Interest Rate Modelling and Derivative Pricing

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Part II

Yield Curves and Linear Products

Outline

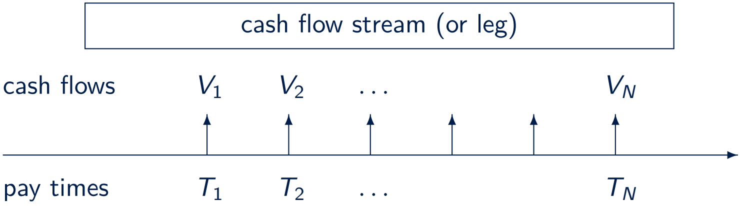
Static Yield Curve Modelling and Market Conventions

Multi-curve Discounted Cash Flow Pricing

Linear Market Instruments

Outline

DCF method requires knowledge of today's ZCB prices



- Assume $t = 0$ and deterministic cash flows, then

$$V(0) = \sum_{i=1}^{N} P(0, T_i) \cdot V_i$$

How do we get today's ZCB prices $P(0, T_i)$?

Yield curve is fundamental object for interest rate modelling

- A yield curve (YC) at an observation time t is the function of zero coupon bonds $P(t, \cdot) : [t, \infty) \to \mathbb{R}^+$ for maturities $T \geq t$.
- YCs are typically represented in terms of interest rates (instead of zero coupon bond prices)
- W.l.o.g. we set observation time $t = 0$ (i.e. consider today's yield curve)
- Discretely compounded zero rate curve $z_p(T)$ with frequency p, such that

$$P(0, T) = \left(1 + \frac{z_p(T)}{p}\right)^{-p \cdot T}$$

- Simple compounded zero rate curve $z_0(T)$ (i.e. $p = 1/T$), such that

$$P(0, T) = \frac{1}{1 + z_0(T) \cdot T}$$

- Continuous compounded zero rate curve $z(T)$ (i.e. $p = \infty$), such that

$$P(0, T) = \exp\left\{-z(T) \cdot T\right\}$$

For interest rate modelling we also need continuous compounded forward rates

- We define forward rates for general observation time t
- For static yield curve modelling we are interested particularly in $t = 0$

Definition (Continuous Forward Rate)

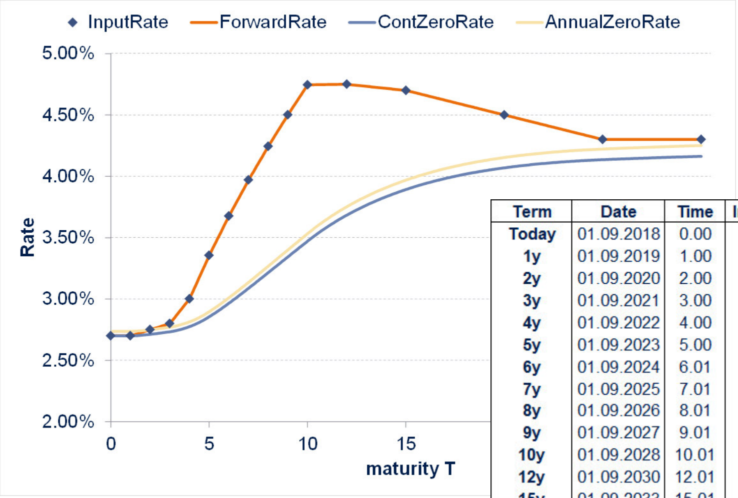
Suppose a given observation time t and zero bond curve $P(t, \cdot) : [t, \infty) \to \mathbb{R}^+$ for maturities $T \geq t$. The continuous compounded forward rate curve is given by

$$f(t, T) = -\frac{\partial \ln (P(t, T))}{\partial T}.$$

From the definition follows

$$P(t, T) = \exp \left\{ -\int_t^T f(t, s) ds \right\}.$$

We show a typical yield curve example



Term	Date	Time	InputRate
Today	01.09.2018	0.00	2.70%
1y	01.09.2019	1.00	2.70%
2y	01.09.2020	2.00	2.75%
3y	01.09.2021	3.00	2.80%
4y	01.09.2022	4.00	3.00%
5y	01.09.2023	5.00	3.36%
6y	01.09.2024	6.01	3.68%
7y	01.09.2025	7.01	3.97%
8y	01.09.2026	8.01	4.24%
9y	01.09.2027	9.01	4.50%
10y	01.09.2028	10.01	4.75%
12y	01.09.2030	12.01	4.75%
15y	01.09.2033	15.01	4.70%
20y	01.09.2038	20.01	4.50%
25y	01.09.2043	25.02	4.30%
30y	01.09.2048	30.02	4.30%

The market data for curve calibration is quoted by market data providers

	Euribor vs 6 mth			3/6 basis Spot Starting Date		Swap Spreads (Gadget)	
1 Yr	-0.226/-0.266	16Yrs	1.295/1.255	1 Yr	4.30		
2 Yrs	-0.128/-0.168	17Yrs	1.334/1.294	2 Yrs	4.80	5y	59.3
3 Yrs	0.010/-0.030	18Yrs	1.367/1.327	3 Yrs	5.35	10y	66.0
4 Yrs	0.154/0.114	19Yrs	1.393/1.353	4 Yrs	5.90		
5 Yrs	0.293/0.253	20Yrs	1.415/1.375	5 Yrs	6.40		
6 Yrs	0.429/0.389			6 Yrs	6.70	Page live in	
7 Yrs	0.558/0.518	21Yrs	1.432/1.392	7 Yrs	6.85	London hours ONLY	
8 Yrs	0.678/0.638	22Yrs	1.446/1.406	8 Yrs	6.90	(between 0700 - 1800)	
9 Yrs	0.790/0.750	23Yrs	1.457/1.417	9 Yrs	6.90		
10Yrs	0.892/0.852	24Yrs	1.465/1.425	10Yrs	6.85		
		25Yrs	1.471/1.431	This page will close 30th April			
11Yrs	0.983/0.943			6.00pm and re open 7.00am 2nd May			
12Yrs	1.064/1.024	26Yrs	1.476/1.436			10X12	0.192/0.152
13Yrs	1.135/1.095	27Yrs	1.480/1.440			10X15	0.378/0.338
14Yrs	1.197/1.157	28Yrs	1.484/1.444			10X20	0.543/0.503
15Yrs	1.250/1.210	29Yrs	1.486/1.446			10X25	0.599/0.559
		30Yrs	1.488/1.448			10X30	0.616/0.576
		35Yrs	1.491/1.451			10X35	0.619/0.579
		40Yrs	1.486/1.446			10X40	0.614/0.574
		45Yrs	1.476/1.436			10X45	0.604/0.564
		50Yrs	1.466/1.426			10X50	0.594/0.554
Disclaimer <IDIS>		60Yrs	1.456/1.416			10X60	0.584/0.544

Outline

Recall the introductory swap example

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2018

Dates

End date: Oct 30, 2038

Market conventions

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(semi-annually, act/360 day count, modified following, Target calendar)

How do we get from description to cash flow stream?

There are a couple of market conventions that need to be taken into account in practice

- Holiday calendars define at which dates payments can be made
- Business day conventions specify how dates are adjusted if they fall on a non-business day
- Schedule generation rules specify how regular dates are calculated
- Day count conventions define how time is meassured between dates

Outline

Dates are represented as triples day/month/year or as serial numbers

	Date	Serial	EUR Payment System (TARGET)	London Bank Holiday
	Friday, July 27, 2018	43308	FALSE	FALSE
	Monday, August 27, 2018	43339	FALSE	TRUE
	Thursday, September 27, 2018	43370	FALSE	FALSE
	Saturday, October 27, 2018	43400	TRUE	TRUE
	Tuesday, November 27, 2018	43431	FALSE	FALSE
	Thursday, December 27, 2018	43461	FALSE	FALSE
	Sunday, January 27, 2019	43492	TRUE	TRUE
	Wednesday, February 27, 2019	43523	FALSE	FALSE
	Wednesday, March 27, 2019	43551	FALSE	FALSE
	Saturday, April 27, 2019	43582	TRUE	TRUE
	Monday, May 27, 2019	43612	FALSE	TRUE
	Sunday, January 1, 1900	1		

A calender specifies business days and non-business days

Holiday Calendar

A holiday calendar $\mathcal{C}$ is a set of dates which are defined as holidays or non-business days

- A particular date d is a non-business day if $d \in \mathcal{C}$
- Holiday calendars are specific to a region, country or market segment
- Need to be specified in the context of financial product
- Typically contain weekends and special days of the year
- May be joint (e.g. for multi-currency products), $\bar{\mathcal{C}} = \mathcal{C}_1 \cup \mathcal{C}_2$
- Typical examples are TARGET calendar and LONDON calendar

Outline

A business day convention maps non-business days to adjacent business days

Business Day Convention (BDC)

- A business day convention is a function $\omega_{\mathcal{C}} : \mathcal{D} \to \mathcal{D}$ which maps a date $d \in \mathcal{D}$ to another date $\bar{d}$.
- It is applied in conjunction with a calendar $\mathcal{C}$
- good business days are unchanged, i.e. $\omega_{\mathcal{C}}(d) = d$ if $d \notin \mathcal{C}$

Following

$\omega_{\mathcal{C}}(d) = \min \left\{ \bar{d} \in \mathcal{D} \backslash \mathcal{C} \mid \bar{d} \geq d \right\}$

Preceding

$\omega_{\mathcal{C}}(d) = \max \left\{ \bar{d} \in \mathcal{D} \backslash \mathcal{C} \mid \bar{d} \leq d \right\}$

Modified Following

$$\omega_{\mathcal{C}}(d) = \begin{cases} \omega_{\mathcal{C}}^{\text{Following}}(d), & \text{if Month}\,[d] = \text{Month}\left[\omega_{\mathcal{C}}^{\text{Following}}(d)\right] \\ \omega_{\mathcal{C}}^{\text{Preceeding}}(d), & \text{else} \end{cases}$$

Outline

Schedules represent sets of regular reference dates

	Annual Frequency	TARGET Calendar	Modified Following
Start	Tue, 30 Oct 2018	FALSE	Tue, 30 Oct 2018
	Wed, 30 Oct 2019	FALSE	Wed, 30 Oct 2019
	Fri, 30 Oct 2020	FALSE	Fri, 30 Oct 2020
	Sat, 30 Oct 2021	TRUE	Fri, 29 Oct 2021
	Sun, 30 Oct 2022	TRUE	Mon, 31 Oct 2022
	Mon, 30 Oct 2023	FALSE	Mon, 30 Oct 2023
	Wed, 30 Oct 2024	FALSE	Wed, 30 Oct 2024
	Thu, 30 Oct 2025	FALSE	Thu, 30 Oct 2025
	Fri, 30 Oct 2026	FALSE	Fri, 30 Oct 2026
	Sat, 30 Oct 2027	TRUE	Fri, 29 Oct 2027
	Mon, 30 Oct 2028	FALSE	Mon, 30 Oct 2028
	Tue, 30 Oct 2029	FALSE	Tue, 30 Oct 2029
	Wed, 30 Oct 2030	FALSE	Wed, 30 Oct 2030
	Thu, 30 Oct 2031	FALSE	Thu, 30 Oct 2031
	Sat, 30 Oct 2032	TRUE	Fri, 29 Oct 2032
	Sun, 30 Oct 2033	TRUE	Mon, 31 Oct 2033
	Mon, 30 Oct 2034	FALSE	Mon, 30 Oct 2034
	Tue, 30 Oct 2035	FALSE	Tue, 30 Oct 2035
	Thu, 30 Oct 2036	FALSE	Thu, 30 Oct 2036
	Fri, 30 Oct 2037	FALSE	Fri, 30 Oct 2037
End	Sat, 30 Oct 2038	TRUE	Fri, 29 Oct 2038

Schedule generation follows some rules/convention as well

1. Consider direction of roll-out: forward or backward (relevant for front/back stubs)
2. Roll out unadjusted dates according to frequency
3. If first/last period is broken consider short stub or long stub
4. Adjust unadjusted dates according to calendar and BDC

Outline

Day count conventions map dates to times or year fractions

Day Count Convention

A day count convention is a function $\tau : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ which measures time period between dates in terms of years.

We give some examples:

Act/365 Fixed Convention

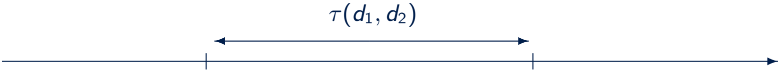
$\tau(d_1, d_2) = (d_2 - d_1)/365$

- Typically used to describe time in financial models

Act/360 Convention

$\tau(d_1, d_2) = (d_2 - d_1)/360$

- Often used for Libor floating rate payments



30/360 methods are slightly more involved

General 30/360 Method

- Consider two dates d_1 and d_2 represented as triples of day/month/year, i.e. $d_1 = [D_1, M_1, Y_1]$ and $d_2 = [D_2, M_2, Y_2]$ with $D_{1/2} \in \{1, \ldots, 31\}$, $M_{1/2} \in \{1, \ldots, 12\}$ and $Y_{1/2} \in \{1, 2, \ldots\}$
- Obviously, only valid dates are allowed (no Feb. 30 or similar)
- Adjust $D_1 \mapsto \bar{D}_1$ and $D_2 \mapsto \bar{D}_2$ according to specific rules
- Calculate

$$\tau(d_1, d_2) = \frac{(Y_2 - Y_1) + 12 \cdot (M_2 - M_1) + 30 \cdot \left(\bar{D}_2 - \bar{D}_1\right)}{360}$$

Some specific 30/360 rules are given below

30/360 Convention (or 30U/360, Bond Basis)

1. $\bar{D}_1 = \min\{D_1, 30\}$
2. If $\bar{D}_1 = 30$ then $\bar{D}_2 = \min\{D_2, 30\}$ else if $\bar{D}_2 = D_2$

30E/360 Convention (or Eurobond)

1. $\bar{D}_1 = \min\{D_1, 30\}$
2. $\bar{D}_2 = \min\{D_2, 30\}$

Outline

Now we have all pieces to price a deterministic coupon leg

ValDate	Sat, 01 Sep 2018

Sum	41,641,052

	Modified Following	_D1	_D2	tau	Rate	Coupon	P(0,T)	E^Q[Cpn] = P(0,T)*Cpn
Start	Tue, 30 Oct 2018							
	Wed, 30 Oct 2019	30	30	1.000	3.00%	3,000,000	0.9691	2,907,348
	Fri, 30 Oct 2020	30	30	1.000	3.00%	3,000,000	0.9429	2,828,753
	Fri, 29 Oct 2021	30	29	0.997	3.00%	2,991,667	0.9171	2,743,628
	Mon, 31 Oct 2022	29	31	1.006	3.00%	3,016,667	0.8904	2,686,104
	Mon, 30 Oct 2023	30	30	1.000	3.00%	3,000,000	0.8622	2,586,453
	Wed, 30 Oct 2024	30	30	1.000	3.00%	3,000,000	0.8319	2,495,579
	Thu, 30 Oct 2025	30	30	1.000	3.00%	3,000,000	0.8003	2,400,837
	Fri, 30 Oct 2026	30	30	1.000	3.00%	3,000,000	0.7677	2,303,214
	Fri, 29 Oct 2027	30	29	0.997	3.00%	2,991,667	0.7347	2,197,910
	Mon, 30 Oct 2028	29	30	1.003	3.00%	3,008,333	0.7011	2,109,002
	Tue, 30 Oct 2029	30	30	1.000	3.00%	3,000,000	0.6686	2,005,657
	Wed, 30 Oct 2030	30	30	1.000	3.00%	3,000,000	0.6375	1,912,636
	Thu, 30 Oct 2031	30	30	1.000	3.00%	3,000,000	0.6080	1,824,110
	Fri, 29 Oct 2032	30	29	0.997	3.00%	2,991,667	0.5800	1,735,139
	Mon, 31 Oct 2033	29	31	1.006	3.00%	3,016,667	0.5532	1,668,789
	Mon, 30 Oct 2034	30	30	1.000	3.00%	3,000,000	0.5280	1,583,997
	Tue, 30 Oct 2035	30	30	1.000	3.00%	3,000,000	0.5041	1,512,275
	Thu, 30 Oct 2036	30	30	1.000	3.00%	3,000,000	0.4814	1,444,198
	Fri, 30 Oct 2037	30	30	1.000	3.00%	3,000,000	0.4600	1,379,911
End	Fri, 29 Oct 2038	30	29	0.997	3.00%	2,991,667	0.4397	1,315,511

Outline

Outline

Recall the introductory swap example

Pays 3% on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(annually, 30/360 day count, modified following, Target calendar)

Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(semi-annually, act/360 day count, modified following, Target calendar)

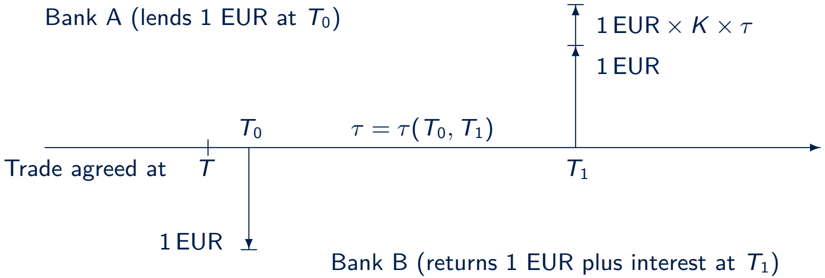
How do we model floating rates?

We start with some introductory remarks

- London Interbank Offered Rates (Libor) currently are the key building blocks of interest rate derivatives
- They are fixed for USD, GBP, JPY, CHF (and EUR)
- EUR equivalent rate is Euribor rate (we will use Libor synonymously for Euribor)
- Libor rate modelling has undergone significant changes since financial crisis in 2008
- This is typically reflected by the term Multi-Curve Interest Rate Modelling
- Recent developments in the market indicate a shift to alternative floating rates in the future
- We will also touch on potential new alternative rates specifications

Let's start with the classical Libor rate model

What is the fair rate K bank A and Bank B can agree on?



We get (via DCF methodology)

$$0 = V(T) = P(T, T_0) \cdot \mathbb{E}^{T_0}\left[-1 \mid \mathcal{F}_T\right] + P(T, T_1) \cdot \mathbb{E}^{T_1}\left[1 + \tau K \mid \mathcal{F}_T\right]$$
$$0 = -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K)$$

Spot Libor rates are fixed daily and quoted in the market

$$0 = -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K)$$

Spot Libor Rate

The fair rate for an interbank lending deal with trade date T, spot starting date T_0 (typically 0d or 2d after T) and maturity date T_1 is

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} - 1 \right] \frac{1}{\tau}.$$

- Panel banks submit daily estimates for interbank lending rates to calculation agent
- Relevant periods (i.e. $[T_0, T_1]$) considered are 1m, 3m, 6m and 12m
- Trimmed average of submissions is calculated and published

Libor rate fixings currently are the most important reference rates for interest rate derivatives

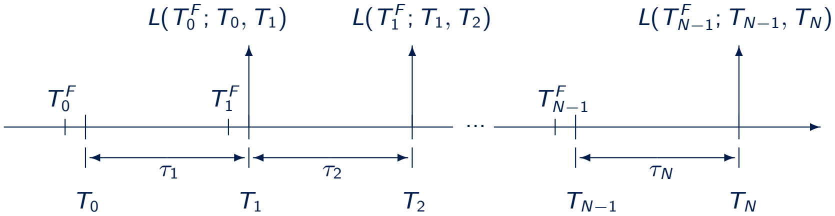
Example publication at Intercontinental Exchange (ICE)

theice.com/marketdata/reports/170

ICE LIBOR Historical Rates

TENOR	PUBLICATION TIME*	USD ICE LIBOR 06-SEP-2018
Overnight	11:55:04 AM	1.91838
1 Week	11:55:04 AM	1.96100
1 Month	11:55:04 AM	2.13256
2 Month	11:55:04 AM	2.20950
3 Month	11:55:04 AM	2.32706
6 Month	11:55:04 AM	2.54419
1 Year	11:55:04 AM	2.84906

A plain vanilla Libor leg pays periodic Libor rate coupons



We get (via DCF methodology)

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[L(T_{i-1}^F, T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t \right] \\
&= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[L(T_{i-1}^F, T_{i-1}, T_i) \mid \mathcal{F}_t \right] \cdot \tau_i
\end{aligned}
$$

Thus all we need is

$$
\mathbb{E}^{T_i} \left[L(T_{i-1}^F, T_{i-1}, T_i) \mid \mathcal{F}_t \right] = ?
$$

Libor rate is a martingale in the terminal measure

Theorem (Martingale property of Libor rate)

The Libor rate $L(T; T_0, T_1)$ with observation/fixing date T, accrual start date T_0 and accrual end date T_1 is a martingale in the T_1-forward measure. That is

$$L(t; T_0, T_1) = \mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right].$$

Proof.

The fair Libor rate at fixing time T is $L(T; T_0, T_1) = \left[P(T, T_0)/P(T, T_1) - 1\right]/\tau$. The zero coupon bond $P(T, T_0)$ is an asset and $P(T, T_1)$ is a numeraire. Thus FTAP yields that the discounted asset price is a martingale, i.e.

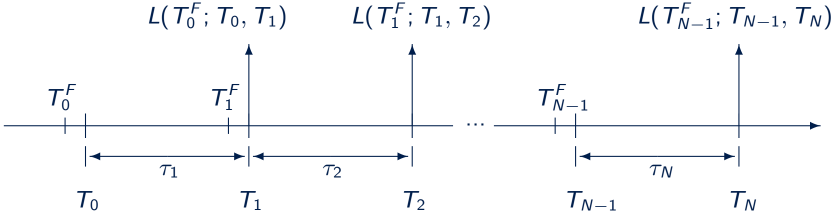
$$\mathbb{E}^{T_1}\left[\frac{P(T, T_0)}{P(T, T_1)} \mid \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)},$$

Linearity of expectation operator yields

$$\mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right]\frac{1}{\tau} = L(t; T_0, T_1).$$

$\square$

This allows pricing the Libor leg based on today's knowledge of the yield curve only



Libor leg becomes

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i} \left[L(T_{i-1}^F, T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t \right] \\ &= \sum_{i=1}^{N} P(t, T_i) \cdot L(t, T_{i-1}, T_i) \cdot \tau_i \end{aligned}$$

Libor leg may be simplified in the current single-curve setting

We have

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L(t, T_{i-1}, T_i) \cdot \tau_i \text{ with } L(t, T_{i-1}, T_i) = \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i}$$

This yields

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i} \cdot \tau_i \\ &= \sum_{i=1}^{N} P(t, T_{i-1}) - P(t, T_i) \\ &= P(t, T_0) - P(t, T_N) \end{aligned}$$

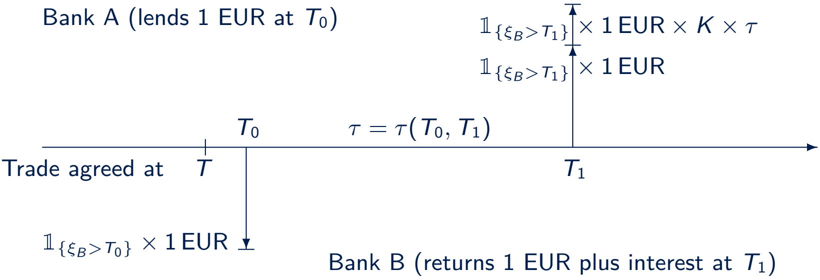
Outline

The classical Libor rate model misses an important detail...

What if a counterparty defaults?

What if Bank B defaults prior to T_0 or T_1?

What is the fair rate K bank A and Bank B can agree on given the risk of default?



- Cash flows are paid only if no default occurs
- We apply a simple credit model
- Denote random variable ξ_B the first time bank B defaults

Credit-risky trade value can be derived using derivative pricing formula

$$\frac{V(T)}{B(T)} = \mathbb{E}^{\mathbb{Q}}\left[-\mathbb{1}_{\{\xi_B > T_0\}} \cdot \frac{1}{B(T_0)} + \mathbb{1}_{\{\xi_B > T_1\}} \cdot \frac{1 + K \cdot \tau}{B(T_0)}\right]$$

(all expectations conditional on $\mathcal{F}_T$)

Assume independence of credit event $\left\{\xi_B > T_{0/1}\right\}$ and interest rate market, then

$$\frac{V(T)}{B(T)} = -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_0\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T_0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_1\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1 + K \cdot \tau}{B(T_0)}\right]$$

Abbreviate survival probability $Q(T, T_{0/1}) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\xi_B > T_{0/1}\right\}} \,|\, \mathcal{F}_T\right]$ and apply change of measure

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}\left[1\right] + P(T, T_0)Q(T, T_1)\mathbb{E}^{T_1}\left[1 + K \cdot \tau\right]$$

This yields the fair spot rate in the presence of credit risk

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}[1] + P(T, T_1)Q(T, T_1)\mathbb{E}^{T_1}[1 + K \cdot \tau]$$

If we solve $V(T) = 0$ we get

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} \cdot \frac{Q(T, T_0)}{Q(T, T_1)} - 1\right]\frac{1}{\tau}$$

We need a model for the survival probability $Q(T, T')$

Consider, e.g., hazard rate model $Q(T, T') = \exp\left\{-\int_T^{T'} \lambda(s)ds\right\}$ with deterministic hazard rate $\lambda(s)$. Then

$$D(T_0, T_1) = \frac{Q(T, T_0)}{Q(T, T_1)} = \exp\left\{-\int_{T_0}^{T_1} \lambda(s)ds\right\}$$

is independent of observation time T

Deterministic hazard rate assumption preserves the martingale property of forward Libor rate

Theorem (Martingale property of credit-risky Libor rate)

Consider the credit-risky Libor rate $L(T; T_0, T_1)$ with observation/fixing date T, accrual start date T_0 and accrual end date T_1. If the forward survival probability $D(T_0, T_1)$ is deterministic such that

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau}$$

then $L(t; T_0, T_1)$ is a martingale in the T_1-forward measure and

$$L(t; T_0, T_1) = \mathbb{E}^{T_1} \left[L(T; T_0, T_1) \mid \mathcal{F}_t \right] = \left[\frac{P(t, T_0)}{P(t, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau}.$$

Proof.

Follows analogous to classical Libor rate martingale property.

□

Outline

Forward Libor rates are typically parametrised via projection curve

- Hazard rate $\lambda(u)$ in $Q(T, T') = \exp\left\{-\int_T^{T'} \lambda(u)du\right\}$ is often considered as a tenor basis spread $s(u)$

- Survival probability $Q(T, T')$ can be interpreted as discount factor

- Suppose we know time-t survival probabilities $Q(t, \cdot)$ for a forward Libor rate $L(t, T_0, T_0 + \delta)$ with tenor δ (typically 1m, 3m, 6m or 12m). Then we define the projection curve

$$P^\delta(t, T) = P(t, T) \cdot Q(t, T)$$

- With projection curve $P^\delta(t, T)$ the forward Libor rate formula is analogous to the classical Libor rate formula, i.e.

$$L^\delta(t, T_0) = L(t, T_0, T_0 + \delta) = \left[\frac{P^\delta(t, T_0)}{P^\delta(t, T_1)} - 1\right]\frac{1}{\tau}$$

This yields the multi-curve modelling framework consisting of discount curve $P(t, T)$ and tenor-dependent projection curves $P^\delta(t, T)$

There is an alternative approach to multi-curve modelling

Define forward Libor rate $L^{\delta}(t, T_0)$ as

$$L^{\delta}(t, T_0) = \mathbb{E}^{T_1}\left[L(T; T_0, T_0 + \delta) \mid \mathcal{F}_t\right]$$

(without any assumptions on default, survival probabilities etc.)

And postulate a projection curve parametrisation

$$L^{\delta}(t, T_0) = \left[\frac{P^{\delta}(t, T_0)}{P^{\delta}(t, T_1)} - 1\right]\frac{1}{\tau}$$

- We will discuss calibration of projection curve $P^{\delta}(t, T)$ later
- This approach alone suffices for linear products (e.g. Libor legs) and simple options
- It does not specify any relation between projection curve $P^{\delta}(t, T)$ and discount curve $P(t, T)$

Projection curves can also be written in terms of zero rates and continuous forward rates

Consider a projection curve given by discount factors $P^{\delta}(0, T)$ (observed today)

- Corresponding continuous compounded zero rates are

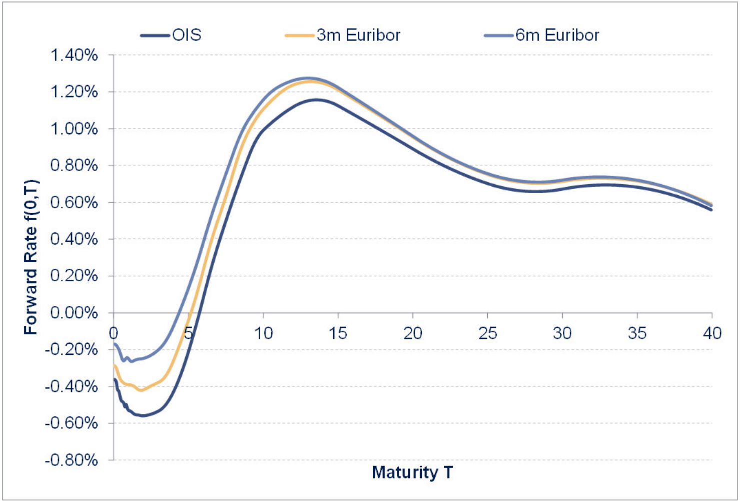
$$z^{\delta}(T) = -\frac{\ln\left[P^{\delta}(0, T)\right]}{T}$$

- Corresponding continuous compounded forward rates are

$$f^{\delta}(T) = -\frac{\partial \ln\left[P^{\delta}(0, T)\right]}{\partial T}$$

We illustrate an example of a multi-curve set-up for EUR

Market data as of July 2016



Libor leg pricing needs to be adapted slightly for multi-curve pricing

Classical single-curve Libor leg price is

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot L(t, T_{i-1}, T_i) \cdot \tau_i \\
&= P(t, T_0) - P(t, T_N)
\end{aligned}
$$

Multi-curve Libor leg pricing becomes

$$
V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L^{\delta}(t, T_{i-1}) \cdot \tau_i
$$

with

$$
L^{\delta}(t, T_0) = \left[\frac{P^{\delta}(t, T_0)}{P^{\delta}(t, T_1)} - 1 \right] \frac{1}{\tau}
$$

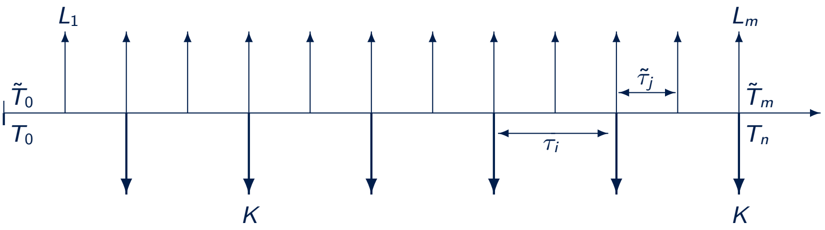
- Note that we need different yield curves for Libor rate projection and cash flow discounting
- Single-curve pricing formula simplification does not work for multi-curve pricing

Outline

Outline

With the fixed leg and Libor leg pricing available we can directly price a Vanilla interest rate swap

float leg (EUR conventions: 6m Euribor, Act/360)



fixed leg (EUR conventions: annual, 30/360)

present value of (fixed rate) payer swap with notional N becomes

$$V(t) = \sum_{j=1}^{m} N \cdot L^{6m}(t, \tilde{T}_{j-1}, \tilde{T}_j) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j) - \sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i)$$

Vanilla swap pricing formula allows us to price the underlying swap of our introductory example

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 30, 2018

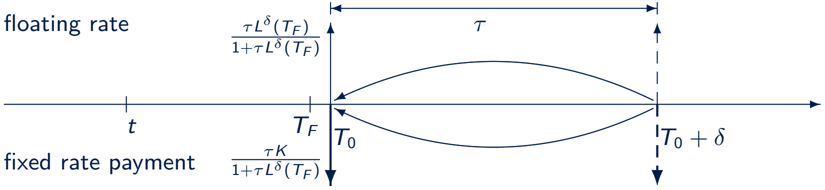
End date: Oct 30, 2038

(semi-annually, act/360 day count, modified following, Target calendar)

We illustrate swap pricing with QuantLib/Excel...

Outline

Forward Rate Agreement yields exposure to single forward Libor rates



- Fixed rate K agreed at trade inception
- Libor rate $L^{\delta}(T_F; T_0, T_0 + \delta)$ fixed at T_F
- Payoff paid at T_0 is difference $\tau \cdot \left[L^{\delta}(T_F; T_0, T_0 + \delta) - K\right]$ discounted from T_1 to T_0 with discount factor $1 + \tau \cdot L^{\delta}(T_F; T_0, T_0 + \delta)$, i.e.

$$V(T_0) = \frac{\tau \cdot \left[L^{\delta}(T_F; T_0, T_0 + \delta) - K\right]}{1 + \tau \cdot L^{\delta}(T_F; T_0, T_0 + \delta)}$$

Time-T_F FRA price can be obtained via deterministic basis spread model

Note that payoff $V(T_0) = \frac{\tau \cdot \left[L^\delta(T_F; T_0, T_0+\delta) - K\right]}{1+\tau \cdot L^\delta(T_F; T_0, T_0+\delta)}$ is already determined at T_F
Thus (via DCF)

$$V(T_F) = P(T_F, T_0) \cdot V(T_0) = P(T_F, T_0) \cdot \frac{\tau \cdot \left[L^\delta(T_F; T_0, T_0+\delta) - K\right]}{1 + \tau \cdot L^\delta(T_F; T_0, T_0+\delta)}$$

Recall that (with $T_1 = T_0 + \delta$)

$$1 + \tau \cdot L^\delta(T_F; T_0, T_0+\delta) = \frac{P^\delta(T_F, T_0)}{P^\delta(T_F, T_1)} = \frac{P(T_F, T_0)}{P(T_F, T_1)} \cdot D(T_0, T_1)$$

Then

$$\begin{aligned} V(T_F) &= P(T_F, T_0) \cdot \tau \cdot \left[L^\delta(T_F; T_0, T_0+\delta) - K\right] \cdot \frac{1}{D(T_0, T_1)} \cdot \frac{P(T_F, T_1)}{P(T_F, T_0)} \\ &= P(T_F, T_1) \cdot \tau \cdot \left[L^\delta(T_F; T_0, T_0+\delta) - K\right] \cdot \frac{1}{D(T_0, T_1)} \end{aligned}$$

Present value of FRA can be obtained via martingale property

Derivative pricing formula in T_1-terminal measure yields

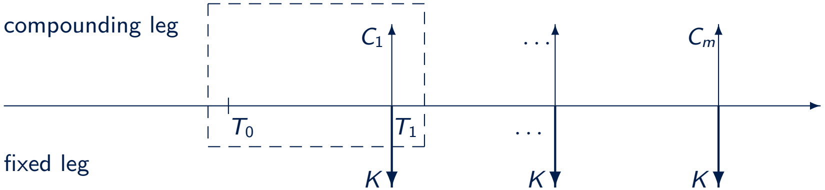
$$\begin{aligned}
\frac{V(t)}{P(t,T_1)} &= \mathbb{E}^{T_1}\left[\frac{P(T_F,T_1)}{P(T_F,T_1)} \cdot \tau \cdot \left[L^\delta(T_F;T_0,T_0+\delta) - K\right] \cdot \frac{1}{D(T_0,T_1)}\right] \\
&= \tau \cdot \left[\mathbb{E}^{T_1}\left[L^\delta(T_F;T_0,T_0+\delta)\right] - K\right] \cdot \frac{1}{D(T_0,T_1)} \\
&= \tau \cdot \left[L^\delta(t;T_0,T_0+\delta) - K\right] \cdot \frac{1}{D(T_0,T_1)}
\end{aligned}$$

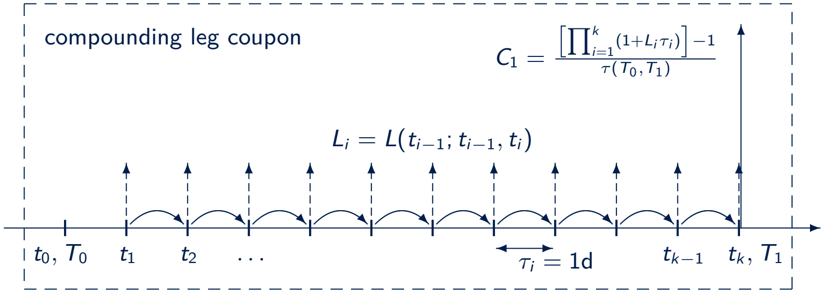
Using $1 + \tau \cdot L^\delta(t;T_0,T_0+\delta) = \frac{P(t,T_0)}{P(t,T_1)} \cdot D(T_0,T_1)$ (deterministic spread assumption) yields

$$\begin{aligned}
V(t) &= P(t,T_0) \cdot \tau \cdot \left[L^\delta(t;T_0,T_0+\delta) - K\right] \cdot \left[\frac{P(t,T_0)}{P(t,T_1)} \cdot D(T_0,T_1)\right]^{-1} \\
&= P(t,T_0) \cdot \frac{\left[L^\delta(t;T_0,T_0+\delta) - K\right] \cdot \tau}{1 + \tau \cdot L^\delta(t;T_0,T_0+\delta)}
\end{aligned}$$

Outline

Overnight index swap (OIS) instruments are further relevant instruments in the market





We need to calculate the compounding leg coupon rate

- Assume overnight index swap (OIS) rate $L_i = L(t_{i-1}; t_{i-1}, t_i)$ is is a credit-risk free Libor rate
- Compounded rate (for a period $[T_0, T_1]$) is specified as

$$C_1 = \left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}$$

- Coupon payment is at T_1
- For pricing we need to calculate

$$\begin{aligned} \mathbb{E}^{T_1}[C_1 \mid \mathcal{F}_t] &= \mathbb{E}^{T_1}\left[\left\{ \left[\prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)} \mid \mathcal{F}_t \right] \\ &= \left\{ \mathbb{E}^{T_1}\left[\prod_{i=1}^{k} (1 + L_i \tau_i) \mid \mathcal{F}_t \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)} \end{aligned}$$

How do we handle the compounding term?

Compounding term

$$\prod_{i=1}^{k} (1 + L_i \tau_i)$$

Overnight rate

$$L_i = L(t_{i-1}; t_{i-1}, t_i) = \left[\frac{P(t_{i-1}, t_{i-1})}{P(t_{i-1}, t_i)} - 1 \right] \frac{1}{\tau_1}$$

We get

$$\prod_{i=1}^{k} (1 + L_i \tau_i) = \prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)}$$

We need to calculate the expectation of $\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)}$

Expected compounding factor can be easily calculated

Lemma (Compounding rate)

Consider a compounding coupon period $[T_0, T_1]$ with overnight observation and maturity dates $\{t_0, t_1, \ldots, t_k\}$, $t_0 = T_0$ and $t_k = T_1$. Then

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_{T_0}\right] = \frac{1}{P(T_0, T_1)}.$$

We proof the result via Tower Law of conditional expectation

$$
\begin{aligned}
\mathbb{E}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\,|\,\mathcal{F}_{T_0}\right] &= \mathbb{E}^{T_1}\left[\mathbb{E}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\,|\,\mathcal{F}_{t_{k-2}}\right]\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\;\mathbb{E}^{T_1}\left[\frac{P(t_{k-1},t_{k-1})}{P(t_{k-1},t_k)}\,|\,\mathcal{F}_{t_{k-2}}\right]\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\;\frac{P(t_{k-2},t_{k-1})}{P(t_{k-2},t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-2}\frac{1}{P(t_{i-1},t_i)}\;\frac{1}{P(t_{k-2},t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
\ldots &= \mathbb{E}^{T_1}\left[\frac{1}{P(t_0,t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
&= \frac{1}{P(T_0,T_1)}
\end{aligned}
$$

Expected compounding rate equals Libor rate

- Expected compounding rate as seen at start date T_0 becomes

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_{T_0}\right] = \left[\frac{1}{P(T_0, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(T_0; T_0, T_1)$$

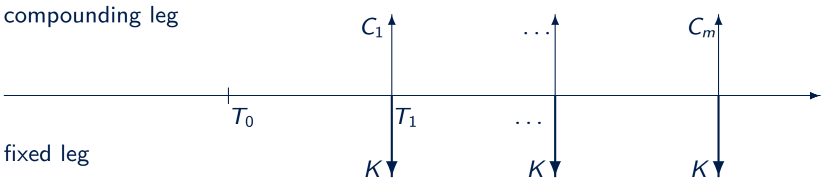
- Consequently, expected compounding rate equals Libor rate for full period
- Moreover, expectations as seen of time-t are

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)}$$

and

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(t; T_0, T_1)$$

Compounding swap pricing is analogous to Vanilla swap pricing



$$
\begin{aligned}
V(t) &= \sum_{j=1}^{m} N \cdot \mathbb{E}^{T_j}\left[C_j \mid \mathcal{F}_t\right] \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j) \\
&= \sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j)
\end{aligned}
$$

Outline

As a summary we give an overview of linear products pricing

Vanilla (Payer) Swap

$$\mathrm{Swap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L^{\delta}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j)}_{\text{float leg}} - \underbrace{\sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i)}_{\text{fixed Leg}}$$

Market Forward Rate Agreement (FRA)

$$\mathrm{FRA}(t) = \underbrace{P(t, T_0)}_{\text{discounting to } T_0} \cdot \underbrace{\left[L^{\delta}(t; T_0, T_0 + \delta) - K\right] \cdot \tau}_{\text{payoff}} \cdot \underbrace{\frac{1}{1 + \tau \cdot L^{\delta}(t; T_0, T_0 + \delta)}}_{\text{discounting from } T_0 \text{ to } T_0 + \delta}$$

Compounding Swap / OIS Swap

$$\mathrm{CompSwap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j)}_{\text{compounding leg}} - \underbrace{\sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j)}_{\text{fixed leg}}$$

Further reading on yield curves, conventions and linear products

- F. Ametrano and M. Bianchetti. Everything you always wanted to know about Multiple Interest Rate Curve Bootstrapping but were afraid to ask (April 2, 2013).
  Available at SSRN: http://ssrn.com/abstract=2219548 or http://dx.doi.org/10.2139/ssrn.2219548, 2013
- M. Henrard. Interest rate instruments and market conventions guide 2.0. Open Gamma Quantitative Research, 2013
- P. Hagan and G. West. Interpolation methods for curve construction. *Applied Mathematical Finance*, 13(2):89–128, 2006

On current discussion of Libor alternatives, e.g.

- M. Henrard. A quant perspective on ibor fallback proposals. `https://ssrn.com/abstract=3226183`, 2018

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