Addendum to HW #1

Note $A_{p\times p}$: p.d. $\Rightarrow$ ① $A = P\Lambda P'$, where $P = (\underset{\sim}{e}_1\ \underset{\sim}{e}_2 \cdots \underset{\sim}{e}_p)$ & → orthonormal eigenvectors

$\Lambda = \mathrm{diag}(\lambda_i)$ — eigenvalue

$\Leftrightarrow$ ② $A = \sum_{i=1}^{p} \lambda_i \underset{\sim}{e}_i \underset{\sim}{e}_i'$

; Spectral decomposition

Let $p = 2$,

$$\underset{\sim}{x}' A \underset{\sim}{x} = \sum_i \sum_j a_{ij} x_i x_j$$

$$= a_{11} x_1^2 + a_{22} x_2^2 + 2 a_{12} x_1 x_2 = \underset{\text{constant distance}}{c^2}$$

$$\underset{\sim}{x}' A \underset{\sim}{x} = \underset{\sim}{x}' (\lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \lambda_2 \underset{\sim}{e}_2 \underset{\sim}{e}_2') \underset{\sim}{x}$$

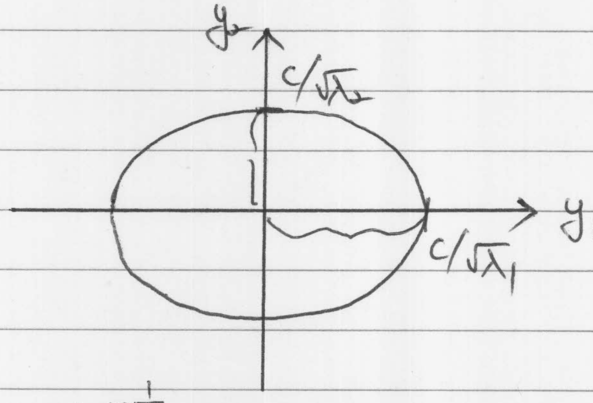
$$= \lambda_1 \underset{\sim}{x}' \underset{\sim}{e}_1 \underset{\sim}{e}_1' \underset{\sim}{x} + \lambda_2 \underset{\sim}{x}' \underset{\sim}{e}_2 \underset{\sim}{e}_2' \underset{\sim}{x}$$

$$= \lambda_1 (\underset{\sim}{x}' \underset{\sim}{e}_1)^2 + \lambda_2 (\underset{\sim}{x}' \underset{\sim}{e}_2)^2$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 = c^2$$

$$= \frac{y_1^2}{c^2/\lambda_1} + \frac{y_2^2}{c^2/\lambda_2} = 1$$

$$\Leftrightarrow \frac{y_1^2}{(c/\sqrt{\lambda_1})^2} + \frac{y_2^2}{(c/\sqrt{\lambda_2})^2} = 1 \quad ; \text{ ellipse in } y_1 \,\&\, y_2$$



Plug in $\underset{\sim}{x} = c\lambda_1^{-\frac{1}{2}} \underset{\sim}{e}_1$

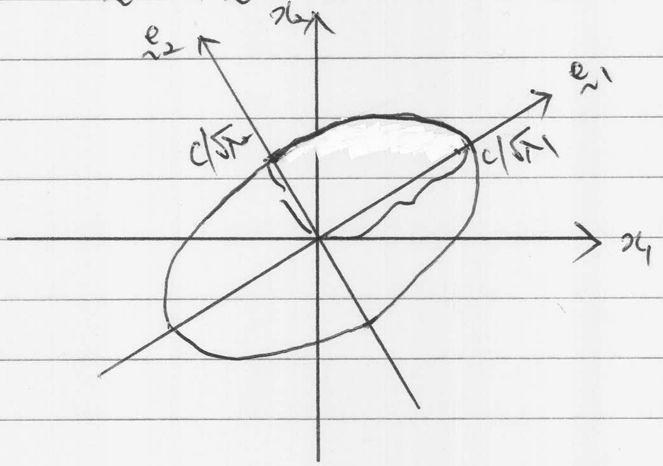
$$\underset{\sim}{x}' A \underset{\sim}{x} = \lambda_1 (c\lambda_1^{-\frac{1}{2}} \overset{=1}{\cancel{\underset{\sim}{e}_1' \underset{\sim}{e}_1}})^2 + \lambda_2 (c\lambda_1^{-\frac{1}{2}} \overset{=0}{\cancel{\underset{\sim}{e}_1' \underset{\sim}{e}_2}})^2$$

since they are orthonormal

$$= \lambda_1 c^2 \lambda_1^{-1} = c^2$$

Similarly, plug in $\underset{\sim}{x} = c\,\lambda_2^{-\frac{1}{2}}\,\underset{\sim}{e}_2$

$$\underset{\sim}{x}' A \underset{\sim}{x} = \cdots = c^2$$



; ellipse with $1 \le \lambda_1 < \lambda_2$

<u>Remark</u> ① $\lambda_1 = \lambda_2 \Rightarrow$ Circle

② The pts at distance c lie on an ellipse whose axes are given by the eigenvectors of A with lengths proportional to the reciprocals of the square roots of the eigenvalues.

③ $(\underset{\sim}{x} - \underset{\sim}{\mu})' A (\underset{\sim}{x} - \underset{\sim}{\mu}) = c^2$; Ellipsoids centered at $\underset{\sim}{\mu}$ & having axes $\pm \frac{c}{\sqrt{\lambda_i}} \underset{\sim}{e}_i$, where $A \underset{\sim}{e}_i = \lambda_i \underset{\sim}{e}_i$

④ $A = \Sigma^{-1}$, i.e. $(\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) = c^2$;

contours of constant density for the p-dim. normal dist

Ellipsoids centered at $\underset{\sim}{\mu}$ & having axes $\pm c\sqrt{\lambda_i}\,\underset{\sim}{e}_i$, where $\Sigma \underset{\sim}{e}_i = \lambda_i \underset{\sim}{e}_i$

Note that $\Sigma \underset{\sim}{e}_i = \lambda_i \underset{\sim}{e}_i$ & $\Sigma^{-1} \underset{\sim}{e}_i = \frac{1}{\lambda_i} \underset{\sim}{e}_i$.

ex) $\underset{\sim}{X} \sim N_2\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{11}\end{pmatrix}\right)$, Note that $\sigma_{11} = \sigma_{22} = \sigma_{11}$

$$|\Sigma - \lambda I| = \begin{vmatrix}\sigma_{11} - \lambda & \sigma_{12}\\ \sigma_{12} & \sigma_{11} - \lambda\end{vmatrix} = (\sigma_{11} - \lambda)^2 - \sigma_{12}^2$$

$$= (\sigma_{11} - \lambda + \sigma_{12})(\sigma_{11} - \lambda - \sigma_{12}) = 0$$

$$\Rightarrow \lambda = \sigma_{11} + \sigma_{12} \quad \text{or} \quad \sigma_{11} - \sigma_{12}$$

i) $\lambda_1 = \sigma_{11} + \sigma_{12}$

$$\Sigma \underset{\sim}{e} = \lambda \underset{\sim}{e}$$

$$\Leftrightarrow \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{11}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = (\sigma_{11} + \sigma_{12})\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

or

$$\begin{cases}\sigma_{11} e_1 + \sigma_{12} e_2 = (\sigma_{11} + \sigma_{12}) e_1\\ \sigma_{12} e_1 + \sigma_{11} e_2 = \quad '' \quad e_2\end{cases}$$

or

$$\begin{cases} e_2 = \frac{\cancel{\sigma_{12}}}{\cancel{\sigma_{12}}} e_1 = e_1\\ e_1 = \frac{\cancel{\sigma_{12}}}{\cancel{\sigma_{12}}} e_2 = e_2\end{cases}$$

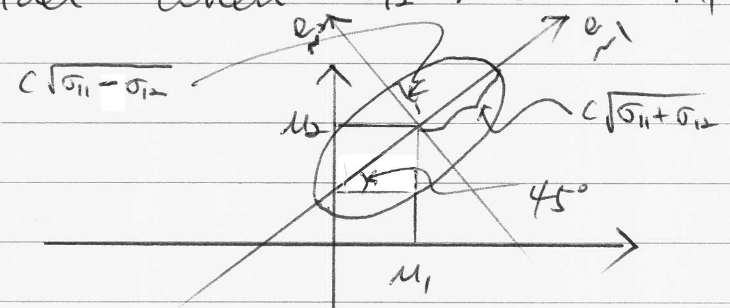
$\Rightarrow$ $\underset{\sim}{e} = \begin{pmatrix}1\\ 1\end{pmatrix}$ $\Rightarrow$ normalize $\quad \|\underset{\sim}{e}\| = \sqrt{\underset{\sim}{e}'\underset{\sim}{e}} = \sqrt{2}$

$$\underset{\sim}{e}_1 = \frac{1}{\|\underset{\sim}{e}\|}\underset{\sim}{e} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

ii) $\lambda_2 = \sigma_{11} - \sigma_{12}$

Similarly, we get $\underset{\sim}{e}_2 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

Consider when $\sigma_{12} > 0 \Rightarrow \lambda_1 > \lambda_2$



i.e. constant density ellipses will be along the 45° line through $\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$

If $\sigma_{12} < 0 \Rightarrow \lambda_1 < \lambda_2$ &
The major axes of the constant density ellipses will lie along a line at right angle to the $45°$ line through $\underset{\sim}{\mu}^T = (\mu_1 \ \ \mu_2)$.

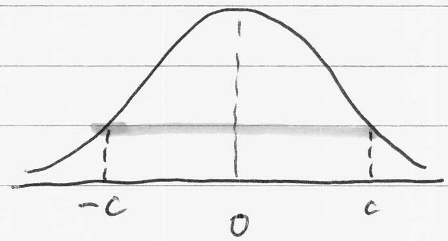
Note ① These results are true only for $\sigma_{11} = \sigma_{22}$

② meaning of constant density

$$\begin{pmatrix} p=1 \\ \mu=0 \\ \sigma=1 \end{pmatrix} \Rightarrow (\underset{\sim}{x}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}) = c^2$$

$$\Leftrightarrow \left(\frac{x-\mu}{\sigma}\right)^2 = c^2$$

$$\Leftrightarrow x^2 = c^2 \qquad \Leftrightarrow x = \pm c$$



i.e. cutted section

For bivariate normal density, the cutted sections (= constant density) are ellipses.