7.1 ~ 7.6

1.7 vector space

Euclidean space 확장

Def 23. "Field $\mathcal{F}$"

$(\mathcal{F}, +, \cdot) = \mathbb{R}$ 이라 생각하면 됨. 약간 subset 개념

$\mathbb{R}$ 덧 곱

<만족하는 성질>

1. $a+b=b+a$, $a\cdot b=b\cdot a$ (+, × 교환법칙)
2. $(a+b)+c=a+(b+c)$, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ (+, × 결합법칙)
3. $0+a=a$, $1\cdot a=a$ 위해 <u>0, 1 존재해야</u> (항등원) · 자신만드는거
4. $a, b\in\mathcal{F}$ 일때 $a+c=0$, $b\cdot d=1$ 만드는 c, d 존재해야 (역원) - 항등원 만드는거
5. $a\cdot(b+c)=a\cdot b+a\cdot c$ (분배법칙)

만족할 때 $\mathcal{F}$라 함. 근데 그냥 $\mathbb{R}$.

ex) $\mathbb{N}$: $\mathcal{F}$ × (0 없음), $\mathbb{Z}$: $\mathcal{F}$ × 역수 없음. $\mathbb{Q}$: 가능. 다항식 : $\mathcal{F}$ × ($x\cdot\left(\frac{1}{x}\right)=1$ 분수식)

Def 24. "Vector space" 새 상막.

$(V, \mathcal{F}, +, \cdot)$ → vector +, sclar multiplication) 4가지 연산 <u>가능해야</u> (아마..?)

$\mathcal{F}$, +, · vector 덧 sclar multiplication 외 2 + ·

<만족하는 성질>

1. $v, w\in V$, $v+w=w+v$ (교환법칙)
2. $u, v, w\in V$, $(u+v)+w=u+(v+w)$ (결합법칙)
3. $v\in V$, $0+v=v$ 만족하는 0 존재해야 (항등원) - 덧셈에 대한
4. $v, w\in V$, $v+w=0$ 만족하는 w 존재해야 (역원) - 덧셈에 대한
5. $v\in V$, $1v=v$ 흠.. trivial
6. $a\in\mathcal{F}$, $v, w\in V$, $a(v+w)=av+aw$
7. $\alpha, \beta\in\mathcal{F}$, $v\in V$, $\alpha(\beta v)=(\alpha\beta)v$
8. $\alpha, \beta\in\mathcal{F}$, $v\in V$, $(\alpha+\beta)v=\alpha v+\beta v$

→ $\mathcal{F}$의 원소 : scalar, vector space V의 성분 : vectors.

ex) V : 다항집합, $\mathcal{F}$: $\mathbb{R}$ → $(V, \mathbb{R}, +, \cdot)$ 즉 $(\mathbb{R}^n, \mathbb{R}, +, \cdot)$: 가능

차수 낮으면 가능한건가..?

* ~~Vector space~~ :

1.8. Basic concepts of Vector Spaces.

Def25. Linear Combination.

$v_1, \cdots v_k \in V$ ($= \mathbb{R}^n$)

$r_1, \cdots, r_k \in F$ ($= \mathbb{R}$)

$r_1 v_1 + \cdots + r_k v_k$: Linear combination.

Def26. Span

(즉. 유한한 벡터로 스팬해서 V를 만들 수 있을때 finitely generate 했다 함)

× $v_1, \cdots, v_k \in V$

$r_1, \cdots, r_k \in F$

$X \subset V$. $\text{span}(v_1, \cdots, v_k) = \text{span}(X)$

k개의 벡터로 만들어진 집합

$\text{span}(X) = \text{span}(v_1 \cdots v_k) = V$ 일때 finite set $X \subset V$, V = finitely generated.

ex) V = {다항식} ($= \mathbb{R}^n$)

$x, x^2 \in V$

$\text{span}(x, x^2) = \{\alpha x + \beta x^2 \mid \alpha, \beta \in \mathbb{R}\}$

(0,0) 지나는 이차다항식, 즉

유한한 벡터(2개)로 스팬할 수 없음 미리놓기 ㅠ

Eg11 (2) p : vector space of all polynomials

$M = \{1, x, x^2, x^3, \cdots\}$

$p = \text{span}(M)$, M은 무한하므로 p is not finitely generated.

Eg11 (1) $M_{m,n}$: vector space of all $m \times n$ matrix.

E : set of E_{ij} (E_{ij} = (i,j) 번째만 1, 나머지는 0 인 $m \times n$ matrix.)

$M_{m,n} = \text{span}(E)$, E는 유한하므로 $M_{m,n} = \text{span}(E)$ is finitely generated.

Def27. Subspace.

- 일반적으로 차원 낮추거나 유사성 분석 때 가능

V : vector space

$W \subset V$ (W is subset V)

V 내에서 덧셈. 곱셈 닫혀있을 때

$v, u \in V$
$v + u \in W$
$\alpha v \in W$

ex) $W = \{s \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid s \in \mathbb{R}\} \subset V = \mathbb{R}^2$

* subspace 따라가서 다시 공부

- 즉 덧으로 나열.

Def 28. Sum space (벡터들을 더해서 만든 공간)

V : vector space

$U, W \subset V$: subspace $\rightarrow$ $\underset{\text{집합+집합}}{U+W} = \{ \underset{\text{원소+원소}}{u+w} \mid u \in U, w \in W \}$

ex) $U = \{ \binom{s}{0} \mid s \in \mathbb{R} \}$, $W = \{ \binom{0}{t} \mid t \in \mathbb{R} \}$

$U+V = \{ \binom{s}{t} \mid s, t \in \mathbb{R} \} = \mathbb{R}^2$

ex) $\{ \binom{1}{2}, \binom{3}{4} \} + \{ \binom{5}{6} \} = \{ \binom{6}{8}, \binom{8}{10} \}$

Def 29. Direct Sum - 교집합 원점뿐인 벡터2개 더한 것

V : vector space

$U, W \subset V$: subspace + $U \cap W = \{0\}$ $\Rightarrow$ $V = U \oplus W$

$U+W = \{ u+w \mid u \in U, w \in W \}$

교집합이 원점뿐!

ex) $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$ (교집합 원점뿐이므로) $\rightarrow$ $\oplus$는 차수 더하는 것.

Def 30. Direct Product - 벡터 두개 쌓은 것

U, V : vector space

$U \otimes V = \{ \binom{u}{v} \mid u \in U, v \in V \}$ ← 덧셈에 닫혀있고 스칼라 곱셈에 닫혀있으면 명백히 Direct product 즉 subspace 면

ex) $U, V = \mathbb{R}$, $U \otimes V = \{ \binom{x}{y} \mid x, y \in \mathbb{R} \} = \mathbb{R}^2$

$$\begin{cases} \underset{\binom{s}{0}}{\mathbb{R}} \oplus \underset{\binom{0}{t}}{\mathbb{R}} = \underset{\binom{s}{t}}{\mathbb{R}^2} & \text{2개짜리 더해서 2개만듦} \\ \mathbb{R} \otimes \mathbb{R} = \underset{\binom{\mathbb{R}}{\mathbb{R}}}{\mathbb{R}^2} & \text{1개짜리 아래로 나열해서 2개만듦} \end{cases}$$

· $\mathbb{R}^n = \mathbb{R}^s \otimes \mathbb{R}^t$ 면 $s+t = n$

ex) $u = \text{span}\left(\binom{1}{0}\right) \subset U$, $v = \text{span}\left(\binom{0}{1}\right) \subset V$, $w = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \subset W$

$U \oplus V = \{ u+v \mid u \in U, v \in V \}$, $U \cap V = \{0\}$ $\rightarrow$ $\mathbb{R}^2$ (1차원 $\oplus$ 1차원 = 2차원)

이렇게 바꾸는법 알기! → $U \otimes W = \left\{ \begin{pmatrix} s \\ 0 \\ 0 \\ t \\ 0 \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$ (1차원 $\otimes$ 1차원 = 2차원)

Def 31. Linear independent - 벡터들이 dependent 하면 어느 한 벡터는 나머지 만들 수 있음.

* vector space : 무한히 많은 linear independent vector 들로 만들어짐

Def 32. Basis .

V : vector space

1) The subset spans V ($V = \text{span}(v_1 \cdots v_n)$) (subset: $v_1 \cdots v_n$)

2) The subset is linearly independent.

$\rightarrow$ A subset of V is a basis for V ($v_1 \cdots v_n$)

1.7 에서는 유한개수로 generated 된 vector space 다룸.

Thm 14. $\text{span}(w_1 \cdots w_k) = V$ & $v_1, \cdots, v_m$: linearly independent $\rightarrow k \geq m$

└ 스팬 개수가 Linearly independent 한 벡터 개수보다 많거나 같아야 함

Col. Invariance of Dimension

- Any two bases of a subspace W of $\mathbb{R}^n$ contain the same number of vectors

Corollary 15.

V : finitely generated vector space. $\rightarrow$ V의 basis 2개의 벡터크기는 동일함.

· $A = \{w_1 \cdots w_k\}$. $B = \{v_1 \cdots v_m\}$ $\rightarrow$ A, B = basis of V

span (A) = V & B is linearly independent $\rightarrow k \geq m$

span (B) = V & A is linearly independent $\rightarrow m \geq k$

$\therefore k = m$

Def 33. Dimension of a vector space

dim(V) = number of basis for V (개수)

└ 차원 만드는 최소 벡터들

ex) $V = \mathbb{R}^2$. $\dim(V) = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$

ex) $V = \left\{ \begin{pmatrix} s \\ 0 \\ 0 \\ t \\ 0 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$= \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$

$\therefore \dim(V) = 2$

Theorem 16.

$U, V \subset \mathbb{R}^n$: subspace of $\mathbb{R}^n$

$U \oplus V = \mathbb{R}^n \Rightarrow \dim(U) + \dim(V) = n$

ex) $U = \operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$

$V = \operatorname{span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$

$U \oplus V = \left\{ s\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$

<prove>

Let $(u_1 \cdots u_s)$ be a basis for U, $(v_1 \cdots v_t)$ be a basis for V

Claim that $(u_1, u_2, \cdots u_s, v_1, v_2, \cdots, v_t)$ is a basis for $\mathbb{R}^n$

1) $\operatorname{span}(u_1 \cdots v_t) = \mathbb{R}^n$

$\Rightarrow$: trivial

$\Leftarrow$: $\forall x \in \mathbb{R}^n$, $x = u + v$ for some $u \in U, v \in V$ $\doteq U \oplus V$

원소하나 뽑았을때
Linear combination이 되면 됨

Note that $u = r_1 u_1 + \cdots + r_s u_s$ for some $r_1 \cdots r_s \in \mathbb{R}$

$v = s_1 v_1 + \cdots + s_t v_t$ for some $s_1 \cdots s_t \in \mathbb{R}$.

Hence, $x = u + v = \underline{r_1 u_1 + \cdots + r_s u_s + s_1 v_1 + \cdots + s_t v_t} = 0$.

모르겠..

2)

Theorem 17.

$U, V \subset \mathbb{R}^n$: subspace of $\mathbb{R}^n$

$\dim(U \otimes V) = \dim(U) + \dim(V)$

의미 $\overset{S}{\overset{\shortparallel}{s\begin{pmatrix}1\\0\end{pmatrix}}}, \overset{T}{\overset{\shortparallel}{t\begin{pmatrix}0\\1\end{pmatrix}}}$ 있을 때

$S \oplus T = \begin{pmatrix}s\\t\end{pmatrix}$

$S \otimes T = \begin{pmatrix}s\\0\\0\\t\end{pmatrix}$

$\begin{pmatrix}u\\v\end{pmatrix} \in U \otimes V$ ($u \subset U$, $v \subset V$)

$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{s} \alpha_i u_i \\ \sum_{i=1}^{t} \beta_i v_i\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{s} \alpha_i u_i \\ 0\end{pmatrix} + \begin{pmatrix}0 \\ \sum_{i=1}^{t} \beta_i v_i\end{pmatrix}$.

$\underset{z_1}{\begin{pmatrix}u_1\\0\end{pmatrix}} \underset{z_2}{\begin{pmatrix}u_2\\0\end{pmatrix}} \cdots \underset{z_s}{\begin{pmatrix}u_s\\0\end{pmatrix}}, \underset{z_{s+1}}{\begin{pmatrix}0\\v_1\end{pmatrix}} \underset{z_{s+2}}{\begin{pmatrix}0\\v_2\end{pmatrix}} \cdots \underset{z_{s+t}}{\begin{pmatrix}0\\v_t\end{pmatrix}}$.

$\therefore \operatorname{span}(z_1, \cdots, z_{s+t}) = U \otimes V$

$+ \therefore z$: linearly independent (차원 $s+t$)

$\left(\sum_{i=1}^{s+t} r_i z_i = 0 \Rightarrow \sum_{i=1}^{s} r_i z_i + \sum_{i=s+1}^{t} r_i z_i = 0 \right.$

$\Rightarrow \sum_{i=1}^{s} r_i \begin{pmatrix}u_i\\0\end{pmatrix} + \sum_{i=s+1}^{t} r_i \begin{pmatrix}0\\v_i\end{pmatrix} = 0$

$\left. \Rightarrow \sum_{i=1}^{s} r_i u_i = 0 \ \& \ \sum_{i=s+1}^{t} r_i v_i = 0 \Rightarrow r_1 = \cdots = r_t = 0 \right)$

span 하는데 있어서

Def 34. Orthogonal basis

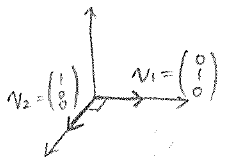
k개까지 벡터 k개

Let $B = \{v_1, \cdots, v_k\}$: basis for $\underset{\text{Subspace}}{V} \subset \mathbb{R}^n$

If $\underbrace{v_i^T v_j}_{\text{수직}} = 0 \quad \forall i \neq j$ (서로다른 v가 수직이면)

then B is orthogonal basis for $V \subset \mathbb{R}^n$

if $\|v_i\| = 1 \ \forall i$ then B is orthonormal basis.



→ V에 대해 v_1, v_2 orthogonal basis.

$\mathbb{R}^2 \ \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}, \ \left\{\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}, \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{pmatrix}\right\}$

$\mathbb{R}^3 \ \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

→ orthonormal basis (무수히 많음!)

Theorem 18.

$\underline{\{v_1, \cdots, v_k\}}$: orthogonal basis for $V \subset \mathbb{R}^n$ 에 더해서 전체 V 만들 수 있음

If $V \neq \mathbb{R}^n$, then there exist $\underline{\{v_{k+1}, \cdots, v_n\} \in \mathbb{R}^n}$ ~~such that~~

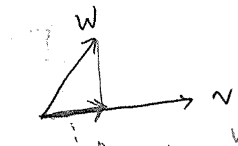
such that $\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ is orthogonal basis for $\mathbb{R}^n$. (많이 있는데 그 중 orthogonal 한 것도 있음)

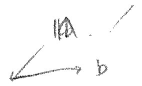
Let $\{v_1, \cdots, v_k, w_{k+1}, \cdots, w_n\}$,

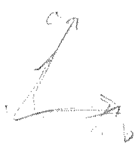
$$v_{k+1} = w_{k+1} - \sum_{i=1}^{k} \frac{\overset{a}{w_{k+1}^T v_i}}{v_i^T v_i} \times v_i$$

then, $$v_1^T v_{k+1} = v_1^T w_{k+1} - \sum_{i=1}^{k} \frac{w_{k+1}^T v_i}{v_i^T v_i} \times v_1^T v_i$$

$$= v_1^T w_{k+1} - \frac{w_{k+1}^T v_1}{v_1^T v_1} \times v_1^T v_1 = 0$$





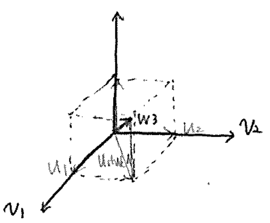


$\sqrt{b}$

$$av = \text{Proj}_v w = \frac{w^T v}{v^T v} \times v \quad \left(\text{Proj}_b a = \frac{a \cdot b}{\|b\|} \times \frac{b}{\|b\|}\right.$$

$$\left. = \frac{a^T b}{\|b\|^2} \times b = \frac{a^T b}{b^T b} \times b \right)$$

$$\text{Proj}_{(b)} a = \frac{a \cdot b}{\|b\|} \times \frac{b}{\|b\|}$$



$w_3 - (u_1 + u_2) = v_3$ ← orthogonal

$$u_1 = \frac{w_3^T v_1}{v_1^T v_1} \cdot v_1$$

$$u_2 = \frac{w_3^T v_2}{v_2^T v_2} \cdot v_2$$

$$\rightarrow v_3 = w_3 - \left(\frac{w_3^T v_1}{v_1^T v_1} \cdot v_1 + \frac{w_2^T v_2}{v_2^T v_2} \cdot v_2\right)$$

$v_1 = w_1$;

$v_2 = w_2 - \frac{w_2^T v_1}{v_1^T v_1} v_1$;

$v_3 = w_3 - \frac{w_3^T v_1}{v_1^T v_1} v_1 - \frac{w_3^T v_2}{v_2^T v_2} v_2$;

$\vdots$

$v_n = w_n - \frac{w_n^T v_1}{v_1^T v_1} v_1 - \cdots - \frac{w_n^T v_{n-1}}{v_{n-1}^T v_{n-1}} v_{n-1}$;

→ $\{v_1 \cdots v_n\}$ orthogonal basis.

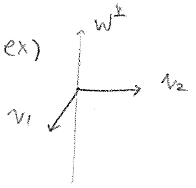
다시 써보기!

Theorem 19.

$W^{\perp} = \{ v \mid v^T w = 0, \forall w \in W \}$ ($\subset \mathbb{R}^n$) $\Rightarrow W \oplus W^{\perp} = \mathbb{R}^n$. thus $\dim(W) + \dim(W^{\perp}) = n$

└ W에 수직인 여집합 (orthogonal complement)

ex)



$\{v_1, v_2, \cdots, v_k\}$: orthogonal basis for W

$\rightarrow \exists \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$: basis for $\mathbb{R}^n$

Chapter 2. Linear Transformation (선형변환)

2.1 Linear Transformation of Euclidean Spaces

$f(x) = Ax$, $A : m \times n$ $\rightarrow$ 정의역 $= \mathbb{R}^n$, 공역 $= \mathbb{R}^m$ in $Ax = b$.

$$\underset{m\times n}{\overset{A}{(\quad)}}\underset{n\times 1}{\overset{x}{(\ )}} = \underset{m\times 1}{\overset{b}{(\ )}}$$

$f(\alpha v_1 + \beta v_2) = (\alpha v_1 + \beta v_2) = \alpha A v_1 + \beta A v_2 = \alpha f(v_1) + \beta f(v_2)$

The functions in vector space that preserve the linear combinations are called linear transformations.

Def 35. function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ($f \rightarrow \mathbb{R}$, $X \rightarrow \mathbb{R}$ ...)

$$\begin{cases} 1. \ T(u+v) = T(u) + T(v) \\ 2. \ T(\alpha v) = \alpha T(v) \end{cases} \quad \forall : u, v \in \mathbb{R}^n, \alpha \in \mathbb{R}$$

만족하면 Linear transformation.

ex) 적분

$T(f) = \int f(x)\,dx$

$T(f+g) = \int (f+g)\,dx = \int f\,dx + \int g\,dx = T(f) + T(g)$

ex) 기댓값

$E(X+Y) = E(X) + E(Y)$

$E(\alpha X) = \alpha E(X)$

$T(r_1 v_1 + \cdots + r_n v_n) = r_1 T(v_1) + r_2 T(v_2) + \cdots + r_n T(v_n)$ — 정의역 속에 Linear Combination 값이 놓이면 각 벡터의 함숫값들의 Linear combination 나옴.

ex) $T\left(2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}\right) = 2T\begin{pmatrix}1\\0\end{pmatrix} + 3T\begin{pmatrix}0\\1\end{pmatrix}$

Eg12.

1. $T(x) = \sin x$. $\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) \neq \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)$

2. $T: \mathbb{R}^2 \to \mathbb{R}^3$. $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_2 \\ x_1 - x_2 \\ 2x_1 + x_2 \end{pmatrix}$ $\to$ T : Linear transformation.

$T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) \overset{?}{=} T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$

$\begin{pmatrix} u_2 + v_2 \\ (u_1+v_1) - (u_2+v_2) \\ 2(u_1+v_1) + (u_2+v_2) \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 - u_2 \\ 2u_1 + u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_1 - v_2 \\ 2v_1 + v_2 \end{pmatrix}$

3. $T: \mathbb{R}^n \to \mathbb{R}^m$, $\underline{T(x) = \overset{\text{유클리디안}}{Ax}}$; $x \in \mathbb{R}^n$, $A: m \times n$. $\to$ T : linear transformation.

$T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$

$T(av) = Aav = aAv = aT(v)$

4. $T: \mathbb{R} \to \mathbb{R}$ be a linear Transformation, $T(1) = a$ $\to$

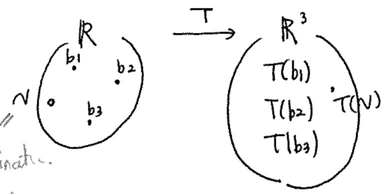
$\to T(x) = T(x \times 1) = xT(1) = ax$, $\forall x \in \mathbb{R}$

└ linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is determined by its value on basis for $\mathbb{R}^n$.

Theorem 20. $T: \mathbb{R}^n \to \mathbb{R}^m$, $B = \{b_1, \cdots, b_n\}$: basis for $\mathbb{R}^n$

<u>$T(v)$ is uniquely generated by the vectors $T(b_1), \cdots, T(b_n)$</u> $\forall v \in \mathbb{R}^n$

└ $v = r_1 b_1 + \cdots + r_n b_n$



b_1, b_2, b_3 // Linear combination (ex) $r_1b_1 + \cdots + r_3b_3$)

$= T(b_1) \cdot T(b_2) \cdot T(b_3)$ 해석 Linear Combi ~ ex) $k_1T(b_1) + k_2T(b_2) + k_3T(b_3)$

Corollary. $T: \mathbb{R}^n \to \mathbb{R}^m$. $A \triangleq (T(e_1), \cdots, T(e_n))$: $m \times n$ $\to T(x) = Ax$ $\forall x \in \mathbb{R}^n$

$x = r_1e_1 + \cdots + r_ne_n = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$

$T(x) = r_1T(e_1) + \cdots + r_nT(e_n)$

$= (T(e_1), \cdots, T(e_n)) \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = Ax.$

ex) $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$: $\mathbb{R}^2 \to \mathbb{R}^2$

$= A\begin{pmatrix} x \\ y \end{pmatrix}$

└ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

CR

Eg 13.

column 개수   row 개수

1. $T : \mathbb{R}^{2} \to \mathbb{R}^{3}$

$$T(e_1) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \qquad A = (T(e_1), \cdots, T(e_n)) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

$$T(x) = Ax = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 4 & -2 \end{pmatrix} x$$

└ $T(e_1), T(e_2)$를 column 으로 가지고 있는 Matrix

2. $T : \mathbb{R}^{4} \to \mathbb{R}^{3}$

$$T\left(\begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_2 - 3x_3 \\ 2x_1 - x_2 + 3x_4 \\ 8x_1 - 4x_2 + 3x_3 + x_1 \end{pmatrix}$$

$$T(e_1) = \begin{pmatrix} 0 \\ 2 \\ 8 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \quad T(e_4) = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

그냥 계수 따는 거

3. $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\Big(\underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{u}\Big) = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad T\Big(\underbrace{\begin{pmatrix} 3 \\ -5 \end{pmatrix}}_{v}\Big) = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$$

$$T(x) = Ax$$

$$T\Big(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x}\Big) = \alpha T\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) + \beta T\left(\begin{pmatrix} 3 \\ -5 \end{pmatrix}\right) = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$$

└ u와 v의 Linear combination (u, v는 $\mathbb{R}^2$의 basis)

└ $x = \alpha u + \beta v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \left(\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \beta = 2 \quad \alpha = 5$$

연습!

ex) $T\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \; T\left(\begin{pmatrix} 3 \\ -5 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right)$를 구하고 A를 찾아라.

Sol) $\begin{pmatrix} -4 \\ 3 \end{pmatrix} = r_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \therefore r_1 = -11, \; r_2 = -5$

$$T\left(\begin{pmatrix} -4 \\ 3 \end{pmatrix}\right) = -11 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - 5 \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 24 \\ -5 \end{pmatrix}$$

$$T(x) = Ax = r_1 T(e_1) + \cdots + r_n T(e_n) = (T(e_1), \cdots, T(e_n)) \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \left(\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & -5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \alpha = 5, \; \beta = 2$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 2 & -5 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 1 & 1 \end{array}\right) \quad \alpha = 3, \; \beta = 1$$

$$T(e_1) = 5 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 2 \end{pmatrix} \qquad T(e_2) = 3 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$$

$$\therefore T(x) = Ax \implies A = \begin{pmatrix} 0 & -1 \\ -9 & -4 \\ 2 & 1 \end{pmatrix}$$

2-1. Summary

수강의

1. A basis for a subspace W is orthogonal if the basis vectors are mutually perpendicular, and it is orthonormal if the vectors also have length 1.

2. Any orthogonal set of vectors in $\mathbb{R}^n$ is basis for the subspace it generates.

3. Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis.
The projection of a vector b in $\mathbb{R}^n$ on W is equal to the sum of the projection of b on each basis vectors.

4. Every non zero subspace W of $\mathbb{R}^n$ has an orthonormal basis. Any basis can be transformed into an orthogonal basis by means the Gram-Schmidt process, in which each vector a_j of the given basis is replaced by the vector v_j obtained by subtracting from a_j its projection on the subspace generated by its predecessors.

5. Any orthogonal set of vectors in a subspace W of $\mathbb{R}^n$ can be expanded, if necessary, to an orthogonal basis for W.

6. Let A be an $n \times k$ matrix of rank k. Then A can be factored as QR, where Q is an $n \times k$ matrix with orthonormal column vectors and R is a $k \times k$ upper-triangu[lar] invertible matrix.

deg: 2제 항이 2보다 작음

1.8 11번 예제

Is $\{1-x+2x^2, -1+x^2, -2-x+5x^2\}$ a linearly independent subset of $\mathbb{P}_2$?

If not, express one of these polynomials as a linear combination of the others.

Solution We need solve

$$a(1-x+2x^2)+b(-1+x^2)+c(-2-x+5x^2)=0$$

This gives 3 equation : $a-b-2c=0$

$(-a-c)x=0$

$(2a+b+5c)x^2=0$

Augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 5 & 0 \end{array}\right) \xrightarrow[R_3 \Rightarrow R_3-2R_1]{R_2 \Rightarrow R_2+R_1} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 3 & 9 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3+3R_2]{} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

무수히 많은 해. → not linearly indepd

$$\xrightarrow[R_2 \to -R_2]{R_1 \to R_1-R_2} \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} a & b & c \end{array} \end{array}$$

$s=t$ $(t \in \mathbb{R})$ then $a=-t$, $b=-3t$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = t \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$\therefore$ Set is not linearly independent.

$\therefore$ $-1(1-x+2x^2)-3(-1+x^2)+(-2-x+5x^2)=0$

$\therefore$ $1-x+2x^2=-3(-1+x^2)+(-2-x+5x^2)$ 이렇게 만들 수 있다니 ㅋ.

2.2 Rank and Nullity of a Matrix and a Linear Transformation

$T : \mathbb{R}^n \to \mathbb{R}^m$

$T(x) = Ax$

homogeneous linear system

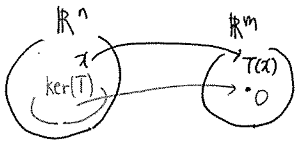
∟ null space of A $= null(A) = ker(A) = \{ Ax = 0 \mid x \in \mathbb{R}^n \}$ kernel 커널   : 커널

$= ker(T) = \{ T(x) = 0 \mid x \in \mathbb{R}^n \}$

┌ A의 column vector 들을 계수 원소로 Linear Combination.

∟ column space of A $= col(A) = Im(A) = \{ Ax \mid x \in \mathbb{R}^n \}$ $= span(A_1, \cdots, A_n)$

$= Im(T) = \{ T(x) \mid x \in \mathbb{R}^n \}$ Image 이미지



ex) $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}$

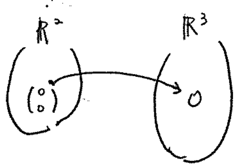
$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

해 스펜

$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right)$ $x_1 = 0,\ x_2 = 0$ $\to null(A) = span\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} = ker(T)$

$\to col(A) = span\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$ — basis 2개인 3차원 : 평면 $(\mathbb{R}^2 \to \mathbb{R}^3)$

↓

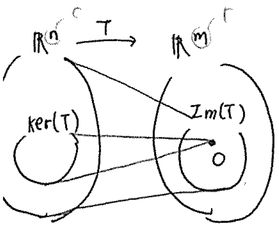


참 $span\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right)$: —

: basis 2개인 3차원 → 평면

(row)	(col)	
3개 짜리	4개	: 3차원 (변항 없음)
	3개	: 3차원
	2개	: 2차원
	1개	: 1차원

< 일반화 >



$T(x) = Ax$, $A = (T(e_1), \cdots, T(e_n))$

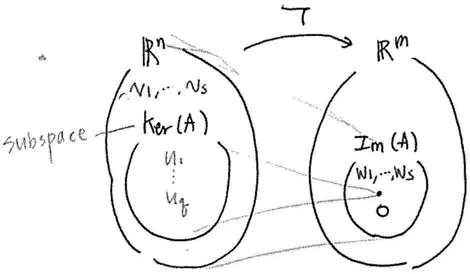
$ker(T) = null(A)$

$Im(T) = col(A)$

$\{ x \mid Ax = 0 \} = null(A) = \{0\}$ — basis 없음 (0벡터는 basis 안 해당)

$dim(ker(T)) = nullity(T)$ $dim(ker(A)) = nullity(A)$

$dim(Im(T)) = rank(T)$ $dim(col(A)) = rank(A)$



- $s + q = n$
- $Av_1 = w_1$

  $\vdots \quad \vdots$

Theorem 21. Rank Equation

$\begin{cases} A : m \times n \\ T : \mathbb{R}^{n} \to \mathbb{R}^m, \ T(x) = Ax \end{cases} \implies \underset{\substack{\| \\ \dim(col(A)) \\ \| \\ Im}}{Rank(A)} + \underset{\substack{\| \\ \dim(null(A)) \\ \| \\ ker}}{nullity(A)} = n \quad (Rank(T) + nullity(T) = n)$

col. 개수

(그림 알면 이해 편함)

- $ker(A) = \{0\} \implies rank(A) = n$
- $ker(A) = \mathbb{R}^n \implies rank(A) = 0 \quad (Im(A) = \{0\})$

증명 읽어보기 (노트)

Remark 20.

$\boxed{A : m \times n}$

- $row(A) = \{A^T y \mid y \in \mathbb{R}^m\} \subset \mathbb{R}^{n}$
- $col(A) = \{Ax \mid x \in \mathbb{R}^n\} = span(A_1, \cdots, A_n) \subset \mathbb{R}^m$
- $null(A) = \{x \mid Ax = 0\} \subset \mathbb{R}^{n}$

$row(A) \oplus null(A) = n$

* 내적 $\underbrace{(A^T y)^T}_{row(A)} \underbrace{x}_{null(A)} = y^T \underbrace{Ax}_{=0} = 0 \to row(A) \perp null(A) \to \dim(row(A)) + \dim(null(A)) = n$

- $rank(A) \le \min\{m, n\}$ (if $rank(A) = \min(m,n)$: A is full rank matrix)
- $rank(A) = rank(A^T)$ — rank가 row 개수랑 같을 때
- $rank(A) = m \iff \exists A^{-1}$, when $m = n$ = invertible
- $rank(AB) = rank(BA) = rank(A)$ for any B that is invertible.
- $rank(AB) \le \max\{rank(A), rank(B)\}$
- $rank(A+B) \le rank(A) + rank(B)$
- $rank(AA^T) = rank(A^TA) = rank(A) = rank(A^T)$

* 책 - Finding Bases for spaces Assosiated with a Matrix.

1. For a basis of row space of A — use the nonzero rows of H.

2. For a basis of column space of A - use the columns of A corresponding to the columns of H containing pivots.

3. For a basis of the null space of A - use H and back substitution to solve $Hx = 0$ in the usual way.

Ex) $A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix}$

$\sim$ $R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 2R_1$

$$\begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 2 & -4 & 2 & 0 \\ 0 & -1 & 3 & -2 & -4 \end{pmatrix} \underset{\substack{R_4 \to R_4 + \frac{1}{2}R_2 \\ R_3 \to R_3 + R_2}}{\sim} \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -4 \end{pmatrix} \underset{\substack{R_2 \to \frac{1}{2}R_2 \\ R_3 \leftrightarrow R_4}}{\sim} \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R_1 \to R_1 + 3R_2 \\ R_2 \to -R_2}}{\sim} \begin{pmatrix} 1 & 0 & 6 & -4 & 2 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\substack{R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 - 6R_3}}{\sim} \begin{pmatrix} 1 & 0 & 0 & 2 & 26 \\ 0 & 1 & 0 & -1 & -8 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = H$$

$\boxed{\text{rank}(A) = 3 \text{ — 피봇 개수와 동일}}$

$\text{row}(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 26 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -4 \end{pmatrix} \right\}$ (0벡터는 basis 포함 X) H에서 그냥 있는 그대로에서 쓰면 됨.

$x_4 = s,\ x_5 = t,\ x_3 = s + 4t \quad x_2 = s + 8t \quad x_1 = -2s - 26t$

$$x = \begin{pmatrix} -2s - 26t \\ s + 8t \\ s + 4t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -26 \\ 8 \\ 4 \\ 0 \\ 1 \end{pmatrix} \to \text{Null}(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -26 \\ 8 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(해 풀어서 나온 애들)

↳ nullity (A) = 2

$\text{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 11 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -4 \\ 3 \end{pmatrix} \right\}$ - 피봇 있는 애들만 원래 A에서 (H에서도 ㄱㅊ)

* 책 - Rank Equation

A : m×n . H : row-echelon Form.

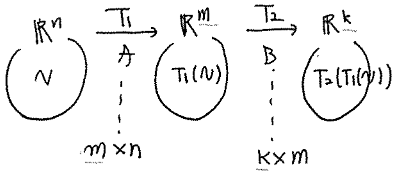
1. nullity (A) = number of pivot-free columns in H.
2. rank (A) = number of pivots in H.
3. (rank equation) : rank (A) + nullity (A) = n = number of columns of A.

2.3 Properties of Linear Transformations. – 합성함수 느낌.

$T_1 : \mathbb{R}^n \to \mathbb{R}^m$

$T_2 : \mathbb{R}^m \to \mathbb{R}^k$

A : invertible , non invertible 상관없이 가능

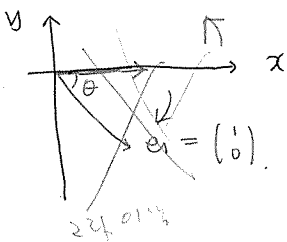


$(T_2 \circ T_1)(v) \triangleq T_2(\underbrace{T_1(v)}_{Av}) = B(Av) = (BA)v$ 순서대로!

Eg 14.

1) counterclockwise rotation

$T : \mathbb{R}^2 \to \mathbb{R}^2$, θ만큼 회전



더해서 돌린 것 · v돌리고 w돌려서 더한

$T(v+w) = T(v) + T(w)$

$T(\alpha v) = \alpha T(v)$

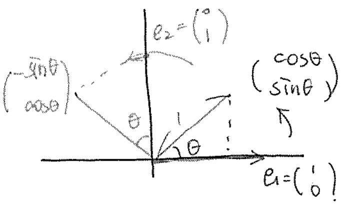
L.T 는 두개 만족

standard matrix representation.

2) Matrix representation of the counterclockwise rotation

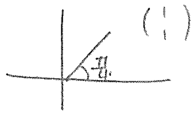
column vector들

$A = (T(e_1), \cdots, T(e_n))$



$$A = \begin{array}{cc} T(e_1) & T(e_2) \end{array}$$
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

ex) $A_{\frac{\pi}{4}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$

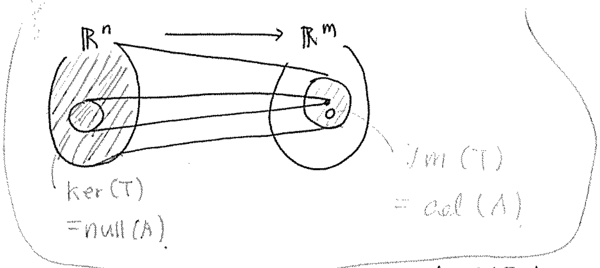


$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $T(v+w) = T(v) + T(w)$, $T(\alpha v) = \alpha T(v)$ 만족.

$\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ & \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$

- $T : \mathbb{R}^n \to \mathbb{R}^n$ Linear transformation.

$\ker(T) = \{0\} \longrightarrow Im(T) = \mathbb{R}^n$



그냥 외우기

- $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$.

$T^{-1}(T(x)) = x \quad (\because \text{rank}(A) = n \to \text{invertible})$

$T^{-1}(r_1 T(v_1) + r_2 T(v_2)) = T^{-1}(T(r_1 v_1 + r_2 v_2)) = r_1 v_1 + r_2 v_2$

$\therefore T^{-1}$: Linear combination.

$\therefore T^{-1} \circ T = I$.

Eg 15. 1) Clockwise rotation.

A : linear transformation that rotates a vector in the plane counterclockwisely by an angle θ

A^{-1} : linear transformation that rotates a vector in the plane clockwisely by an angle θ.

= clockwisely by an angle $-\theta$.

$$\therefore A^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

4사분면. cosθ 양+

2) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T([x_1, x_2, x_3]) = [x_1 - 2x_2 + x_3,\ x_2 - x_3,\ 2x_2 - 3x_3]$

$\hookrightarrow A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix}$ ← standard matrix representation.

Sol) $\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & -1 \\ 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right)$

$\to T^{-1}(x) = A^{-1}x = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 4x_2 - x_3 \\ 3x_2 - x_3 \\ 2x_2 - x_3 \end{pmatrix}$.

* standard matrix representation

* column space A = range of T_A.

책 154 - 164.

1. Non invertible Transformation (2x2)

① rank(A) = 0. : entire plane is collapsed to a single point - the origin. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

② rank(A) = 1 : one-dimensional subspace of $\mathbb{R}^2$ ($\mathbb{R}^2$의 1차원)

: line through the origin. (원점 지남)

: 적어도 한개의 non zero column vector 가. 둘다. non zero라면 2놈은 Linear dependent해야

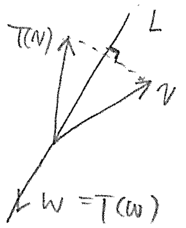
ex) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

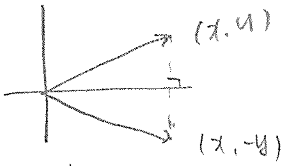
projection on x-axis, " y-axis, Collapse onto y=2x

* projection
$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ ex) $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: collapse onto along the y=2x

ex) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. projection on

* reflection in the Line (대칭일때 (수직))





L Reflection in the x-axis. = $\begin{array}{cc} e_1 & e_2 \end{array}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

더쉬운

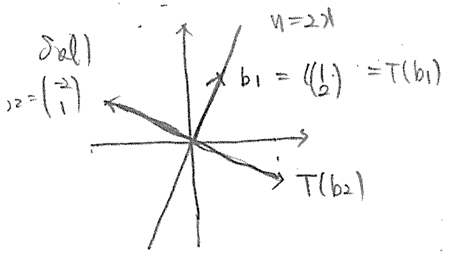
즉 회전하면서 의미만 - $\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$ 꼴과 동일

e_1은 가만히 나두고 ($T(e_1) = e_1$)

e_2만 - 로 만들어줌 ($T(e_2) = -e_2$)

. L.T ($T: \mathbb{R}^2 \to \mathbb{R}^2$) preserve lengths of all vectors in $\mathbb{R}^2$ (ex) $\|T(x)\| = \|x\|$ (for all $x \in \mathbb{R}^n$)

중요하니까 꼭!
ex) Find a standard matrix representation A for the reflection of the plane in the line y=2x.



$T(b_1) = b_1$ & $T(b_2) = -b_2$

$\begin{array}{cc} b_1 & b_2 \end{array}$
$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$: reflection in line y=2x.

∴ standard matrix representation : $T(e_1), T(e_2)$에 관한 행렬이므로 계산 해야

$\left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} \end{array}\right)$ $\to e_1 = \frac{1}{5}b_1 - \frac{2}{5}b_2$, $\to e_2 = \frac{2}{5}b_1 + \frac{1}{5}b_2$.

$T_E\left(\left[\begin{matrix} x \\ y \end{matrix}\right]\right) = \begin{pmatrix} x \\ -y \end{pmatrix}$

$T(e_1) = \frac{1}{5}T(b_1) - \frac{2}{5}T(b_2) = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{2}{5} \times (-1)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$

$T(e_2) = \frac{2}{5}T(b_1) + \frac{1}{5}T(b_2) = \frac{2}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{5} \times (-1)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$

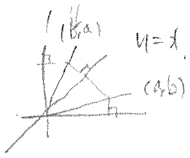
∴ Standard matrix representation $A = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

ERO

* reflection in x-axis : $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix}$

* reflection in y-axis : $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} -x \\ y \end{pmatrix}$

* interchange rows : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}$ $T_E\left(\left[\begin{matrix} x \\ y \end{matrix}\right]\right) = \begin{pmatrix} -x \\ y \end{pmatrix}$



$T_E\left(\left[\begin{matrix} x \\ y \end{matrix}\right]\right) = \begin{pmatrix} y \\ x \end{pmatrix}$

* $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ 형태

- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} rx \\ y \end{pmatrix} \leftarrow \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ y고정

$r>1$ — horizontal expansion

$0<r<1$ — horizontal constraction

$-1<r<0$ — horizontal constraction followed by a reflection in the y-axis.

$r<-1$ — horizontal expansion followed by a reflection in the y-axis

- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} x \\ ry \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ x고정

$r>1$ - vertical expansion

$0<r<1$ — vertical constraction.

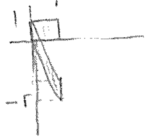
$-1<r<0$ — vertical constraction followed by a reflection in the x-axis.

$r<-1$ — vertical expansion followed by a reflection in the x-axis.

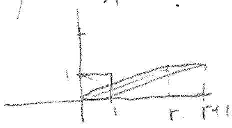
- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} x \\ rx+y \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ y고정. (밑변위치)

$r>0$ — The vertical shear

$r<0$ — The vertical shear

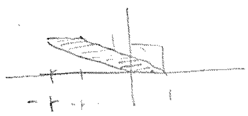


- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} x+ry \\ y \end{pmatrix} \leftarrow \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ x고정.



$r>0$ - The horizontal shear.

$r<0$ - The horizontal shear



<정리>

Geometric Description of Invertible Transformations of $\mathbb{R}^2$.

A linear transformation T of plane $\mathbb{R}^2$ into itself is invertible if and only if (필.충).
T consists of a finite sequence of:

Reflections in the x-axis, the y-axis or the line $y=x$;

Vertical or horizontal expansions or contractions; and

Vertical or horizontal shears.