II. 벡터와 공간기하학

#11.2 벡터

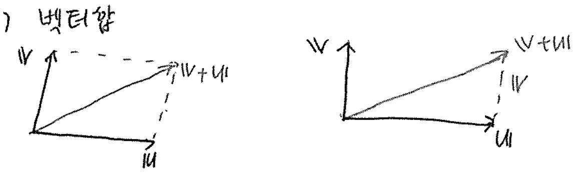
- 벡터 (vector) : 크기와 방향이 주어진 양

  $\overrightarrow{AB}$. $\vec{v}$, $\mathbb{v}$

  (1) 크기와 방향이 같으면 같은 벡터.

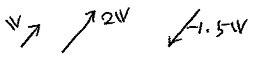
  (2) 영벡터 : 크기 0

  (3) 벡터합



  (4) 스칼라곱

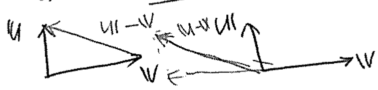
  $c>0$: 같은 방향, $c<0$: 반대 방향



  ˙ 한 벡터가 다른 벡터의 스칼라곱이면 두 벡터는 평행하다.

  (5) 벡터의차

  $u-v$ ← 순서따라가기. ($u \sim v \sim \rightarrow$)



Remark $\mathbb{v} = \langle v_1, v_2 \rangle$ 의 크기는 $\|\mathbb{v}\| = \sqrt{v_1^2+v_2^2}$ (원소끼리 연산.)

1. $\mathbb{u} = \mathbb{v} \iff u_1 = v_1$ & $u_2 = v_2$

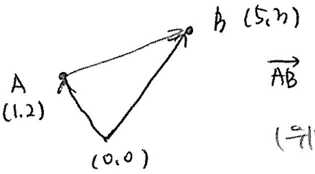
2. $\mathbb{0} = \langle 0.0 \rangle$

3. $\mathbb{u}+\mathbb{v} = \langle u_1+v_1, u_2+v_2 \rangle$

4. $k\mathbb{u} = \langle ku_1, ku_2 \rangle$

5. $\mathbb{u}-\mathbb{v} = \langle u_1-v_1, u_2 v_2 \rangle$.

Remark



$\overrightarrow{AB} = \langle 5-1, 3-2 \rangle = \langle 4.1 \rangle$

(뒤에서 앞 뺌.)

* $V_n = \{ (v_1 \cdots v_n) \mid v_i \in \mathbb{R} \}$: n차원 벡터 전체집합

정의 $\mathbb{R}^3$에서 $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$를 표준 기저 벡터라 한다.

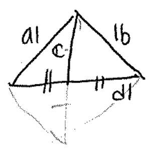
(임의의 벡터를 i, j, k로 나타낼 수 있음) 받아들이기!

정의 크기가 1인 벡터 : 단위벡터 (Unit vector), ($\frac{a}{\|a\|}$ 가 a와 같은 방향 가지는 단위벡터)

ex) $2i - j - 2k$ 와 같은 방향의 단위벡터

Sol) $a = \langle 2, -1, 2 \rangle$, $\frac{a}{\|a\|} = \frac{\langle 2, -1, 2 \rangle}{\sqrt{4+1+4}} = \frac{1}{3} \langle 2, -1, 2 \rangle$

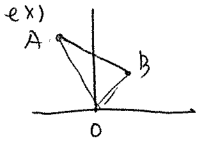
ex)



$\to c = \frac{a+b}{2}$

$d = \frac{1}{2}(b - a)$

ex)



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

ex) a와 방향이 같고 크기가 5인 벡터는?

Sol) $5 \times \frac{a}{\|a\|}$

11.3 내적 (Inner product = scalar product)

정의 $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$

$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

(1) $a \cdot a = \|a\|^2$

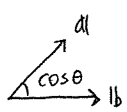
(2) $a \cdot (b + c) = a \cdot b + a \cdot c$

(3) $a \cdot b = b \cdot a$

(4) $(ka) \cdot b = k(a \cdot b)$

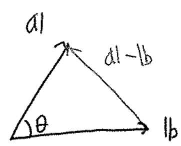
(5) $0 \cdot a = 0$

정리



$a \cdot b = \|a\| \|b\| \cos\theta \to \cos\theta = \frac{a \cdot b}{\|a\| \|b\|}$

증명)



$\|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\| \cos\theta$ ($\because$ cos 제 2법칙)

ex) $\mathbf{a} = \langle 2, 2, -1 \rangle$. $\mathbf{b} = \langle 5, -3, 2 \rangle$ 사이의 각

sol) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \dfrac{10-6-2}{\sqrt{9}\times\sqrt{38}} = \dfrac{2}{3\sqrt{38}} \rightarrow \theta = \cos^{-1}\left(\dfrac{2}{3\sqrt{38}}\right)$

$\theta = 0$ or π

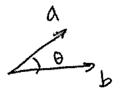
$a\cdot b = \dfrac{\cos\theta}{\|a\|\,\|b\|}$ $\quad a\cdot b = \pm$

$a\cdot b \rightarrow$

Remark '각' 하면 내적!

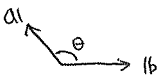
1. a . b 평행 $\Leftrightarrow \theta = 0$ or $\pi \Leftrightarrow a\cdot b = \pm \|\mathbf{a}\|\cdot\|\mathbf{b}\|$ $\quad \cos\theta = 1$ or -1

2. a . b 수직 or 직교 $\Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0 \Leftrightarrow a\cdot b = 0$



$\Leftrightarrow \mathbf{a}\cdot\mathbf{b} > 0$

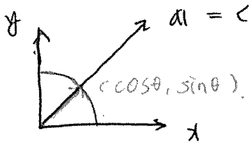
$\Leftrightarrow \mathbf{a}\cdot\mathbf{b} = 0$



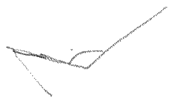
$\Leftrightarrow \mathbf{a}\cdot\mathbf{b} < 0$

ex) 2차원 벡터 $\mathbf{a}$. $\mathbf{a}$와 같은 방향 단위벡터? $= \dfrac{a}{|a|}$

sol)



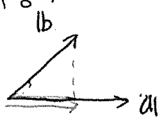
$\dfrac{a}{\|a\|} = \langle \cos\theta, \sin\theta \rangle$, $\quad \cos\theta = \dfrac{a_1}{\|a\|}$, $\sin\theta = \dfrac{a_2}{\|a\|}$

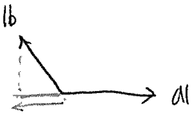


Comp$_a$ b $\|b\| \times \cos\theta$. $\dfrac{a\cdot b}{\|a\|\,\|b\|} \times \dfrac{a}{|a|}$

<사영>

1.





— $\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \dfrac{a\cdot b}{\|a\|}$; $\mathbf{a}$ 위로의 $\mathbf{b}$의 스칼라 사영.

→ $\text{proj}_{\mathbf{a}}\mathbf{b} = \underbrace{\|\mathbf{b}\|\cos\theta}_{\text{크기}} \times \underbrace{\dfrac{\mathbf{a}}{\|\mathbf{a}\|}}_{\text{방향}} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\times\mathbf{a}$; $\mathbf{a}$ 위로의 b의 벡터 사영

ex) $a = \langle -2, 3, 1 \rangle$ 위로 $\mathbf{b}$ $\langle 1, 1, 2 \rangle$의 사영

sol) $\text{Comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \dfrac{a\cdot b}{|\mathbf{a}|} = \dfrac{3}{\sqrt{14}}$

$\text{proj}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta \times \dfrac{a}{\|\mathbf{a}\|} = \dfrac{3}{\sqrt{14}}\times\dfrac{1}{\sqrt{14}}\langle -2, 3, 1 \rangle$

#11.4 벡터곱 (외적)

<u>정의</u> $\mathbb{a}, \mathbb{b} \in \mathbb{R}^3$

← 벡터

$$\mathbb{a} \times \mathbb{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \in \mathbb{R}^3$$; $\mathbb{a}$와 $\mathbb{b}$의 벡터곱 (외적)

(cf. $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3 = \|a\| \cdot \|b\| \cdot \cos\theta$)

← 스칼라

<u>Remark</u> 행렬식.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

①
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

②
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - ahf + bfg - bdi + chd - ceg$$

ex) $a = \langle 1, 3, 4 \rangle$ $b = \langle 2, 7, -5$ $a \times b$

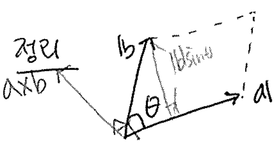
sol) $$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \langle -15 - 28,\ -(-5 - 8),\ 7 - 6 \rangle$$

a, b 의 정사각 벡터
a·b 인경우
a×b ≠ 0 ...

<u>정리</u> $\mathbb{a} \times \mathbb{b}$는 $\mathbb{a}$와 $\mathbb{b}$와 모두 수직이다. (오른손의 법칙)

↳ $\mathbb{a}, \mathbb{b}$, $\mathbb{a} \times \mathbb{b}$는 오른손 법칙을 만족함.

정리



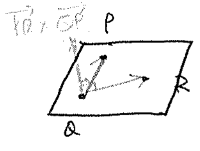
$\langle \ \|a \times b\| = \underbrace{\|a\|}_{\text{밑변}} \underbrace{\|b\| \sin\theta}_{\text{높이}} = \text{▱ 면적}. \ \rangle$

. <u>$\mathbb{a} \times \mathbb{b} = 0$ 이면 $\mathbb{a}$와 $\mathbb{b}$는 평행하다.</u>

ex) $P(1, 4, 6)$ $Q(-2, 5, -1)$ $R(1, -1, 1)$ 을 지나는 평면에 수직인 벡터.

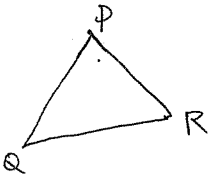
뒤-앞

sol)



$$\overrightarrow{QP} \times \overrightarrow{QR} = (3, -1, 7) \times (3, -6, 2)$$
$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 7 \\ 3 & -6 & 2 \end{vmatrix} = \langle -2 - (-42),\ -(6 - 21),\ -18 + 3 \rangle$$
$$= \langle 40, 15, -15 \rangle$$

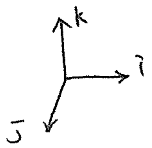
ex) P(1, 4, 6) Q(-2, 5, -1) R(1, -1, 1) . △PQR 넓이



sol) $\frac{1}{2}|\overrightarrow{QP} \times \overrightarrow{QR}| = \frac{1}{2}\sqrt{40^2+15^2+(-15)^2}$

$a \times b \neq b \times a$
u 결합법칙

Remark $i \times j = -k$
$j \times i = k$



< 외적 성질 >

→ $a \times b \neq b \times a$

→ $a \times (b \times c) \neq (a \times b) \times c$ (순서 중요!)

→ 분배법칙, 스칼라 교환법칙, 내적 교환법칙은 성립

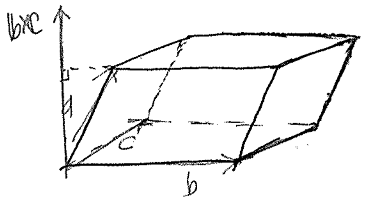
$a \times (b+c) = a\times b + a \times c$ $\quad (ca \times b) = a \times (cb)$ $\quad a \cdot (b \times c) = (a \times b) \cdot c$

정의 스칼라 삼중곱 $a \cdot (b \times c)$

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

← 네 식으면 절댓값인듯

* 벡터 a, b, c에 의해 결정되는 평행육면체의 부피 $V = |a \cdot (b \times c)|$ (스칼라 삼중곱의 크기)



기하학적 의미

부피 = 밑면 × 높이
$= \|b \times c\| \times \|a\| \cos\theta$
$= \|a\| \cdot (\|b \times c\|)$
$= \|a \cdot (b \times c)\|$ ← 절댓값

$a \cdot (b \times c) = 0$ 이면
세 벡터가 같은 평면위에 있음을 나타냄.

Remark 행렬식

(1) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$: 스칼라삼중곱 = a, b, c로 만들어지는 평행육면체의 부피 $\|a \cdot (b \times c)\|$

$|a \times b| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

(2) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$: $\|a \times b\|$ = a와 b로 만들어지는 평행사변형의 넓이
$= a_1b_1 - a_2b_1$

↳ 2차원도 가능.

$a \langle a_1, a_2, 0 \rangle \quad b \langle b_1, b_2, 0 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \langle 0, 0, a_1b_2 - a_2b_1 \rangle$$

$$|a \times b| = \sqrt{(a_1b_2 - a_2b_1)^2} = a_1b_2 - a_2b_1$$

ex) $a = (1, 4, -7)$, $b = (2, -1, 4)$, $c = \langle 0, -9, 18 \rangle$ 이 같은 평면에 있음을 보여라

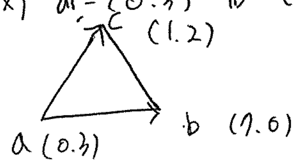
Sol) $a \cdot (b \times c) = 0$ 이면 평면에 있음.

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18+36) - 4(36) - 7(-18)$$

$$= 18 - 144 + 126 = 0$$

* 삼각형의 넓이

ex) $a = (0, 3)$ $b = (1, 0)$ $c = (1, 2)$

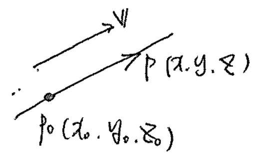


Sol) $\frac{1}{2} \left| \begin{matrix} 1 & -1 \\ 1 & -3 \end{matrix} \right| = 2$

11.5 직선과 평면의 방정식.

* 직선. (참) $\langle 1, m \rangle$ → "기울기 m 인 직선"

$P_0(x_0, y_0, z_0)$을 지나고 $v = \langle a, b, c \rangle$ 에 평행한 직선.



(1) $\overrightarrow{P_0P} = tv$: 벡터방정식 $(t \in \mathbb{R})$

(2) $P - P_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$

$= t \langle a, b, c \rangle$

$$\therefore \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad t \in \mathbb{R}$$

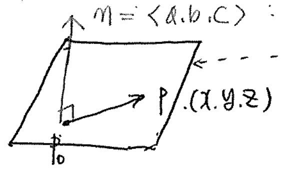
∴ 직선의 매개방정식.

(3) $(t =)$ $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$: 대칭방정식.

Remark if $v = \langle a, b, 0 \rangle$ 이면

$\frac{x - x_0}{a} = \frac{y - y_0}{b}$, $z = z_0$ (2) 대입.

* 평면의 방정식. — 점 $P_0(x_0, y_0, z_0)$을 지나고 벡터 $n = \langle a, b, c \rangle$ (법선벡터)에 수직인 평면.



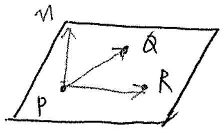
(1) $\overrightarrow{P_0P} \cdot n = 0$ (수직이니까!)

(2) $\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$\therefore \underline{a(x-x_0) + b(y-y_0) + z(z-z_0) = 0}$

ex) $P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$을 포함하는 평면의 방정식.

sol)



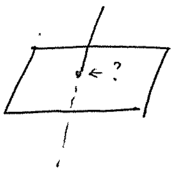
$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, -4, 4 \rangle \times \langle 4, -1, -2 \rangle$

$= \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \langle 8+12, -(-4-16), -2+16 \rangle$

$\therefore 20(x-1) + 20(y-3) + 14(z-2) = 0.$

ex) $\frac{x-2}{3} = \frac{y}{-4} = z-5$ 가 평면 $4x+5y-2z=18$과 만나는 점.

sol) $\begin{cases} x = 3t+2 \\ y = -4t \\ z = t+5 \end{cases}$



$4x+5y-2z = 18$

대입

$4(3t+2) + 5(-4t) - 2(t+5) = 18.$

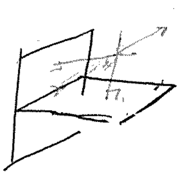
$-10t = 20$ $\therefore t = -2$ ⟶ 교점 $\langle -4, 8, 3 \rangle$.

점과 점 ⟶ 직선

직선과 직선 ⟶ 평면.

ex) ① $x+y+z=1$ 과 $x-2y+3z=1$ 사이의 cos 각 = ?

sol)



법선벡터들의 cosine.

↳ $\cos\theta = \frac{a \cdot b}{\|a\| \|b\|}$ $a = (1, 1, 1)$, $b = (1, -2, 3)$

$= \frac{2}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$

② 이 두 평면의 교선의 방정식.

↳ 2차원 - 기울기, 한 점, 3차원 - 벡터, 한 점

sol) a, b의 외적 (수직벡터)

$\langle 1, 1, 1 \rangle \times \langle 1, -2, 3 \rangle = \langle 5, -2, 3 \rangle$ ← 벡터.

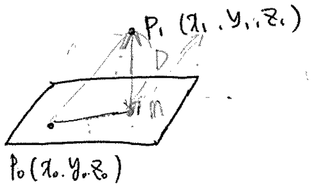
한점 → $x+y+z=1$
$-\ \underline{x-2y+3z=1.}$

$3y - 2z = 0.$ → $y = \frac{2}{3}z$
→ $x = 1 - \frac{5}{2}y$ → $y = -\frac{2}{5}(x-1)$

$\therefore -\frac{2}{5}(x-1) = y = \frac{2}{3}z$

ex) 점 $P_1(x_1, y_1, z_1)$ 에서 평면 $ax+by+cz+d=0$ 까지의 거리 D에 관한 공식 구해라.

sol)



$$k = \| Comp_{n} \overrightarrow{P_0P_1} \| = \| \overrightarrow{P_0P_1} \| \cos\theta = \left\| \frac{\overrightarrow{P_0P_1} \cdot n}{\|n\|} \right\| = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$(ax_0+by_0+z_0=-d)$
평면 위 점이므로

즉! 점과 직선 사이의 거리랑 동일

$$\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \sim \frac{|ax_1+b}{\sqrt{a^2+b^2+c^2}}$$