새로운 관점!

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$

sol) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \underset{R_2 \to R_2 - 3R_1}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \underset{R_2 \to -\frac{1}{2}R_2}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \underset{R_1 \to R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\underset{E_3}{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}} \underset{E_2}{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}} \underset{E_1}{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} = I$

$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

vertical shear ← expansion & reflection ← horizontal shear

Chapter 3. Properties of Square Matrices

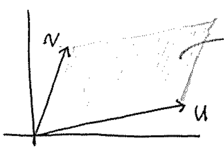
3.1. Determinants of the second order and of the third order.

$A = \left(\overset{u}{\begin{pmatrix} a \\ c \end{pmatrix}} \overset{v}{\begin{pmatrix} b \\ d \end{pmatrix}} \right)$

$\det(A) = |A| = \underset{\text{행렬식}}{\underline{ad - bc}}$ $\neq 0 \longrightarrow \exists A^{-1}$

$= 0 \longrightarrow \nexists A^{-1}$

| 행렬식 | = 평행사변형 넓이



$S = \| u \| \times \| v - \overset{\text{밑변}}{\frac{v^T u}{u^T u}} \cdot u \| = | ad - bc |$

\ in 응용 $\| u \| \times \| v \| \cdot \sin\theta = \| u \times v \|$

$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

(외적은 다루지 않음!

면적 = 0 → 두 벡터 평행

L Linearly dependent

L ERO $\nsim I$.

L Not invertible

3.2 Determinant of a square matrix.

$A = \begin{pmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

$\det(A) = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - b_2c_1)$

\ in 응용 $a \cdot (b \times c)$: 스칼라 삼중곱 = 평행육면체의 부피

부피 = 0 → 면 or 선

L Linearly dependent

L ERO $\nsim I$

L Not invertible.

\ 외적은 다루지 않음!

+ $\det(A^T) = \det(A)$

$\boxed{\text{Def 37}}$. Minor matrix & cofactor. ($A : n \times n$)

Minor matrix A_{ij} = $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column

a_{ij} 행,열 없애고 만든 Matrix

$A_{ij} = \begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ \vdots & & a_{ij} & \\ a_{n1} & & \cdots & a_{nn} \end{pmatrix}$

이인자.. 공동인자 cofactor $= (-1)^{i+j} \cdot \det(A)$

Def 38. Determinant. – 역행렬 뿐만 아니라 다양히 쓰임! (역행렬은 ERO가 더 빠름)

$A : n \times n$ $\quad n = 1 \to \det(A) = 1$

$\quad n > 1 \to \underline{\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \cdot \det(A_{ij})}$

Γ 아무거나 보통 1

0이나 1이 많다거나 이런게 아님 보통 (1,1)을 선정함.

L 이게 Thm 22. Expansion by minors.

$\boxed{a_{ij} \neq 0}$

㊟ Volume of n-Box. (벡터 n개로 이루어진 부피)면적)

The volume of the n-box in $\mathbb{R}^m$ determined by independent vectors $a_1, \cdots, a_n$ is given by

$\text{Volume} = \sqrt{\det(A^TA)}$ where A is the $m \times n$ matrix with a_j as jth colum vector.

$= \sqrt{(\det(A))^2} = \pm\det(A)$

Thm22. Expansion by Minor.

cofactor : $C_{ij} = (-1)^{i+j}\det(A_{ij})$

$\forall r,\ \det(A) = \sum_{j=1}^{n} a_{rj} C_{rj} = \sum_{j=1}^{n} a_{rj}(-1)^{i+j}\det(A_{ij})$

겹치는 성분

Eg17.

1. $\det(A)$

$$A = \begin{pmatrix} 3 & 2 & 0 & 1 & 3 \\ -2 & 4 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 & -5 \\ -1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\det(A) = \sum_{j=1}^{n} a_{rj} C_{rj} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(A_{ij})$$

$$= 2\,(-1)^{5+5}\begin{vmatrix} 3 & 2 & 0 & 1 \\ -2 & 4 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 0 & -1 \end{vmatrix}$$

0 | | 0 | | … 없애자!

$$= 2\cdot\left(1\cdot(-1)^{2+3}\begin{vmatrix} 3 & 2 & 1 \\ \cancel{0} & \cancel{-1} & \cancel{1} \\ -1 & 2 & -1 \end{vmatrix}\right)$$

$$= -2\,\left(-1\,(-1)^{2+2}(-3+1) + 1\cdot(-1)^{2+3}(6+2)\right)$$

$$= -2\,(2+8) = 12.$$

<참> 1의 $\det(A) = 12$를 통해 알 수 있는 것.

$12 \neq 0 \rightarrow$ 역 O

$\rightarrow$ linearly independent ——→ 해당 안되면 (pivot free colum) 있으면 det(0)

$\rightarrow$ col(A)

$\rightarrow$ unique solution

$\rightarrow$ ERO $\sim I$

$\rightarrow$ row space dim $= 5$

$\rightarrow$ nulity $= 0$

$\rightarrow$ rank $= 5$

2. The determinant of upper-lower triangular square matrix is product of its diagonal elements.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & \\ \vdots & & \\ 0 & & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} \end{vmatrix} = \cdots = \underline{a_{11} a_{22} a_{33} \cdots a_{nn}}$$

3. $\det(A) = \det(A^T)$

Remark 22 Properties of determinants as a function of row vectors.

default : $A = \begin{pmatrix} A_1^T \\ \vdots \\ A_n^T \end{pmatrix}$

1) $B = \begin{pmatrix} rA_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$ → $ra_{11}, ra_{12}, \cdots, ra_{1n}$, $\det(B) = \sum_{j=1}^{n} r a_{1j} \overset{(-1)^{1+j}}{\vee} \det(B_{1j}) = r\det(A)$ +) $\det(rA) = r^n \det(A)$

2) $B = \begin{pmatrix} A_1^T + C^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$ → $a_{11}+c_1, a_{12}+c_2, \cdots, a_{1n}+c_n$, $\det(B) = \sum_{j=1}^{n} (a_{1j} + c_j) \overset{(-1)^{1+j}}{\vee} \det(B_{1j}) = \sum_{j=1}^{n} (a_{1j} + c_j) \det(A_{1j})$

$= \det(A) + \sum_{j=1}^{n} c_j (-1)^{1+j} \det(A_{1j})$ ↖ 즉 $\begin{pmatrix} A_1^T \\ \vdots \\ A_n^T \end{pmatrix}$ 의 det + $\begin{pmatrix} C_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$ 의 det

ex) $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underbrace{\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}}_{\text{항상 되니까}} + \underbrace{\det\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}}_{\text{만 계산해야 함}}$

→ 한 행의 배수만큼 빼서 0 만들 수 있는거 냅두고 나머지만 계산하는 식으로 숫자 줄임!

3) $B = \begin{pmatrix} A_2^T \\ A_1^T \\ \vdots \\ A_n^T \end{pmatrix}$ 바뀐 것 , $\det(B) = \sum_{j=1}^{n} b_{2j} (-1)^{2+j} \det(B_{2j}) = \sum_{j=1}^{n} a_{1j} (-1)^{2+j} \det(A_{1j})$

$= -\det(A)$

4) $B = \begin{pmatrix} A_1^T \\ A_1^T \\ A_3^T \\ \vdots \\ A_n^T \end{pmatrix}$ $C = \begin{pmatrix} A_1^T \\ A_1^T \\ A_3^T \\ \vdots \\ A_n^T \end{pmatrix}$, $\det(B) = -\det(C) = -\det(B)$

$\therefore \det(B) = \det(C) = 0$

+) $B = \begin{pmatrix} A_1^T \\ rA_1^T \\ A_3^T \\ \vdots \\ A_n^T \end{pmatrix}$, $\det(B) = 0$

5) $B = \begin{pmatrix} A_1^T + rA_2^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$, $\det(B) = \det(A) + 0 = \det(A)$

└ ERO 해도 det 변화 X (바뀐거 빼고, r배 빼고)

* $\det(EA) = \det(E) \cdot \det(A)$ ⟶ ERO 해서 (◺) 꼴 만들고 determinant 구함

Remark 23 More Properties of Determinants.

1. A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

2. A, B are $n \times n$ $\rightarrow$ $\det(AB) = \det(A) \cdot \det(B)$

-1) $\det(A^{-1}) = \dfrac{1}{\det(A)} = (\det(A))^{-1}$

-1) $\det(I) = 1$

ex) $\det(A) = ?$

$$A = \begin{pmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 & 2 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 2 & -3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 2 & -4 \\ 1 & 3 & 2 \\ -2 & 2 & -3 \end{vmatrix} = 2 \begin{vmatrix} 0 & 2 & -4 \\ 1 & 3 & 2 \\ 0 & 8 & 1 \end{vmatrix} = 2(-1)(2+32) = -68$$

참 linear system $Ax = b$, where $A = [a_{ij}]$ is $n \times n$ invertible

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

a_n 열에 b 대입

This system has a unique solution given by

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for } k = 1, \dots, n$$

, B_k is the matrix obtained from A by replacing the k-th column vector of A by the column vector b.

$\hookrightarrow \det(A) \cdot \det(X_k) = \det(B_k)$

ex)
$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 &= 3 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

~~-1) $Ae_1 = a_1$.~~

$A = \begin{pmatrix} & \end{pmatrix}$

$Ae_1 = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Ae_1$ 첫번째 column vector

Sol) $\det(A) = \begin{vmatrix} 5 & -2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -15$

$\det(B_1) = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -5$

$\rightarrow x_1 = \dfrac{-5}{-15} = -\dfrac{1}{3}.$

3.7 Eigenvalues and Eigenvectors.

Def 39. Eigenvalues & Eigenvectors

$$\left\{\begin{array}{l} \underline{A v = \lambda v} \\ A : n\times n \\ v \neq 0 \\ \lambda \in \mathbb{R} \end{array}\right. \Longrightarrow \left\{\begin{array}{l} \lambda : \text{Eigen value of } A \\ v : \text{Eigen vector of } A \end{array}\right.$$

* Deposition.
$A = PDP^{-1}$ 꼴로 만들기.

계산.

$A v = \lambda v$

$(A - \lambda I) v = 0$

$\underbrace{B \not\sim I}$ $= \det B = 0$ $\quad \boxed{\therefore \det(A - \lambda I) = 0}$

ex) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\det\left(\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - 4 = 0$$

· $\underline{P(\lambda) = \det(A - \lambda I) = 0}$

· multiplicity : 중복도 $\quad$ ex) $P(\lambda) = (\lambda-4)^2(\lambda-1)$

$\left\{\begin{array}{l} \lambda_1 = \lambda_2 = 4 \quad : \text{중복도} 2 \\ \lambda_3 = 1 \end{array}\right.$

eg2 1. $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$. A의 Eigenvalue, 각 Eigenvalue에 일치하는 Eigen vector.

$$\rightarrow P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} = -1\{3(2-\lambda) - 1\} + (1-\lambda)\{(2-\lambda)\cdot(-\lambda) + 1\}$$

$$= -3(2-\lambda) + 1 + (1-\lambda)(-\lambda)\cancel{(\lambda-2)} + 2 - \lambda$$

$$= (2-\lambda)\{-3 + (1-\lambda)(-\lambda) + 1\} \qquad \lambda^2 - \lambda - 2$$

$$= (2-\lambda)(\lambda-2)(\lambda+1) \qquad \left\{\begin{array}{l} \lambda_1 = \lambda_2 = 2 \rightarrow \text{multiplicity} = 2 \\ \lambda_3 = -1 \end{array}\right.$$

→ Find Eigenvector

$\lambda_3 = -1 \rightarrow (A - (-1)I)v = 0$: we solve the homogeneous system.

$$A + I = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & -8 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} x_3 = r \\ x_2 = -\frac{3}{4}r \\ x_1 = \frac{5}{4} \end{array}$$

Solution set, $r\begin{pmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 2 \rightarrow (A - (2)I)v = 0$

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$x_1 = x_3$

$x_2 = 0$

Solution set $r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Remark 25. properties of Eigenvalues & Eigenvectors.

$A : n \times n$

1. $Av = \lambda v \longrightarrow A^k w = \lambda^k w$ (w도 vector)

2. $Av = \lambda v$, $\exists A^{-1} \longrightarrow \underbrace{A^{-1}v = \frac{1}{\lambda}v}_{\lambda^{-1}v}$ $(\lambda \neq 0)$

$\rightarrow A^{-k}v = \frac{1}{\lambda^k}v$

3. Definition of Eigen space.

$Av = \lambda v$

$E_\lambda \triangleq \{0\} \cup \{v \mid Av = \lambda v\}$ 0 U 만족하는 모든 Eigen vector들

$\underline{E_\lambda \text{ is a subspace}} \rightarrow u, v \in E_\lambda \rightarrow u+v \in E_\lambda$ 즉 $A(u+v) = \lambda(u+v)$

$v \in E_\lambda \rightarrow \alpha v \in E_\lambda$ $A(\alpha v) = \lambda(\alpha v)$

$\rightarrow E_\lambda = null(A - \lambda I)$

4. linear transformation. $\underline{T(v) = \lambda v}$ (T is paralled to v)

└ 만족 시 λ는 T의 Eigen value

5. 모든 행렬이 다 Eigen vetor 갖는것은 아님.

+) 노트 필기

$\lambda_1, \dots, \lambda_n$: eigen value of A $\Rightarrow \begin{cases} \sum_{i=1}^{n} \lambda_i = tr(A) \\ \prod_{i=1}^{n} \lambda_i = det(A) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n. \end{cases}$

$P(\lambda) = det(A - \lambda I)$

$= a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda^1 + a_0$ for some $a_n \cdots a_0$

$= a_n((\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n))$

$\rightarrow P(0) = \prod_{i=1}^{n} \lambda_i \cdot a_n (-1)^n$

ex) $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix} \rightarrow \lambda_1 = \lambda_2 = 2$, $\lambda_3 = -1$ $\longleftarrow tr(A) = 3$

$\Rightarrow 2 + 2 - 1 = 3.$

$det(A) = 2 \cdot 2 \cdot -1 = -4$

뒷 내용.

+) 추가

$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$: diagonal matrix

$P_D(\lambda) = det(D - \lambda I)$

$= \begin{vmatrix} d_1 - \lambda & & & \\ & d_2 - \lambda & & \\ & & \ddots & \\ & & & d_n - \lambda \end{vmatrix}$

$= (d_1 - \lambda)(-1)^{1+1} \times (d_2 - \lambda)(-1)^{1+1} \times \cdots \times (d_n - \lambda)(-1)^{1+1}$

$= (d_1 - \lambda) \cdots (d_n - \lambda) = 0$

$\therefore \underbrace{\lambda = d_1 \cdots d_n}_{\text{diagonal 들의 合}}$

· D의 0이 아닌 원소가 Eigen value 들

Def 40.

Eipen vector들이 basis for $\mathbb{R}^n$

A : diaponalizable 대각화 가능한

if $D = C^{-1}AC$ for some $\begin{cases} D : \text{diagonal matrix} \\ C : \text{invertible} \end{cases}$

$\rightarrow A = CDC^{-1}$

$C = (v_1, v_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\parallel$

$CDC^{-1} = A$

만족하면 A가 diaponalizable

Def 41.

P is similar to Q , $C^{-1}PC = Q$ for some C : invertible.
$n \times n$ $n \times n$

$\Updownarrow$

$P = CQC^{-1}$ $(\det(Q) = \det(P))$

"Trace"

Def. $tr(A) = \sum_{i=1}^{n} a_{ii}$: 대각원소의 합

1) $tr(AB) = tr(BA)$

2) $tr(Q) = tr(C^{-1}PC) = tr(PC^{-1}C) = tr(P)$ $\Rightarrow$ similar한 행렬은 det도 같고 tr도 같음

Thm 23.

A : $n \times n$, $1 \le m \le n$

$v_1, \cdots, v_m$: Eipen vetors of A

$\lambda_1, \cdots, \lambda_m$: Eipen values of A , $\lambda_1 \neq \cdots \neq \lambda_m$.

$\rightarrow v_1, \cdots, v_m$: Linear independent

Remark 21.

1. 대각화가능하다고 Eipen value 하나인것은 아님. 같은게 있을수도!
2. 대수적 중복도 (alpebraic multiplicity) $m_g(\lambda) = \dim(E_\lambda) = \dim(\text{null}(A - \lambda I)) = \text{nullity}(A - \lambda I)$
3. A : diaponalizable $\Leftrightarrow$ $m_g(\lambda) = m_a(\lambda)$ 기하학적 중복도
   대수적 중복도
4. real 실수 Symmetric matrix는 real eipenvalues 로 diaponalizable 하다.

+ Eipen value들이 distinct하면 diaponazable

$\rightarrow$ 별다른게 없으면 λ 구해보기

Lemma. A is similar to B

1. $P_A(\lambda) = P_B(\lambda)$
2. $A - \lambda I$ & $B - \lambda I$ = similar to each other.
3. nullity (A) = nullity (B)

시험백퍼!

① Block wise inverse.

ex) $B = \begin{pmatrix} \overset{A_{11}}{\begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix}} & \overset{A_{12}}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \\ \underset{A_{21}}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} & \underset{A_{22}}{\begin{pmatrix} 3 & 6 \\ -4 & 2 \end{pmatrix}} \end{pmatrix}$, $\det(B) = \det(A_{11})\det(A_{22}) - \det(A_{12})\det(A_{21})$ (양쪽 0으로 나뉜 경우)

$= 14 \cdot 30 = 420.$

ex) $B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & 0 & -3 & 2 & 4 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$ $= 2(-7) - (3) = -21 \quad -210$

Def4.0 확장버전

공식 $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}$ A와 D는 $n \times n$, inverse 있어야

② A: $(n+1)\times(n+1)$ real symmetric matrix (주대각선에 대칭인 행렬)

$A(v_1, v_2, \cdots, v_{n+1}) = (v_1, v_2, \cdots, v_{n+1})\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{n+1} \end{pmatrix} \leftarrow AC = CD$ where $C = (v_1, \cdots, v_{n+1})$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{pmatrix}$

$A(v_1, \cdots, v_{n+1}) = (v_1, \cdots, v_{n+1})\begin{pmatrix} \lambda_1 & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ D: diagonal

$A = (v_1, \cdots, v_{n+1})\begin{pmatrix} \lambda_1 & D_{12} \\ D_{21} & D_{22} \end{pmatrix}(v_1, \cdots, v_{n+1})^{-1}$

$= (v_1, \cdots, v_{n+1})\begin{pmatrix} \lambda_1 & D_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} v_1^T \\ \vdots \\ v_{n+1}^T \end{pmatrix}$

$A = \underset{C}{(v_1 \cdots v_{n+1})}\begin{pmatrix} \lambda_1 & 0 \\ 0 & PDP^{-1} \end{pmatrix}\begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix}$

$\downarrow$

$C\square\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix}\square^{-1}C^{-1}$

$= C\square\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix}(C\square)^{-1}$

$AC = C\underset{\text{diagonal}}{D}$

ex) $\begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

$\downarrow$

$\lambda_1 = 4$

$\lambda_2 = -1.$

$\therefore \begin{pmatrix} \lambda_1 & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & PD \end{pmatrix}\begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix}$

* ex) $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ $\lambda_1 = 4, \lambda_2 = -1$, $v_1 = ?$

$\rightarrow \lambda_1 = 4 \rightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2s \\ s \end{pmatrix}$ ex) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = v_1$

$\rightarrow \lambda_2 = -1 \rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s \\ s \end{pmatrix}$ ex) $\begin{pmatrix} -1 \\ 2 \end{pmatrix} = v_2$