1. Distribution of quadratic forms

(1) $N(0, I)$ case $p(\geq n)$-dimensional.

$X \sim N(0, I)$

$X'AX \sim \chi^2(n)$ if $A^2 = A,\ n = \mathrm{tr}(A)$

<u>Proof</u>

- $A^2 = A$: real symmetric $\Longrightarrow A = P \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} P',$

  $PP' = P'P = I,\ n = \mathrm{tr}(A)$

- Let $Z = P'X \sim N(0, I)$ (Recall the distribution of a linear transformation of MVN random vector.)

- $X'AX = Z' \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} Z = \sum_1^n Z_i^2,\ \sim \chi^2(n)$

  $Z_i \sim N(0, 1)$ independent $(i = 1, \cdots, n)$

(2) Non-singular $N(0, \Sigma)$ case

$X \sim N(0, \Sigma),\ \Sigma :$ non-singular.

$X'AX \sim \chi^2(n)$ if $A\Sigma :$ idempotent, $n = \mathrm{tr}(A\Sigma)$

<u>Proof</u>

- $X \overset{d}{\equiv} \Sigma^{1/2} Z,$

  $Z \sim N(0, I),\ \Sigma^{1/2}\Sigma^{1/2} = \Sigma,\quad \Sigma^{1/2}$: symm. non-singular

- $X'AX = Z'\Sigma^{1/2}A\Sigma^{1/2}Z,\ Z \sim N(0, I)$

- From (1),

  $X'AX \sim \chi^2(n)$ if $\Sigma^{1/2}A\Sigma^{1/2} \cdot \Sigma^{1/2}A\Sigma^{1/2} = \Sigma^{1/2}A\Sigma^{1/2}$,

$$n = \mathrm{tr}(\Sigma^{1/2}A\Sigma^{1/2})$$

$$\iff A\Sigma A = A,\ n = \mathrm{tr}(A\Sigma)$$

$$\iff A\Sigma : \text{idempotent},\ n = \mathrm{tr}(A\Sigma)$$

2. Independence of Quadratic Forms

(1) $N(0, I)$ case

$X \sim N(0, I)$.

$X'AX$ and $X'BX$ are independent $\iff AB = 0$

Proof (sketch)

- $X'AX, X'BX$: independent

  $\iff \mathrm{mgf}_{X'AX,X'BX}(t_1, t_2)$

$$= \mathrm{mgf}_{X'AX}(t_1) \cdot \mathrm{mgf}_{X'BX}(t_2) \quad \text{for } |t_1| < \epsilon \ \& \ |t_2| < \epsilon$$

- $\mathrm{mgf}_{X'AX,X'BX}(t_1, t_2)$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{\frac{p}{2}} \exp\left(t_1 x'Ax + t_2 x'Bx - \frac{1}{2}x'x\right) dx_1 \cdots dx_p$$

$$= (\det(I - 2t_1A - 2t_2B))^{-\frac{1}{2}}$$

- $\mathrm{mgf}_{X'AX}(t_1) = (\det(I - 2t_1A))^{-\frac{1}{2}}$

  $\mathrm{mgf}_{X'BX}(t_2) = (\det(I - 2t_1B))^{-\frac{1}{2}}$

- $(\det(I - 2t_1A - 2t_2B))^{-\frac{1}{2}} = (\det(I - 2t_1A))^{-\frac{1}{2}}(\det(I - 2t_1B))^{-\frac{1}{2}}$

  $\iff \det(I - 2t_1A - 2t_2B) = \det(I - 2t_1A - 2t_2B + 4t_1t_2AB)$

(2) Non-singular $N(0, \Sigma)$ case

$X \sim N(0, \Sigma)$ with Σ : non-singular.

$X'AX, X'BX$: independent $\iff A\Sigma B = 0$

Idea of Proof

- $X \overset{d}{\equiv} \Sigma^{1/2} Z$, $Z \sim N(0, I)$

  $\Sigma^{1/2}$: symmetric non-singular $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$

- $$\begin{cases} X'AX = Z'\Sigma^{1/2} A \Sigma^{1/2} Z \ , \ Z \sim N(0, I) \ , \quad \eta = \Sigma^{-1/2}\mu \\ X'BX = Z'\Sigma^{1/2} B \Sigma^{1/2} Z, \end{cases}$$

- $X'AX, X'BX$: indep $\Longleftrightarrow \Sigma^{1/2} A \Sigma^{1/2} \Sigma^{1/2} B \Sigma^{1/2} = 0$ by (1)

  $\Longleftrightarrow A\Sigma B = 0$ ($\Sigma^{1/2}$: non-singular)

3. Independence between quadratic and linear forms

(1) $N(0, I)$ case : $X \sim N(0, I)$.

$X'AX,\ BX$: independent $\Longleftrightarrow AB' = 0$

Idea of Proof

$X'AX, BX$: independent

$$\begin{aligned}
&\Longleftrightarrow \quad X'AX, X'B'BX : \text{independent},\ X \sim N(0, I) \\
&\Longleftrightarrow \quad AB'B = 0 \quad \text{by 2-(1)} \\
&\Longleftrightarrow \quad AB'BA' = 0 \\
&\Longleftrightarrow \quad AB' = 0 \quad (\because CC' = 0 \Leftrightarrow C = 0)
\end{aligned}$$

(2) Non-singular $N(0, \Sigma)$ case : $X \sim N(0, \Sigma)$, Σ : non-singular.

$X'AX, BX$: independent $\Longleftrightarrow A\Sigma B' = 0$

Idea of Proof

- $X \overset{d}{\equiv} \Sigma^{1/2} Z$, $Z \sim N(0, I)$,

  $\Sigma^{1/2}$: symmetric non-singular, $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$

- $$\left.\begin{aligned} X'AX &= Z'\Sigma^{1/2} A \Sigma^{1/2} Z \\ BX &= B\Sigma^{1/2} Z \end{aligned}\right\} \text{independent},\quad Z \sim N(\Sigma^{-1/2}\mu, I)$$

  $\Longleftrightarrow \Sigma^{1/2} A \Sigma^{1/2} (B\Sigma^{1/2})' = 0$ by (1)

  $\Longleftrightarrow A\Sigma B' = 0$