2}a^2\left(T_N - T_0\right)^2\right] / \left(T_N - T_0\right)} \\
&= \frac{1 - \frac{1}{2}a\left(T_M - T_0\right)}{1 - \frac{1}{2}a\left(T_N - T_0\right)} \approx \frac{e^{-\frac{1}{2}a(T_M - T_0)}}{e^{-\frac{1}{2}a(T_N - T_0)}} \\
&= e^{-\frac{1}{2}a(T_M - T_N)}
\end{aligned}$$

Finally, we end up with

$$\frac{\sigma_N^{T_0, T_M}}{\sigma_N^{T_0, T_N}} \approx e^{-\frac{1}{2}a(T_M - T_N)}$$

The relation $\sigma_N^{T_0,T_M}/\sigma_N^{T_0,T_N} \approx e^{-\frac{1}{2}a(T_M-T_N)}$ can be verified numerically

- Use flat short rate volatility σ - calibrated to 10y-10y swaption with 100bp volatility
- Mean reversion $a \in \{-5\%, 0\%, 5\%\}$



increasing flat decreasing

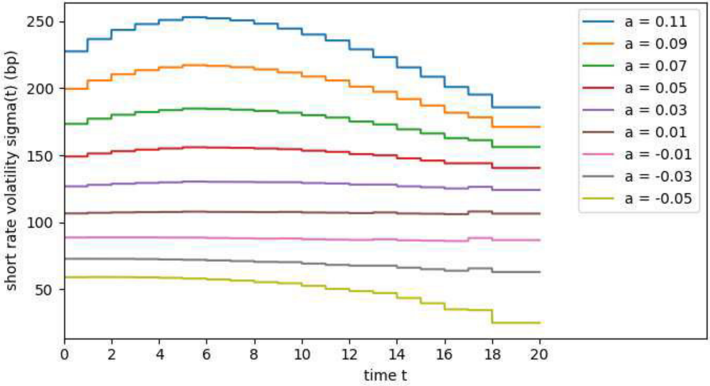
We can use volatility ratio property with co-terminal swaption volatility calibration

- Consider improvement of overall fit to ATM volatility surface as general calibration objective
- Calibrate mean reversion to ratio of
  - first exercise and co-terminal swap rate and
  - first exercise and short-term swap rate

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y	20y
1y	26.2	32.5	38.2	43.1	46.8	49.5	51.8	54.0	55.2	56.0	56.5	56.9	57.3	57.8	58.2	58.3	58.4	58.4	58.5	58.5
2y	38.5	43.3	47.2	50.3	52.7	54.9	56.7	58.0	59.0	59.7	59.9	60.0	60.2	60.3	60.5	60.5	60.5	60.6	60.6	60.6
3y	50.6	52.7	54.7	56.5	58.0	59.5	60.6	61.6	62.3	62.9	62.7	62.5	62.2	62.0	61.8	61.7	61.6	61.5	61.4	61.3
4y	57.7	58.9	59.7	60.9	61.8	62.6	63.4	64.0	64.5	64.8	64.4	63.9	63.4	62.9	62.5	62.3	62.1	61.9	61.7	61.5
5y	62.1	62.6	63.1	63.7	64.3	64.8	65.3	65.8	66.1	66.2	65.6	64.9	64.3	63.6	63.0	62.6	62.3	61.9	61.6	61.2
6y	64.4	64.7	65.0	65.3	65.6	65.9	66.2	66.4	66.6	66.6	65.8	65.0	64.1	63.3	62.5	62.1	61.7	61.4	61.0	60.6
7y	66.3	66.6	66.8	66.8	66.8	66.9	67.0	67.1	67.0	66.9	66.0	65.1	64.2	63.3	62.4	61.9	61.4	61.0	60.5	60.0
8y	66.4	66.7	66.9	67.1	67.1	67.1	67.1	67.0	66.9	66.6	65.7	64.7	63.8	62.8	61.9	61.4	60.9	60.3	59.8	59.3
9y	66.5	66.8	67.1	67.2	67.3	67.3	67.2	67.0	66.7	66.3	65.3	64.3	63.4	62.4	61.4	60.9	60.3	59.8	59.2	58.6
10y	66.6	66.9	67.1	67.2	67.2	67.2	67.1	66.8	66.5	66.0	65.0	64.0	63.0	61.9	60.9	60.3	59.8	59.2	58.6	58.0
11y	65.7	66.0	66.2	66.3	66.3	66.3	66.2	65.9	65.6	65.2	64.1	63.1	62.0	61.0	60.0	59.3	58.7	58.1	57.5	56.8
12y	64.7	65.0	65.3	65.4	65.4	65.4	65.3	65.1	64.7	64.3	63.3	62.2	61.1	60.1	59.0	58.3	57.7	57.0	56.3	55.7
13y	63.8	64.1	64.4	64.5	64.5	64.5	64.4	64.2	63.9	63.5	62.4	61.3	60.2	59.1	58.1	57.3	56.6	55.9	55.2	54.5
14y	62.9	63.2	63.4	63.6	63.6	63.6	63.6	63.3	63.0	62.6	61.5	60.4	59.3	58.2	57.1	56.3	55.6	54.8	54.0	53.3
15y	61.9	62.2	62.5	62.7	62.7	62.8	62.7	62.5	62.2	61.7	60.6	59.5	58.4	57.3	56.2	55.3	54.5	53.7	52.9	52.1
16y	61.0	61.3	61.6	61.7	61.8	61.8	61.7	61.5	61.2	60.7	59.6	58.5	57.4	56.3	55.2	54.4	53.5	52.7	51.9	51.0
17y	60.1	60.4	60.6	60.8	60.9	60.9	60.8	60.5	60.2	59.7	58.6	57.5	56.4	55.3	54.2	53.4	52.5	51.7	50.8	50.0
18y	59.1	59.4	59.7	59.9	60.0	60.0	59.8	59.6	59.2	58.7	57.6	56.5	55.4	54.3	53.2	52.4	51.5	50.7	49.8	49.0
19y	58.2	58.5	58.8	59.0	59.1	59.1	58.9	58.6	58.2	57.7	56.6	55.5	54.4	53.3	52.2	51.4	50.5	49.7	48.8	47.9
20y	57.3	57.6	57.8	58.1	58.1	58.1	57.9	57.6	57.2	56.6	55.6	54.5	53.4	52.3	51.3	50.4	49.5	48.6	47.8	46.9

Another calibration objective is time-stationarity of the model

- Based on mean reversion the calibrated term-structure of short rate volatilities changes



We can choose mean reversion such that calibrated short rate volatility is as close to constant as possible.

An alternative view on mean reversion is obtained via auto-correlation

Consider

$$F(T_0, T_0, T_M) = F(t, T_0, T_M) + \frac{G(T_0, T_M)x(T_0) - \frac{1}{2}G(T_0, T_M)^2 y(T_0)}{T_M - T_0}.$$

Then

$$\text{Corr}\left[F(T_0, T_0, T_M), F(T_1, T_1, T_N)\right] = \text{Corr}\left[x(T_0), x(T_1)\right].$$

We have

$$x(T) = e^{-a(T-t)}\left[x(t) + \int_t^T e^{a(u-t)}\left(y(u)du + \sigma(u)dW(u)\right)\right].$$

It follows for $T_1 > T_0$ (see exercises or literature)

$$\text{Corr}\left[x(T_0), x(T_1)\right] = e^{-2a(T_1 - T_0)}\sqrt{\frac{1 - e^{-2aT_0}}{1 - e^{-2aT_1}}}$$

Auto-correlation (or inter-temporal correlation) is independent of volatility $\sigma(t)$ and maturities T_M and T_N.

Auto-correlation property is sometimes used to calibrate mean reversion to interest rate time series

Consider limit $T_0 \to \infty$ then

$$\mathrm{Corr}\left[x(T_0), x(T_1)\right] \approx e^{-2a(T_1 - T_0)}.$$

- Use a time-series of proxy rates $\{R(t_k)\}_{k=1,2,\ldots}$ and estimate $\rho(\Delta) = \mathrm{Corr}\left[R(t_k), R(t_k + \Delta)\right]$
- Find mean reversion a such that

$$\rho(\Delta) \approx e^{-2a\Delta}$$

- However, method strongly depends on the choice of proxy rate and estimation time window
- Also, mean reversion in risk-neutral measure needs to be distinguished from mean reversion in real-world measure, see e.g. sec. 18 in
  - R. Rebonato. *Volatility and Correlation.* John Wiley & Sons, 2004

Outline

Summary on Hull-White model calibration

- Hull-White model calibration is distinguished between
  - short rate volatility calibration
  - mean reversion parameter calibration
- Short rate volatility is calibrated product-specific to match relevant Vanilla options
  - for Bermudan swaptions these are co-terminal European swaptions
- Mean reversion calibration involves subjective judgement regarding calibration objective
  - fit to reference exotic prices (e.g. Bermudans) if available
  - improve overall calibration fit to ATM swaption volatilities or time-stationarity of model

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