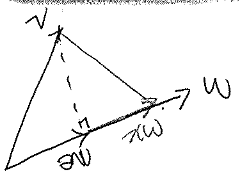
4. Orthogonality



$\therefore aw = \dfrac{v^T w}{w^T w} w$

$f(x) = \|v - xw\|^2 = v^T v - 2v^T w x + w^T w x^2$

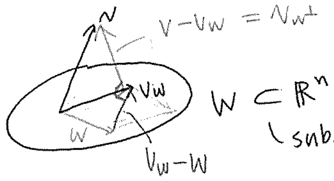
$\underline{\operatorname{argmin} f(x)} = \dfrac{v^T w}{w^T w}$ $\Leftrightarrow$ $f'(x) = 0$ 되는 점

└ $-2v^T w + 2w^T w = 0$

f(x)를 최소값으로 만들기 위한 x를 구함

||

거리 제일 짧은게 수직.



$W \subset \mathbb{R}^n$
└ subspace

$W \oplus W^\perp = \mathbb{R}^n$

① $\forall v \in \mathbb{R}^n$, $v = V_W V_{W^\perp}$, for some $\begin{cases} V_W \in W \\ V_W \in W^\perp \end{cases}$ ...? // 바닥에 있는 모든 벡터와 수직

② $v - v_m \in W^\perp$ $\therefore \forall w \in W$, $(v - v_w)^T w = 0$ $\therefore \underline{v - v_w \perp W}$

③ $\forall w \in W$, $\|v - w\|^2 = \|\underbrace{v - v_w}_{\in W^\perp} + \underbrace{v_w - w}_{\in W}\|^2 = \|v - v_m\|^2 + \|v_m - w\|^2 \geq \|v - v_m\|^2$

→ projection 은 가장 가까운 벡터를 찾는 것이다.

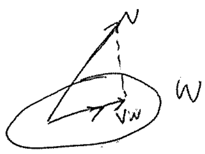
Remark 28. Definition of projection.

$W \subset \mathbb{R}^n$: a subspace.

$\Pi_W : \mathbb{R}^n \to W$

$\Pi_W(v) = v_W$: v를 W에 프로젝션

= v를 W와 가장 가깝게 하는 벡터 中 W에 들어있는 거.

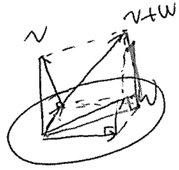


$\therefore \Pi_W(v) = \dfrac{v^T w}{w^T w} w$

1. Π_W is linear transformation.

1) $\Pi_W(u + v) = \Pi_W(u) + \Pi_W(v)$

2) $\Pi_W(av) = a\Pi_W(u)$



$\Rightarrow \Pi_W(au + \beta v) = a\Pi_W(u) + \beta \Pi_W(v)$

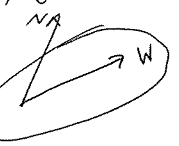
2. Matrix representation of Π_W

$\therefore \Pi_W(v) = X(X^T X)^{-1} X^T v \left(= \dfrac{v^T X}{X^T X} X. \right)$

└ 바닥 벡터

ex) 전체 $\mathbb{R}^3$

$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



$W = \mathbb{R}^2$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

바닥의 basis

$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$X(X^T X)^{-1} X^T$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \underbrace{\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right]^{-1}}_{I} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \times \overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ← 수직인 벡터!

$\boxed{\text{정리}}$ 바닥과 떠있는 벡터 사이의 가장 가까운 벡터

→ 바닥공간의 basis 찾고 $X(X^T X)^{-1} X^T v$ 계산.

3. $\Pi_W(v) = \underbrace{X(X^TX)^{-1}X^T}_{\text{unique}} v$

(basis는 여러개여도 이들의 Linear combination은 유일하다)

ex 27) in $\mathbb{R}^3$

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, 2x_1 - x_2 - 3x_3 = 0 \right\}$ 위에 projection한 벡터 찾아라.

1) $W = \mathrm{span}\left(\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}\right)^{\perp} \rightarrow \dim(W) = 2$ $\quad \dim(W) + \dim(W^{\perp}) = 3$ 전체

$2x_1 - x_2 - 3x_3 = 0$

$\mathrm{span}\left(\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}\right)$

2) W의 basis : $\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ (2개)

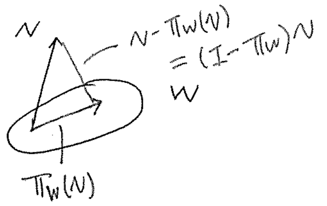
스팬, Linearly independent

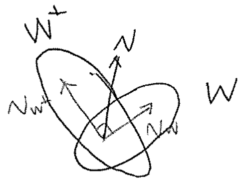
$2x_1 - x_2 - 3x_3 = 0$ 만족하는거 아무거나.

3) $X = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{pmatrix}$, $X(X^TX)^{-1}X = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 6 \end{pmatrix} \left\{ \begin{pmatrix} 0 & 3 & -1 \\ 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 6 \end{pmatrix} \right\}^{-1} \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 6 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

└ v를 projection 한거 구하는거라 v 곱했었는데 얘 2개는 W 위에 projection 한거라 안 곱함.

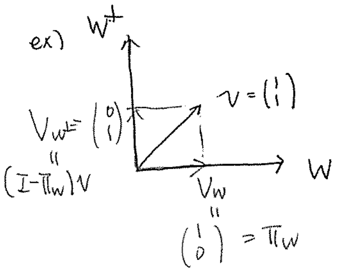
+)





$\therefore \Pi_{W^{\perp}} = (I - \Pi_W)v$

ex)



$\therefore \begin{pmatrix} v_{W^{\perp}} = (I - \Pi_W)v \\ v_W = \Pi_W \end{pmatrix}$

Remark 29.

$\{x_1, \cdots, x_k\}$: basis for W

$X = x_1 \cdots x_k \rightarrow \mathrm{rank}(X) = k$ — 선형독립인 column vector 개수

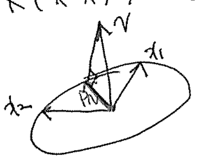
$P = X(X^TX)^{-1}X^T$ onto W $\quad \boxed{\text{v 안 곱한거!}}$

$\|$

$\{x_1 \cdots x_k\}$: linearly independent.

$X : x_1, \cdots, x_k$.

$P = X(X^TX)^{-1}X^T$ onto $\mathrm{col}(X)$



① P is idempotent ≜ $P^2 = P$

② P is symmetric ≜ $P^T = P$

③ idemp & symmetric → projection

식 복잡하니까 ①② 를 이용해

프로젝션 찾음

└ column space의 orthogonal space에 하는 projection.

Eg24.

$y_i = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_p x_{pi} + \varepsilon_i$

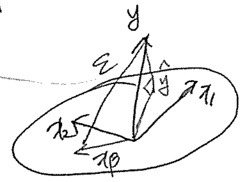
$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{1i} - \cdots - \beta_p x_{ip})^2$. ∨ OLS때. 최소제곱법

$= (Y - X\beta)^T (Y - X\beta) = \|Y - X\beta\|^2$

$\downarrow$

$X^T X \beta = X^T y$

$\beta = (X^T X)^{-1} X^T y$



span $(x_1, x_2$.

<구는 방법>

① 편미분

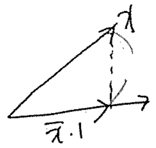
② 프로젝션

$\therefore \|y - X\beta\|^2 = \|\varepsilon\|^2 \longrightarrow \hat{\beta} = \arg\min \|y - X\beta\|^2$.

ex) $\sum_{i=1}^{n} (x_i - \bar{x})^2$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\bar{x} 1$ = x를 1로 projection $= \frac{x^T 1}{1^T 1} \times 1$
$= \bar{x} \cdot 1$



$\|x - \bar{x}\cdot 1\|^2 = \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - \bar{x} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$.

4.2. Properties of $\boxed{\text{Symmetric}}$ Matrices.

Def4.2. $A : n \times n$. $A^T A = I$ 면 A is orthogonal.

Remark70.

1. $A = (A_1, \cdots, A_n)$: basis for $\mathbb{R}^n$. (orthonormal basis) — column
2. $A^{-1} = A^T$
3. $A^T A = \begin{pmatrix} A_1^T \\ \vdots \\ A_n^T \end{pmatrix} (A_1, \cdots, A_n) = \begin{pmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & \ddots & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{pmatrix} = I$
4. $A = \begin{pmatrix} A_1^T \\ \vdots \\ A_n^T \end{pmatrix}$: basis for $\mathbb{R}^n$ (orthonormal basis) — row
5. $(Av)^T (Aw) = v^T A^T A w = v^T w$ 내적을 해면 내적이 나옴! 내적 보존!
6. $\|Av\| = \|v\|$

Thm24. Eigenvectors of $n \times n$ symmetric matrix A are orthogonal to each other.

Corollary25. An $n \times n$ symmetric matrix A is diagonalizable with an orthogonal matrix C of which columns are eigenvectors of A.

Eg26. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ 의 orthogonal diagonalization

① $\det(A - \lambda I) = \lambda^2 - 5\lambda \longrightarrow \lambda_1 = 5,\ \lambda_2 = 0$

Eigenspace $\quad \frac{1}{\sqrt{\lambda_1 + \lambda_2}}$

② $A + 5I = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \xrightarrow{+5I} A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow E_5 = \text{span}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

$\xrightarrow{+0I} A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \Rightarrow E_0 = \text{span}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$

$$\therefore \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}^{-1}}_{C^{-1}} \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}_{C}.$$

$D =$

Thm 26. A : $n \times n$ symmetric matrix,

$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ ← 절댓값.

$$A = CDC^{-1} = CDC^T = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T \quad (v_1, \cdots, v_n) : \text{basis for } \mathbb{R}^n.$$
$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

$C : (v_1 \cdots v_n)$

$D : \text{diag}(\lambda_1, \cdots, \lambda_n)$

Remark 31.

1. $v_i v_i^T$ is projection matrix onto span(v_i), since $\|v_j\| = 1$.
2. $A^k = \sum_{j=1}^{n} \lambda_j^k v_j v_j^T$.

Eg 27. The spectral decomposition of the matrix A.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 5\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) + 0\begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

Def 43. A : $n \times n$ symmetric matrix.

1. $w^T A w \geq 0 \quad \forall w \neq 0 \rightarrow A$: ~~po~~ semi positive definite matrix (" non negative .)
2. $w^T A w > 0 \quad \forall w \neq 0 \rightarrow A$: × positive "
3. $w^T A w < 0 \quad \forall w \neq 0 \rightarrow A$: " negative defi "
4. $w^T A w \leq 0 \quad \forall w \neq 0 \rightarrow A$: semi "
5. ①~④로 정의 안되면 indefinite.

Eg 28.

$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$: positive definite for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

$$x^T A x = (x_1\ x_2\ x_3)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$
$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0 \quad (x_1 \neq x_2 \neq x_3 \text{ 일 때})$$

3. $\begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix}$.

$\begin{pmatrix} -1-\lambda & 2 \\ 4 & 5-\lambda \end{pmatrix}$

$(-1-\lambda)(5-\lambda)-8=0$

$-5+\lambda-5\lambda+\lambda^2+8=0$

$\lambda^2-4\lambda+3=0$
-3
-1

$\lambda_1=1 \to v=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2=3 \to v=\begin{pmatrix} \ \end{pmatrix}$.

17.
$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-3y \\ -3x+2y \end{pmatrix}$ $\lambda=2$

$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

1. $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$ $D=C^{-1}AC$.
$A=CDC^{-1}$

$\det\begin{pmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{pmatrix}=0$

$9-6\lambda+\lambda^2-16=0$ $\lambda=$ ~~4 or -1~~.
$\lambda^2-6\lambda-7=0$
7

$\lambda_1=7 \to \left(\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}-\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right)v=0$

$\begin{pmatrix} -5 & \\ & -5 \end{pmatrix}$

$\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}-\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

$\begin{pmatrix} -10 & 4 \\ 4 & -4 \end{pmatrix}v=0$

$\begin{pmatrix} -6 & 0 \\ 4 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$x_1=1$ $x_2=-1$

$\to v_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $C=\begin{matrix} -2 & 1 \\ 1 & \end{matrix}$

~~$A\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $Av=v\lambda$~~

$P=\begin{pmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{pmatrix}$ diagonalizable 하다!
$A=CDC^{-1}$ 꼴

$\det\begin{pmatrix} 1-\lambda & 2 & 6 \\ 2 & -\lambda & -4 \\ 6 & -4 & 3-\lambda \end{pmatrix}=0$

$1-\lambda(-3\lambda+\lambda^2-16)-2(6-2\lambda+24)+6(-8+6\lambda)$

$-3\lambda+\lambda^2-16+3\lambda^2-\lambda^3+16\lambda-12+4\lambda-48-48+36\lambda=0$

$\lambda^3-4\lambda^2-53\lambda-136=0$

$1-\lambda\{-\lambda(3-\lambda)-16\}-2\{2(3-\lambda)+24\}+6(-8+6\lambda)$

$-\lambda(1-\lambda)(3-\lambda)-16(1-\lambda)-4(3-\lambda)-48-48+36\lambda$

$(3-\lambda)(-\lambda(1-\lambda)-4))-16(1-\lambda)+36\lambda-96$

$-\lambda^2-\lambda-4$
$-(\lambda+1)\lambda+4$

$2\,|\,136$
$2\,|\,68$
$2\,|\,34$
17

$1 \quad -4 \quad -53 \quad -136$

$\to$ symmetric matrix $\to$ diagona

11. $\begin{pmatrix} -1 & 4 & 2 & 7 \\ 0 & 5 & -3 & 6 \\ 0 & & -5 & 1 \\ 0 & & & -11 \end{pmatrix}$

1. (1 2 1) on span $\left(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right)$ in $\mathbb{R}^3$.

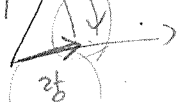
$2x_1+x_2-x_3=0$ $\to$ dim 2

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ —

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$ ...... $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $P=\frac{1}{6}\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$

$3\times 3 - 3\times 1$

projection 은

$\frac{1}{6}\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} =$ $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$  각.

Remark 3.2.

1. All eigenvalues of a positive definite matrix are positive.
2. All eigenvalues of a negative definite matrix are negative.
3. 〃 a positive semi-definite matrix are non-negative.
4. Every positive definite matrix is invertible and its inverse is also positive definite.

   Eigenvalue > 0