HW #1 Solution

#1 (1) $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$$|A - \lambda I| = 0 \iff \begin{vmatrix} 1-\lambda & r \\ r & 1-\lambda \end{vmatrix} = 0$$

$\iff (1-\lambda)^2 - r^2 = 0$

$\iff (1-\lambda+r)(1-\lambda-r) = 0$

$\iff \lambda = 1+r$ or $1-r$, where $-1 < r < 1$

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \iff \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$

$$\iff \begin{cases} (1-\lambda)x_1 + r x_2 = 0 \\ r x_1 + (1-\lambda) x_2 = 0 \end{cases}$$

① $\lambda = 1+r$

$$\left.\begin{aligned} -r x_1 + r x_2 &= 0 \\ r x_1 - r x_2 &= 0 \end{aligned}\right\} \iff x_1 = x_2 \text{ if } r \neq 0$$

$$\underset{\sim}{p}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

② $\lambda = 1-r$

$$\left.\begin{aligned} r x_1 + r x_2 &= 0 \\ r x_1 + r x_2 &= 0 \end{aligned}\right\} \iff x_1 = -x_2 \text{ if } r \neq 0$$

$$\underset{\sim}{p}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(2)

$$\underset{\sim}{p}_1^T \underset{\sim}{p}_2 = (1 \quad 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 1 = 0$$

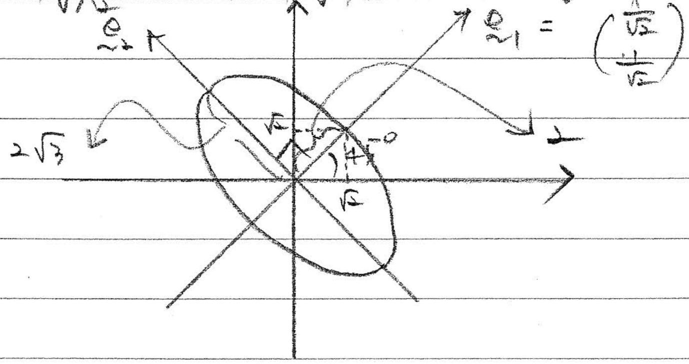
(3) $\underset{\sim}{x}^T A \underset{\sim}{x} = 6$ for $r = 1/2$

$$\lambda_1 = \frac{3}{2} > \lambda_2 = \frac{1}{2} > 0 \ \& \ \underset{\sim}{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \ \underset{\sim}{e}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \underset{\sim}{x} = 6 = c^2$$

$$c/\sqrt{\lambda_1} = \sqrt{6}/\sqrt{3/2} = 2$$

$$c/\sqrt{\lambda_2} = \sqrt{6}/\sqrt{1/2} = 2\sqrt{3}$$



#2 skip.

#4 In exercises 1,

$\lambda_1 = \frac{3}{2} > \lambda_2 = \frac{1}{2}$ & corresponding eigenvectors are

$\underset{\sim}{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\underset{\sim}{e}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

#7. For (2), use $AB = (I \otimes J)(J \otimes I)$

$= IJ \otimes JI = J \otimes J$

3. $q(\underset{\sim}{x}) = 4x_1^2 + 4x_2^2 + 6x_1x_2 = (x_1 \;\; x_2)\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Note that $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ is p.d.

$$(\because) \quad |4| = 4 > 0, \quad \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16 - 9 > 0$$

By the maximization of Quadratic forms for points on the unit sphere,

$$\max_{\underset{\sim}{x} \neq \underset{\sim}{0}} \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} = \lambda_1 \quad (\text{attained when } \underset{\sim}{x} = \underset{\sim}{e}_1)$$

$$\min_{\underset{\sim}{x} \neq \underset{\sim}{0}} \frac{\underset{\sim}{x}' A \underset{\sim}{x}}{\underset{\sim}{x}' \underset{\sim}{x}} = \lambda_p \quad (\text{attained when } \underset{\sim}{x} = \underset{\sim}{e}_p),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$

Eigenvalues of $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ are $\lambda_1 = 7 > 1 = \lambda_2$

Hence $\quad \max_{\underset{\sim}{x} \neq \underset{\sim}{0}} q(\underset{\sim}{x}) = \lambda_1 = 7$ &

$$\min_{\underset{\sim}{x} \neq \underset{\sim}{0}} q(\underset{\sim}{x}) = \lambda_2 = 1$$

5. $n=3$, $S = I - \frac{1}{n}U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} \frac{2}{3}-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3}-\lambda & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda(\lambda-1)^2 = 0$$

i) $\lambda = 0$ $\quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$x_1 = x_2 = x_3 \qquad \underset{\sim}{x}^T = (1 \;\; 1 \;\; 1)$

ii) $\lambda = 1$ $\quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$x_1 + x_2 + x_3 = 0 \quad \begin{cases} \underset{\sim}{x}^T = (1 \;\; -1 \;\; 0) \\ \underset{\sim}{x}^T = (1 \;\; 0 \;\; -1) \text{ or } \underset{\sim}{x}^T = (1 \;\; 1 \;\; -2) \end{cases}$

<u>Note</u> orthonormalized eigenvectors

$$\underset{\sim}{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad \underset{\sim}{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \underset{\sim}{x}_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

HW #2 Solution

1. $l = x_1 - x_2 = (1 \ -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ & $m = -x_1 + 3x_2 = (-1 \ 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$El = (1 \ -1)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mu_1 - \mu_2$

$Vl = (1 \ -1)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma_{11} + \sigma_{22} - (\sigma_{12} + \sigma_{21})$

$= \sigma_{11} + \sigma_{22} - 2\sigma_{12}$

$\mathrm{cov}(l, m) = (1 \ -1)\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = -\sigma_{11} - 3\sigma_{22} + 4\sigma_{12}$

$\mathrm{corr}(l, m) = \dfrac{\mathrm{cov}(l, m)}{\sqrt{Vl}\ \sqrt{Vm}} = \dfrac{-\sigma_{11} - 3\sigma_{22} + 4\sigma_{12}}{\sqrt{\sigma_{11}+\sigma_{22}-2\sigma_{12}}\ \sqrt{\sigma_{11}+9\sigma_{22}-6\sigma_{12}}}$

2. Do similar stuff as #1.

3. (i) $\bar{x} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

(ii) $X_c = \begin{pmatrix} -2 & 0 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 2 \end{pmatrix}$

(iii) $S = \dfrac{1}{2} X_c^T X_c$, where X_c is at (ii)

$R = \hat{\rho} = \mathrm{Diag}\left(\dfrac{1}{\sqrt{S_{ii}}}\right) S \ \mathrm{Diag}\left(\dfrac{1}{\sqrt{S_{ii}}}\right)$,

where S_{ii} is the ith main diagonal element of S.

4. (i) $\underset{\sim}{l} = Xb = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ 14 \\ 8 \end{pmatrix}$

$\bar{l} = (8+14+14+8)/4 = 11$

$S_l^2 = \frac{1}{n-1} \Sigma (l_i - \bar{l})^2 = \cdots$

$\underset{\sim}{m} = Xc = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 5 \\ -5 \end{pmatrix}$

$\bar{m} = -2$

$S_m^2 = \frac{1}{n-1} \Sigma (m_i - \bar{m})^2 = \cdots$

$S_{ij} = \frac{1}{n-1} \Sigma (l_i - \bar{l})(m_i - \bar{m})$

(iii)

$\underset{\sim}{l} = X\underset{\sim}{b}$ & $\underset{\sim}{m} = X\underset{\sim}{c}$

$\bar{l} = \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{l} = \frac{1}{n} \underset{\sim}{1}^T X \underset{\sim}{b} = \bar{\underset{\sim}{x}}^T \underset{\sim}{b}$

$S_{ij} = \widehat{\text{cov}(\underset{\sim}{l}, \underset{\sim}{m})} = \underset{\sim}{b}^T S \underset{\sim}{c}$

Use $\bar{\underset{\sim}{x}}$ & S which are calculated at (ii).

5. $x \sim N_p(\underset{\sim}{\mu}, \Sigma)$

(i) $\underset{\sim}{x}_2 \mid \underset{\substack{\uparrow \\ t\text{-dim}}}{\underset{\sim}{x}_1} \sim N_{p-t}(\underset{\sim}{\mu}_{2.1}, \Sigma_{2.1})$, where

$\underset{\sim}{\mu}_{2.1} = \underset{\sim}{\mu}_2 + \Sigma_{21} \Sigma_{11}^{-1} (\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)$ &

$\Sigma_{2.1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{p-1} \\ \hdashline x_p \end{pmatrix} = \begin{pmatrix} \underset{\sim}{x}_1 \\ \hdashline \underset{\sim}{x}_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}, \left(\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array} \right) \right)$$

$\therefore\ x_p | x_1 \cdots x_{p-1} \sim N(\underset{\sim}{\mu}_{2.1}, \Sigma_{2.1})$,

Note that $\mu_2 = \mu_p$; $\Sigma_{22} = \sigma_{pp}$ &

$$\Sigma_{12} = \underset{\sim}{\sigma}_{1p} = \begin{pmatrix} \sigma_{1p} \\ \sigma_{2p} \\ \vdots \\ \sigma_{p-1,p} \end{pmatrix}$$

where $\underset{\sim}{\mu}_{2.1} = \mu_p + \underset{\sim}{\sigma}_{1p}^T \Sigma_{11}^{-1} (\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)$ &

$$\Sigma_{2.1} = \sigma_{pp} - \underset{\sim}{\sigma}_{1p}^T \Sigma_{11}^{-1} \underset{\sim}{\sigma}_{1p}$$

(ii) $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B\underset{\sim}{x}$

$\sim N_2 (B\underset{\sim}{\mu}, B\Sigma B^T)$, where

$$B\underset{\sim}{\mu} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 + \mu_2 \\ -\mu_1 + \mu_2 \end{pmatrix} \ \&$$

$$B\Sigma B^T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \cdots$$

(iii) $\underset{\sim}{y} \sim N_2 (B\underset{\sim}{\mu}, B\Sigma B^T)$, where

$$B\underset{\sim}{\mu} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 + \mu_2 \\ -\mu_1 + \mu_2 \end{pmatrix} \ \&$$

$$B\Sigma B^T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11} + \sigma_{21} & \sigma_{12} + \sigma_{22} \\ -\sigma_{11} + \sigma_{21} & -\sigma_{12} + \sigma_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11} + \sigma_{22} + 2\sigma_{12} & \sigma_{22} - \sigma_{11} \\ \sigma_{22} - \sigma_{11} & \sigma_{11} + \sigma_{22} - 2\sigma_{12} \end{pmatrix}$$

4)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 + \mu_2 \\ \mu_2 - \mu_1 \end{pmatrix}, \begin{pmatrix} \sigma_{11} + \sigma_{22} + 2\sigma_{12} & \sigma_{22} - \sigma_{11} \\ \sigma_{22} - \sigma_{11} & \sigma_{11} + \sigma_{22} - 2\sigma_{12} \end{pmatrix} \right)$$

$\Rightarrow$ $y_2 \mid y_1 \sim N(\mu_{2.1}, \Sigma_{2.1})$, where

$$\mu_{2.1} = (\mu_2 - \mu_1) + (\sigma_{22} - \sigma_{11})(\sigma_{11} + \sigma_{22} + 2\sigma_{12})^{-1}(y_1 - (\mu_1 + \mu_2))$$

$$= \mu_2 - \mu_1 + \frac{\sigma_{22} - \sigma_{11}}{\sigma_{11} + \sigma_{22} + 2\sigma_{12}}(y_1 - (\mu_1 + \mu_2)) \quad \&$$

$$\Sigma_{2.1} = \sigma_{11} + \sigma_{22} - 2\sigma_{12} - \frac{(\sigma_{22} - \sigma_{11})^2}{\sigma_{11} + \sigma_{22} + 2\sigma_{12}}$$