Exercises

(p.140) Determine whether the given matrix is invertible, by finding its rank.

7. $\begin{pmatrix} 0 & -9 & -9 & 2 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{pmatrix}$

9. $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix}$

(p.141-142) Answer the followings.

11. Mark each of the following True or False.
    a. The number of independent row vectors in a matrix is the same as the number of independent column vectors.
    b. If H is a row-echelon form of a matrix A, then the nonzero column vectors in H form a basis for the column space of A.
    c. If H is a row-echelon form of a matrix A, then the nonzero row vectors in H form a basis for the row space of A.
    d. If an $n \times n$ matrix A, is invertible then $rank(A) = n$.
    e. For every matrix A, we have $rank(A) > 0$.
    f. For positive integers m and n, the rank of an $m \times n$ matrix might be any number from 0 to the minimum of m and n.
    g. For all positive integers m and n the nullity of an $m \times n$ matrix might be any number from 0 to the minimum of m and n.
    h. For all positive integers m and n the nullity of an $m \times n$ matrix might be any number from 0 to m.
    i. For all positive integers m and n the nullity of an $m \times n$ matrix might be any number from 0 to n.
    j. For all positive integers m and n with $m \geq n$ the nullity of an $m \times n$ matrix might be any number from 0 to n.
12. Prove that, if A is a square matrix, the nullity of A is the same as the nullity of A^T.

(p152) Assume that T is a linear transformation. Answer the followings.

5. If $T([1,0]) = [3,-1]$ and $T([0,1]) = [-2,5]$, find $T([4,-6])$.
7. If $T([1,0,0]) = [3,1,2]$, $T([0,1,0]) = [2,-1,4]$, and $T([0,0,1]) = [6,0,1]$, find $T([2,-5,1])$.
9. If $T([-1,2]) = [1,0,0]$ and $T([2,1]) = [0,1,2]$, find $T([0,10])$.
11. If $T([1,2,-3]) = [1,0,4,2]$, $T([3,5,2]) = [-8,3,0,1]$, and $T([-2,-3,-4]) = [0,2,-1,0]$, find $T([5,-1,4])$.

(p153) The given formula defines a linear transformation. Give its standard matrix representation.

13. $T([x_1, x_2]) = [x_1 + x_2, x_1 - 3x_2]$
18. $T([x_1, x_2, x_3]) = x_1 + x_2 + x_3$

(p153) Answer the followings.

19. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by T($[x_1, x_2]$) = $[2x_1 + x_2, x_1, x_1 - x_2]$ and T': $\mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1, x_2, x_3])$=$[x_1 - x_2 + x_3, x_1 + x_2]$, find the standard matrix representation $T' \circ T$ that carries $\mathbb{R}^2$ onto $\mathbb{R}^2$. Find a formula for $T' \circ T([x_1, x_2])$
20. Referring to Exercise 19, find the standard matrix representation for the linear transformation $T \circ T'$ that carries $\mathbb{R}^3$ into $\mathbb{R}^3$. Find a formula for $T \circ T'([x_1, x_2, x_3])$

(p.153) Determine whether the indicated linear transformation T is invertible. If it is, find a formula for T^{-1} in row notation. If is not, explain why it is not.

21. $T([x_1, x_2]) = [x_1 + x_2, x_1 - 3x_2]$
22. $T([x_1, x_2]) = [2x_1 - x_2, x_1 + x_2, x_1 + 3x_2]$
23. $T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, x_1 + x_2, x_1]$

(p.153) Answer the followings.

29. Mark each of the following True or False.
    a. Every linear Transformation is a function.
    b. Every function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is a linear Transformation.
    c. Composition of linear Transformations corresponds to multiplication of their standard matrix representations.
    d. Function composition is associative.
    e. An invertible linear Transformation mapping $\mathbb{R}^n$ into itself has a unique inverse.
    f. The same matrix may be the standard matrix representation for several different linear Transformation.
    g. A linear Transformation having an $m \times n$ matrix as standard matrix representation maps $\mathbb{R}^n$ into $\mathbb{R}^m$.
    h. If T and T' are different linear Transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for some standard basis vector $\mathbf{e}_i$ of $\mathbb{R}^n$.
    i. If T and T' are different linear Transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for all standard basis vector $\mathbf{e}_i$ of $\mathbb{R}^n$.
    j. If $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear Transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1, 2, \cdots, n$.
33. Let A be an $m \times n$ matrix with row-echelon form H, and let V be the row space of A (and thus of H). Let $W_k = sp(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_k)$ be the subspace of $\mathbb{R}^n$ generated by the first k rows of the $n \times n$ identity matrix. Consider $T_k : V \to W_k$ defined by $T_k([x_1, x_2, \cdots, x_n]) = [x_1, x_2, \cdots, x_k, 0, \cdots, 0]$.

a. Show that T_k is a linear transformation of V into W_k and that $T_k(V) = \{T_k(\mathbf{v}) : \mathbf{v} \in V\}$ is a subspace of W_k.

b. If $T_k(V)$ has dimension d_k, show that for each $j < n$, we have either $d_{j+1} = d_j$ or $d_{j+1} = d_j + 1$.

c. Assume that A has four columns. Referring to part (b), suppose that $d_1 = d_2 = 1$ and $d_3 = d_4 = 2$. Find the number of pivots in H, and give the location of each.

d. Repeat part (c) for the case where A has six columns and $d_1 = 1$, $d_2 = d_3 = d_4 = 2$ and $d_5 = d_6 = 3$.

e. Argue that, for any matrix A, the number of pivots and the location of each pivot in any row-echelon form of A is always the same.

f. Show that the reduced row-echelon form of A is unique. [Hint: Consider the nature of the basis for the row space of A given by the nonzero rows of H.]

(p. 165) Answer the followings.

2. Give the standard matrix representation of the rotation of the plane counterclockwise about the origin through an angle of

   a. 45^o
   b. 90^o
   c. 135^o

8. show that the linear Transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ affects the plane $\mathbb{R}^2$ as follows.

   a. A vertical expansion if $r >1$.
   b. A vertical contraction if $0 < r < 1$.
   c. A vertical expansion followed by a reflection in the x-axis if $r < -1$.
   d. A vertical contraction follows by a reflection in the x-axis if $-1 < r < 0$.

(p.165) (optional) Answer the followings.

1. Explain why the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, where $A = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$, has the line $y = 2x$ as range, but is not the projection of $\mathbb{R}^2$ onto that line.