Interest Rate Modelling and Derivative Pricing

Sebastian Schlenkrich

HU Berlin, Department of Mathematics

WS, 2018/19

Part III

Vanilla Option Models

Outline

Vanilla Interest Rate Options

SABR Model for Vanilla Options

Summary Swaption Pricing

Outline

Now we have a first look at the cancellation option

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

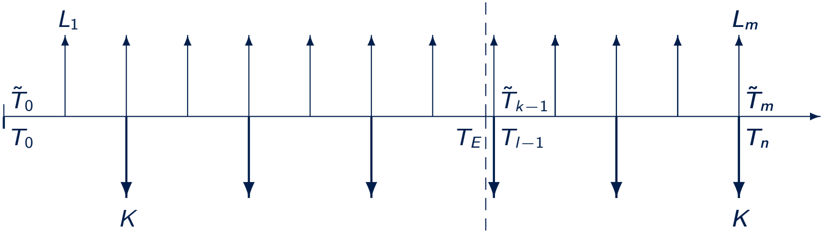
Start date: Oct 30, 2018

End date: Oct 30, 2038

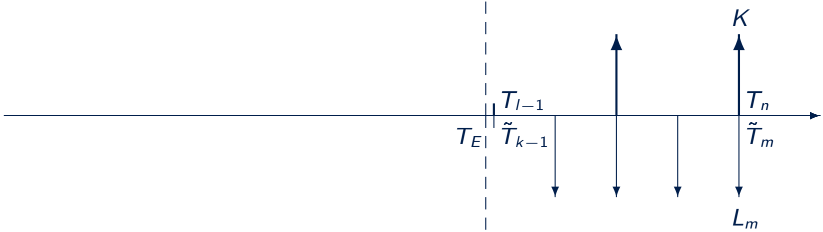
(semi-annually, act/360 day count, modified following, Target calendar)

Bank A may decide to early terminate deal in 10, 11, 12,.. years

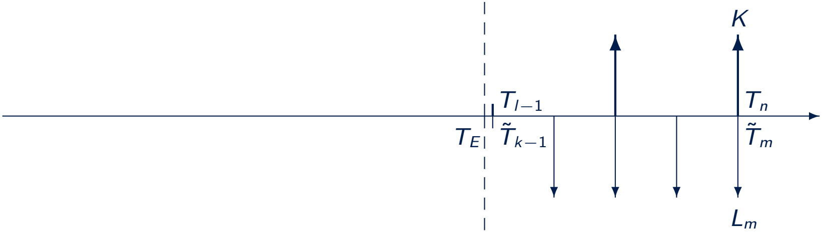
We represent cancellation as entering an opposite deal



[cancelled swap] = [full swap] - [opposite forward starting swap]



Option to cancel is equivalent to option to enter opposite forward starting swap



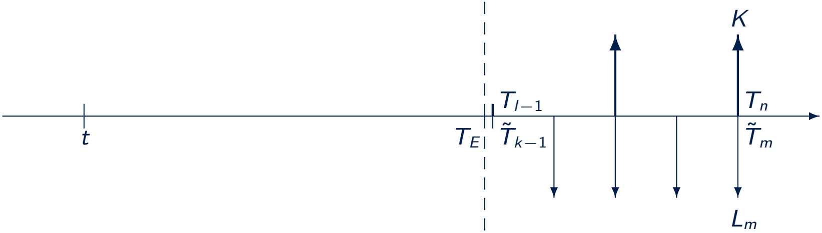
- At option exercise time T_E present value of remaining (opposite) swap is

$$V^{\text{Swap}}(T_E) = \underbrace{K \cdot \sum_{i=l}^{n} \tau_i \cdot P(T_E, T_i)}_{\text{future fixed Leg}} - \underbrace{\sum_{j=k}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta) \cdot \tilde{\tau}_j \cdot P(T_E, \tilde{T}_j)}_{\text{future float leg}}$$

- Option to enter represents the right but not the obligation to enter swap
- Rational market participant will exercise if swap present value is positive, i.e.

$$V^{\text{Option}}(T_E) = \max\left\{V^{\text{Swap}}(T_E), 0\right\}$$

Option can be priced via derivative pricing formula



- Using risk-neutral measure, today's time-t present value of option is

$$V^{\text{Option}}(t) = B(t) \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{V^{\text{Option}}(T_E)}{B(T_E)} \right] = B(t) \cdot \mathbb{E}^{\mathbb{Q}} \left[\frac{\max \left\{ V^{\text{Swap}}(T_E), 0 \right\}}{B(T_E)} \right]$$

- Requires dynamics of future zero bonds $P(T_E, T)$ and numeraire $B(T_E)$

Option pricing requires specific model for interest rate dynamics

Outline

A European Swaption is an option to enter into a swap

Physically Settled European Swaption

A physically settled European Swaption is an option with exercise time T_E. It gives the option holder the right (but not the obligation) to enter into a

- fixed rate payer (or receiver) Vanilla swap with specified
- start time T_0 and end time T_n ($T_E \leq T_0 < T_n$),
- floating rate Libor index payments $L^\delta(T^F_{j-1}, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)$ paid at $\tilde{T}_j$, and
- fixed rate K paid at T_i

All properties are specified at inception of the deal

At exercise time T_E swaption value is

$$V^{\mathrm{Swpt}}(T_E) = \left[\phi\left(\sum_{j=0}^{m} L^\delta(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)\tilde{\tau}_j P(T_E, \tilde{T}_j) - K \sum_{i=0}^{n} \tau_i P(T_E, T_i)\right)\right]^+$$

$\phi = \pm 1$ is payer/receiver swaption, $[\cdot]^+ = \max\{\cdot, 0\}$

A European Swaption is also an option on a swap rate

$$V^{\mathrm{Swpt}}(T_E) = \left[\phi\left(\sum_{j=0}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(T_E, \tilde{T}_j) - K\sum_{i=0}^{n} \tau_i P(T_E, T_i)\right)\right]^+$$

$$= \sum_{i=0}^{n} \tau_i P(T_E, T_i)\left[\phi\left(\frac{\sum_{j=0}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(T_E, \tilde{T}_j)}{\sum_{i=0}^{n} \tau_i P(T_E, T_i)} - K\right)\right]^+$$

Float leg, annuity and swap rate

float leg $\quad Fl(T_E) = \sum_{j=0}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(T_E, \tilde{T}_j)$

annuity $\quad An(T_E) = \sum_{i=0}^{n} \tau_i P(T_E, T_i)$

swap rate $\quad S(T_E) = \frac{\sum_{j=0}^{m} L^{\delta}(T_E, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(T_E, \tilde{T}_j)}{\sum_{i=0}^{n} \tau_i P(T_E, T_i)} = \frac{Fl(T_E)}{An(T_E)}$

$$V^{\mathrm{Swpt}}(T_E) = An(T_E)\cdot\left[\phi\left(S(T_E) - K\right)\right]^+$$

Swap rate is the key quantity for Vanilla option pricing

- Swap rate $S(T_E)$ always needs to be interpreted in the context of its underlying swap with float schedule $\left\{\tilde{T}_j\right\}_j$, Libor index rates $L^\delta(\cdot)$ and fixed schedule $\{T_i\}_i$

- We omit swap details if underlying swap context is clear

- Fixed rate K is the strike rate of the option

- At-the-money strike $K = S(T_E)$ is the fair fixed rate as seen at T_E which prices underlying swap at par (i.e. zero present value)

- Float leg can be considered an asset with time-t value ($t \leq T_E$)

$$Fl(t) = \sum_{j=0}^{m} L^\delta(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)\tilde{\tau}_j P(t, \tilde{T}_j)$$

- Annuity can be considered a positive asset with time-t value ($t \leq T_E$)

$$An(t) = \sum_{i=0}^{n} \tau_i P(t, T_i)$$

Libor rates can be seen as one-period swap rates

- Consider single period swap rate $S(T_E)$ with $m = n = 1$ and $\tau = \tilde{\tau}_1 = \tau_1$, then

$$S(T_E) = \frac{L^\delta(T_E, \tilde{T}_0, \tilde{T}_0 + \delta)\tilde{\tau}_1 P(t, \tilde{T}_0)}{\tau_1 P(t, T_1)} = L^\delta(T_E, \tilde{T}_0, \tilde{T}_0 + \delta)$$

- Option on Libor rate $L^\delta(T_E)$ is called Caplet ($\phi = +1$) or Floorlet ($\phi = -1$) with strike K, pay time T_1 and payoff

$$\tau \cdot \left[\phi\left(L^\delta(T_E, \tilde{T}_0, \tilde{T}_0 + \delta) - K\right)\right]^+$$

- Time-T_E price of caplet/floorlet (i.e. optionlet) is

$$V^{\mathrm{Opl}}(T_E) = \tau \cdot P(T_E, T_1) \cdot \left[\phi\left(L^\delta(T_E, \tilde{T}_0, \tilde{T}_0 + \delta) - K\right)\right]^+$$

- Optionlet payoff is analogous to swaption payoff

Pricing caplets and floorlets is analogous to pricing swaptions

We focus on swaption pricing

Swap rate is a martingale in the annuity measure

Definition (Annuity measure)

Consider a swap rate $S(\cdot)$ with corresponding underlying swap details. The annuity $An(t)$ $(t \leq T_E)$ is a numeraire. The annuity measure is the equivalent martingale measure corresponding to $An(t)$. Expectation under the annuity measure is denoted as $\mathbb{E}^A[\cdot]$.

Theorem (Swap rate martingale property)

The swap rate $S(t)$ is a martingale in the annuity measure and for $t \leq T \leq T_E$

$$S(t) = \mathbb{E}^A\left[S(T) \,|\, \mathcal{F}_t\right] = \frac{\sum_{j=0}^{m} L^\delta(t, \tilde{T}_{j-1}, \tilde{T}_{j-1} + \delta)\tilde{\tau}_j P(t, \tilde{T}_j)}{\sum_{i=0}^{n} \tau_i P(t, T_i)} = \frac{Fl(t)}{An(t)}.$$

Swap rate $S(t)$ is denoted forward swap rate.

Proof.

Annuity measure is well defined via FTAP. The swap rate $S(T) = Fl(T)/An(T)$ is a discounted asset. Thus martingale property follows directly from definition of equivalent martingale measure.

□

Swaption becomes call/put option in annuity measure

$$V^{\text{Swpt}}(T_E) = An(T_E) \cdot [\phi(S(T_E) - K)]^+$$

Derivative pricing formula yields

$$\frac{V^{\text{Swpt}}(t)}{An(t)} = \mathbb{E}^A\left[\frac{V^{\text{Swpt}}(T_E)}{An(T_E)} \,|\, \mathcal{F}_t\right] = \mathbb{E}^A\left[[\phi(S(T_E) - K)]^+ \,|\, \mathcal{F}_t\right]$$

- $[\phi(S(T_E) - K)]^+$ is call ($\phi = +1$) or put ($\phi = -1$) option payoff
- Requires modeling of terminal distribution of $S(T_E)$
- Must comply with martingale property, i.e. $S(t) = \mathbb{E}^A[S(T_E) \,|\, \mathcal{F}_t]$

Put-call-parity for options is an important property

We can decompose a forward payoff into a long call and a short put option

$$S(T_E) - K = [S(T_E) - K]^+ - [K - S(T_E)]^+$$

$$\mathbb{E}^A [S(T_E) - K \mid \mathcal{F}_t] = \mathbb{E}^A \left[[S(T_E) - K]^+ \mid \mathcal{F}_t \right] - \mathbb{E}^A \left[[K - S(T_E)]^+ \mid \mathcal{F}_t \right]$$

$$\underbrace{S(t) - K}_{\text{forward minus strike}} = \underbrace{\mathbb{E}^A \left[[S(T_E) - K]^+ \mid \mathcal{F}_t \right]}_{\text{undiscounted call}} - \underbrace{\mathbb{E}^A \left[[K - S(T_E)]^+ \mid \mathcal{F}_t \right]}_{\text{undiscounted put}}$$

Put-call-parity is a general property and not restricted to Swaptions

General swap rate dynamics are specified by martingale representation theorem

Theorem (Swap rate dynamics)

Consider the swap rate $S(t)$ and a Brownian motion $W(t)$ in the annuity measure. There exists a volatility process $\sigma(t,\omega)$ adapted to the filtration $\mathcal{F}_t$ generated by $W(t)$ such that

$$dS(t) = \sigma(t,\omega)dW(t).$$

Proof.

$S(t)$ is a martingale in annuity measure. Thus, statement follows from martingale representation theorem.

$\square$

- Theorem provides a general framework for all swap rate models
- Swap rate models (in annuity measure) only differ in specification of volatility function $\sigma(t,\omega)$

We will discuss several models and their volatility specification

Outline

Normal model is the most basic swap rate model

Assume a fixed absolute volatility parameter σ and $W(t)$ a scalar Brownian motion in annuity measure

$$dS(t) = \sigma \cdot dW(t)$$

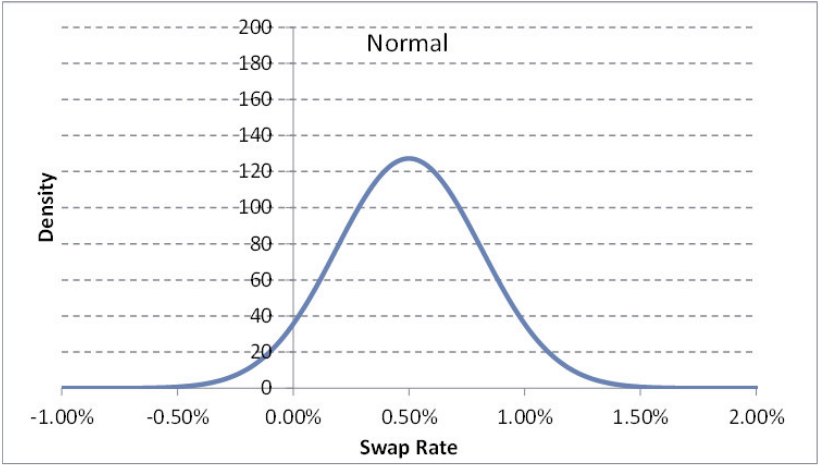
Swap rate $S(T)$ for $t \leq T$ becomes

$$S(T) = S(t) + \sigma \cdot [W(T) - W(t)]$$

Swap rate is normally distributed with

$$S(T) \sim N\left(S(t), \sigma\sqrt{T-t}\right)$$

Normal model terminal distribution of $S(T)$ for $S(0) = 0.50\%$, $T = 1$, $\sigma = 0.31\%$



Option price in normal model is calculated via Bachelier formula

Theorem (Bachelier formula)

Suppose $S(t)$ follows the normal model dynamice

$$dS(t) = \sigma \cdot dW(t).$$

Then the forward Vanilla option price becomes

$$\mathbb{E}^{A}\left[\left[\phi\left(S(T_E) - K\right)\right]^{+} \mid \mathcal{F}_t\right] = \textit{Bachelier}\left(S(t), K, \sigma\sqrt{T - t}, \phi\right)$$

with

$$\textit{Bachelier}(F, K, \nu, \phi) = \nu \cdot \left[\Phi\left(\frac{\phi\left[F - K\right]}{\nu}\right) \cdot \frac{\phi\left[F - K\right]}{\nu} + \Phi'\left(\frac{\phi\left[F - K\right]}{\nu}\right)\right]$$

and $\Phi(\cdot)$ being the cumulated standard normal distribution function.

We derive the Bachelier formula... (1/2)

$$\mathbb{E}^A\left[[S(T_E)-K]^+ \mid \mathcal{F}_t\right] = \int_K^\infty \underbrace{[s-K]}_{\text{payoff}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left\{-\frac{[s-S(t)]^2}{2\sigma^2(T-t)}\right\}}_{\text{density}} ds$$

Substitute $x = [s-S(t)] / \left(\sigma\sqrt{T-t}\right)$, then

$$\mathbb{E}^A[.] = \int_{[K-S(t)]/\left(\sigma\sqrt{T-t}\right)}^\infty \left[\sigma\sqrt{T-t}x + S(t) - K\right] \underbrace{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}}_{\Phi'(x)} dx$$

$$= \sigma\sqrt{T-t}\int_{[K-S(t)]/\left(\sigma\sqrt{T-t}\right)}^\infty \left[x + \frac{S(t)-K}{\sigma\sqrt{T-t}}\right]\Phi'(x)dx$$

Use

$$\int x\Phi'(x)dx = -\Phi'(x)$$

We derive the Bachelier formula... (2/2)

$$\mathbb{E}^A[.] = \sigma\sqrt{T-t}\int_{[K-S(t)]/\left(\sigma\sqrt{T-t}\right)}^{\infty}\left[x + \frac{S(t)-K}{\sigma\sqrt{T-t}}\right]\Phi'(x)dx$$

$$= \sigma\sqrt{T-t}\left[-\Phi'(x) + \frac{S(t)-K}{\sigma\sqrt{T-t}}\Phi(x)\right]_{[K-S(t)]/\left(\sigma\sqrt{T-t}\right)}^{+\infty}$$

$$= \sigma\sqrt{T-t}\left[0 + \Phi'\left(\frac{K-S(t)}{\sigma\sqrt{T-t}}\right) + \frac{S(t)-K}{\sigma\sqrt{T-t}}\left[1 - \Phi\left(\frac{K-S(t)}{\sigma\sqrt{T-t}}\right)\right]\right]$$

$$= \sigma\sqrt{T-t}\left[\Phi'\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \frac{S(t)-K}{\sigma\sqrt{T-t}}\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right]$$

Log-normal model is the classical swap rate model

Assume a fixed relative volatility parameter σ and $W(t)$ a scalar Brownian motion in annuity measure

$$dS(t) = \sigma \cdot S(t) \cdot dW(t)$$

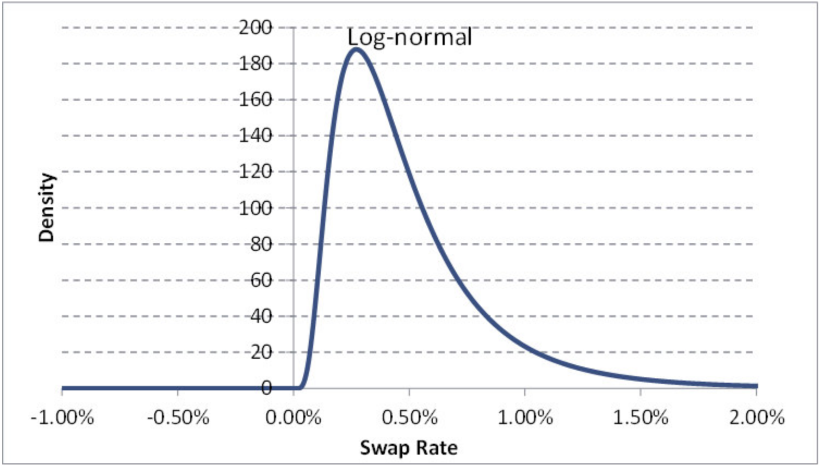
We can substitute $X(t) = \ln(S(t))$, and get with Ito formula

$$dX(t) = -\frac{1}{2}\sigma^2 \cdot dt + \sigma \cdot dW(t)$$

Log-swap rate $\ln(S(T))$ is normally distributed with

$$\ln(S(T)) \sim N\left(-\frac{1}{2}\sigma^2 \cdot (T - t), \sigma \cdot \sqrt{T - t}\right)$$

Log-normal model terminal distribution of $S(T)$ for $S(0) = 0.50\%$, $T = 1$, $\sigma = 63.7\%$



Option price in log-normal model is calculated via Black formula

Theorem (Black formula)

Suppose $S(t)$ follows the log-normal model dynamice

$$dS(t) = \sigma \cdot S(t) \cdot dW(t).$$

Then the forward Vanilla option price becomes

$$\mathbb{E}^A \left[[\phi \left(S(T_E) - K \right)]^+ \mid \mathcal{F}_t \right] = Black\left(S(t), K, \sigma\sqrt{T-t}, \phi \right)$$

with

$$Black\left(F, K, \nu, \phi\right) = \phi \cdot \left[F \cdot \Phi\left(\phi \cdot d_1\right) - K \cdot \Phi\left(\phi \cdot d_2\right) \right],$$
$$d_{1,2} = \frac{\ln\left(F/K\right)}{\nu} \pm \frac{\nu}{2}$$

and $\Phi(\cdot)$ being the cumulated standard normal distribution function.

Proof see exercises.

Shifted log-normal model allows *interpolating* between log normal and normal model

Assume a fixed relative volatility parameter σ, a positive shift parameter λ and a scalar Brownian motion $W(t)$ in annuity measure

$$dS(t) = \sigma \cdot [S(t) + \lambda] \cdot dW(t)$$

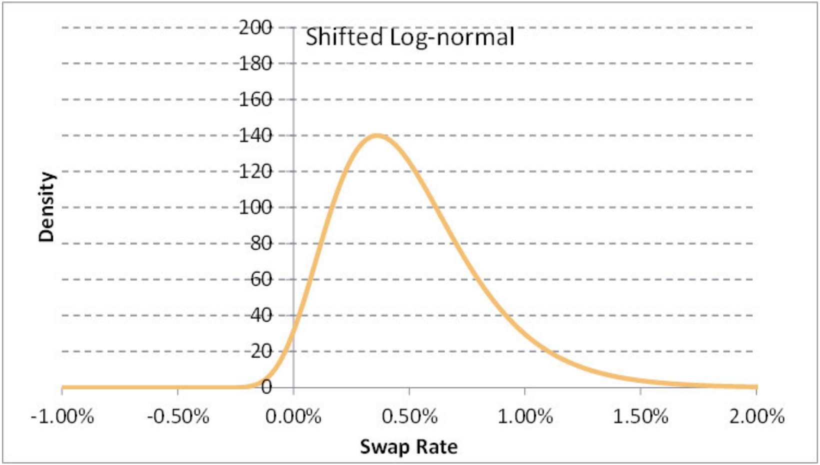
We can substitute $X(t) = \ln(S(t) + \lambda)$, and get with Ito formula

$$dX(t) = -\frac{1}{2}\sigma^2 \cdot dt + \sigma \cdot dW(t)$$

Log of shifted swap rate $\ln(S(T) + \lambda)$ is normally distributed with

$$\ln(S(T) + \lambda) \sim N\left(-\frac{1}{2}\sigma^2 \cdot (T - t), \sigma \cdot \sqrt{T - t}\right)$$

Shifted log-normal model terminal distribution of $S(T)$ for $S(0) = 0.50\%,\ T = 1,\ \lambda = 0.5\%\ \sigma = 31.5\%$



In general option pricing formula in shifted model can be obtain via un-shifted pricing formula

Theorem (Shifted model pricing formula)

Suppose an underlying process $S(t)$ with a Vanilla call option pricing formula $\mathbb{E}\left[(S(T)-K)^+ \mid \mathcal{F}_t\right] = V\left(S(t), K\right)$. For a shift parameter λ and a shifted underlying process $\tilde{S}(t)$ with

$$\tilde{S}(t) = S(t) - \lambda$$

we get the Vanilla call option pricing formula

$$\mathbb{E}\left[\left(\tilde{S}(T)-K\right)^+ \mid \mathcal{F}_t\right] = V\left(\tilde{S}(t)+\lambda, K+\lambda\right).$$

The same result holds for put option.

We prove shifted model pricing formula

Proof.
With $\tilde{S}(t) = S(t) - \lambda$ we get

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{S}(T) - K\right)^{+} \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\left(S(T) - [K + \lambda]\right)^{+} \mid \mathcal{F}_t\right] \\
&= V\left(S(t), K + \lambda\right) \\
&= V\left(\tilde{S}(t) + \lambda, K + \lambda\right)
\end{aligned}$$

□

- Shifted pricing formula result is model-independent
- We will apply it to several model

We apply the previous result to shifted log-normal model

Corollary (Shifted Black formula)

Suppose $S(t)$ follows the log-normal model dynamics

$$d\tilde{S}(t) = \sigma \cdot \left(\tilde{S}(t) + \lambda\right) \cdot dW(t).$$

Then the forward Vanilla option price becomes

$$\mathbb{E}^{A}\left[\left[\phi\left(\tilde{S}(T_E) - K\right)\right]^{+} \mid \mathcal{F}_t\right] = Black\left(\tilde{S}(t) + \lambda, K + \lambda, \sigma\sqrt{T - t}, \phi\right).$$

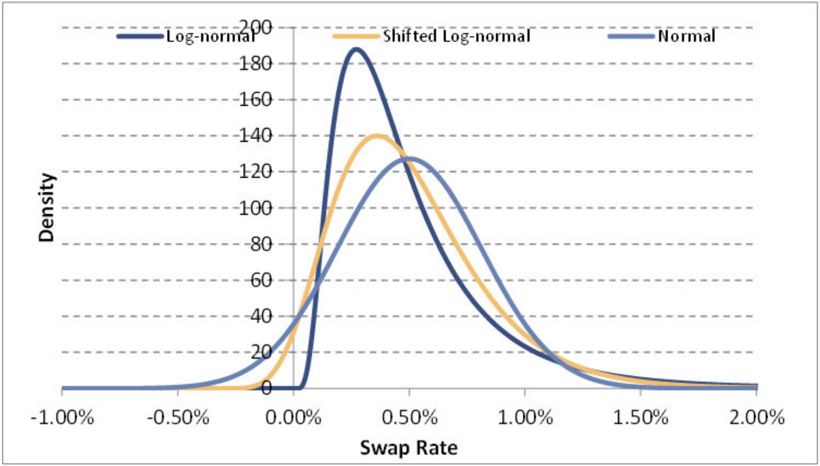
Proof.

Set $S(t) = \tilde{S}(t) + \lambda$. Then $S(T)$ is log-normally distributed and Vanilla options are priced via Black formula. Pricing formula for shifted log-normal model follows from previous theorem.

$\square$

We compare the distribution examples for models calibrated to same forward ATM price

$$\mathbb{E}^A\left[[S(T) - S(t)]^+\right] = 0.125\%,\ S(0) = 0.50\%,\ T = 1,\ \lambda = 0.5\%$$



Outline

Implied Volatilities are a convenient way of representing option prices

Definition (Implied volatility)

Consider a Vanilla call ($\phi = 1$) or put option ($\phi = -1$) on an underlying $S(T)$ with strike K, and time to option expiry $T - t$. Assume that $S(t)$ is a martingale with $S(t) = \mathbb{E}\left[S(T) \,|\, \mathcal{F}_t\right]$. For a given forward Vanilla option price

$$V(K, T-t) = \mathbb{E}\left[\left(\phi\left[S(T) - K\right]\right)^+ \,|\, \mathcal{F}_t\right]$$

we define the

1. implied normal volatility σ_N such that

$$V(K, T-t) = \text{Bachelier}\left(S(t), K, \sigma_N \cdot \sqrt{T-t}, \phi\right)$$

2. implied log-normal volatility

$$V(K, T-t) = \text{Black}\left(S(t), K, \sigma_{LN} \cdot \sqrt{T-t}, \phi\right)$$

3. implied shifted log-normal volatility for a shift parameter λ

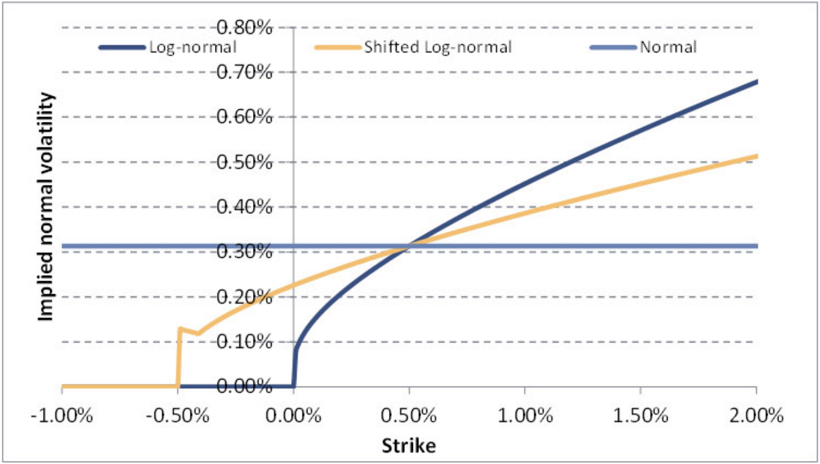
$$V(K, T-t) = \text{Black}\left(S(t) + \lambda, K + \lambda, \sigma_{SLN} \cdot \sqrt{T-t}, \phi\right)$$

We give some remarks on implied volatilities

- Implied (normal/log-normal/shifted-log-normal) volatility is only defined for attainable forward prices $V(\cdot, \cdot)$
- Implied volatility (for swaptions) is independent from notional and annuity
- For a given (arbitrage-free) model, implied volatilities are equal for respective call and put options
- Typically model or market prices are expressed in terms of implied volatilities for comparison

In rates markets prices are often expressed in terms of implied normal volatilities

$$\mathbb{E}^A\left[[S(T)-S(t)]^+\right]=0.125\%,\ S(0)=0.50\%,\ T=1,\ \lambda=0.5\%$$



Market participants quote ATM swaptions and skew

EUR ATM Swaption Straddles - BP Volatilities (Calendar day vols)
Please call +44 (0)20 7532 3080 for further details

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	25Y	30Y
1M Opt	45.3	45.0	48.0	53.8	55.3	61.6	70.1	78.6	85.2	88.7	90.0
2M Opt	38.8	40.9	44.8	48.3	51.4	58.6	67.0	76.3	82.5	84.5	85.5
3M Opt	35.6	37.3	41.7	46.8	50.9	58.3	66.7	75.0	80.5	82.5	84.1
6M Opt	34.9	37.7	42.1	46.9	51.0	59.3	66.3	74.1	78.7	80.1	81.3
9M Opt	35.4	38.0	43.1	47.3	51.5	59.1	66.9	73.8	77.5	78.7	79.0
1Y Opt	37.0	40.3	44.3	48.1	52.4	59.8	67.0	73.2	76.0	77.2	77.4
18M Opt	41.3	44.7	48.0	50.6	55.0	61.6	68.3	72.6	74.8	75.7	76.1
2Y Opt	46.5	49.4	52.6	55.0	58.2	63.9	69.8	73.0	74.2	75.1	75.5
3Y Opt	56.9	58.8	60.6	62.5	64.4	68.3	72.6	73.2	72.9	73.4	73.7
4Y Opt	64.1	65.5	66.0	67.4	68.6	71.1	73.9	72.4	71.5	71.1	71.0
5Y Opt	68.7	69.2	70.0	70.8	71.5	73.0	74.7	71.8	70.2	69.3	69.0
7Y Opt	73.0	73.3	73.6	73.8	74.1	74.5	74.8	70.1	67.6	66.4	66.0
10Y Opt	73.2	73.8	74.1	74.1	73.8	73.8	72.9	67.7	64.9	64.1	63.3
15Y Opt	70.8	71.2	71.1	71.0	70.7	69.9	68.4	62.9	59.1	57.8	56.8
20Y Opt	67.7	68.4	68.0	67.3	66.6	65.5	63.6	58.5	54.3	53.0	51.8
25Y Opt	64.6	64.8	64								
30Y Opt	60.4	60.9	59								

EUR Vega - Normal Vol Skews

	Receivers						Payers				
	-200	-150	-100	-50	-25	ATM	+25	+50	+100	+150	+200
1y2y		22.29	14.02	5.40	1.84	40.72	0.91	4.20	13.83	24.45	
1y5y		0.20	-2.25	-2.44	-1.59	52.79	2.29	5.14	11.97	19.60	
1y10y		0.24	-1.69	-2.09	-1.40	67.86	2.10	4.80	11.53	19.32	
1y20y		13.45	7.57	2.33	0.63	76.97	0.67	2.64	9.62	18.89	
1y30y		7.75	4.34	1.43	0.46	79.14	0.16	1.00	4.59	10.15	
2y2y		11.95	6.40	1.54	0.14	49.98	1.32	3.90	11.32	19.98	
2y5y		-3.21	-3.26	-2.23	-1.28	58.62	1.61	3.52	8.09	13.38	
2y10y		-3.50	-2.97	-1.83	-1.01	70.41	1.21	2.63	6.04	10.10	
2y20y		1.10	0.20	-0.30	-0.28	75.04	0.57	1.44	4.09	7.81	
2y30y		4.86	2.51	0.58	0.07	76.50	0.46	1.48	5.08	10.29	
5y2y		-1.06	-1.41	-1.15	-0.70	69.84	0.95	2.14	5.18	8.91	
5y5y		-3.97	-2.93	-1.73	-0.94	72.02	1.11	2.39	5.42	9.00	
5y10y		-3.73	-2.55	-1.40	-0.74	75.23	0.84	1.80	4.04	6.69	
5y20y		-1.66	-1.12	-0.68	-0.38	70.67	0.49	1.10	2.72	4.85	
5y30y		-1.51	-0.99	-0.61	-0.35	69.54	0.47	1.07	2.69	4.86	
10y2y		-3.45	-2.56	-1.43	-0.75	74.34	0.83	1.74	3.79	6.11	
10y5y		-4.90	-3.28	-1.70	-0.87	74.37	0.92	1.89	4.00	6.33	
10y10y		-3.04	-1.95	-1.03	-0.54	73.36	0.60	1.29	2.91	4.89	
10y20y		-2.31	-1.32	-0.64	-0.33	65.38	0.36	0.78	1.79	3.08	
10y30y		-1.95	-1.17	-0.65	-0.36	63.77	0.46	1.02	2.53	4.54	

EUR market data as of Feb2016

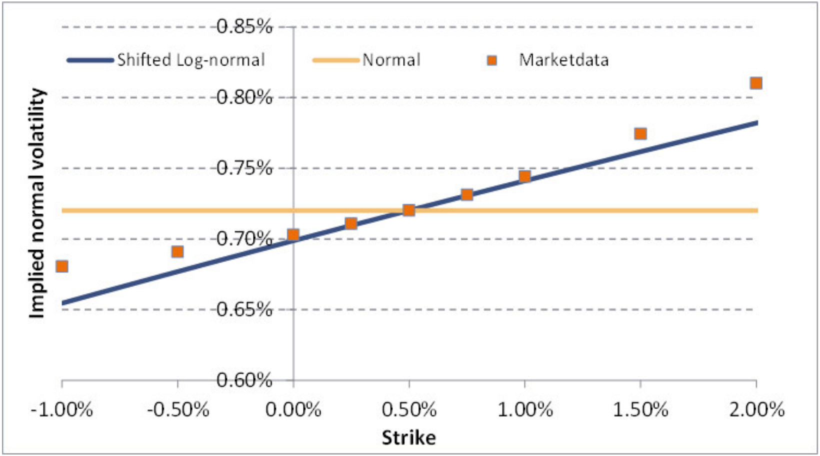
How do the market data compare to our basic swaption pricing models?

- We pick the skew data for 5y (expiry) into 5y (swap term) swaption
- Quoted data: relative strikes and normal volatility spreads in bp

	Receiver					Payer			
	-150	-100	-50	-25	ATM	+25	+50	+100	+150
5y5y	-3.97	-2.93	-1.73	-0.94	72.02	1.11	2.39	5.42	9.00
Vols	68.05	69.09	70.29	71.08	72.02	73.13	74.41	77.44	81.02

- Assume 5y into 5y forward swap rate $S(t)$ at 50bp (roughly corresponds to Feb'16 EUR market data)

We can fit ATM and volatility skew (i.e. slope at ATM) with a shifted log-normal model and 8% shift



However, there is no chance to fit the smile (i.e. curvature at ATM) with a basic model.

In practice Vanilla option pricing is about interpolation

Suppose we want to price a swaption with $7.6y$ expiry, on an $8y$ swap with strike 3.15%

1. Interpolate ATM volatilities in expiry dimension
   - Typically use linear interpolation in variance $\sigma_N^2 (T - t)$
2. Interpolate ATM volatilities in swap term dimension
   - Typically use linear interpolation

This yields interpolated ATM volatility σ_N^{ATM}. Then

3. Calibrate models for available skew market data
   - We will discuss modells with sufficient flexibility
4. Interpolate smile models and combine with ATM volatility
   - This yields a Vanilla model for the smile section $7.6y$ expiry, on an $8y$ swap term
5. Use interpolated model to price swaption with strike 3.15%

Outline

The SABR model was the de-facto market standard for Vanilla interest rate options until the financial crisis 2008

- Stochastic Alpha Beta Rho model is named after (some of) the parameters involved
- Original reference is: P. Hagan, D. Kumar, A. S. Lesniewski and D. E. Woodward: *Managing Smile Risk*. Wilmott Magazine, July 2002, 86-108
- Motivation for SABR model was less smile fit but primarily modelling smile dynamics
  - Smile fit could (in principle) also be realised via local volatility model

    $$dS = \sigma(S) \cdot dW(t)$$

    with sufficiently complex local volatility function $\sigma(S)$
  - We will address smile dynamics later
- We discuss the model based on the original reference

Outline

The SABR model extends log-normal model by local volatility term and stochastic volatility term

Swap rate dynamics in annuity meassure in SABR model are

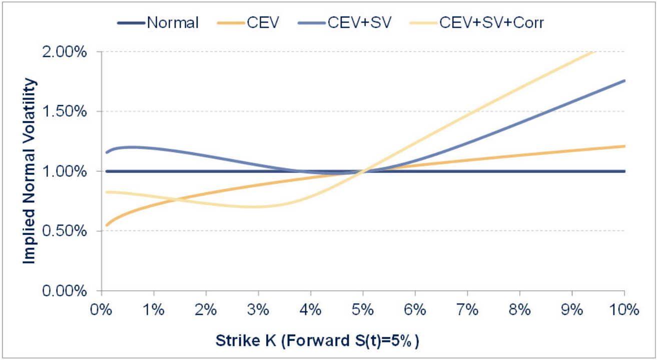
$$
\begin{aligned}
dS(t) &= \hat{\alpha}(t) \cdot S(t)^{\beta} \cdot dW(t) \\
d\hat{\alpha}(t) &= \nu \cdot \hat{\alpha}(t) \cdot dZ(t) \\
\hat{\alpha}(0) &= \alpha \\
dW(t) \cdot dZ(t) &= \rho \cdot dt
\end{aligned}
$$

Initial condition for $S(0)$ is given by today's yield curve

- Elasticity parameter $\beta \in (0, 1)$ (extends local volatility)
- stochastic volatility $\hat{\alpha}(t)$ with volatility-of-volatility $\nu > 0$ and initial condition $\alpha > 0$
- $W(t)$ and $Z(t)$ Brownian motions, correlated via $\rho \in (-1, 1)$

There is no analytic formula for Vanilla options. We analyse classical approximations.

First we give some intuition of the impact of the model parameters on implied volatility smile



	SABR	Normal	CEV	CEV+SV	CEV+SV+Corr
$S(t) = 5\%$	α	1.00%	4.50%	4.05%	4.20%
	β	0	0.5	0.5	0.5
$T = 5y$	ν	0	0	50%	50%
	ρ	0	0	0	70%

Outline

Approximation result is formulated for auxilliary model

Consider *a small* $\varepsilon > 0$ and a model with general local volatility function $C(S)$

$$
\begin{aligned}
dS(t) &= \varepsilon \cdot \alpha(t) \cdot C\left(S(t)\right) \cdot dW(t) \\
d\hat{\alpha}(t) &= \varepsilon \cdot \nu \cdot \hat{\alpha}(t) \cdot dZ(t)
\end{aligned}
$$

- In the original SABR model $C(S)$ is specalised to $C(S) = S^{\beta}$
- Approximation is accurate in the order of $\mathcal{O}\left(\varepsilon^2\right)$

Vanilla option is approximated via Bachelier formula

$$
\mathbb{E}^A\left[\left[\phi\left(S(T_E) - K\right)\right]^+ \mid \mathcal{F}_t\right] = \text{Bachelier}\left(S(t), K, \sigma_N \cdot \sqrt{T_E - t}, \phi\right)
$$

- Black formula implied log-normal volatility approximation σ_{LN} is also derived
  - Actually, log-normal volatility approximation was primarily used

Key aspect for us is approximation of implied normal volatility $\sigma_N = \sigma_N\left(S(t), K, T_E - t\right)$.

We start with the original approximation result

The approximate implied normal volatility is[2]

$$\sigma_N\left(S(t),K,T\right) = \frac{\varepsilon\alpha\left(S(t)-K\right)}{\int_K^{S(t)}\frac{dx}{C(x)}} \cdot \frac{\zeta}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t),K\right)\cdot\varepsilon^2 T\right]$$

with

$$S_{av} = \sqrt{S(t)\cdot K}, \quad \zeta = \frac{\nu}{\alpha}\cdot\frac{S(t)-K}{C(S_{av})}, \quad \hat{\chi}(\zeta) = \ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right),$$

$$I^1\left(S(t),K\right) = \frac{2\gamma_2-{\gamma_1}^2}{24}\alpha^2 C(S_{av})^2 + \frac{\rho\nu\alpha\gamma_1}{4}C(S_{av}) + \frac{2-3\rho^2}{24}\nu^2,$$

$$\gamma_1 = \frac{C'(S_{av})}{C(S_{av})}, \quad \gamma_2 = \frac{C''(S_{av})}{C(S_{av})}$$

There are some difficulties with above formula which we discuss subsequently.

[2] Eg. A.59 in Hagen et.al, 2002.

We adapt the original approximation result

Geometric average $S_{av} = \sqrt{S(t) \cdot K}$

- Inspiried by assumption that rates are more log-normal than normal
- Not applicable if forward rate $S(t)$ or strike K is negative

We use arithmetic average

$$S_{av} = [S(t) + K] / 2$$

- Arithmetic average is also suggested as viable alternative in Hagan et.al., 2002

Approximation for $\zeta = \nu/\alpha \cdot [S(t) - K] / C(S_{av})$

- Eq. (A.57c) in Hagan et.al., 2002 specifies

$$\zeta = \frac{\nu}{\alpha} \int_K^{S(t)} \frac{dx}{C(x)} \approx \frac{\nu}{\alpha} \cdot \frac{S(t) - K}{C(S_{av})}$$

- We use integral representation; consistent with an improved SABR approximation[3]

---

[3] See J. Obloj, *Fine-tune your smile*. Imperial College working paper. 2008

Adapting the ζ term allows simplifying the volatility formula

With

$$\zeta = \frac{\nu}{\alpha} \int_K^{S(t)} \frac{dx}{C(x)}$$

we get

$$\begin{aligned}
\sigma_N\left(S(t), K, T\right) &= \frac{\varepsilon\alpha\left(S(t) - K\right)}{\int_K^{S(t)} \frac{dx}{C(x)}} \cdot \frac{\zeta}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot \varepsilon^2 T\right] \\
&= \frac{\varepsilon\alpha\left(S(t) - K\right)}{\int_K^{S(t)} \frac{dx}{C(x)}} \cdot \frac{\frac{\nu}{\alpha}\int_K^{S(t)} \frac{dx}{C(x)}}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot \varepsilon^2 T\right] \\
&= \nu \cdot \frac{\varepsilon\left(S(t) - K\right)}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot \varepsilon^2 T\right]
\end{aligned}$$

Further, we set $\varepsilon = 1$, i.e. omit small time expansion

This yields normal volatility SABR approximation

SABR model normal volatility $\sigma_N(K, T)$

The approximated implied normal volatility $\sigma_N(K, T)$ in the SABR model with general local volatility function $C(S)$ is given by

$$\sigma_N\left(S(t), K, T\right) = \nu \cdot \frac{S(t) - K}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot T\right]$$

with

$$S_{av} = \frac{S(t) + K}{2}, \quad \zeta = \frac{\nu}{\alpha} \cdot \int_K^{S(t)} \frac{dx}{C(x)}, \quad \hat{\chi}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right),$$

$$I^1(K) = \frac{2\gamma_2 - \gamma_1^2}{24}\alpha^2 C(S_{av})^2 + \frac{\rho\nu\alpha\gamma_1}{4} C(S_{av}) + \frac{2 - 3\rho^2}{24}\nu^2,$$

$$\gamma_1 = \frac{C'(S_{av})}{C(S_{av})}, \quad \gamma_2 = \frac{C''(S_{av})}{C(S_{av})}$$

More concrete, we get with $C(S) = S^\beta$ and $\beta \in (0, 1)$

$$\zeta = \frac{\nu}{\alpha} \cdot \frac{S(t)^{1-\beta} - K^{1-\beta}}{1 - \beta}, \quad \gamma_1 = \frac{\beta}{S_{av}}, \quad \gamma_2 = \frac{\beta\left(\beta - 1\right)}{S_{av}^2}$$

SABR model ATM volatility needs special treatment

- Implementing $\sigma_N\left(S(t), K, T\right) = \nu \cdot \frac{S(t)-K}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1\left(S(t), K\right) \cdot T\right]$ yields division by zero for $K = S(t)$, i.e. $\zeta = 0$
- Use L'Hôpital's rule for $\lim_{S(t)\to K}\left(\sigma_N\left(S(t), K, T\right)\right)$

$$\lim_{S(t)\to K}\left(\frac{S(t)-K}{\hat{\chi}(\zeta)}\right) = \frac{1}{\left[\hat{\chi}'(\zeta)\cdot\frac{d\zeta}{dS}\right]_{S(t)=K}}$$

$$\hat{\chi}'(\zeta) = \frac{1}{\sqrt{\zeta^2 - 2\rho\zeta + 1}}, \quad \hat{\chi}'(0) = 1$$

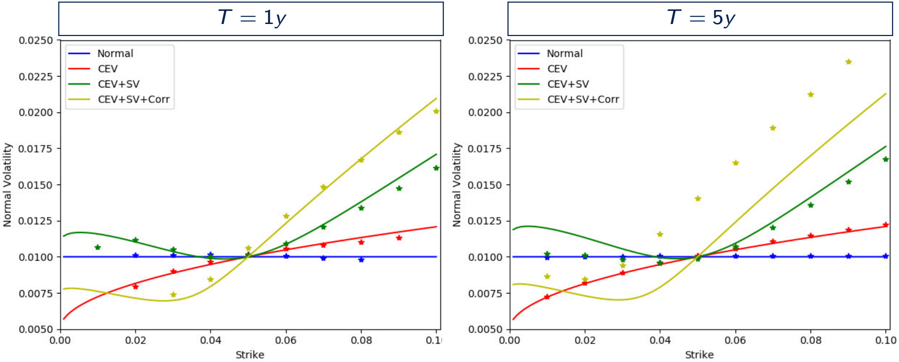
$$\left.\frac{d\zeta}{dS}\right|_{S(t)=K} = \frac{\nu}{\alpha}\cdot\frac{d}{dS}\left[\int_K^{S(t)}\frac{dx}{C(x)}\right]_{S(t)=K} = \frac{\nu}{\alpha C\left(S(t)\right)}$$

- This yields ATM volatility aproximation

$$\sigma_N\left(S(t), T\right) = \alpha \cdot C\left(S(t)\right)\cdot\left[1 + I^1\left(S(t), S(t)\right)\cdot T\right]$$

Outline

We compare analytic approximation with Monte-Carlo simulation



- $S(0) = 5\%$, $\sigma_N^{ATM} = 100bp$, $\beta = 0.5$ (CEV), $\nu = 0.5$ (SV), $\rho = 0.7$ (Corr)
- 10^3 Monte-Carlo paths, 100 time steps per year
- Approximation less accurate for longer maturities, low strikes, non-zero ν and ρ

Poor approximation accuracy is less problematic in practice since SABR model is primarily used as parametric interpolation of implied volatilities.

Terminal distribution of swap rate $S(T)$ can be derived from put prices

Consider the forward put price

$$V^{\text{put}}(K) = \mathbb{E}^A\left[(K - S(T))^+\right] = \int_{-\infty}^{K} (K - s) \cdot p_{S(T)}(s) \cdot ds$$

Here $p_{S(T)}(s)$ is the density of the terminal distribution of $S(T)$.

We get (via Leibniz integral rule)

$$\begin{aligned}
\frac{\partial}{\partial K} V^{\text{put}}(K) &= (K - K) \cdot p_{S(T)}(K) \cdot 1 - \lim_{a \downarrow -\infty} \left[(K - a) \cdot p_{S(T)}(a) \cdot 0\right] \\
&\quad + \int_{-\infty}^{K} \frac{\partial}{\partial K} \left[(K - s) \cdot p_{S(T)}(s)\right] \cdot ds \\
&= \int_{-\infty}^{K} p_{S(T)}(s) \cdot ds \ = \ \mathbb{P}^A \left\{S(T) \leq K\right\}
\end{aligned}$$

and

$$\frac{\partial^2}{\partial K^2} V^{\text{put}}(K) = p_{S(T)}(K)$$

We may also use call prices for density calculation

Recall put-call parity

$$V^{\text{call}}(K) - V^{\text{put}}(K) = \mathbb{E}^A\left[(S(T)-K))^+\right] - \mathbb{E}^A\left[(K - S(T))^+\right] = S(t) - K$$

Differentiation yields

$$\frac{\partial}{\partial K}\left[V^{\text{call}}(K) - V^{\text{put}}(K)\right] = -1$$

and

$$\frac{\partial^2}{\partial K^2}\left[V^{\text{call}}(K) - V^{\text{put}}(K)\right] = 0$$

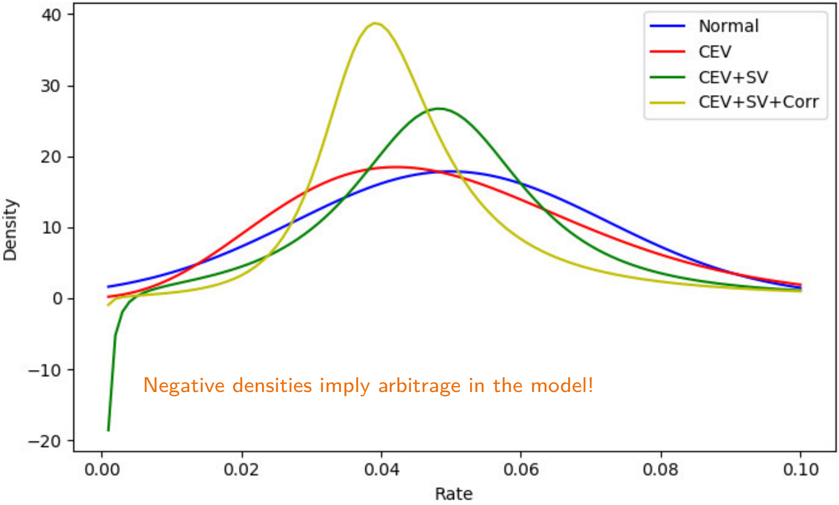
Consequently

$$\frac{\partial}{\partial K} V^{\text{call}}(K) = \frac{\partial}{\partial K} V^{\text{put}}(K) - 1 = \mathbb{P}^A\left\{S(T) \le K\right\} - 1$$

and

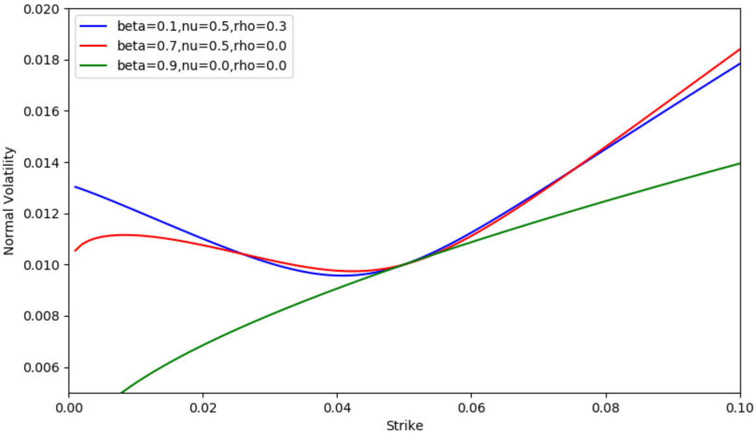
$$\frac{\partial^2}{\partial K^2} V^{\text{call}}(K) = \frac{\partial^2}{\partial K^2} V^{\text{put}}(K) = p_{S(T)}(K)$$

Implied Densities for example models illustrate difficulties of SABR formula for longer expiries and small strikes



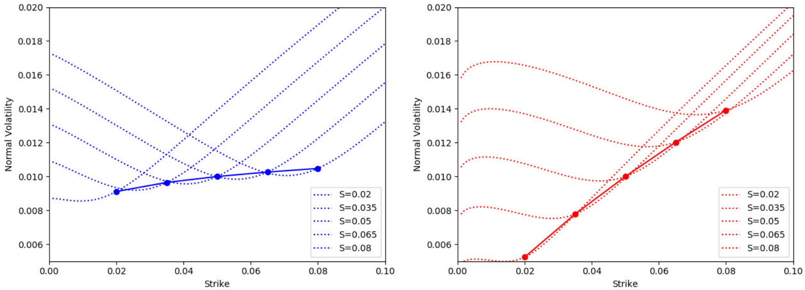
Outline

Static skew can be controlled via β and ρ



- Pure local volatility (i.e. CEV) model does not exhibit curvature
- We can model similar skew/smile with low and high β and adjusted correlation ρ
- What is the difference between both stochastic volatility models?

How does ATM volatility and skew/smile change if forward moves?

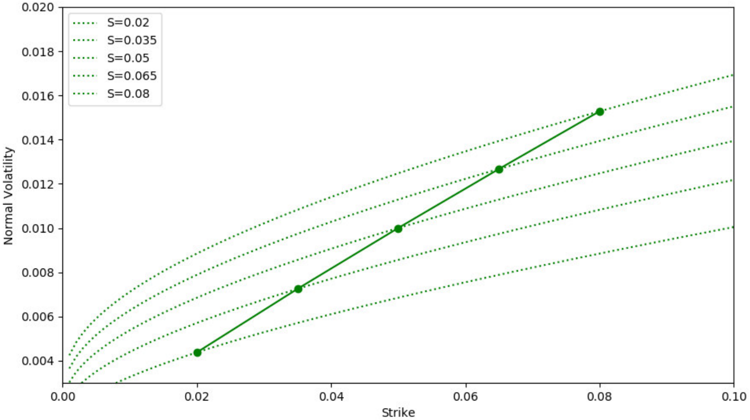


- Low β yields horizontal shift, high β moves smile upwards
- Observation is consistent with expectation about *backbone function* $\sigma_N^{ATM}(S(t))$ (solid lines in graphs)

$$\sigma_N^{ATM}(S(t)) \approx \alpha \cdot C(S(t)) = \alpha S(t)^{\beta}$$

- β also impacts smile on the wings (i.e. low and high strikes)

What is the picture in the pure local volatility model?



- Again, high β moves smile upwards
- Vol shape yields appearance the smile moves left if forward moves right
- Observation is sometimes considered contradictory to market observations

Backbone also impacts sensitivities of the option

Recall e.g. option price

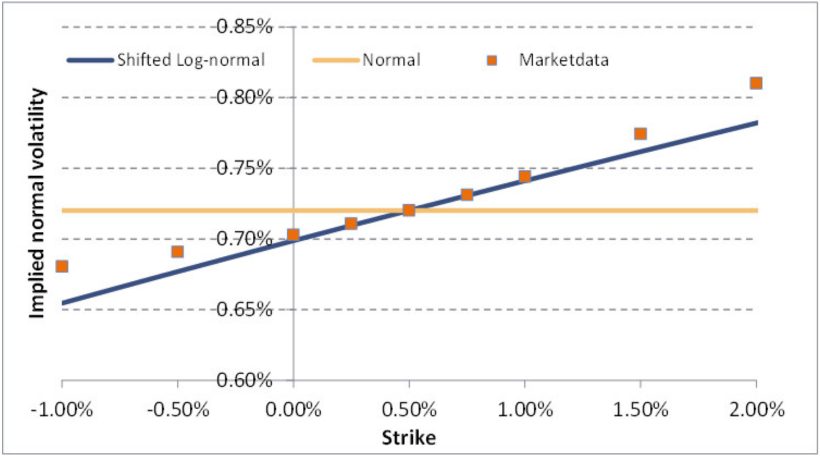
$$V(0) = \text{Bachelier}\left(S(t), K, \sigma_N\left(S(t), K, T_E\right) \cdot \sqrt{T_E}, \phi\right)$$

We get for the Delta sensitivity

$$\begin{aligned}\Delta =& \frac{dV(0)}{dS(0)} \\ =& \underbrace{\frac{\partial}{\partial S}\text{Bachelier}\left(S(t), K, \sigma_N\left(S(t), K, T_E\right) \cdot \sqrt{T_E}, \phi\right)}_{\text{Bachelier-Delta}} + \\ & \underbrace{\frac{\partial}{\partial \sigma}\text{Bachelier}\left(S(t), K, \sigma_N\left(S(t), K, T_E\right) \cdot \sqrt{T_E}, \phi\right)}_{\text{Bachelier-Vega}} \cdot \underbrace{\frac{d\sigma_N\left(S(t), K, T_E\right)}{dS}}_{\text{related to backbone slope}}\end{aligned}$$

Outline

Recall market data example from basic Swaption pricing modells



Model needs to allow negative interest rates. SABR model with $C(S) = S^{\beta}$ does not allow negative rates (unless $\beta = 0$)

Shifted SABR model allows extending the model domain to negative rates

Define $\tilde{S}(t) = S(t) - \lambda$ where $S(t)$ follows standard SABR model. Then

$$
\begin{aligned}
d\tilde{S}(t) = dS(t) &= \alpha(t) \cdot \left[\tilde{S}(t) + \lambda\right]^{\beta} \cdot dW(t) \\
d\hat{\alpha}(t) &= \nu \cdot \hat{\alpha}(t) \cdot dZ(t) \\
\hat{\alpha}(0) &= \alpha \\
dW(t) \cdot dZ(t) &= \rho \cdot dt
\end{aligned}
$$

- Initial condition for $S(0)$ is given by today's yield curve
- Shift parameter $\lambda \geq 0$ extends model domain to $[-\lambda, +\infty)$
- Elasticity parameter $\beta \in (0, 1)$ (extends local volatility)
- stochastic volatility $\hat{\alpha}(t)$ with volatility-of-volatility $\nu > 0$ and initial condition $\alpha > 0$
- $W(t)$ and $Z(t)$ Brownian motions, correlated via $\rho \in (-1, 1)$

We can apply SABR model pricing result to shifted local volatility function $C(S) = [S + \lambda]^\beta$

Vanilla option is approximated via Bachelier formula

$$\mathbb{E}^A \left[\left[\phi \left(\tilde{S}(T_E) - K \right) \right]^+ \mid \mathcal{F}_t \right] = \text{Bachelier}\left(\tilde{S}(t), K, \sigma_N(K, T_E - t) \cdot \sqrt{T_E - t}, \phi \right)$$

and

$$\sigma_N(K, T) = \nu \cdot \frac{\tilde{S}(t) - K}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1(K) \cdot T \right]$$

Details of normal volatility formula need to be adjusted for $C(S) = [S + \lambda]^\beta$ compared to $C(S) = S^\beta$ in original SABR model

Shifted SABR normal volatility approximation is straight forward

Recall general approximation result

$$\sigma_N(K,T) = \nu \cdot \frac{\tilde{S}(t) - K}{\hat{\chi}(\zeta)} \cdot \left[1 + I^1(K) \cdot T\right]$$

with

$$S_{av} = \frac{\tilde{S}(t) + K}{2}, \quad \zeta = \frac{\nu}{\alpha} \cdot \int_K^{\tilde{S}(t)} \frac{dx}{C(x)}, \quad \hat{\chi}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right),$$

$$I^1(K) = \frac{2\gamma_2 - \gamma_1^2}{24}\alpha^2 C(S_{av})^2 + \frac{\rho\nu\alpha\gamma_1}{4} C(S_{av}) + \frac{2 - 3\rho^2}{24}\nu^2,$$

$$\gamma_1 = \frac{C'(S_{av})}{C(S_{av})}, \quad \gamma_2 = \frac{C''(S_{av})}{C(S_{av})}$$

For shifted SABR with $C(S) = [S + \lambda]^\beta$ and $\beta \in (0, 1)$ we get

$$\zeta = \frac{\nu}{\alpha} \cdot \frac{\left[\tilde{S}(t) + \lambda\right]^{1-\beta} - [K + \lambda]^{1-\beta}}{1 - \beta}, \quad \gamma_1 = \frac{\beta}{S_{av} + \lambda}, \quad \gamma_2 = \frac{\beta(\beta - 1)}{(S_{av} + \lambda)^2}$$

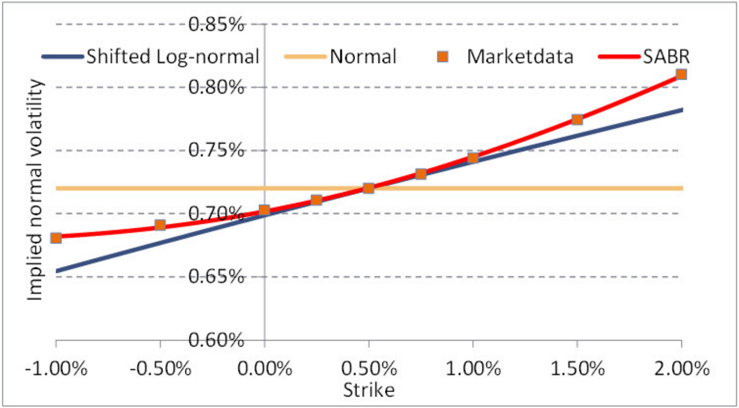
Some care is required when marking λ and β

Linearisation yields

$$\begin{aligned} C(S) &= [S+\lambda]^{\beta} \\ &\approx [S_0+\lambda]^{\beta} + \beta\,[S_0+\lambda]^{\beta-1}\,[S-S_0] \\ &= \beta\,[S_0+\lambda]^{\beta-1}\cdot\left[S+\frac{S_0+\lambda}{\beta}-S_0\right] \end{aligned}$$

- Both λ and β impact volatility skew
- Increasing λ is similar to decreasing β (w.r.t. skew around ATM)
- However, only λ controls domain of modelled rates

Shifted SABR model can match example market data



- $T = 5y$, $S(t) = 0.5\%$
- Shifted SABR: $\lambda = 5\%$, $\alpha = 5.38\%$, $\beta = 0.7$, $\nu = 23.9\%$, $\rho = -2.1\%$

Outline

European Swaption pricing can be summarized as follows

1. Determine underlying swap start date T_0, end date T_n, schedule details and expity date T_E

2. Calculate annuity (as seen today) $An(t) = \sum_{i=0}^{n} \tau_i P(t, T_i)$

3. Calculate forward swap rate (as seen today)
$S(t) = \frac{\sum_{j=0}^{m} L^{\delta}(t, \tilde{T}_{j-1}, \tilde{T}_{j-1}+\delta)\tilde{\tau}_j P(t, \tilde{T}_j)}{\sum_{i=0}^{n} \tau_i P(t, T_i)} = \frac{Fl(t)}{An(t)}$

4. Apply a model for the swap rate to evaluate
$V^{\text{Swpt}}(t) = An(t) \cdot \mathbb{E}^{A}\left[\left[\phi\left(S(T_E) - K\right)\right]^{+} \mid \mathcal{F}_t\right]$; with (shifted) SABR model

   4.1 Determine/calibrate SABR parameters; typically depending on time to expiry $T_E - t$ and time to maturity $T_n - T_0$

   4.2 Calculate approximate normal volatility $\sigma_N\left(S(t), K, T\right)$

   4.3 Use Bachelier's formula

$$V^{\text{Swpt}}(t) = An(t) \cdot \text{Bachelier}\left(S(t), K, \sigma_N \cdot \sqrt{T_E - t}, \phi\right)$$

We illustrate Swaption pricing with QuantLib/Excel...

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 30, 2018

End date: Oct 30, 2038

(annually, 30/360 day count, modified following, Target calendar)

Pays 6-months Euribor floating rate on 100mm EUR

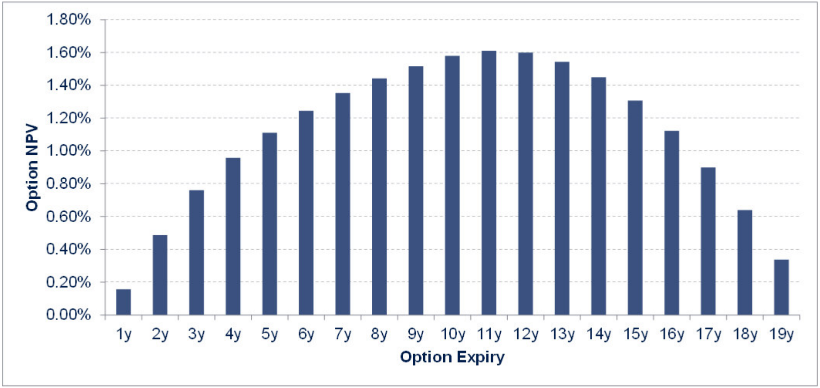
Start date: Oct 30, 2018

End date: Oct 30, 2038

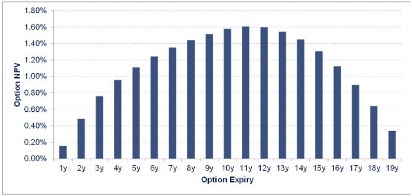
(semi-annually, act/360 day count, modified following, Target calendar)

Bank A may decide to early terminate deal in 10, 11, 12,.. years

We typically see a concave profile of European exercises



Our final swap cancellation option is related to the set of European exercise options



- Denote $V_i^{\text{Swpt}}(t)$ present value of swaption with exercise time $T_i \in \{1y, \ldots, 19y\}$
- Denote $V^{\text{Berm}}(t)$ present value of a *Bermudan* option which allows to
  - choose any exercise time $T_i \in \{1y, \ldots, 19y\}$ and the corresponding option
  - (as long as not exercised) postpone exercise decision on remaining options

It follows

$$V^{\text{Berm}}(t) \geq V_i^{\text{Swpt}}(t) \quad \forall i \quad \Rightarrow \quad V^{\text{Berm}}(t) \geq \underbrace{\max_i \left\{ V_i^{\text{Swpt}}(t) \right\}}_{\text{MaxEuropean}}$$

or

$$V^{\text{Berm}}(t) = \text{MaxEuropean} + \text{SwitchOption}$$

Further reading on Vanilla models and SABR model

- P. Hagan, D. Kumar, A. Lesniewski, and D. Woodward. Managing smile risk.
  Wilmott magazine, September 2002

- M. Beinker and H. Plank. New volatility conventions in negative interest environment.
  d-fine Whitepaper, available at www.d-fine.de, December 2012

- There are a variety of SABR extensions
  - No-arbitrage SABR (P. Hagan et.al.)
  - Free boundary SABR (A. Antonov et.al.)
  - ZABR model (J. Andreasen et.al.)

- Alternative local volatility-based approach

  - S. Schlenkrich. Approximate local volatility model for vanilla rates options.
    `https://ssrn.com/abstract=3150689`, 2018

Contact

Dr. Sebastian Schlenkrich
Office: RUD25, R 1.211
Mail: sebastian.schlenkrich@hu-berlin.de

d-fine GmbH
Mobile: +49-162-263-1525
Mail: sebastian.schlenkrich@d-fine.de