## 1.1 Vectors in Euclidean spaces

뒤에 조건에 따라 나타내는 것 다름 (직선 등)

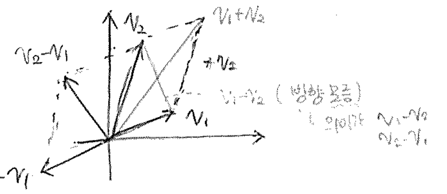
Eg1. $\mathbb{R}^2 = \{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1 \in \mathbb{R} \text{ and } v_2 \in \mathbb{R} \}$ : 평면 $\quad \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow (x, y)$ 좌표

$\hookrightarrow x \in \mathbb{R}^n \rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Def1. $v = w \iff v_i = w_i$

Def2. $v \pm w = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \pm \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$

$\alpha v = \alpha \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}$




공간에서

합 :

차 : 둘을 이으면 됨 ( 대신 방향 모름 )

( 빼는 대상에 (-) 곱한 후 더함.

㉠ $(\alpha+\beta)v$ 증

$(\alpha+\beta)v = (\alpha+\beta)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)v_1 \\ \vdots \\ (\alpha+\beta)v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta v_1 \\ \vdots \\ \alpha v_n + \beta v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix} + \begin{pmatrix} \beta v_1 \\ \vdots \\ \beta v_n \end{pmatrix}$

$= \alpha v + \beta v$

$\rightsquigarrow (\alpha+\beta)v_i = \alpha v_i + \beta v_i$ 실수성 이용

Remark1. $v = \alpha w \begin{cases} \alpha > 0 & : \text{same direction} \\ \alpha < 0 & : \text{opposite direction.} \end{cases}$

㉠ $2 = 6r, \quad 1 = 3r \rightsquigarrow$ parallel

$\boxed{\text{Def3}}$ Linear combination. (선형 결합)

$v_1, \cdots, v_k \in \mathbb{R}^n,\ r_1, \cdots, r_k \in \mathbb{R}$, $\underline{r_1v_1 + r_2v_2 + \cdots + r_kv_k}$ is a linear combination of the vector $v_1, \cdots, v_k$ with coefficients $r_1, \cdots, r_k$

$\Leftrightarrow \sum_{i=1}^{k} r_i v_i$

Remark2. $e_j \in \mathbb{R}^n$ : a vector of zeros but the $j^{th}$ element is one.

ex) $e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ; 2. 3차원일 경우 벡터가 축이랑 겹침 ex)

Remark3. $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $v^T = (v_1, \cdots, v_n)$

$\boxed{\text{Def4}}$ linear span (선형생성 부분공간)

└ 주어진 벡터를 선형결합해서 모아놓은 집합

㉠ 그 공간을 스팬할 수 있는 최소한의 값이 공간의 차원

$sp(v_1, \cdots, v_k) = \{ r_1v_1 + \cdots + r_kv_k \mid r_1, \cdots, r_k \in \mathbb{R} \}$

ex) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$sp(v_1, v_2) = \{ r_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid r_1, r_2 \in \mathbb{R} \} = \{ \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \mid r_1, r_2 \in \mathbb{R} \} = \mathbb{R}^2$

ex) $sp\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ 의미 없음 ( 이미 2개로도 $\mathbb{R}^2$ 형성 ) $\rightsquigarrow$ 즉, 이차원 벡터 2개가 평면. 세개 노의미 / 3차원 벡터 3개일 때 공간.

마찬가지로 3차원 벡터 4개도 노의미

$\rightsquigarrow$ 벡터의 개수와 차수의 조화 필요

. 일반적인 span은 전체 범위 이지만 범위 설정 가능

ex) $sp(v, w) = \{ av + bw \mid a \geq 0, b \geq 0 \}$


→ (반직선으로 놓인 곳에 둘러싸인 공간 )

. $\{ av + bw \mid$ ____ $\}$ 조건을 이용하여 모양 만들기 ex) $\{ av + bw \mid a + b = 1 \}$ $\begin{cases} ① \ a > 0 \rightarrow \overline{vw} \text{ 선분} \\ ② \ b > 0 \rightarrow \text{반직선} \end{cases}$

closed

. $\mathbb{R}^2$ 위해서는 2개 이상의 벡터가 필요하며 이 둘은 평행해선 안된다.

# 1.2 Usual Euclidean Norm and Inner Product

Norm └ 벡터의 길이 (n차원)

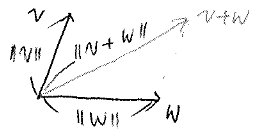
$\boxed{\text{Def5}}$ $v \in \mathbb{R}^n$. $\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$ (노옴) - 거리 구하듯이!

$$= \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$$

참) $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$ p-norm (양수가 p승 합은 $\frac{1}{p}$) (1-norm은 제곱합)

properties of the norm in $\mathbb{R}^2$

1. $\forall v \in \mathbb{R}^n$, $\|v\| \geq 0$. (The equality holds when $v = 0$)
2. $\forall v \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$, $\|\lambda v\| = |\lambda| \|v\|$.
3. $\forall v, w \in \mathbb{R}^n$, $\|v + w\| \leq \|v\| + \|w\|$ — triangular inequality




* $v^T v = v \cdot v = \|v\|^2$

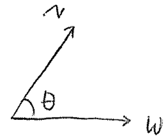
$\boxed{\text{Def6.}}$ Inner Product (내적)

$v, w \in \mathbb{R}^n$. $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots v_n w_n = \sum_{i=1}^{n} v_i w_i$

$\langle v, w \rangle = \underline{v^T w} = w^T v$

Remark5.

1. The inner product of two vectors is a sclar.
2. $\underline{v \cdot w = \|v\| \|w\| \cos\theta}$ ($\theta$ is the angle between two vector $v$ and $w$)




$$\underline{\cos\theta = \frac{v \cdot w}{\|v\| \cdot \|w\|}}, \quad -1 \leq \cos\theta \leq 1, \quad |v \cdot w| \leq \|v\| \|w\| \text{ - Schwarz inequality}$$

$$= \frac{\sum_{i=1}^{n} v_i \times w_i}{\sqrt{\sum_{i=1}^{n} v_i^2} \cdot \sqrt{\sum_{i=1}^{n} w_i^2}} \quad \leftarrow \text{두 벡터의 상관계수}$$

properties of the inner product in $\mathbb{R}^n$

1. $v \cdot w = w \cdot v$
2. $u \cdot (v + w) = u \cdot v + u \cdot w$
3. $\alpha (v \cdot w) = (\alpha v) \cdot w = v \cdot (\alpha w)$. $\forall \alpha \in \mathbb{R}$
4. $\|v\|^2 = v \cdot v = v^T v$

Theorem1. Properties of Vector Algebra in $\mathbb{R}^n$

$u, v, w \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

- properties of vector addition
  1. $(u + v) + w = u + (v + w)$
  2. $v + w = w + v$
  3. $0 + v = v$
  4. $v + (-v) = 0$ (벡터) (and $-v = -1\, v$)
- properties involving scalar multiplication
  1. $\alpha(v + w) = \alpha v + \alpha w$
  2. $(\alpha + \beta) v = \alpha v + \beta v$
  3. $\alpha(\beta v) = (\alpha\beta) v$

Def7. Orthogonal Vectors

$v, w \in \mathbb{R}^n$. if $v \cdot w = 0$ and we wrote $v \perp w$

$(v \cdot w = \|v\| \|w\| \cos\theta)$ < parallel : $\theta = 0 \rightsquigarrow \cos\theta = 1$ ⟶
perpendicular : $\theta = 90 \rightsquigarrow \cos\theta = 0$

센지 나올듯

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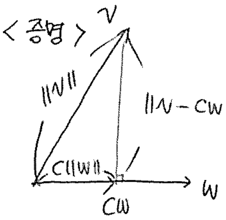



$$\|v\|^2 = c^2 \|w\|^2 + \|v - cw\|^2$$

$$\cancel{v^T v} = c^2 w^T w + (v - cw)^T (v - cw) = c^2 w^T w + \cancel{v^T v} - c v^T w - c w^T v + c^2 w^T w$$

$$c \cdot v^T w = c^2 w^T w$$

$$\therefore c(c \cdot w^T w - v^T w) = 0$$

$c \neq 0$ → 아예 0이 되버림

$$\rightarrow c = \frac{v^T w}{w^T w}$$

$$\therefore \cos\theta = \frac{c\|w\|}{\|v\|} = \frac{v^T w}{w^T w} \times \frac{\sqrt{w^T w}}{\sqrt{v^T v}} = \frac{v^T w}{\|w\| \|v\|}$$

참) $v^T w = w^T v$
$u^T (v + w) = u^T v + u^T w$

참) $\|v\|^2 + \|w\|^2 = \|v - w\|^2 + 2\|v\| \|w\| \cos\theta$
$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$

$\boxed{\text{Theorem 2}}$ Schwarz Inequality (코시 - 슈바르츠 부등식)

$v, w \in \mathbb{R}^n$, $|v \cdot w| \le \|v\| \|w\|$

proof) $v = 0 \rightsquigarrow$ trivial (당연하다)

$v \ne 0$

$\downarrow$ Let $f(x) = \| xv - w \|^2$, then $f(x) \ge 0$, $\forall x \in \mathbb{R}$

Note $f(x) = (xv - w)^T (xv - w)$

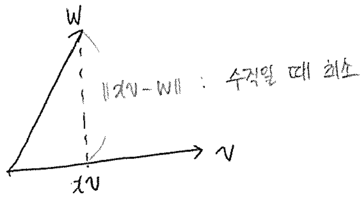
$= (v^T v) x^2 - (2 v^T w) x + w^T w$

$D = (v^T w)^2 - (v^T v)(w^T w) \le 0$

$(v^T w)^2 \le (v^T v)(w^T w)$

$\therefore |v^T w| \le \|v\| \|w\|$

└ $v = 0$ or $v, w$ 이 평행할 때 등호성립




Remark 7. $\left(\sum_{i=1}^{n} v_i w_i\right)^2 \le \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$

+) The triangle Inequality

$v, w \in \mathbb{R}^n$, $\|v + w\| \le \|v\| + \|w\|$

proof) $\|v + w\|^2 = (v + w)^T (v + w) = v^T v + 2 v^T w + w^T w \le v^T v + 2\|v\|\|w\| + w^T w$

$= \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$

$= (\|v\| + \|w\|)^2$

$\therefore \|v + w\|^2 \le (\|v\| + \|w\|)^2$

$\therefore \|v + w\| \le \|v\| + \|w\|$ $(\because \forall v \in \mathbb{R}, \|v\| \ge 0)$

<Summary>

$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n$

1. The norm of $v$ is $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$
2. The norm satisfies the properties given below Def5.
3. A unit vector is a vector of magnitude 1.
4. The dot product of $v$ and $w$ is $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = v^T w$
5. The dot product satisfies the properties given below Remark5.
6. Moreover, we have $v \cdot v = \|v\|^2$ and $|v \cdot w| \le \|v\| \|w\|$ and also $\|v + w\| \le \|v\| + \|w\|$
7. The angle $\theta$ between the vectors $v$ and $w$ can be found by using the relation $v \cdot w = \|v\| \|w\| \cos\theta$
8. The vectors $v$ and $w$ are orthogonal (perpendicular) if $v \cdot w = 0$

## 1.3 Matrix

Q. what is a matrix?

$$A = \begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} u_1 \\ \\ u_n \end{array} \\ \begin{array}{ccc} v_1 & \quad & v_n \end{array} & \end{array} = (v_1, v_2 \cdots v_n) = \begin{pmatrix} u_1^T \\ \vdots \\ u_m^T \end{pmatrix}$$

$(m \times n)$

↑ 이 형태 多 사용

ex) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = (v_1, v_2, v_3) = \begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} \rightarrow u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad u_1^T = (1, 2, 3)$

$u_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad u_2^T = (4, 5, 6)$

$v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

* 벡터는 무조건 세로!

**Remark 8.** A column vector in $\mathbb{R}^n$ can be regarded as an $n \times 1$ matrix and a row vector in $\mathbb{R}^n$ can be regarded as an $1 \times n$ matrix.

**Def 8.** Matrix Algebra

$A = (A_{ij})_{i=1,\dots,m;\, j=1,\dots,n}$ and $B = (B_{ij})_{i=1,\dots,m;\, j=1,\dots,n}$ be $m \times n$ matrixes.

1. $A \pm B \triangleq (A_{ij} + B_{ij})$ ← can be defined two matrices with the same size
2. $\alpha A \triangleq (\alpha A_{ij})$ for $\alpha \in \mathbb{R}$

**Remark 9.** Properties of matrix algebra

1. $A + B = B + A \quad [(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = (B+A)_{ij}]$
2. $(A+B) + C = A + (B+C)$
3. $A + O = O + A = A \quad [(A+O)_{ij} = A_{ij} + O_{ij} = A_{ij}]$
4. $\alpha(A+B) = \alpha A + \alpha B \quad [\{\alpha(A+B)\}_{ij} = \alpha(A_{ij} + B_{ij}) = \alpha A_{ij} + \alpha B_{ij} = (\alpha A + \alpha B)_{ij}]$
5. $(\alpha + \beta)A = \alpha A + \beta A$
6. $(\alpha\beta)A = \alpha(\beta A)$

$\boxed{\text{Def 9.}}$ Matrix Product

$A = (A_{ij}) \quad 1 \le i \le m,\ 1 \le j \le n$, $m \times n$ matrix

$B = (B_{jk}) \quad 1 \le j \le n,\ 1 \le k \le s$, $n \times s$ matrix

$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} \cdot B_{jk}$

$(AB)_{ij} = \sum_{k}^{n} A_{ik} B_{kj}$

ex) $\underset{2 \times 3}{\overset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}} \times \underset{3 \times 2}{\overset{B}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}} = \overset{AB}{\begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix}}$

$(AB)_{11} = \sum_{j=1}^{?} A_{1j} \cdot B_{j1}$

Remark 10.

1. matrix multiplication can be defined only ....
   첫 행렬 열 개수와 두번째 행렬 행 개수 같을 때 정의, 곱해서 나온 Matrix는 첫 행렬 행개수 × 두 번째 행렬 열 개수로 나타남.

   ⁎ commute 교환하다.

2. The matrix product is not commutative. so $AB \neq BA$ usually. (If both $AB$, $BA$ are defined)

3. If matrix has the same number of rows and columns, it is called a <u>square matrix.</u> (정방행렬, $n \times n$)

   If a square matrix has zero entry except possibly on the diagonal, it is called a <u>diagonal matrix</u>

   . (대각제외 '0', $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ 대각 diagonal entry, diagonal entry 제외 나머지 원소 = off - diagonal = 비대각

   $\begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix}$ low triangular matrix

   → triangular matrix (삼각행렬)

   $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$ upper triangular matrix

4. If $A$ is square matrix, $A^3 = A^2A = AA^2 \rightarrow A^r = AA^{r-1} = A^2A^{r-2} \cdots$ can be defined.

5. $n \times n$ identity matrix : $I = (a_{ij})$ where $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ otherwise. (항등행렬) ex) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. $v \cdot u = v^T u = \langle v, u \rangle$ = scalar

7. $(aA)B = A(aB) = a(AB)$

8. $(AB)C = A(BC)$ 앞에 순서 바꾸는 게 아니면 가능

9. $A(B+C) = AB + AC$ $\left\langle \begin{array}{l} \{A(B+C)_{ij}\} = \sum_k A_{ik}(B+C)_{kj} = \sum_k A_{ik}(B_{kj} + C_{kj}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad // \\ \{AB+AC\}_{ij} = AB_{ij} + AC_{ij} = \sum A_{ik}B_{kj} + \sum_k A_{ik}C_{kj} \end{array} \right.$

   $(AI)_{ij} = \sum_k A_{ik} I_{kj}$
   $= A_{ij} I_{jj}$
   $= A_{ij}$

10. $(A+B)C = AC + BC$

11. $AI = A$ and $BI = B$ with $m \times n$ matrix $A$, $n \times s$ matrix $B$, $I$ is $n \times n$ matrix.

Def 10. Transpose of a Matrix (행번호 열번호 바뀌는 것)

$A = (a_{ij})$ $1 \le i \le m$, $1 \le j \le n$

$A^T = (a_{ji})$ $1 \le j \le n$, $1 \le i \le m$

. $(A^T)^T = A$

Remark 11.

1. $A = (v_1, \cdots, v_n)$, an $m \times n$, $A^T = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}$, an $n \times m$ matrix.
2. If <u>$A^T = A$</u>, then the matrix <u>A is symmetric</u>. (대칭행렬)
3. $(A^T)^T = A$
4. $(A+B)^T = A^T + B^T$ $\{(A+B)^T\}_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T + B^T$
5. $(AB)^T = B^T A^T$ $\{(AB)^T\}_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$
   $\{B^T A^T\}_{ij} = \sum_k (B^T)_{ik} (A^T)_{kj} = \sum_k B_{ki} A_{jk}$

<참 !
⁎ 대칭행렬 필요충분조건 $AB = BA$

⁎ skew symmetric : 교대대칭
$A^T = -A$ 형태

< Blockwise Product > (참)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$\Downarrow$

$3 \times 4$

└ 참) 회귀분석에 ~~유~~ 사용

$$\underset{n\times 1}{y} = \underset{n\times p}{X}\underset{p\times 1}{\beta} + \varepsilon$$

$X = (X_1, X_2, \cdots, X_p)$ ; $n\times p$

$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$ ; $p\times 1$

$X\beta = X_1\beta_1 + \cdots + X_p\beta_p$

ex) $$\underset{3\times 4}{\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}} \underset{4\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}} = 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 3\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} + 4\begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

Eg3. $\underset{n\times m}{A} \sim \underset{m\times 1}{v}$

1) $(A_1, \cdots, A_m)\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = A_1v_1 + A_2v_2 + \cdots + A_mv_m$

2) $\begin{pmatrix} A_1^T \\ \vdots \\ A_m^T \end{pmatrix} v = \begin{pmatrix} A_1^Tv \\ \vdots \\ A_m^Tv \end{pmatrix}$

- 미지수 개수 > 차수 일 때 행렬 ~~유~~ 쓰임

## 1.4 System of Linear Equations

- $ax = b \longleftrightarrow x = \begin{cases} \frac{b}{a} & (a\neq 0) \\ \times & (a=0,\ b\neq 0) \\ \mathbb{R} & (a=0,\ b=0) \end{cases}$

- $ax + by = c$
  $ax = -by + c \longrightarrow x = \begin{cases} -\frac{by-c}{a} & a\neq 0 \\ \times & a=0,\ -by+c\neq 0 \\ \mathbb{R} & a=0,\ -by+c=0 \end{cases}$

  └ $\{(x,y) \mid x = \frac{-by+c}{a}\}$

⇒ 식 많고 미지수 적을수록 해가 없을 확률 ~~유~~, 식 적고 미지수 많을수록 해가 많을 확률 ~~유~~ → 감각

$$\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \ \end{pmatrix}$$

└ Matrix 특징에 따라 해 개수, 종류 알 수 있음.

**Def 11.** Systems of Linear equation

ex) $2x_1 + 3x_2 + x_3 = 1$
$x_1 + 0 \cdot x_2 + 2x_3 = 0$
$0 \cdot x_1 + x_2 - x_3 = 1$

$$\Leftrightarrow \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

* $Ax = b$
↳ $Ax$ is equal to a linear combination of the column vectors of $A$.

Note that finitely many application of the following operations with system <u>do not change the solution sets</u>

R1: Interchange two equations in system  식들 순서 바꿔도 해는 동일

ex) $x_1 + 0 \cdot x_2 + 2 \cdot x_3 = 0$
$2x_1 + 3x_2 + x_3 = 1$
$0 \cdot x_1 + x_2 - x_3 = 1$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

R2: Multiply an equation in system by a nonzero constant. 각 식 혹은 한개의 식에 같은 값을 곱해줘도 해 동일

ex) $\begin{pmatrix} 4 & 6 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 4 & 6 & 2 \\ 2 & 0 & 4 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

R3: Replace an equation in system with the sum of itself and a different equation of the system.
원래 행에 다른 어느 행의 상수배를 더해도 (다른행들은 그대로 두고) 해는 동일!

$$\begin{pmatrix} A_1^T \\ A_2^T \\ A_3^T \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \xrightarrow{\text{바꿔보자}} \begin{pmatrix} A_1^T + rA_2^T \\ A_2^T \\ A_3^T \end{pmatrix} x = \begin{pmatrix} b_1 + rb_2 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{동일}$$

$$= \begin{pmatrix} A_1^T x \\ A_2^T x \\ A_3^T x \end{pmatrix} + \begin{pmatrix} rA_2^T x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} rb_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 + rb_2 \\ b_2 \\ b_3 \end{pmatrix}$$

ex) $x + y = 1$
$3x - 2y = 0$
$\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 을 푸는거나

$\Updownarrow$

$x + y = 1$
$5x + 0y = 2$
$\begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 을 푸는거나 해 동일

$2(x+y) = 2$ 와 $3x - 2y = 0$ 을 더함

↳ $R_1\ R_2\ R_3$ 은 $a_{11}x_1 + \cdots + a_{1n}x_n = b_1$, $\vdots$, $a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$ 식에 적용되는 법칙이고 이게 행렬에 적용되면 ERO가 되는것! 이미 행렬에 적용하는 걸로 공부했지만..ㅎ

Remark 12.

$Ax = b$.

$$\underset{m \times n}{\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & & A_{mn} \end{pmatrix}} \underset{n \times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{m \times 1}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}} \rightarrow \overset{(A|b)}{\left(\begin{array}{ccc|c} A_{11} & \cdots & A_{1n} & b_1 \\ \vdots & & & \vdots \\ A_{m1} & \cdots & A_{mn} & b_m \end{array}\right)}$$

∴ Augmented matrix or partitioned matrix

**Def 12.** Elementary Row Operations

1. (Row interchanging) Interchange two row vectors in a matrix. 어느 한 행과 다른 행을 바꾼다
2. (Row scaling) Multiply a row vector in a matrix by a nonzero constant. 어떤 행에 상수배
3. (Row addition) Replace row vector in a matrix with the sum of itself and a different row vector of the matrix. 한 행을 자기자신과 다른행과의 합으로 대체

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 11 & 13 & 15 \\ 7 & 8 & 9 \end{pmatrix}$$

**Def13** Row Equivalent Matrices

Matrices A and B are <u>row equivalent</u> if each matrix can be obtained from the other by <u>finetely many application of elementary row operations</u>, which we denote by <u>$A \sim B$</u>.

ex) $\underset{A}{\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right)} \underset{\text{행바꿈}}{\sim} \left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right) \iff A \sim B$ (A와 B가 equivalent 하다)

Theorem 3. Invariance under the row equivalence

If $(A|b)$ and $(H|c)$ are <u>row equivalent augmented matrices</u>, then the linear systems $Ax = b$ and $Hx = c$ have the <u>same solution set</u>.

$(A|b) \overset{ERO}{\sim} (H|c)$
$Ax = b$ $Hx = c$ 해 동일

ex) $\begin{matrix} 2x+3y=1 \\ x-y=2 \end{matrix} \rightarrow \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & 0 & 7 \\ 1 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & 0 & 7 \\ 0 & -1 & \frac{3}{5} \end{array}\right)$ $2-\frac{7}{5}$

$\therefore x = \frac{7}{5}$
$\therefore y = -\frac{3}{5}$ 바로 알 수 있음!

<u>즉, 풀기 쉬운 형태로 바꾸기 위해 ERO를 하는 것!</u>

<u>1로 만드는 거! 피봇 위아래는 0 ....</u> — Gauss Jordan

**Def14** Row-Echelon Form, Pivot <정확히> (1로 만들고 나머지 0 → Reduced Row-echelon form)

A matrix is row-echelon form if it satisfies two condition

1. All rows containing <u>only zeros</u> appear <u>below</u> rows with <u>nonzero entries</u>. (성분)

   0만 있는 row는 반드시 숫자보다 아래에 나와야 (맨 아래일 필요 x)

   ex) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ x $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ o $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2. The first nonzero entry in any row appear in a column to the right of the first nonzero entry in any preceding row.

   위에오는 row에서 첫번째 나오는 숫자보다 그 밑 row에 첫번째 나오는 숫자가 뒤에 있어야

   ex) $\begin{pmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 6 \end{pmatrix}$ pivot

   * 각 행에서 처음으로 나오는 0이 아닌 숫자 : 피봇

ex))

The linear system $(H|c)$ in <u>row-echelon form</u> that is equivalent to <u>linear system</u> $(A|b)$ is the "simpler" equations with the same solution set.

→ How to solve a linear system in row-echelon form? "<u>Back substitution</u>" 역 대입법

마지막 줄부터 차례로 올라옴

- $\underset{3\times3\ \ 3\times1}{Ax = b} \overset{ERO}{\sim} Hx = C$

  $\underset{3\times1}{x}$ $= \left(\begin{array}{ccc|c} -5 & -1 & 3 & 3 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 2 & -4 \end{array}\right) \longrightarrow$

  $x_3 = -2$
  $x_2 = \frac{8+10}{3} = 6$
  $x_1 = \frac{15}{-5} = -3.$

  $\therefore x = \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}.$

- $Hx = c.$

  $(H|c) = \left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right) \rightarrow$ No solution

  * The left side of a final augmented matrix is ~~in~~ reduced row-echelon form.

Def15. <u>Consistent Linear System</u>

A Linear system having no solution is <u>inconsistent</u>. 해 x

If it has one or more solutions, the linear system is said to be <u>consistent</u>. 해 o

**Remark 13** Gauss Reduction (◺ 꼴 만드는 방법 중 하나) <정확히>

procedure for solving $Ax=b$ is known as Gauss reduction with back substitution.

ERO를 이용해서 피봇을 1로 만들고 각 피봇 밑으로는 숫자 없게 하는 방법 (맨위에 0이 있으면 바꿈.)

ex)
$$\begin{pmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

: $R_1 \to \frac{1}{2}R_1$

: $R_2 \to R_2 - 2R_1$
: $R_3 \to R_3 + R_2$

: $R_3 \to R_3 + R_2$
: $R_4 \to R_4 + R_2$

(위에 있는 row 사용해야!)

The last matrix is row-echelon form with both pivots equal to 1.

ex) Solve the linear system - 피봇 위아래 다 0으로 만들기('I')

$$\begin{aligned} x_2 - 3x_3 &= -5 \\ 2x_1 + 3x_2 - x_3 &= 7 \\ 4x_1 + 5x_2 - 2x_3 &= 10 \end{aligned} \rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 0 & -4 \end{array}\right)$$

: $R_1 \leftrightarrow R_2$ ; $R_1 \to \frac{1}{2}R_1$ ; $R_3 \to R_3 - 4R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 4 & \text{[illegible]} \\ 0 & 1 & -3 & \text{[illegible]} \\ 0 & 0 & \text{[illegible]} & \text{[illegible]} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$R_3 \to R_3 + R_2$ ; $R_1 \to R_1 - \frac{3}{2}R_2$ ; [illegible] ; $R_2 \to R_2 - R_3$, $R_3 \to -\frac{1}{3}R_3$

$\therefore x = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ 한번에 찾을수 있음!

- 각 피봇이 그 값을 구해줌. (피봇이 없는 변수: free variable - 아무거나 다 됨)
- solution set 은 집합 형태로 적을 것 ex) $\left\{ \begin{pmatrix} 2s+1 \\ s \\ 3 \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}$

**Remark 14.**

the linear system $Ax=b$ is consistent (if and only if the vector $b \in \mathbb{R}^n$ is in the span of the column vector of $A$)

$\rightarrow Ax = b$

$(A_1, A_2, \cdots, A_n)\, x = b$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\rightarrow A_1x_1 + A_2x_2 + \cdots + A_nx_n = b$ (벡터, 스칼라)

즉, A의 column vector 들의 linear combination 이 b가 됨.
즉, 〃 ⟹ Span 된 집합안에 b가 있으면 해가 있는것 (consistent)

**Theorem 4.** Solutions of $Ax=b$

1. The system is <u>inconsistent</u> if and only if the augmented matrix $(H|c)$ has a row with all entries zero to the left of the partition and a nonzero entry to the right of the partition. $(0\,0\,0\,|\,\text{숫자})$ 꼴일때 해 없음.
2. If the system is <u>consistent</u> and every column of H contains a pivot, the system has a unique solution. 피봇있으면 해 하나
3. If the system is <u>consistent</u> and some columns of H have no pivot, the system has infinitely many solutions, with as many free variable as the number of pivot-free columns in H. 피봇 없는 변수도 해 ○

ex) $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

이들을 Linear Combination 해서 $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ 만드는것
↳ 3차원 만들 수 있으니 해가 100% 있음.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
↳ 2차원 평면 밖에 안됨
→ $c \neq 0$ 일때 해 없음

$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right) \rightarrow \begin{aligned} x_2 &= s \in \mathbb{R} \\ x_5 &= t \in \mathbb{R} \\ x_1 &= 2s \end{aligned}$ $\left\{ \begin{pmatrix} 2s \\ s \\ 2 \\ 3 \\ t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$

pivot × → $x_2, x_5$ 해 ○

$\therefore$ span $(A_1, \cdots, A_n) \ni b \rightarrow$ consistent
$\not\ni b \rightarrow$ inconsistent

**Def 16.** Elementary Matrix <정확히>

A matrix obtained from an identity matrix by an elementary row operation is elementary matrix.
I로 ERO해서 만들 수 있는 매트릭스 (ERO 1번 한거!)

**Theorem 5.** $A \sim B \Rightarrow B = EA$ ($E$ : elementary matrix)

ex) $\underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \underset{ERO}{\sim} \underset{E}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \rightarrow \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \sim \underset{B}{\begin{pmatrix} c & d \\ a & b \end{pmatrix}}$

ex) $\underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ E.M (o)
$\sim \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
$\sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ E.M (X)
L 아래행 2배의 값을 더하고 즉 2번 시행

$Ax = b \sim Hx = c$
$\downarrow$
$H = \underbrace{E_r \cdots E_2 E_1}_{\text{유한번}} A$ ↙ A와 H의 관계 즉, $A \sim B \Leftrightarrow B = E_r \cdots E_2 E_1 A$.

## 1.5 Inverse of Square Matrices.

Q1. $CA = I$ or $AD = I$ 인 것 Exist? (존재성)

Q2. $CA = I$ or $AD = I \longrightarrow C = D$ ?

**Theorem 6.** Uniqueness of an Inverse 역행렬의 유일성

If $AC = DA = I$ for an $n \times n$ matrix, then $C = D$.

proof) $D = DI = DAC = IC = C$

**Def 17.** Invertible Matrix

$\left.\begin{array}{l} A : n \times n \\ C : n \times n \\ AC = CA = I \end{array}\right\} \rightarrow$ we say $C$ is a inverse of $A$ demoting $C$ as $A^{-1}$.

· invertible = non-singular / not invertible = singular ⇝ $Ax = b$ 못풀어
consistent 없어

**Remark 15.**

1. $r > 0$, $A^{-r} = A^{-1} \cdot A^{-(r-1)}$, $A^0 = I$
2. ☆ Elementary Matrix : invertable

1) $I = \begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \end{pmatrix} \sim E = \begin{pmatrix} e_3^T \\ e_2^T \\ e_1^T \end{pmatrix}$ $F = \begin{pmatrix} e_3^T \\ e_2^T \\ e_1^T \end{pmatrix} = (e_3, e_2, e_1)$

$EF = \begin{pmatrix} e_3^T \\ e_2^T \\ e_1^T \end{pmatrix}(e_3, e_2, e_1) = I$.

2) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim E\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $F = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

$EF = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

→ 어떤 row에다 r배해준 행렬은 그행을 $\frac{1}{r}$ 곱으로 바꾸면 역행렬이 됨

3) 일반화
$E = \begin{pmatrix} e_1^T \\ re_k^T \\ e_n^T \end{pmatrix}$ $F = \begin{pmatrix} e_1^T \\ \frac{1}{r}e_k^T \\ e_n^T \end{pmatrix} = (e_1, e_2, \cdots, \frac{1}{r}e_k, \cdots e_n)$

$EF = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = I$

**Theorem 7.** Inverses of Products

$A, B$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

③ $E = \begin{pmatrix} e_1^T + e_2^T \\ e_2^T \\ \vdots \\ e_n^T \end{pmatrix} \sim E^{-1}$ 존재!?

Remark 15-2 보충.

1. Identity Matrix 의 i행과 j행을 바꾼것의 역행렬은 그 자신임.

ex) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Identity Matrix 의 어떤 Row에 r배 해준 행렬의 역행렬은 그 행렬의 row에 $\frac{1}{r}$ 곱해준 것임.

ex) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. Identity Matrix 의 어떤 (i)행에 r배 후 k행에 더해준 행렬의 역행렬은 I에 i행에 −r배한 후 k행에 더해준 것임.

ex) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3행에 4배후 1행더함

→ Every elementary Matrix is invertible.

Remark 16.

$A \sim I \iff \underbrace{E_r \cdots E_2 E_1}_{A^{-1}} A = I$

$E_1^{-1} E_2^{-1} \cdots E_r^{-1} = (E_r \cdots E_1)^{-1}$

Lemma1. The linear system $Ax = b$ with an $n \times n$ matrix $A$ has a solution for every choice of $b \in \mathbb{R}^n$ if and only if $A$ is row equivalent to the $n \times n$ identity matrix $I$.

→ $A$ : invertible $\iff A \sim I$
= consistent (해 유일하게 되니까)

proof) $(\Leftarrow)$ $(A|b) \sim (I|c)$ $\exists c$ → 임의의 c 에대해 항상 c가 있음 → $Ax=b$ 에서도 b에 관계없이 c를 가지게 됨.

$Ax=b$ $\quad Ix=c$ 해 $\quad x=c$

$(\Rightarrow)$ 대우명제증명 $A \sim H \neq I \longrightarrow H(n) = 0$

we write $H = E_r \cdots E_2 E_1 \cdot A \rightarrow (A|b) \sim (H|E_r \cdots E_1 b)$

let $b = (E_r \cdots E_1)^{-1} e_n$, $e_n = (0, \cdots 0, 1)^T$ $\longrightarrow$ $H(n)$ $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \vdots \\ & & & & 1 \end{array}\right)$

└ No solution!

→ $A \sim H \neq I$ 는 틀림 ($\neq I$ 면 No solution)

역행렬 있는 행렬은 계속 하다 보면 I 가 될 수 밖에 없음. 즉 역행렬이 없다는 것은 0인 줄이 존재한 다는 것임.

Theorem 8. $A, C$ : $n \times n$ matrix, Then $CA = I$ if and only if $AC = I$ $(CA = I \iff AC = I)$

proof) $AC = I \longrightarrow x = Cb$ is a solution of $Ax = b$ $\forall b \in \mathbb{R}^n$
($A^{-1}$)

→ by Lemma 1. $A \sim I$

→ by Remark 16. $E_r \cdots A_1 = I$

Q3. How can we compute the inverse of an $n\times n$ Matrix $A$ if it exist?

$$AX = I$$

$$X = (X_1, \cdots, X_n)$$

$$AX = A(X_1, \cdots, X_n) = (AX_1, \cdots, AX_n) = I = (e_1, \cdots, e_n)$$

ex) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \sim I$

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{A} = I.$$

$\therefore X = E_2 E_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$AX_1 = e_1 \Leftrightarrow (A|e_1) \sim (I|C_1) \qquad X_1 = C_1 = E_r \cdots E_1 e_1$

$\vdots \qquad\qquad \vdots$

$AX_n = e_n \Leftrightarrow (A|e_n) \sim (I|C_n) \qquad X_n = C_n = E_r \cdots E_1 e_n$

$$X = (X_1, \cdots, X_n) = (C_1, \cdots, C_n) = E_r \cdots E_1 (e_1, \cdots, e_n) = E_r \cdots E_1 = A^{-1}$$

즉, $(A|I) \sim (I|C) \qquad A^{-1} = C$

Eg10. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad-bc \neq 0$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \sim \underset{R_1 \to \frac{1}{a}R_1}{\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right)} \sim \underset{R_2 \to R_2 - c\times R_1}{\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right)} \sim \underset{R_2 \to \frac{a}{ad-bc}R_2}{\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)}$$

$$\sim \underset{R_1 \to R_1 - \frac{b}{a}R_2}{\left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)}$$

ex) $\left(\begin{array}{cc|cc} 2 & 9 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 9 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -4 & 9 \\ 0 & 1 & 1 & -2 \end{array}\right)$ $\hookrightarrow A^{-1}$

Remark17. Equivalent Conditions with Invertibility

1. $\exists A^{-1}$ ($A$ is invertible)
2. $A \sim I$ ($A$ is row equivalent to $I$)
3. $A = E_r \cdots E_1$ ($A$ can be expressed as a product of elementary matrix)
4. $Ax = {}^\forall b$ : consistent (The system $Ax = b$ has a solution for each $\in \mathbb{R}^n$)
5. span $(A_1, \cdots, A_n) = \mathbb{R}^n$ (The span of column vectors of $A$ is $\mathbb{R}^n$) 어떤 b든 가져와도 $Ax=b$를 만족하는 b존재 뜻

## 1.6. Homogeneous Systems.

$\boxed{\text{Def 18}}$ A homogeneous system

$\begin{cases} Ax = b \text{ is linear system} \\ b = 0 \end{cases}$ : homogeneous

· homogeneous linear system $Ax = b$ is always consistent since $x = 0$ is a solution.
└ we called 'trivial solution'

→ then non trivial solution exist?

ex) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

피봇 X

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left\{ \begin{pmatrix} -2s \\ s \end{pmatrix} \middle| s \in \mathbb{R} \right\}$ ∴ 해가 무수히 많음.

**Theorem 9.** Structure of the Solution Set of $Ax = 0$

If $h_1$ and $h_2$ are solutions of $Ax = 0$, then any linear combination $rh_1 + sh_2$ solve the linear system $Ax = 0$

proof. $A(rh_1 + sh_2) = rAh_1 + sAh_2 = r0 + s0 = 0$

→ Every linear combination of solutions of a homogeneous system $Ax = 0$ is again a solution of the system.

$\boxed{\text{Def 19}}$ Subspace of $\mathbb{R}^n$

(homogeneous system 은 subspace 임)

$W \subset \mathbb{R}^n$ ($W$ is subset of $\mathbb{R}^n$)

$u, v \in W$ $\begin{cases} u + v \in W \\ av \in W \ \forall a \in \mathbb{R} \end{cases}$ 일 때 $W$ : subspace of $\mathbb{R}^n$
└ 일반적으로 차원을 낮추거나 원점을 지날 때 가능

덧셈에도 닫혀있고 스칼라 곱에도 닫혀 있을 때

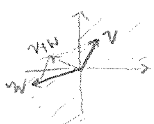
Eg8. 1. $\mathbb{R}^m$ is subspace of $\mathbb{R}^n$ for positive integer $m \le n$

2. $v \in \mathbb{R}^n$, $W = \{ av \mid a \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^n$ $(a_1v + a_2v = (a_1 + a_2)v)$

Thm 3) $v_1 \cdots v_k \in \mathbb{R}^n$ $(k > 0)$, span $(v_1, \cdots, v_k)$ is subspace of $\mathbb{R}^n$ → $u = r_1v_1 + \cdots + r_nv_n$, $w = s_1v_1 + \cdots + s_nv_n$

4. The set $\{0\}$ where $0 \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$. (0벡터)

5. The set $\{ (x, y) \in \mathbb{R}^2 : xy \ge 0 \}$ is not subspace of $\mathbb{R}^2$




ex) $\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ 3 & -5 & 5 & -4 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ ?

$x = \begin{pmatrix} -5r + 3s \\ -2r + s \\ r \\ s \end{pmatrix} = sp\left( \begin{pmatrix} -5 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$

**Def 20** Row space, Column Space, Null space $\qquad A^T_{(1)} = 1\times n$ 짜리 벡터

$A$ is $m\times n$ Matrix, $A = (\underset{m\times 1 \text{짜리 벡터}}{A_1}, \cdots, A_n) = \begin{pmatrix} A^T_{(1)} \\ \vdots \\ A^T_{(m)} \end{pmatrix}$

- Row space of $A \triangleq \text{span}(\underset{n\times 1 \text{짜리}}{A_{(1)}}, \cdots, A_{(m)}) \triangleq \text{row}(A)$ → Row space of $A$ is a subspace of $\mathbb{R}^n$
  반대라고 생각하자! row면 column 수
- Column space of $A \triangleq \text{span}(A_1, \cdots, A_n) \triangleq \text{Col}(A)$ → Column space of $A$ is a subspace of $\mathbb{R}^m$
- Null space of $A \triangleq \{x \mid Ax = 0\} \triangleq \text{Null}(A)$ → Null space of $A$ is a subspace of $\mathbb{R}^n$
  └ homogeneous linear system 을 만족하는 해들의 집합
  └ $A^{-1}$ 존재하면 $x=0$

ex) $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

- row space of $A \triangleq \text{span}((1,0,3)^T, (0,1,-1)^T)$ in $\mathbb{R}^3$
- Column space of $A \triangleq \text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}3\\-1\end{pmatrix}\right)$ in $\mathbb{R}^2$
- Null space of $A \triangleq \text{span}\left(\begin{pmatrix}-3\\1\\1\end{pmatrix}\right)$ in $\mathbb{R}^3$ $\qquad x = \left\{ s\begin{pmatrix}-3\\1\\1\end{pmatrix} \middle| s\in\mathbb{R} \right\}$ 직선
  └ 풀어봐야 함!

$\begin{pmatrix} 1 & 0 & 3 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{pmatrix}$ $\qquad \mathbb{R}^3 \quad \begin{pmatrix} -3s \\ s \\ s \end{pmatrix}$

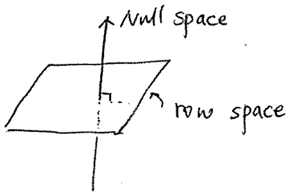
* $\text{span}\left(\begin{pmatrix}a\\b\\c\end{pmatrix}\begin{pmatrix}d\\e\\f\end{pmatrix}\right)$ : 평면 → 즉!
  - 3개짜리 4개 : 3차원 (변함없음)
  - 3개 : 3차원
  - 2개 : 2차원 (평면)
  - 1개 : 1차원 (직선)

* 행렬 기본

$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$A_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad A_{(1)} = \begin{pmatrix}1\\0\\3\end{pmatrix} \qquad A^T_{(1)} = (1, 0, 3)$

- row space vector ⊥ Null space Vector.




$x \in \text{Null}(A) \to \{x \mid Ax = 0\} = \begin{pmatrix} A^T_{(1)} \\ A^T_{(m)} \end{pmatrix} x = 0$

$y \in \text{row}(A) \to \text{span}(A_{(1)}, \cdots, A_{(m)}) = r_1 A_{(1)} + \cdots + r_m A_{(m)} \quad (r \in \mathbb{R})$

$x \cdot y = y^T x = r_1 \underline{A^T_{(1)} x} + \cdots + r_m \underline{A^T_{(m)} x} = 0 \qquad \therefore$ orthogonal

ex) $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ $\qquad \text{row}(A) = \text{span}\left(\begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}\right)$ : 2차원
└ Null space 는 이 평면에 수직인 것! (원점 지나면서, $x=0$ 은 trivial 하니까)
근데 내적해서 0 되려면 Null space 는 0벡터 뿐이 안됨

**Remark 18.** Column Space Criterion

Linear system $Ax = b$ is consistent if $b$ is a linear combination of the column vectors of $A$. (Remark 14)

$\Updownarrow$

Linear system $Ax = b$ is consistent if $b$ is in the column space of $A$.

(T/F 문제 마구출제 예정. 이해 수준 이상으로!)

ex) $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = (A_1, A_2)$

$\to x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = (A_1, A_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}$
$\qquad = x_1A_1 + x_2A_2$ $\qquad$ R14.

$\to \text{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix}\right) = \left\{ x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$
$\qquad = x_1A_1 + x_2A_2 = (A_1, A_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}$

## Def 21. Linear independence (선형독립)

$w_1, \cdots, w_k \in \mathbb{R}^n$.

$\begin{cases} r_1 w_1 + \cdots + r_k w_k = 0 \\ r_1 = \cdots = r_k = 0 \end{cases}$ Linear combination ⇒ the vectors $w_1, \cdots, w_k$ are linearly independent.

만족.

계수들 모두 0 일때

ex) 

(두개이용) 0 아님 0벡터 불가

ㄴ 독립

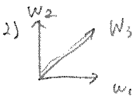
ex) 

계수 0 아니어도 가능

ㄴ 독립 X (종속)

ex) $w_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $w_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $w_3 = \begin{pmatrix}1\\1\end{pmatrix}$ 별로 안좋음 ($w_1 + w_2$)

1) $r_1 w_1 + r_2 w_2 = 0 \rightarrow \begin{pmatrix}r_1\\r_2\end{pmatrix} = 0 \rightarrow$ 선형독립

2)  $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = 0 \rightarrow$ 선형독립 X (계수 0 아님)

ex) $\begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}3\\4\end{pmatrix} \begin{pmatrix}4\\6\end{pmatrix}$

↳ $r_1\begin{pmatrix}1\\2\end{pmatrix} + r_2\begin{pmatrix}3\\4\end{pmatrix} + r_3\begin{pmatrix}4\\6\end{pmatrix} = 0$, $r_1 = r_2 = r_3 = 0$ 이면 독립

$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right)$

ㄴ pivot free → 해 多

→ nontrivial solution 존재

= 0이 아닌 해 존재

- 선형독립인 벡터들의 일부를 조합해도 그 다른애 못만듦. (만들 수 있으면 선형 종속)
- 우리는 선형독립인 애들만 필요로 함! (종속인 애들 별로 안좋음. 떡거지들..)

## Def 22. Basis for a subspace of $\mathbb{R}^n$

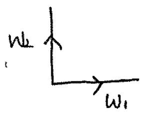
Let $W$ be a subspace of $\mathbb{R}^n$.

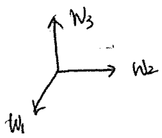
↳ A subset $\{w_1 \cdots w_k\}$ of $W$ is a basis for $W$

1) $W = \text{span}(w_1, \cdots, w_n)$

2) The vectors $w_1, \cdots, w_k$ are linearly independent.

ex) $w_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $w_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $w_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $w_4 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ in $\mathbb{R}^3$

①  → $\{w_1, w_2\}$ 는 $W$의 basis 가 아님 ($\mathbb{R}^3$인데 평면밖에 안나옴)

②  → $\{w_1, w_2, w_3\}$ 은 $W$의 basis.

③ $\{w_1, w_2, w_3, w_4\}$ 는 basis 아님 ($w_1 + w_2 + w_3 = w_4$)

## Theorem 10. Unique Linear Combinations.

The set $\{w_1, \ldots, w_k\}$ is a basis for a subspace $W \subset \mathbb{R}^n$ if and only if every vector in $W$ can be expressed uniquely as a linear combination of $w_1, \ldots, w_k$

( $\{w_1, \ldots, w_k\}$ basis for $W \subset \mathbb{R}^n$ $\iff$ $\forall w \in W$. $\exists\, r_1, \ldots, r_k \in \mathbb{R}$.
임의의 원소 w에 대해 어떤 K개의 스칼라가 있어서
s.t. $w = r_1 w_1 + \cdots + r_k w_k$ unique )
그 결과 이런 선형결합이 컵.

+

참 proof (⇒). Assume that $w = s_1 w_1 + \cdots + s_k w_k$ , $s_i \neq w_i$ $\exists\, i$

$0 = (r_1 - s_1) w_1 + \cdots + (r_k - s_k) w_k$

$\Rightarrow r_1 - s_1 = \cdots = r_k - s_k = 0.$

$\Rightarrow r_i = s_i$ (but $s_i \neq w_i$에 모순)

$w_1, \ldots w_k$ : linearly independent, $\rightarrow r_1 w_1 + \cdots + r_k w_k = 0$, $r_i \neq 0$ $\exists\, i$

$\rightarrow (2r_1) w_1 + \cdots + (2r_k) w_k = 0$ ($2r_i \neq r_i$ 이어야 되는데 둘다 0인 선형 결합을 만들었으므로 모순)

which contradicts the assumption.

< Lemma*. If 0 is unique linear combination of $w_1, \ldots, w_k$, then $w_1, \ldots, w_k$ are linearly independent >

(⇐) $\mathrm{span}(w_1, \ldots, w_k) = W$ ⇒ trivial

Theorem 10이 말하고자 하는 것. — basis의 조건!
벡터가 너무 많아서도 적어서도 안되며, 그 공간을 만들었을 때 그 방법이 하나여야함.

## Theorem 11. Let $A$ be $n \times n$ matrix. The following are equivalent.
↳ $A\,(A_1, \cdots, A_n)$

1. The linear system $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$
2. The matrix $A$ is row equivalent to the $I$ ($A \sim I$)
3. The matrix $A$ is invertible
4. The column vectors of $A$ form a basis for $\mathbb{R}^n$.

$A^{-1}$ 있으면 $\{A_1, \ldots, A_n\}$ linearly independent 함! → 해 unique → $A \sim I$ ....

(1) ~ (4) 자유롭게 왔다갔다 하며 증명 가능해야 ㅠ.ㅠ

Theorem 12. Let $A$ be $m \times n$ matrix, The following are equivalent.

1. Each consistent system $Ax = b$ has a unique solution.
   ($Ax=b$ : consistent $\rightarrow x$ is unique.)

2. The column vector of $A$ are linearly independent.

0이 아닌 애들로 $Ay=0$ 즉 선형종속.

proof) $(1 \rightarrow 2)$ If it not linear independent $\exists y \neq 0$, st $Ay = 0$.

얘는 부정하는게 아님! $Ax = b$ is consistent system (by (1)) $\exists x$ s.t $Ax = b$.

$Ay + Ax = A(y+x) = b \rightarrow y+x$ is a solution of $Ax=b$

근데 $y \neq 0$ 이므로 위의 계산 아래 $x$ 다름 $\rightarrow$ unique 해 아님! $\therefore$ 모순.

$(2 \rightarrow 1)$ $Ax=b$ is consistent system, $\exists x$, s.t $Ax=b$.

Assume $x_1$ is not unique.

then $\exists x_2 \neq x_1$, s.t $Ax_2 = b$.

$\therefore A(x_1 - x_2) = 0 \rightarrow x_1 - x_2 = 0$ $\therefore$ 모순.

← 식 많으면 해 있을 가능성↓

개수 많으면 해가 있을 가능성↑ (무한히 많겠지)

$m=n$ 유일한 해 가질 가능성 있음.

Remark 19 Corollaries of Theorem 12.

1. When $m \geq n$ consistent system $Ax=b$ has a unique solution.

2. if $m < n$ $Ax = b$ is consistent, it has infinitely many solutions.
   pivot free

3. if $m < n$ ; homogeneous linear system $Ax = 0$ has non trivial solution.

4. When $m = n$, square homogeneous system $Ax = 0$ has nontrivial solution, if and only if $A$ is not invertible.

벡터개수 = 차원

5. A basis for $\mathbb{R}^n$ cannot contain more than $n$ vectors (∵ 차원개수)
   - If $v_1, \cdots, v_n \in \mathbb{R}^m$ are linearly independent, then $n \leq m$.
   - If $v_1, \cdots, v_n \in \mathbb{R}^m$ span $\mathbb{R}^m$, then $n \geq m$

   $\Rightarrow v_1, \cdots, v_n \in \mathbb{R}^m$ form a basis for $\mathbb{R}^m$, then $n = m$