Multivariate Normal Distribution

Mean and covariance of random vector　Cov(X, Y) = E(X−μX)(Y−μY)ᵀ 일때 성립

(1) $A,\ B$: matrices of non-random constants, X : r.vector

linear 해서 Σ 와 순서바꾸기 가능 $E(AXB) = A(EX)B$

(2) $A,\ B,\ b$: constants, $X,\ Y$: r.vector

$$\begin{cases} E(AX+b) = AE(X) + b \\ \mathrm{Cov}(X+b,\ Y) = \mathrm{Cov}(X,\ Y) \\ \mathrm{Cov}(AX,\ BY) = A\,\mathrm{Cov}(X,\ Y)B^{\mathsf{T}} \\ \underset{\text{Cov}(AX,\,AX)}{\underset{\shortparallel}{\mathrm{Cov}(AX)}} = A\,\mathrm{Cov}(X)A^{\mathsf{T}} \end{cases}$$

Non-singular linear transformation of $N_k(0, I)$ r.vector

표준정규분포　독립　k차원 정규분포　서로 독립

$$\left\{ \begin{array}{l} Z_1, \cdots, Z_k \ : \text{i.i.d. } N(0,\ 1)\ ,\ Z = (Z_1, \cdots, Z_k)' \\ X = AZ + \mu\ ,\ A : \text{non-singular} \end{array} \right\} \quad \begin{array}{l} Z \sim N_k(\,0\,,\ \underline{I}\,) \\ \Leftrightarrow Z_i \overset{\text{iid}}{\sim} N(0,\ 1) \end{array}$$

$$\Longrightarrow \begin{cases} \mathrm{pdf}_X(x) = |2\pi \textstyle\sum|^{-\frac{1}{2}} \exp\left(-\dfrac{1}{2}\,(x-\mu)' \textstyle\sum^{-1}(x-\mu)\right),\ x \in R^k \\ \mathrm{mgf}_X(t) = \exp\left(\mu'\,t + \dfrac{1}{2}\,t\,'\textstyle\sum t\right),\ t \in R^k \end{cases}$$

where $\sum = \mathrm{Cov}(X) = AA',\ \ \mu = EX$

$(\because)\ \ z \longrightarrow Az + \mu = h(z)$ 1-1 from R^k onto R^k $\ (A : \text{non-singular})$

안봐도 ㄱㅊ

$$\begin{aligned} \mathrm{pdf}_X(x) &= \mathrm{pdf}_Z(z)\ \left|\det\left(\frac{\partial z}{\partial x}\right)\right|\ ,\quad x = Az + \mu \\ &= |\det(A^{-1})|\ (2\pi)^{-\frac{k}{2}}\ \exp\left(-\frac{1}{2}\,(x-\mu)'(AA')^{-1}(x-\mu)\right) \\ &= |\det(2\pi \textstyle\sum)|^{-\frac{1}{2}}\ \exp\left(-\frac{1}{2}\,(x-\mu)' \textstyle\sum^{-1}(x-\mu)\right) \end{aligned}$$

$$\begin{aligned} \mathrm{mgf}_X(t) &= E\left\{\exp(t'(AZ+\mu))\right\} \\ &= \mathrm{mgf}_Z(A't)\ \exp(\mu' t) \end{aligned}$$

event space

* 확률공간 ⊢ 사건공간 합친 것.

* $X : \Omega \to \mathbb{R}$: 확률변수 (사건 공간 – 실수)
$X : \Omega \to \mathbb{R}^k$: 확률벡터 (사건 공간 – 벡터)

* $Cov(X) = E(X - \underbrace{\mu_x}_{=EX})(X-\mu_x)^T$ 자기자신과의 공분산

$$= E\begin{pmatrix} X_1-\mu_1 \\ \vdots \\ X_t-\mu_t \end{pmatrix}(X_1-\mu_1, \cdots, X_t-\mu_t) = E\begin{pmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & & (X_t-\mu_t)^2 \end{pmatrix} = \begin{pmatrix} E(X_1-\mu_1)^2 & & \\ & \ddots & \\ & & E(X_t-\mu_t)^2 \end{pmatrix}$$

분산
2개 entry간 공분산

→ symmetric
→ positive definite

* $cov(AX, BY) = E(AX - A\mu_X)(BY - B\mu_Y)^T$
$= EA(X-\mu_X)\{B(Y-\mu_Y)\}^T$
$= EA(X-\mu_X)(Y-\mu_Y)^T B^T$

* 서로 독립이면 Joint 분포 쉽게 구할 수 있음. –각 density의 곱으로 표현.

* 모든 정규분포 $\xleftrightarrow{\text{적당한 trans}}$ 표준정규분포.

$X = AZ + \mu$ (A 역할이 적당한 trans

$E(X) = AE(Z) + \mu = \mu$

$cov(X) = cov(AZ+\mu, AZ+\mu) = cov(AZ, AZ) = A\underbrace{cov(Z)}_{I}A^T = AA^T = \Sigma$

symmetric, positive definite
→ 항상분해가능
$= PDP^T = PD^{\frac{1}{2}}D^{\frac{1}{2}}P^T$
(by Spectral 정리)

안봐도_7조

$$= \exp\left(\mu' t + \frac{1}{2}\,(A't)'(A't)\right)$$

$$= \exp\left(\mu'\, t + \frac{1}{2} t\,' \textstyle\sum t\right)$$

<u>Remark</u>

It says $\displaystyle\int \exp\left(-\frac{1}{2}\,z'Bz\right)dz = |2\pi B^{-1}|^{\frac{1}{2}}$, $\;B$: p.d.

<u>Definition of $N_k(\mu, \sum)$</u>

$X \sim N_k(\mu, \sum)$, $\;\sum$: symmetric, n.n.d.

$\Longleftrightarrow$ (i) $X \overset{d}{\equiv} AZ + \mu$ with $Z \sim N_\ell(0, I)$, $\; AA' = \sum \;\; (A : k \times \ell)$

$$\begin{pmatrix} A \text{ can be taken as } \sum^{\frac{1}{2}} \;:\; \text{real symm.,} \quad \sum^{\frac{1}{2}} \sum^{\frac{1}{2}} = \sum \\ A \text{ can be taken as full column rank } (\ell : \text{rank of } \sum) \\ \ell \text{ may not be equal to } k \end{pmatrix}$$

$\Longleftrightarrow$ (ii) $\displaystyle \mathrm{mgf}_X(t) = \exp\left(\mu' t + \frac{1}{2}\,t' \textstyle\sum t\right)$, $\;\sum = AA'$

$\Longleftrightarrow$ (iii) $\displaystyle \mathrm{cgf}_X(t) = \left(\mu' t + \frac{1}{2}\,t' \textstyle\sum t\right)$, $\;\sum = AA'$

$\Longleftrightarrow$ (iv) $c'X \;\sim\; N_1\left(c'\mu,\; c' \sum c\right)$ $\quad$ for all c

$\Longleftrightarrow$ (v) (When $\sum$ is non-singular)

$$\mathrm{pdf}_X(x) = |2\pi \textstyle\sum|^{-\frac{1}{2}} \exp\left\{-\dfrac{1}{2}\,(x-\mu)' \textstyle\sum^{-1}(x-\mu)\right\},\; x \in R^k$$

$(\because)$ (i)$\Leftrightarrow$(ii)

$$\mathrm{mgf}_X(t) = \mathrm{mgf}_Z(A't)\,\exp\,(\mu' t)$$

$$= \exp\left(\mu' t + \frac{1}{2}\,t' \textstyle\sum t\right)$$

(ii)$\Leftrightarrow$(iv)

$$\mathrm{mgf}_X(t) = \exp\left(\mu' t + \frac{1}{2} t' \sum t\right) \quad \forall t \in \mathbb{R}^k$$

$$\Longleftrightarrow \mathrm{mgf}_{c'X}(u) = \mathrm{mgf}_X(cu)$$

$$= \exp\left\{(c'\mu)u + \left(\frac{1}{2}c'\sum c\right)u^2\right\} \quad \forall u \in \mathbb{R}^1,\ \forall c \in \mathbb{R}^k$$

$$\Longleftrightarrow c'X \sim N_1(c'\mu, c'\sum c) \quad \forall c \in \mathbb{R}^k$$

<u>Properties of MVN</u> 다차원 정규분포의 성질 (앞으로 계속 사용)

(1) (Linear transformation preservation) ① 어떤 확률벡터가 k차원 정규분포 따르면 임의의 선형변환도 정규분포 따름

$X \sim N_k(\mu, \sum) \Longrightarrow AX + b \sim N_m(A\mu + b,\ A\sum A'),\ A : m \times k$ └ 곱셈이 정의되기 위해

(2) (Independence among components)

If $X \sim N_k(\cdot,\cdot)$ & $X = ({X_1}',\ {X_2}')'$, then ★ 두 확률변수 독립 $\substack{\xrightarrow{\;O\;} \\ \xleftarrow{\;\times\;}}$ COV = 0

$X_1,\ X_2$: indep. $\Longleftrightarrow$ Cov$(X_1,\ X_2) = 0$ (될때) 정규분포 따를 때

(3) ★ (Independence between linear transforms)

If $X \sim N_k(\cdot,\cdot)$, then

$AX + c,\ BX + d$: indep. e)

$\Longleftrightarrow$ Cov$(AX + c\ ,\ BX + d) \ = \ A\sum B^{\mathsf{T}} = 0$

계산에 익숙해지기

(4) ★★ (Conditional distribution in non-singular case)

$$X_{2|X_1=x_1} \sim N(\ \mu_2 + \textstyle\sum_{21}\sum_{11}^{-1}(x_1 - \mu_1)\ ,\ \sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12}\ )$$

$$\textstyle\sum_{22\cdot 1} \equiv \sum_{22} - \sum_{21}\sum_{11}^{-1}\sum_{12}$$

or equivalently

$$X_2 - \mu_2 - \textstyle\sum_{21}\sum_{11}^{-1}(X_1 - \mu_1) \sim N(0,\ \sum_{22\cdot 1})\ , \quad \text{indep. of } X_1$$

$$(\because) \begin{pmatrix} I & 0 \\ -\sum_{21}\sum_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix}$$

$$= \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 - \sum_{21}\sum_{11}^{-1}(X_1 - \mu_1) \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -\sum_{21}\sum_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix} \begin{pmatrix} I & -\sum_{11}^{-1}\sum_{12} \\ 0 & I \end{pmatrix}$$

$$
= \begin{pmatrix} \sum_{11} & 0 \\ 0 & \sum_{22\cdot1} \end{pmatrix}
$$

<u>Remark</u>

$$
\det \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix} = \det(\textstyle\sum_{11}) \cdot \det(\textstyle\sum_{22\cdot1})
$$

$$
\begin{pmatrix} Z_1{}', & Z_2{}' \end{pmatrix} \begin{pmatrix} \sum_{11} & \sum_{12} \\ \sum_{21} & \sum_{22} \end{pmatrix}^{-1} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}
$$

$$
\equiv Z_1{}' \textstyle\sum_{11}^{-1} Z_1 + (Z_2 - \sum_{21} \sum_{11}^{-1} Z_1)\,' \sum_{22\cdot1}^{-1} (Z_2 - \sum_{21} \sum_{11}^{-1} Z_1)
$$

해석 if stationary gaussian process

→ 과거 알면 미래 예측 $\underbrace{\text{조건부 확률로}}_{\text{과거들의 선형함수}}$ 할수 있음

계산에 익숙해지기

(4) (Conditional distribution in non-singular case)

$Cov(X_2, X_1)$, if 독립 → X_1에 상관없이 평균 μ_1 (0으로 뒤에 사라지니까)

독립X → x_1에 비례하게 μ_1 변함 → 즉, X_1의 선형변환

$$
X_{2|X_1=x_1} \sim N(\, \mu_2 + \textstyle\sum_{21} \sum_{11}^{-1} (x_1 - \mu_1) \,,\ \sum_{22} - \sum_{21} \underbrace{\sum_{11}^{-1}}_{(X^T X)^{-1}} \sum_{12} \,)
$$

SSE 와 비슷, 조건부 분산

$$
\textstyle\sum_{22\cdot1} \equiv \sum_{22} - \sum_{21} \sum_{11}^{-1} \sum_{12}
$$

x_1이 X_2에 미치는 영향 (ACF. AVCF 로도 계산가능)

(4) $X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \\ X_{k+1} \\ \vdots \\ X_t \end{pmatrix}$ $\begin{matrix} \to X_1 \\ \\ \to X_2 \end{matrix}$

* 조건부 기댓값 (정규분포에 한해)

$X_2 | X_1 = x_1$ (X2가 X_1 이 특정한 값으로 주어졌을 때 어떤 분포를 따르는지)

→ 똑같이 정규분포 따르는데 평균이 바뀜!