Exercises

(p.15) In exercises 5-8, let $\mathbf{u} = [-1, 3, -2], \mathbf{v} = [4, 0, -1]$, and $\mathbf{w} = [-3, -1, -2]$. Compute the indicated vector.

5. $3\mathbf{u} - 2\mathbf{v}$
6. $\mathbf{u} + 2(\mathbf{v} - 4\mathbf{w})$
7. $\mathbf{u} + \mathbf{v} - \mathbf{w}$
8. $4(3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w})$

(p.16) In Exercises 21-30, find all scalars c, if any exist, such that the given statement is true. Try to do some of these problems without using pencil and paper.

21. The vector $[2, 6]$ is parallel to the vector $[c, -3]$.
22. The vector $[c^2, -4]$ is parallel to the vector $[1, -2]$.
23. The vector $[c, -c, 4]$ is parallel to the vector $[-2, 2, 20]$.
24. The vector $[c^2, c^3, c^4]$ is parallel to the vector $[1, -2, 4]$ with same direction.

(p.31) In Exercises 1-17, let $\mathbf{u} = [-1, 3, 4], \mathbf{v} = [2, 1, -1]$, and $\mathbf{w} = [-2, -1, 3]$. Find the indicated quantity.

1. $\| - \mathbf{u}\|$
3. $\|\mathbf{u} + \mathbf{v}\|$
5. $\|3\mathbf{u} - \mathbf{v} + w\mathbf{w}\|$
7. The unit vector parallel to $\mathbf{u}$, having the same direction
11. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$
12. The angle between $\mathbf{u}$ and $\mathbf{v}$
14. The value of x such that $[x, -3, 5]$ is perpendicular to $\mathbf{u}$
16. A nonzero vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$
17. A nonzero vector perpendicular to both $\mathbf{u}$ and $\mathbf{w}$

(p.31) In Exercises 25-30, classify the vectors as parallel, perpendicular, or neither. If they are parallel, state whether they have the same direction or opposite direction.

25. $[-1, 4]$ and $[8, 2]$
26. $[-2, -1]$ and $[5, 2]$
27. $[3, 2, 1]$ and $[-9, -6, -3]$

31. The **distance** between points $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ in $\mathbb{R}^n$ is the norm $\|\mathbf{v} - \mathbf{w}\|$, where $\mathbf{u} = [v_1, \ldots, v_n]$ and $\mathbf{w} = [w_1, \ldots, w_n]$. Why is this a reasonable definition of distance?

(p.31) In Exercises 32-35, use the definition given in Exercise 31 to find the indicated distance.

33. The distance from $[2, -1, 3]$ to $[4, 1, -2]$ in $\mathbb{R}^3$
35. The distance from $[-1, 2, 1, 4, 7, -3]$ to $[2, 1, -3, 5, 4, 5]$ in $\mathbb{R}^6$

(p.33) Answer the followings.

43. For vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, prove that $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are perpendicular if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$.
44. For vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and for scalars r and s, prove that if $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{w}$ is perpendicular to $r\mathbf{u} + s\mathbf{v}$.

(p.46) In Exercises 1-17, let A, B, C, D, E and F be

$$\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

respectively. Compute the indicated quantity, if it is defined.

3. $A + B$
7. AB
9. $(2A)(5C)$
11. A^2
15. $(A^T)A$
17. E^2 and E^7
18. F^2 and F^7

(p.47) Answer the followings.

19. For the vectors $\mathbf{x} = [-2, 3, -1]^T$ and $\mathbf{y} = [4, -1, 3]^T$, compute the matrix products $\mathbf{x}^T\mathbf{y}$ and $\mathbf{y}\mathbf{x}^T$.
20. Fill in the missing entries in the 4×4 matrix $\begin{bmatrix} 1 & -1 & & \\ & 4 & & 8 \\ 2 & -7 & -1 & \\ & & 6 & 3 \end{bmatrix}$ so that the matrix is symmetric.
21. Mark each of the following True or False. The statements involve matrices A, B, and C that are assumed to have appropriate size.
    a. $A = B \Rightarrow AC = BC$
    b. $AC = BC \Rightarrow A = B$

c. $AB = O \Rightarrow A = O$ or $B = O$

d. $A + C = B + C \Rightarrow A + B$

e. $A^2 = I \Rightarrow A = \pm I$

f. $B = A^2$ and A is $n \times n$ symmetric $\Rightarrow b_{ii} \geq 0,\ \forall i \leq n$

g. $AB = C$, and A and B are square $\Rightarrow C$ is square

h. $AB = C$ and C is a column vector $\Rightarrow B$ is a column vector

i. $A^2 = I \Rightarrow A^n = I,\ \forall n \geq 2$

j. $A^2 = I \Rightarrow A^n = I,\ \forall$ even $n \geq 2$

23. Let A be an $m \times n$ matrix and let $\mathbf{b}$ and $\mathbf{c}$ be column vectors with n components. Express the dot product $(A\mathbf{b}) \cdot (\mathbf{Ac})$ as a product of matrices.

(p.47) Answer the followings.

35. If B is an $m \times n$ matrix and if $B = A^T$, find the size of A, AA^T and A^TA.

36. Let $\mathbf{v}$ and $\mathbf{w}$ be column vectors in $\mathbb{R}^n$. What is the size of $\mathbf{vw}^T$? What relationships hold between $\mathbf{vw}^T$ and $\mathbf{wv}^T$?

38. Prove that, if A is a square matrix, then the matrix $A + A^T$ is symmetric.

39. Prove that, if A is a matrix, then the matrix AA^T is symmetric.

41. Let A and B be $m \times n$ matrices, $\mathbf{e}_j$ be the $n \times 1$ vector whose jth element is 1 and the others are 0. Answer the followings.

    (a) Show that $A\mathbf{e}_j$ is the jth column vector of A.

    (b) Prove that, if $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$, then $A = O$. [Hint: Use part (a)].

    (c) Prove that, if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$, then $A = B$. [Hint: Use part (b)]

(p.48) Answer the followings.

42. Let A and B be square matrices. Is $(A + B)^2 = A^2 + B^2 + 2AB$? If so, prove it; if not, give a counter example and state under what conditions the equation is true.

43. Let A and B be square matrices. Is $(A - B)(A + B) = A^2 - B^2$? If so, prove it; if not, give a counter example and state under what conditions the equation is true.

44. A square matrix C is **skew symmetric** if $C^T = -C$. Prove that every square matrix A can be written *uniquely* as $A = B + C$ where B is symmetric and C is skew symmetric.

45. Find all values of r for which A commutes with B, where $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

46. Find all values of r for which A commutes with B, where $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(p.68) In Exercises 1-6, reduce the matrix to (a) row-echelon form, and (b) reduced row-echelon form. Answers to (a) are not unique, so your answer may differ from the one at the back of the text.

1. $\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 3 & -1 & 6 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 2 & -4 \end{bmatrix}$

(pp.68-69) In Exercises 7-12, describe all solutions of a linear system whose corresponding augmented matrix can be row-reduced to the given matrix. If requested, also give the indicated particular solution, if it exist.

7. $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 4 & 2 \end{array}\right]$, solution with $x_3 = 2$

11. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(p.69) In Exercises 13-20, find all solutions of the given linear system, using the Gauss method with back substitution.

13. $2x - y = 8$
    $6x - 5y = 32$

17. $x_1 - 2x_2 = 3$
    $3x_1 - x_2 = 14$
    $x_1 - 7x_2 = -2$

20. $x_1 - 3x_2 + 2x_3 - x_4 = 8$
    $3x_1 - 7x_2 + x_4 = 0$

(p.69) In Exercises 21-24, find all solutions of the linear system, using the Gauss-Jordan method.

21. $3x_1 - 2x_2 = -8$
    $4x_1 + 5x_2 = -3$

24. $x_1 + 2x_2 - 3x_3 + x_4 = 2$
    $3x_1 + 6x_2 - 8x_3 - 2x_4 = 1$

(p.69) In Exercises 25-28, determine whether the vector $\mathbf{b}$ is in the span of the vectors $\mathbf{v}_i$.

25. $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}$

28. $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ -2 \\ -8 \\ -9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$

(pp.69-70) Answer the followings.

29. Mark each of the following True or False.
    (a) Every linear system with the same number of equations as unknowns has a unique solution.
    (b) Every linear system with the same number of equations as unknowns has at least one solution.
    (c) A linear system with more equations than unknowns may have an infinite number of solutions.
    (d) A linear system with fewer equations than unknowns may have no solution.
    (e) Every matrix is row equivalent to a unique matrix in row-echelon form.
    (f) Every matrix is row equivalent to a unique matrix in reduced row-echelon form.
    (g) If $[A|\mathbf{b}]$ and $[B|\mathbf{c}]$ are row-equivalent partitioned matrices, the linear systems $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{c}$ have the same solution set.
    (h) A linear system with a square coefficient matrix A has a unique solution if and only if A is row equivalent to the identity matrix.
    (i) A linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row-reduced to an echelon matrix that includes some column containing no pivot.
    (j) A consistent linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row-reduced to an echelon matrix that includes some column containing no pivot.

38. Determine all values of the b_i that make the linear system

$$\begin{aligned} x_1 + 2x_2 &= b_1 \\ 3x_1 + 6x_2 &= b_2 \end{aligned}$$

consistent.

39. Determine all values b_1 and b_2 such that $\mathbf{b} = [b_1, b_2]$ is a linear combination of $\mathbf{v}_1 = [1, 3]$ and $\mathbf{v}_2 = [5, -1]$.

41. Determine all values b_1, b_2, and b_3 such that $\mathbf{b} = [b_1, b_2, b_3]$ lies in the span of $\mathbf{v}_1 = [1, 1, 0]$, $\mathbf{v}_2 = [3, -1, 4]$, and $\mathbf{v}_3 = [-1, 2, -3]$.

42. Find an elementary matrix E such that

$$E \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 3 & 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & 2 & -11 \end{bmatrix}.$$

(p.84) In Exercises 1-8, (a) find the inverse of the square matrix, if it exists, and (b) express each invertible matrix as a product of elementary matrices.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$

(pp.84-86) Answer the followings.

9. Find the inverse of the matrix, if it exists.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

11. Determine whether the span of the column vectors of the given matrix is $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & -2 \\ -3 & 0 & 0 & -1 \end{bmatrix}$$

13. Answer the followings.

    (a) Show that the matrix $A = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$ is invertible, and find its inverse.

    (b) Use the result in (a) to find the solution of the system of equations

$$2x_1 - 3x_2 = 4, \quad 5x_1 - 7x_2 = -3.$$

17. Let $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$. If possible, find a matrix C such that $ACA = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix}$.

21. Find all numbers r such that $\begin{bmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is invertible.

22. Let A and B be two $m \times n$ matrices. Show that A and B are row equivalent if and only if there exists an invertible $m \times m$ matrix C such that $CA = B$.

23. Mark each of the following True or False. The statements involve matrices A, B, and C, which are assumed to be of appropriate size.

    (a) If $AC = BC$ and C is invertible, then $A = B$.

    (b) If $AB = O$ and B is invertible, then $A = O$.

    (c) If $AB = C$ and two of the matrices are invertible, then so is the third.

    (d) If $AB = C$ and two of the matrices are singular, then so is the third.

    (e) If A^2 is invertible, then A^3 is invertible.

(f) If A^3 is invertible, then A^2 is invertible.

(g) Every invertible matrix is an elementary matrix.

(h) If A and B are invertible matrices, then so is $A+B$, and $(A+B)^{-1} = A^{-1} + B^{-1}$.

(i) If A and B are invertible, then so is AB, and $(AB)^{-1} = A^{-1}B^{-1}$.

24. Show that, if A is an invertible $n \times n$ matrix, then A^T is invertible. Describe $(A^T)^{-1}$ in terms of A^{-1}.

25. Answer the followings.

(a) If A is invertible, is $A + A^T$ always invertible?

(b) If A is invertible, is $A + A$ always invertible?

26. Let A be a matrix such that A^2 is invertible. Prove that A is invertible.

27. Let A and B be $n \times n$ matrices with A invertible.

(a) Show that $AX = B$ has the unique solution $X = A^{-1}B$.

(b) Show that $X = A^{-1}B$ can be found by the following row reduction:

$$[A|B] \sim [I|X].$$

That is, if the matrix A is reduced to the identity matrix I, then the matrix B will be reduced to $A^{-1}B$.

29. An $n \times n$ matrix A nilpotent if $A^r = O$ (the $n \times n$ zero matrix) for some positive r.

(a) Give an example of a nonzero nilpotent 2×2 matrix.

(b) Show that, if A is an invertible $n \times n$ matrix, then A is not nilpotent.

30. A square matrix A is said to be idempotent if $A^2 = A$.

(a) Give an example of an idempotent matrix other than O and I.

(b) Show that, if a matrix A is both idempotent and invertible, then $A = I$.

(p.99) In Exercises 1-10, determine whether the indicated subset is a subspace of the given Euclidean space $\mathbb{R}^n$.

1. $\{[r, -r] | r \in R\}$ in $\mathbb{R}^2$

3. $\{[n, m] | n, m \text{ are interges}\}$ in $\mathbb{R}^2$

5. $\{[x, y, z] | x, y, z \in \mathbb{R}, x, y \geq 0\}$ in $\mathbb{R}^3$

6. $\{[x, y, z] | x, y, z \in \mathbb{R}, z = 3x + 2\}$ in $\mathbb{R}^3$

9. $\{[2x_1, 3x_2, 4x_3, 5x_4] | x_i \in \mathbb{R}, i \leq 4\}$ in $\mathbb{R}^4$

(p.99) Answer the followings.

11. Prove that the line $y = mx$ is a subspace of $\mathbb{R}^2$.

12. Let a, b and c be scalars such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$.

14. Prove that every subspace of $\mathbb{R}^n$ contains the zero vector.

15. Is the zero vector a basis for the subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$? why or why not?

(pp.99–100) In Exercises 16-21, find a basis for the solution set of the given homogeneous linear system.

16. $x - y = 2x - 2y = 0$

17. $3x_1 + x_2 + x_3 = 6x_1 + 2x_2 + 2x_3 = -9x_1 - 3x_2 - 3x_3 = 0$

19. $2x_1 + x_2 + x_3 + x_4 = x_1 - 6x_2 = x_3 = 3x_1 - 5x_2 + 2x_3 + x_4 = 5x_1 - 4x_2 + 3x_3 + 2x_4 = 0$

(pp.100–101) In Exercises 22-30, determine whether the set of vectors is a basis for the subspace of $\mathbb{R}^n$ that the vectors span.

23. $\{[-1, 3, 1], [2, 1, 4]\}$ in $\mathbb{R}^2$

27. The set of row vectors of the matrix $\begin{bmatrix} 2 & -6 & 1 \\ 1 & -3 & 4 \end{bmatrix}$ in $\mathbb{R}^3$

28. The set of column vectors of the matrix in Exercise 27 in $\mathbb{R}^2$

(pp.100–101) Answer the followings.

31. Find a bassi for the null space of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}$

35. Solve the linear system, $2x_1 - x_2 + 3x_3 + 3 = 4x_1 + 2x_2 - x_4 - 1 = 0$

38. Mark each of the following True or False

    (a) A linear system with fewer equations than unknowns has an infinite number of solutions.

    (b) A consistent linear system with fewer equations than unknowns has an infinite number of solutions.

    (c) If a square linear system $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of column vector $\mathbf{b}$, then the solution is unique for each $\mathbf{b}$.

    (d) If a square linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

    (e) If a linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

    (f) The sum of two solution vectors of any linear system is also a solution vector of the system.

    (g) the sum of two solution vectors of any homogeneous linear system is also a solution vector of the system.

    (h) A scalar multiple of a solution vector of any homogenous linear system is also a solution vector of the system.

    (i) Every line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

    (j) Every line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

45. Let $\mathbf{v_1}, \mathbf{v_2}$ be vectors in $\mathbb{R}^n$. Prove the following set equalities by showing that each of the spans is contained in the other.

   (a) $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_2)$

   (b) $sp(\mathbf{v}_1, \mathbf{v}_2) = sp(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)$

47. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace.

(p.185) Theorem 3.1 (Elementary Properties of Vector Spaces) Every vector space V has the following properties:

1. The vector $\mathbf{0}$ is the unique vector $\mathbf{x}$ satisfying the equation $\mathbf{x} + \mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$ in V.
2. For each vector $\mathbf{v}$ in V, the vector $-\mathbf{v}$ is the unique vector $\mathbf{y}$ satisfying $\mathbf{v} + \mathbf{x} = \mathbf{0}$.
3. If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$ for vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in V, then $\mathbf{v} = \mathbf{w}$.
4. $0\mathbf{v} = \mathbf{0}$ for all vectors in V.
5. $r\mathbf{0} = \mathbf{0}$ for all scalars r in $\mathbb{R}$.
6. $(-r)\mathbf{v} = r(-\mathbf{v}) = -(r\mathbf{v})$ for all scalars r in $\mathbb{R}$ and vectors in V.

(p.189) In Exercises 1-8, decide whether or not the given set, together with the indicated operations of addition and scalar multiplication, is a (real) vector space.

1. The set $\mathbb{R}^2$, with the usual addition but with scalar multiplication defined by $r[x, y] = [ry, rx]$.
3. The set $\mathbb{R}^2$, with the usual scalar multiplication but with addition defined by $[x, y] \oplus [r, s] = [y + s, x + r]$.
5. The set of all 2×2 matrices, with the usual addition but with scalar multiplication defined by $rA = O$, the 2×2 zero matrix.

(p.189) In Exercises 9-16, determine whether the given set is closed under the usual operations of addition and scalar multiplication, and is a (real) vector space.

9. The set of all upper-triangular $n \times n$ matrices.
11. The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, a, b \in \mathbb{R}$.

(pp.201–202) Let P be the vector space of all polynomials with coefficients in $\mathbb{R}$. Answer the followings.

1. Determine whether the set of all polynomials of degree greater than 3 together with the zero polynomial is a subspace of P.
11. Prove whether $\{x^2 - 1, x^2 + 1, 4x, 2x - 3\}$ in P is dependent or independent.
25. Mark each of the following True or False.

   (a) The set consisting of the zero vector is a subspace for every vector space.

(b) Every vector space has at least two distinct subspaces.

(c) Every vector space with a nonzero vector has at least two distinct subspaces.

(d) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subset of a vector space V, then $\mathbf{v}_i \in sp(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for all i from 1 to n.

(e) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subset of a vector space V, then $\mathbf{v}_i + \mathbf{v}_j \in sp(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for all i and j from 1 to n.

(f) If $\mathbf{u} + \mathbf{v}$ lies in a subspace W of a vector space V, then both $\mathbf{u}$ and $\mathbf{v}$ lie in W.

(g) Two subspaces of a vector space V may have empty intersection.

(h) If S is independent, each vector in V can be expressed uniquely as a linear combination of vectors in S.

(i) If S is independent and generates V, each vector in V can be expressed uniquely as a linear combination of vectors in S.

(j) If each vector in V can be expressed uniquely as a linear combination of vectors in S, then S is an independent set.

(p.203) Answer the followings.

29. Let V be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Prove that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is also a basis for V.

30. Let V be a vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and let $W = sp(\mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_n)$. If $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_3$ is in W, show that $\mathbf{w} = \mathbf{0}$.

31. Let V be a vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Prove that the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_1 - \mathbf{v}_3$ do not generate V.

33. Let V be a vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and let $\mathbf{w} = t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k$, with $k \neq 0$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{w}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is a basis for V.

(pp.347–348) Answer the followings.

5. Find an orthonormal basis for the plane $2x + 3y + z = 0$.

6. Find an orthonormal basis for the subspace $W = \{[x_1, \ldots, x_4] | x_1 = x_2 + 2x_3, x_4 = -x_2 + x_3\}$ of $\mathbb{R}^4$.

7. Find an orthonormal basis for the subspace $sp([0, 1, 0], [1, 1, 1])$ of $\mathbb{R}^3$.

8. Find an orthonormal basis for the subspace $sp([1, 1, 0], [-1, 2, 1])$ of $\mathbb{R}^3$.

21. Find an orthonormal basis for the subspace $sp([2, 1, 1], [1, -1, 2])$ that contains $(1/\sqrt{6})[2, 1, 1]$.