Exercises

(p.248) In Exercise 1-4, find the indicated determinant.

1. $\begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix}$

3. $\begin{vmatrix} 0 & -3 \\ 5 & 0 \end{vmatrix}$

(p.248) In Exercise 6-9, find the indicated determinant.

6. $\begin{vmatrix} 1 & 4 & -2 \\ 5 & 13 & 0 \\ 2 & -1 & 3 \end{vmatrix}$

9. $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & -4 \end{vmatrix}$

(p.248) In exercise 13-18, find $\mathbf{a} \times \mathbf{b}$.

13. $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{3k}, \mathbf{b} = \mathbf{i} + \mathbf{2j}$

15. $\mathbf{a} = -\mathbf{i} + \mathbf{2j} + \mathbf{4k}, \mathbf{b} = \mathbf{2i} - \mathbf{4j} - \mathbf{8k}$

(p.248) In exercise 10-12, show the followings by direct computation.

10. a. $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

    b. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$

11. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$

12. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(p.249) In Exercise 25-32, find the area of the given geometric configuration.

25. The triangle with vertices $(-1, 2), (3, -1)$, and $(4, 3)$

27. The triangle with vertices $(2, 1, -3), (3, 0, 4)$, and $(1, 0, 5)$

31. The parallelogram with vertices $(1, 0, 1), (3, 1, 4), (0, 2, 9)$, and $(-2, 1, 6)$

(p.249) In Exercise 45-48, use a determinant to ascertain wether the given points lie on a line in $\mathbb{R}^2$. [HINT : What is the area of a "parallelogram" with collinear vertices?]

45. $(0,0),(3,5),(6,9)$
46. $(0,0),(4,2),(-6,-3)$

(p.249) In Exercise 49-52, use a determinant to ascertain wether the given points lie in a plane in $\mathbb{R}^3$. [HINT : What is the "volume" of a box with coplanar vertices?]

49. $(0,0,0),(1,4,3),(2,5,8),(-1,2,-5)$
50. $(0,0,0),(2,1,1),(3,-2,1),(-1,2,3)$

(p.248) Answer the followings.

19. Mark each of the following True or False.
    a. The determinant of a 2×2 matrix is a vector.
    b. If two rows of a 3×3 matrix are interchanged, the sign of the determinant is changed.
    c. The determinant of a 3×3 matrix is zero if two rows of the matrix are parallel vector in $\mathbb{R}^3$.
    d. In order for the determinant of a 3×3 matrix to be zero, two rows of the matrix must be parallel vectors in $\mathbb{R}^3$
    e. The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ given by rows of the matrix lie in plane.
    f. The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ given by rows of the matrix lie in plane trough the origin.
    g. The parallelogram in $\mathbb{R}^2$ determined by nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is a square if and only if $\mathbf{a}\cdot\mathbf{b}=0$.
    h. The box in $\mathbb{R}^3$ determined by vectors $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ is a cube if and only if $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{c}=0$ and $\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}$.
    i. If the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ in $\pi/4$, then $\|\mathbf{a}\times\mathbf{b}\| = |\mathbf{a}\cdot\mathbf{b}|$.
    j. For any vector $\mathbf{a}$ in $\mathbb{R}^3$, we have $\|\mathbf{a}\times\mathbf{a}\| = \|\mathbf{a}\|^2$.

(p.261) In Exercise 1-10, find the determinant of the given matrix.

1. $\begin{bmatrix} 5 & 2 & 1 \\ 1 & -1 & 4 \\ 3 & 0 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & 4 & 6 \\ 2 & 0 & -9 & 6 \\ 4 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & 4 \\ 6 & 2 & 8 & 1 & -1 & 1 \\ 4 & 2 & 1 & 2 & 2 & -5 \\ 4 & 5 & 4 & 5 & 1 & 2 \\ 1 & 2 & 0 & -1 & 2 & 4 \\ 1 & 0 & 1 & 8 & 1 & 5 \end{bmatrix}$

(p.262) In Exercise 15-20, let$A$ be a 3×3 matrix with $det(A) = 2$.

15. find $det(A^2)$
16. find $det(A^k)$
17. find $det(3A)$
18. find $det(A + A)$
19. find $det(A^{-1})$
20. find $det(A^T)$

(p.262) Answer the followings.

21. Mark each of the followings True or False.
    a. The determinant $det(A)$ is defined for any matrix A.
    b. The determinant $det(A)$ is defined for each square matrix A.
    c. The determinant of a square matrix is a scalar.
    d. If a matrix A is multiplied by a scalar c, the determinant of resulting matrix is $c \cdot det(A)$.
    e. If an $n \times n$ matrix A is multiplied by a scalar c, the determinant of the resulting matrix is $c^n \cdot det(A)$.
    f. For every square matrix A, we have $det(AA^T) = det(A^T A) = [det(A)]^2$.
    g. If two rows and also two columns of a square matrix A are interchanged, the determinant changes sign.
    h. The determinant of an elementary matrix is nonzero.
    i. If $det(A) = 2$ and $det(B) = 3$, then $det(A + B) = 5$.
    j. If $det(A) = 2$ and $det(B) = 3$, then $det(A + B) = 6$.

(p.262) In Exercise 26-29, find the values of λ for which the given matrix is singular.

27. $\begin{bmatrix} -\lambda & 5 \\ 2 & 3-\lambda \end{bmatrix}$

29. $\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 3 \\ 0 & 4 & -\lambda \end{bmatrix}$

(p.262) Answer the followings.

30. If A and B are $n \times n$ matrices and if A is singular, prove (without using Theorem 4.4) that AB is also singular. [HINT : Assume that AB is invertible, and derive a contradiction.]

32. if A and C are $n \times n$ matrices, with C invertible, prove that $det(A) = det(C^{-1}AC)$.

(p.271) In Exercise 1-10, find the determinant of the given matrix.

1. $\begin{bmatrix} 2 & 3 & -1 \\ 5 & -7 & 1 \\ -3 & 2 & -1 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ -5 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 8 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 & 1 \\ -1 & 2 & 4 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 5 & 1 & 5 & 0 & 0 \end{bmatrix}$

(p.271) Answer the followings.

11. The matrix in Exercises 9 has zero entries except for entries in an $r \times r$ submatrix R and a separates $s \times s$ submatrix S whose main diagonals lie on the main diagonal of the whole $n \times n$ matrix, and where $r + s = n$. Prove that, if A is such a matrix with submatrices R and S, then $det(A) = det(R) \cdot det(S)$.

12. The matrix A in exercise 10 has a structure similar to that discussed in exercise11, except that the square submatrices R and S lie along the order diagonal. State and prove a result similar to that in exercise 11 for such a matrix.

13. State and prove a generalization of the result in exercise 11, when the matrix A has zero entries except for entries in k submatrices positioned along the diagonal.

(p.271-272) In Exercise 14-19, find A^{-1} if A is invertible.

17. $\begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

(p. 272) Answer the followings.

35. Let A be a square matrix. Mark each of the following True or False.

   a. The determinant of a square matrix is the product of the entries on its main diagonal.
   b. The determinant of an upper-triangular square matrix is the product of the entries on its main diagonal.
   c. The determinant of a lower-triangular square matrix is the product of the entries on its main diagonal.
   d. A square matrix is nonsingular if and only if its determinant is positive.
   e. The column vectors of an $n \times n$ matrix are independent if and only if the determinant of the matrix is nonzero
   f. A homogeneous square linear system has a nontrivial solution if and only if the determinant of its coefficient matrix is zero.
   g. The product of a square matrix and its adjoint is the identity matrix.
   h. The product of a square matrix and its adjoint is equal to some scalar times the identity matrix.
   i. The transpose of the adjoint of A is the matrix of cofactors of A.
   j. The formula $A^{-1} = (1/det(A))adj(A)$ is of practical use in computing the inverse of a large nonsingular matrix.

(p.300) In Exercise 2-16, find the characteristic polynomial, find the real eigenvalues, and the corresponding eigenvectors of the given matrix.

3. $\begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}$

(p.300-301) In Exercise 17-22, find the eigenvalue λ, and the corresponding eigenvectors $\mathbf{v}_j$ of the linear transformation T.

17. T defined on $\mathbb{R}^2$ by $T([x, y]) = [2x - 3y, -3x + 2y]$

19. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1 + x_3, x_2, x_1 + x_3]$

21. T defined on $\mathbb{R}^3$ by $T([x_1, x_2, x_3]) = [x_1, -5x_1 + 3x_2, -3x_1 - 2x_2]$

(p.301) Answer the followings.

23. Mark each of the following True or False.

    a. Every square matrix has real eigenvalues.
    b. Every $n \times n$ matrix has n distinct (possibly complex) eigenvalues.
    c. Every $n \times n$ matrix has n not necessarily distinct and possibly complex eigenvalues.
    d. There can be only one eigenvalue associated with an eigenvalue of a linear transformation.
    e. There can be only one eigenvector associated with an eigenvalue of a linear transformation.
    f. if $\mathbf{v}$ is an eigenvector of a matrix A, then $\mathbf{v}$ is an eigenvector of $A + cI$ for all scalars c.
    g. if λ is an eigenvalue of a matrix A, then λ is an eigenvalue of $A + cI$ for all scalars c.
    h. if $\mathbf{v}$ is an eigenvector of an invertible matrix A, then c$\mathbf{v}$ is an eigenvector of A^-1 for all nonzero scalars c.
    i. Every vector in a vector space V is an eigenvector of the identity transformation of V into V.
    j. Every nonzero vector in a vector space V is an eigenvector of the identity transformation of V into V.

(p.300-301) Answer the followings.

1. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

   and the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

   List the vectors that are eigenvectors of A_1 and the ones that are eigenvectors of A_2. Give the eigenvalue in each case.

25. Prove that if A is a square matrix, then AA^T and A^TA have same eigenvalues.

32. Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Compare the eigenvectors and eigenvalues of A with those of $A + rI$ for a scalar r.

34. Let A be an $n \times n$ real matrix. An eigenvector $\mathbf{w}$ in $\mathbb{R}^n$ and a corresponding eigenvalue α of A^T are also called a left eigenvector and eigenvalue of A. Explain the reason for this name.

35. (Principle of biorthogonality) Let A be an $n \times n$ real matrix. Let $\mathbf{v}$ in $\mathbb{R}^n$ be an eigenvector of A with corresponding eigenvector λ, and let $\mathbf{w} \in \mathbb{R}^n$ be an eigenvector of A^T with corresponding eigenvalue α. Prove that if $\lambda \neq \alpha$, then $\mathbf{v}$ and $\mathbf{w}$ are perpendicular vectors. [HINT : Refer to exercise 34, and compute $\mathbf{w}^T A\mathbf{v}$ in two ways, using associativity of matrix multiplication.]

36. Answer the followings.

    a. Prove that the eigenvalues of an $n \times n$ real matrix A are the same as the eigenvalues of A^T.
    b. With reference to part(a), show by a counterexample that an eigenvector of A need not be an eigenvector of A^T.

39. Cayley-Hamilton theorem: Every square matrix A satisfies its characteristic equation. That is, if the characteristic equation is $p_n\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0$, then $p_nA^n + p_{n-1}A^{n-1} + \cdots + p_1A + p_0I = O$, the zero matrix. Illustrate the Cayley-Hamilton theorem for the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.