Неделя 1. I, II, III, IV

I. ОПРЕДЕЛИТЕЛИ

14.4.2

$$|A_6| = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 3\cdot 7 - 5\cdot 4 = 21 - 20 = 1$$

14.4.5

$$|A_{77}| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2 + \sin^2\alpha = 1$$

14.7.1

$$|A_{200}| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$$

14.7.3

$$|A_{202}| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = 1(1-4) - 2(2-(-4)) + 2(-4-2) =$$

$$= -3 - 12 - 12 = -27$$

II. МАТРИЦЫ

15.2.1

$$3\begin{Vmatrix} 1 & 2 \\ 1 & 2 \end{Vmatrix} - \begin{Vmatrix} 3 & 2 \\ 3 & 2 \end{Vmatrix} - 4\begin{Vmatrix} 0 & 1 \\ 0 & 1 \end{Vmatrix} = \begin{Vmatrix} 3 & 6 \\ 3 & 6 \end{Vmatrix} - \begin{Vmatrix} 3 & 2 \\ 3 & 2 \end{Vmatrix} - \begin{Vmatrix} 0 & 4 \\ 0 & 4 \end{Vmatrix} =$$

$$= \begin{Vmatrix} 0 & 4 \\ 0 & 4 \end{Vmatrix} - \begin{Vmatrix} 0 & 4 \\ 0 & 4 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}$$

15.5.1

$$\underbrace{\begin{Vmatrix} 2 & -3 & 0 \end{Vmatrix}}_{1\times 3} \underbrace{\begin{Vmatrix} 4 \\ 3 \\ 1 \end{Vmatrix}}_{3\times 1} = \begin{Vmatrix} 2\cdot 4 - 3\cdot 3 - 0\cdot 1 \end{Vmatrix} = \begin{Vmatrix} -1 \end{Vmatrix}$$

15.5.2

$$\underbrace{\begin{Vmatrix} 4 \\ 3 \\ 1 \end{Vmatrix}}_{3\times 1} \underbrace{\begin{Vmatrix} 2 & -3 & 0 \end{Vmatrix}}_{1\times 3} = \begin{Vmatrix} 8 & -12 & 0 \\ 6 & -9 & 0 \\ 2 & -3 & 0 \end{Vmatrix}$$

15.10.1

$$\begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix} \begin{Vmatrix} 1 & 2 \end{Vmatrix} \begin{Vmatrix} 2 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix}$$

произведение $\neg\exists$

$2\times 2 \quad 1\times 2 \quad 2\times 1$

15.10.2

$$\begin{Vmatrix} 2 \\ 4 \end{Vmatrix} \begin{Vmatrix} 1 & 2 \end{Vmatrix} \begin{Vmatrix} 2 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 2 & 4 \\ 4 & 8 \end{Vmatrix} \begin{Vmatrix} 2 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 2\cdot 2+4\cdot 1 \\ 4\cdot 2+8\cdot 1 \end{Vmatrix} = \begin{Vmatrix} 8 \\ 16 \end{Vmatrix}$$

$2\times 1 \quad 1\times 2 \quad 2\times 1 \qquad 2\times 2 \quad 2\times 1$

15.11.1

$$\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}^n =$$

$$\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} = \begin{Vmatrix} 2 & 2 \\ 2 & 2 \end{Vmatrix} = 2\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}$$

$$= 2^1 \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}^{n-1} = 2^k \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}^{n-k} = 2^{n-1} \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} = \begin{Vmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{Vmatrix}$$

III. Системы линейных ур-ий

17.1.4

$$\begin{cases} y + 3z = -1 \\ 2x + 3y + 5z = 3 \\ 3x + 5y + 7z = 6 \end{cases} \qquad \begin{cases} y = -3z - 1 \\ 2x - 9z - 3 + 5z = 3 \\ 3x - 15z - 5 + 7z = 6 \end{cases} \qquad \begin{cases} x - 2z = 3 \\ 3x - 8z = 11 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -2 \\ 3 & -8 \end{vmatrix} = -8 - (-6) = -2 \qquad x = \frac{\Delta_x}{\Delta} = \frac{-2}{-2} = 1$$

$$\Delta_x = \begin{vmatrix} 3 & -2 \\ 11 & -8 \end{vmatrix} = -24 - (-22) = -2 \qquad z = \frac{\Delta_z}{\Delta} = \frac{2}{-2} = -1$$

$$\Delta_z = \begin{vmatrix} 1 & 3 \\ 3 & 11 \end{vmatrix} = 11 - 9 = 2 \qquad y = -3z - 1 = 3 - 2 = 2$$

Ответ: $(1, 2, -1)$

15.1.5*

$$\begin{cases} x+2y+3z=-4 \\ 2x+3y+4z=1 \\ 3x+4y+5z=6 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 6 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 1 & 1 & 1 & 5 \\ 3 & 4 & 5 & 6 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 1 & 1 & 1 & 5 \\ \cancel{2} & 2 & 2 & 10 \end{array}\right)$$

одинаков.

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 1 & 1 & 1 & 5 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 2 & -9 \\ 1 & 1 & 1 & 5 \end{array}\right)$$

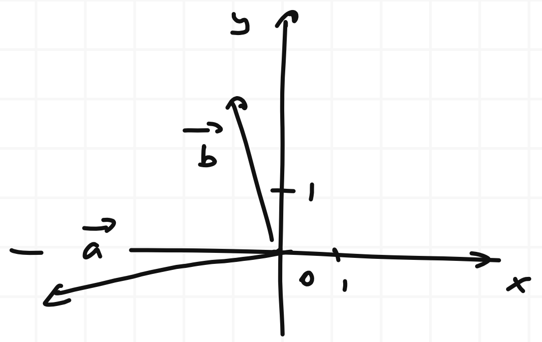
$$\begin{cases} y+2z+9=0 \\ x+y+z-5=0 \end{cases}$$ ← ответ

IV. ЛИНЕЙНЫЕ ОПЕРАЦИИ И ЛИНЕЙНАЯ ЗАВИСИМОСТЬ

1.6.

$\vec{a}(-5,-1)$

$\vec{b}(-1;3)$



$\vec{a} \not\parallel \vec{b}$, значит, обр. базис

$\vec{c}(-1,2)$

$\vec{d}(2,-6)$

$\vec{c}: \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \alpha_1 \begin{pmatrix} -5 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\Delta = \begin{vmatrix} -5 & -1 \\ -1 & 3 \end{vmatrix} = -15-1=-16$

$\Delta_1 = \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} = -3+2=-1$

$\Delta_2 = \begin{vmatrix} -5 & -1 \\ -1 & 2 \end{vmatrix} = -10-1=-11$

$\alpha_1 = \frac{\Delta_1}{\Delta} = \frac{1}{16} \qquad \alpha_2 = \frac{\Delta_2}{\Delta} = \frac{11}{16}$

Ответ: $\left(\frac{1}{16}, \frac{11}{16}\right)$

$\vec{d}: \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \beta_1 \begin{pmatrix} -5 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\Delta = -16$

$\Delta_1 = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 6-6=0$

$\Delta_2 = \begin{vmatrix} -5 & 2 \\ -1 & -6 \end{vmatrix} = 30+2=32$

$\beta_1 = 0 \qquad \beta_2 = \frac{-32}{16} = -2$

Ответ: $(0;-2)$

1.11.2

$$\begin{cases} l = a+b+c \\ m = b+c \\ n = -a+c \end{cases}$$

Если компланарны, то

$\vec{l}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\vec{m}\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\vec{n}\begin{pmatrix}-1\\0\\1\end{pmatrix}$ – линейно зависима

Подберем коэф. α_i:

$$\alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}-1\\0\\1\end{pmatrix} = 0$$

$$\begin{cases} \alpha_1 - \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \qquad \begin{aligned} \alpha_3 &= \alpha_1 \\ \alpha_2 &= -\alpha_1 \end{aligned}$$

$$\alpha_1 - \alpha_1 + \alpha_1 = \alpha_1 = 0;$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

Система решается единств. образом, $\forall i$: $\alpha_i = 0$, а зн. система линейно независима.

Ответ: нет

1.11.3

$$\begin{cases} l = c \\ m = a-b-c \\ n = a-b+c \end{cases}$$

Если компланарны, то

$\vec{l}\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\vec{m}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $\vec{n}\begin{pmatrix}1\\-1\\1\end{pmatrix}$ – лин. зав.

Подберем коэф α_i:

$$\alpha_1\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$$

$$\begin{cases} \alpha_2 + \alpha_3 = 0 \\ -\alpha_2 - \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases} \qquad \begin{cases} \alpha_3 = -\alpha_2 \\ \alpha_1 - 2\alpha_2 = 0 \end{cases} \qquad \begin{cases} \alpha_1 = 2\alpha_2 \\ \alpha_3 = -\alpha_2 \end{cases}$$

Решений много. $\exists\ \alpha_2 = 1$

$$\begin{cases} \alpha_1 = 2 \\ \alpha_2 = 1 \\ \alpha_3 = -1 \end{cases} \leftarrow \text{например}$$

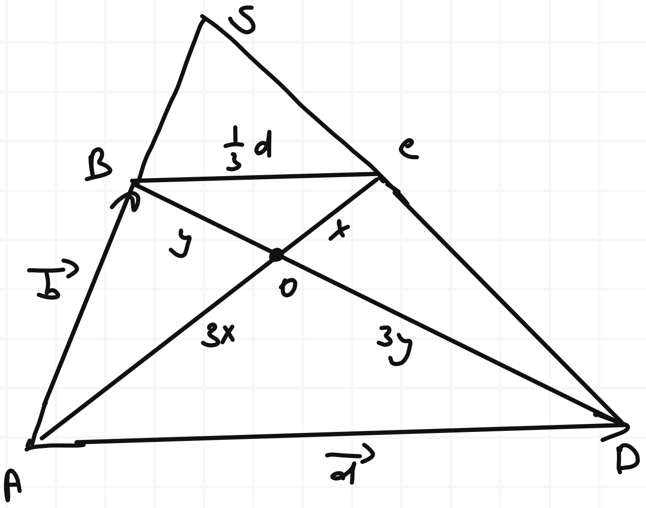
Значит, векторы компланарны.

Ответ: да; $2\vec{l} + \vec{m} - \vec{n} = 0$

1.16.

$AD : BC = 3 : 1$

базис: $\vec{d}, \vec{b}$



$\vec{AC} = \vec{AB} + \vec{BC} = \vec{b} + \frac{1}{3}\vec{d}$ $\qquad \vec{AC}\left(\frac{1}{3}; 1\right)$

$\vec{AO} = \frac{3}{4}\vec{AC} = \frac{3}{4}\vec{b} + \frac{1}{4}\vec{d}$ $\qquad \vec{AO}\left(\frac{1}{4}; \frac{3}{4}\right)$ $\qquad (\triangle BOC \sim \triangle DOA)$

$\vec{AS}$ =

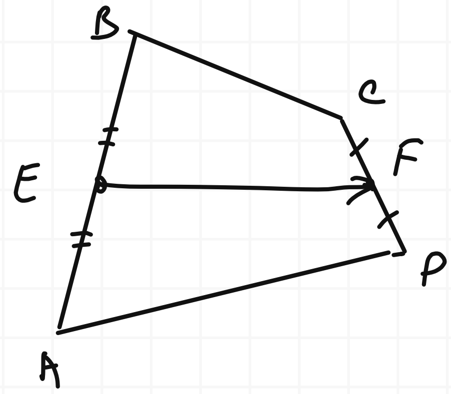
// $\vec{d} = \vec{AS} - \vec{DS} = \left[\frac{AS}{AB} = \frac{DS}{CD} = k\right] = k\vec{b} + k\vec{CD} = \left[\vec{CD} = -\frac{1}{3}\vec{d} - \vec{b} + \vec{d} = \frac{2}{3}\vec{d} - \vec{b}\right] =$

подобие

$= k\left(\cancel{\vec{b}} + \frac{2}{3}\vec{d} - \cancel{\vec{b}}\right) = \frac{2}{3}k\vec{d}$; $\quad \vec{d} = \frac{2}{3}k\vec{d}$; $\quad k = \frac{3}{2}$ //

= $k\vec{b} = \frac{3}{2}\vec{b}$ $\qquad \vec{AS}\left(0; \frac{3}{2}\right)$

1.17



$\vec{EF} \overset{?}{=} \frac{1}{2}(\vec{BC} + \vec{AD})$

$\vec{EF} = \frac{1}{2}\vec{AB} + \vec{BC} - \frac{1}{2}\vec{CD}$

$+$

$\vec{EF} = -\frac{1}{2}\vec{AB} + \vec{AD} + \frac{1}{2}\vec{CD}$

$2\vec{EF} = \vec{AD} + \vec{BC}$

$\vec{EF} = \frac{1}{2}(\vec{AD} + \vec{BC})$ q.e.d.

N1.24(1)

$\vec{OM}$

$M \in [AB]$

$\frac{|AM|}{|BM|} = \frac{m}{n}$

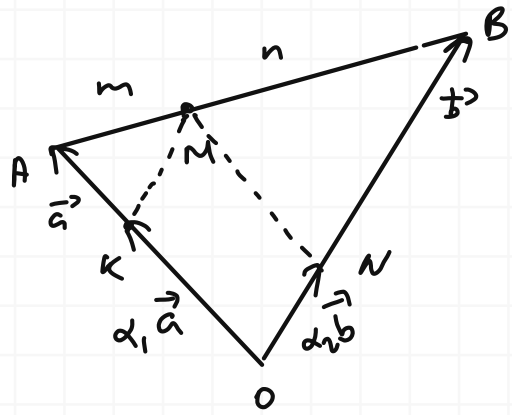
] $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

$\triangle ABO \sim \triangle MBN$

$\frac{MN}{OA} = \frac{OK}{OA} = \alpha_1 = \frac{n}{n+m}$

$\triangle ABO \sim \triangle AMK$

$\frac{MK}{OB} = \frac{ON}{OB} = \alpha_2 = \frac{m}{n+m}$

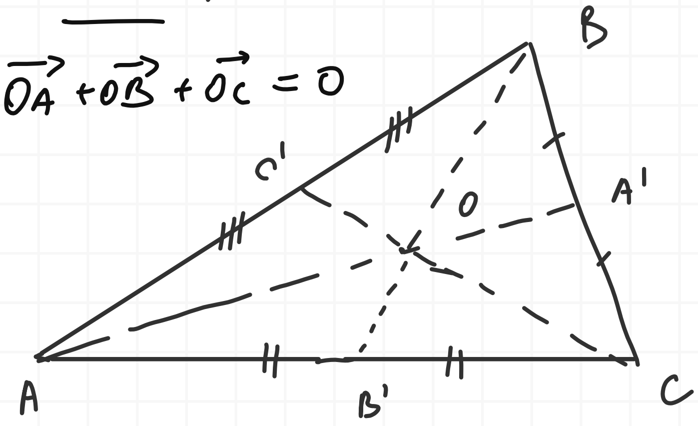


$\vec{OM} = \frac{n}{n+m}\vec{a} + \frac{m}{n+m}\vec{b}$

Ответ: $\left(\frac{n}{n+m}; \frac{m}{n+m}\right)$

1.37 (1)

$\vec{OA}+\vec{OB}+\vec{OC}=0$



Да, внутри треуг. такая точка — — точка пересеч. медиан.

$$\vec{OA}=\frac{2}{3}\vec{A'A}=\frac{1}{3}\vec{BA}+\frac{1}{3}\vec{CA}$$

$$+\quad \vec{OB}=\frac{1}{3}\vec{AB}+\frac{1}{3}\vec{CB}$$

$$+\quad \vec{OC}=\frac{1}{3}\vec{AC}+\frac{1}{3}\vec{BC}$$

$$\vec{OA}+\vec{OB}+\vec{OC}=\vec{0}$$

q.e.d.

А еще точку O, удовл. данным усл., можно найти так:

$$\vec{CO}=\vec{OA}+\vec{OB}$$

$$\vec{CO}=\vec{CB}+\vec{BO}$$

$$\vec{CO}=\vec{CA}+\vec{AO}$$

$$3\vec{CO}=\vec{CA}+\vec{CB}$$

$$\vec{OC}=\frac{1}{3}\vec{AC}+\frac{1}{3}\vec{BC}$$

аналогично:

в базисе $\vec{AC}$, $\vec{BC}$
$\quad\ \ \vec{a}\quad\ \ \vec{b}$

$$\vec{OA}=\frac{1}{3}(b-a)-\frac{1}{3}a=-\frac{2}{3}a+\frac{1}{3}b \qquad \begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$$

$$\vec{OB}=\frac{1}{3}(a-b)-\frac{1}{3}b=-\frac{2}{3}b+\frac{1}{3}a \qquad \begin{pmatrix}\frac{1}{3}\\ -\frac{2}{3}\end{pmatrix}$$

$$\vec{OC}=\frac{1}{3}a+\frac{1}{3}b \qquad \begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

(2) Как видно из вывода, она единственная. А значит, других таких точек O, в т.ч. вне треуг., нет.

Ответ: нет.

V. ЗАМЕНА БАЗИСА И СИСТ. КООРДИНАТ

<u>4.3</u>

$S(O, e_1, e_2)$

$S'(O', e_1', e_2')$

$(\overrightarrow{OO'})_e = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$(e_1')_e = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$(e_2')_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$e_1' = 2e_1 + 3e_2$
$e_2' = e_1 + e_2$

$C = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad e' = eC$

1) $M_{S'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $M_S - ?$

$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$

$e(\overrightarrow{OM})_e = e(\overrightarrow{OO'})_e + e'(\overrightarrow{O'M})_{e'}$

$(\overrightarrow{OM})_e = (OO')_e + C(O'M)_{e'}$

$M_S = O'_S + C \cdot M_{S'} =$

$= \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x' + y' - 1 \\ 3x' + y' + 3 \end{pmatrix}$

2) $M_S = \begin{pmatrix} x \\ y \end{pmatrix}$, $M_{S'} - ?$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x' + y' - 1 \\ 3x' + y' + 3 \end{pmatrix}$; $\begin{cases} 2x' + y' = x + 1 \\ 3x' + y' = y - 3 \end{cases}$

решим методом Крамера

$\Delta = 2 \cdot 1 - 3 \cdot 1 = -1$

$\Delta_x = \begin{vmatrix} x+1 & 1 \\ y-3 & 1 \end{vmatrix} = x + 1 - y + 3 = x - y + 4$

$\Delta_y = \begin{vmatrix} 2 & x+1 \\ 3 & y-3 \end{vmatrix} = 2y - 6 - 3x - 3 = -3x + 2y - 9$

$\begin{cases} x' = -x + y - 4 \\ y' = 3x - 2y + 9 \end{cases}$ $\quad M_{S'} = \begin{pmatrix} -x + y - 4 \\ 3x - 2y + 9 \end{pmatrix}$

3) $e_1' = 2e_1 + 3e_2$
$e_2' = e_1 + e_2$

$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$

$\Delta_{e_1} = \begin{vmatrix} e_1' & 3 \\ e_2' & 1 \end{vmatrix} = e_1' - 3e_2'$

$\Delta_{e_2} = \begin{vmatrix} 2 & e_1' \\ 1 & e_2' \end{vmatrix} = 2e_2' - e_1'$

$e_1 = -e_1' + 3e_2'$

$e_2 = e_1' - 2e_2'$

$e = \begin{pmatrix} -e_1' + 3e_2' \\ e_1' - 2e_2' \end{pmatrix} = e' \underbrace{\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}}_{\tilde{C}}$

ответ:

$\begin{cases} e_1(-1; 3) \\ e_2(1; -2) \end{cases}$

$$e\left(\overrightarrow{OO'}\right)_e = \overrightarrow{OO'} = -\overrightarrow{O'O} = -e'\left(\overrightarrow{O'O}\right)_{e'}$$

$$e\begin{pmatrix}-1\\3\end{pmatrix}_e = -\cancel{e'}\begin{pmatrix}x_0\\y_0\end{pmatrix}_{e'}$$

$$\|$$

$$\cancel{e'}\tilde{C}$$

$$\begin{pmatrix}x_0\\y_0\end{pmatrix} = -\begin{pmatrix}-1&1\\3&-2\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}1&-1\\-3&2\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}-1-3\\3+6\end{pmatrix} = \begin{pmatrix}-4\\9\end{pmatrix}$$

Ответ: $O_{S'} = \begin{pmatrix}-4\\9\end{pmatrix}$

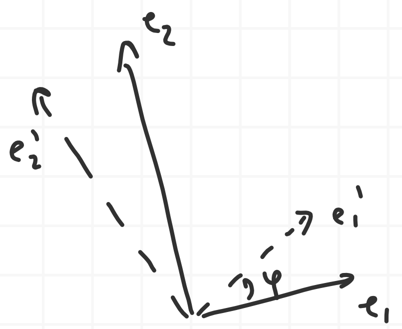
4.26 (1)

$S(O, e_1, e_2)$

$S'(O', e_1', e_2')$

$O'_S = \begin{pmatrix}1\\3\end{pmatrix}$

$\varphi = 60^\circ$



в полярных:

$e_1(1, \varphi_{01})$ $\quad e_1'(1, \varphi_{01}+\varphi)$

$e_2(1, \varphi_{02})$ $\quad e_2'(1, \varphi_{01}+\varphi)$

в базисе e:

$$e_1'\begin{pmatrix}\cos\varphi\\ \frac{|e_1|}{|e_2|}\sin\varphi\end{pmatrix}_e \qquad e_2'\begin{pmatrix}-\frac{|e_2|}{|e_1|}\sin\varphi\\ \cos\varphi\end{pmatrix}_e$$

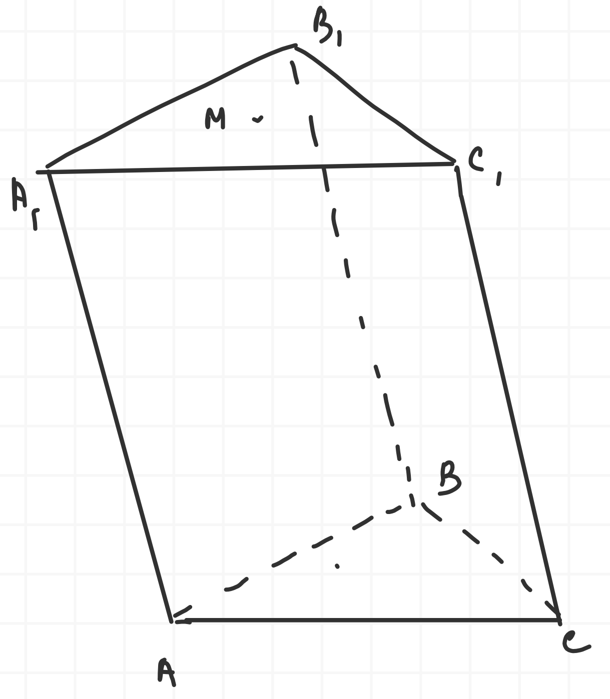
$\varphi = 60^\circ \quad \cos\varphi = \frac{1}{2} \quad \sin\varphi = \frac{\sqrt{3}}{2}$

$$e_1'\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\cdot\frac{|e_1|}{|e_2|}\end{pmatrix} \qquad e_2'\begin{pmatrix}-\frac{\sqrt{3}}{2}\frac{|e_1|}{|e_1|}\\ \frac{1}{2}\end{pmatrix}; \qquad C = \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\frac{|e_2|}{|e_1|}\\ \frac{\sqrt{3}}{2}\frac{|e_1|}{|e_2|} & \frac{1}{2}\end{pmatrix}$$

$$M_S = O'_S + CM_{S'}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + C\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\frac{1}{2}x' - \frac{\sqrt{3}}{2}\frac{|e_2|}{|e_1|}y' + 1\\ \frac{\sqrt{3}}{2}\frac{|e_1|}{|e_2|}x' + \frac{1}{2}y' + 3\end{pmatrix} = \begin{bmatrix}\text{если}\\ |e_1| = |e_2|\end{bmatrix} = \begin{pmatrix}\frac{1}{2}x' - \frac{\sqrt{3}}{2}y' + 1\\ \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' + 3\end{pmatrix}$$

4.19



$$S(A, \vec{AB}, \vec{AC}, \vec{AB_1}) \qquad S'(A_1, \vec{A_1B}, \vec{A_1C}, \vec{A_1M})$$

$$\vec{AB} = e_1,\ \vec{AC} = e_2,\ \vec{AB_1} = e_3 \qquad \vec{A_1B} = e_1',\ \vec{A_1C} = e_2',\ \vec{A_1M} = e_3'$$

$$e_1' = \vec{A_1B} = -\vec{AA_1} + \vec{AB} = \left[\vec{AA_1} = \vec{AB_1} - \vec{AB} = e_3 - e_1\right] =$$

$$= -\vec{AB_1} + 2\vec{AB} = 2e_1 - e_3$$

$$e_2' = \vec{A_1C} = -\vec{AA_1} + \vec{AC} = e_1 + e_2 - e_3$$

$$e_3' = \vec{A_1M} = \frac{1}{3}(\vec{AB} + \vec{AC}) = \frac{1}{3}e_1 + \frac{1}{3}e_2$$

$$e_1' = 2e_1 - e_3$$
$$e_2' = e_1 + e_2 - e_3$$
$$e_3' = \frac{1}{3}e_1 + \frac{1}{3}e_2$$

$$e' = eC = e\begin{pmatrix} 2 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ -1 & -1 & 0 \end{pmatrix}$$

$$A_{1S} = (AA_1)_e = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$X_S = A_{1S} + CX_{S'}$$

$$\underline{X_S} = \underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underline{\begin{pmatrix} 2x' + y' + 1/3 \cdot z' - 1 \\ y' + 1/3 \cdot z' \\ -x' - y' + 1 \end{pmatrix}}$$

VI. СКАЛЯРНОЕ ПРОИЗВЕДЕНИЕ

2.27(2)

$a(-1, -1, 2)$

$b(1; 1; 2)$

$Pr_a b - ?$

$$(a, b) = |a|\underbrace{|b|\cos\langle a, b\rangle}_{Pr_a b} = |a|\, Pr_a b$$

$$|a| = \sqrt{(a,a)} = \sqrt{1+1+4} = \sqrt{6}$$

$$|b| = \sqrt{(b,b)} = \sqrt{1+1+4} = \sqrt{6}$$

$$(a, b) = -1 - 1 + 4 = 2$$

$$Pr_a b = \frac{(a,b)}{|a|} = \frac{2}{\sqrt{6}} = \frac{\cancel{2}\sqrt{6}}{\cancel{6}\,3} = \frac{\sqrt{6}}{3}$$

2.35

$a(1,-1,1)$

$b(5,1,1)$

$|c|=1$

$c \perp a$

$\angle(b,c) = \arccos\left(\sqrt{\frac{2}{27}}\right)$

$c - ?$

сколько решений?

$\exists c(x,y,z)$

$(a,c) = 0 = x - y + z; \quad y = x+z$

$c(x, x+z, z)$

$(b,c) = |b||c|\cos\angle(b,c) = \left[|b| = \sqrt{25+1+1} = \sqrt{27}\right] = \sqrt{27}\cdot\sqrt{\frac{2}{27}} = \sqrt{2}$

$(b,c) = \sqrt{2} = 5x + (x+z) + z = 6x + 2z \quad (1)$

$|c| = 1$

$x^2 + (x+z)^2 + z^2 = 1$

$x^2 + xz + z^2 = \frac{1}{2} \quad (2)$

$$\begin{cases} 3x + z = \frac{1}{\sqrt{2}}; \quad z = \frac{1}{\sqrt{2}} - 3x \\ x^2 + xz + z^2 = \frac{1}{2} \end{cases}$$

$\frac{1}{2} = x^2 + x\left(\frac{1}{\sqrt{2}} - 3x\right) + \left(\frac{1}{\sqrt{2}} - 3x\right)^2$

$\cancel{\frac{1}{2}} = x^2 + \frac{1}{\sqrt{2}}x - 3x^2 + \cancel{\frac{1}{2}} + 9x^2 - 3\sqrt{2}x$

$$\left[\begin{array}{l} x = 0 \\ \underline{x} + \frac{1}{\sqrt{2}} \underline{-3x} + \underline{9x} - 3\sqrt{2} = 0 \end{array}\right.$$

$$\left[\begin{array}{l} x = 0 \\ 7x = \frac{5\sqrt{2}}{2} \end{array}\right. \qquad \left[\begin{array}{l} x = 0 \\ x = \frac{5\sqrt{2}}{14} \end{array}\right.$$

$z = \frac{\sqrt{2}}{2} \qquad y = \frac{\sqrt{2}}{2}$

$z = \frac{\sqrt{2}}{2}^{\backslash 7} - \frac{15\sqrt{2}}{14} = \frac{-\cancel{8}^{4}\sqrt{2}}{\cancel{14}_{7}} = -\frac{4\sqrt{2}}{7}$

$y = \frac{5\sqrt{2}}{14} - \frac{8\sqrt{2}}{14} = -\frac{3\sqrt{2}}{14}$

Ответ: 1) $\left[\begin{array}{l} \left(0; \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ \left(\frac{5\sqrt{2}}{14}, -\frac{3\sqrt{2}}{14}; -\frac{4\sqrt{2}}{7}\right) \end{array}\right.$

2) 2 решения

2.21

$|e_1| = 3$ $\quad \angle(e_1, e_2) = 45^\circ$ $\quad a(1, -3, 0)$ $\quad |a|, |b| - ?$

$|e_2| = \sqrt{2}$ $\quad \angle(e_2, e_3) = 45^\circ$ $\quad b(-1, 2, 1)$ $\quad \varphi = \angle(a, b) - ?$

$|e_3| = 4$ $\quad \angle(e_1, e_3) = 60^\circ$

$$|a|^2 = (a, a) = a_e^T G a_e =$$

$$(e_1, e_2) = 3\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 3$$

$$(e_2, e_3) = 4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 4 \qquad G = \begin{pmatrix} 9 & 3 & 6 \\ 3 & 2 & 4 \\ 6 & 4 & 16 \end{pmatrix}$$

$$(e_1, e_3) = 12 \cdot \frac{1}{2} = 6$$

$$= (1\ -3\ 0) \begin{pmatrix} 9 & 3 & 6 \\ 3 & 2 & 4 \\ 6 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = (1\ -3\ 0) \begin{pmatrix} 0 \\ -3 \\ -6 \end{pmatrix} = 9 \ ; \quad \underline{|a| = 3}$$

$$|b|^2 = (b, b) = b_e^T G b_e = (-1\ 2\ 1) \begin{pmatrix} 9 & 3 & 6 \\ 3 & 2 & 4 \\ 6 & 4 & 16 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = (-1\ 2\ 1) \begin{pmatrix} -9+6+6 \\ -3+4+4 \\ -6+8+16 \end{pmatrix} =$$

$$= (-1\ 2\ 1) \begin{pmatrix} 3 \\ 5 \\ 18 \end{pmatrix} = -3 + 10 + 18 = 25 \ ; \quad \underline{|b| = 5}$$

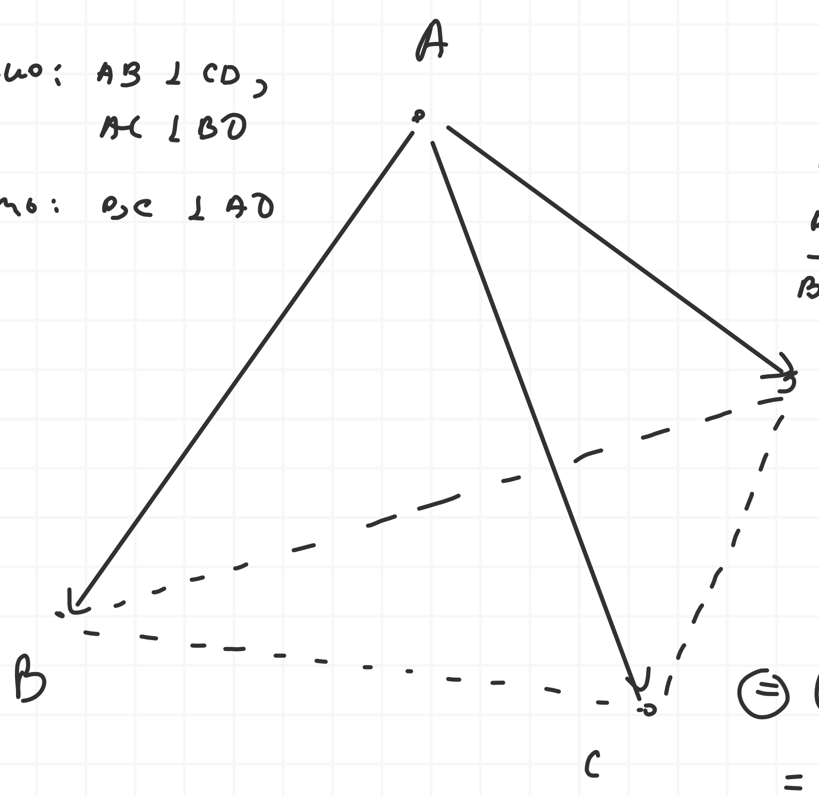
$$\cos\varphi = \frac{(a, b)}{|a||b|} = \frac{(1, -3, 0)(-1, 2, 1)}{3 \cdot 5} = \frac{-1 - 6}{15} = -\frac{7}{15}$$

$$\underline{\angle(a, b) = \arccos\left(-\frac{7}{15}\right)}$$

2.45

дано: $AB \perp CD$, $AC \perp BD$

д-ть: $BC \perp AD$



Заведем базис: $\vec{AB}, \vec{AC}, \vec{AD}$
$\quad \quad \quad \quad \quad \quad \quad \ \ e_1 \quad e_2 \quad e_3$

$$\vec{CD} = -\vec{AC} + \vec{AD} = -e_2 + e_3$$

$$\vec{BD} = -\vec{AB} + \vec{AD} = -e_1 + e_3$$

$$\vec{BC} = -\vec{AB} + \vec{AC} = -e_1 + e_2$$

$$(\vec{AB}, \vec{CD}) = e_1(-e_2 + e_3) = -e_1e_2 + e_1e_3 = 0$$

$$(\vec{AC}, \vec{BD}) = e_2(-e_1 + e_3) = -e_1e_2 + e_2e_3 = 0$$

$$(\vec{BC}, \vec{AD}) = e_3(-e_1 + e_2) = \underbrace{-e_1e_3 + e_2e_3}_{(1)} \overset{?}{=} 0$$

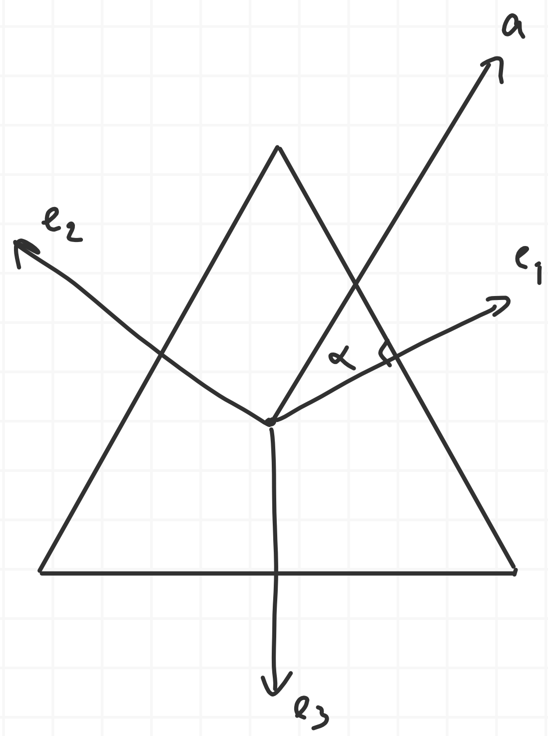
$$(1) = (-\cancel{e_1e_2} + e_2e_3) - (-\cancel{e_1e_2} + e_1e_3) = (\vec{AC}, \vec{BD}) - (\vec{AB}, \vec{CD}) =$$

$$= 0 - 0 = 0$$

ч.т.д.

T.1.

a



Заведем базис из единичных векторов ⊥ сторонам треугольника.

Проекц. a на e_i численно равна соот. пр. на треугольн.

$\exists \angle(e_1, a) = \alpha$, тогда

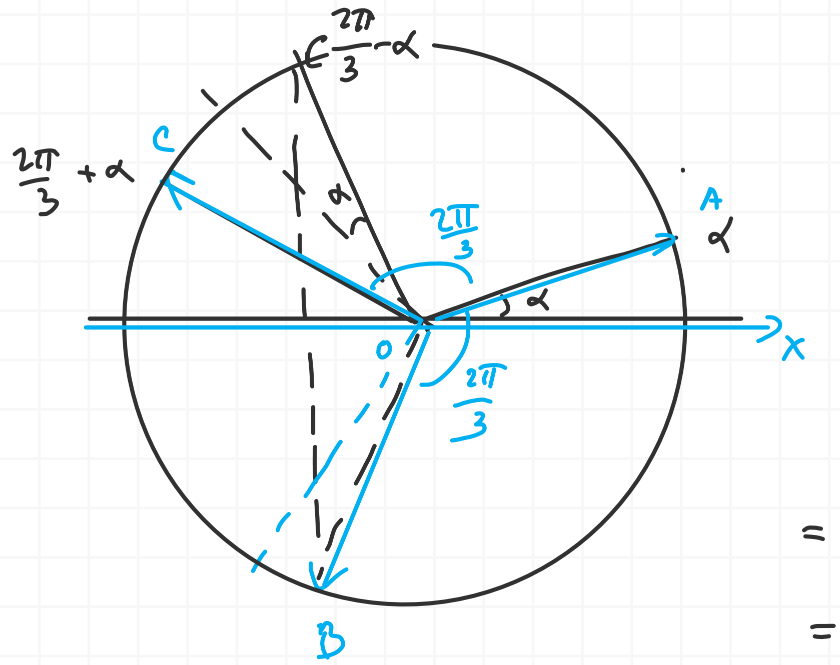
$\angle(e_2, a) = \frac{2\pi}{3} - \alpha$

$\angle(e_3, a) = \frac{2\pi}{3} + \alpha$

возможно, равны по mod 2π — это не влияет на знач. cos

$$\sum_{i=1}^{3} Pr_{e_i} a = \left[Pr_b a = \frac{(a,b)}{|b|} \right] = \frac{(a,e_1)}{|e_1|} + \frac{(a,e_2)}{|e_2|} + \frac{(a,e_3)}{|e_3|} =$$

$$= \left[|e_1| = |e_2| = |e_3| = 1 \right] = (a,e_1) + (a,e_2) + (a,e_3) = a\left(\cos\alpha + \cos\left(\frac{2\pi}{3} - \alpha\right) + \cos\left(\frac{2\pi}{3} + \alpha\right)\right) =$$



$$= \left[\cos\left(\frac{2\pi}{3} - \alpha\right) = \cos\left(\alpha - \frac{2\pi}{3}\right) \right] =$$

$$= a\left(\cos\alpha + \cos\left(\alpha - \frac{2\pi}{3}\right) + \cos\left(\frac{2\pi}{3} + \alpha\right)\right)$$

← см. геометр. интерпретацию

$$= a\left(\overrightarrow{OA}_x + \overrightarrow{OB}_x + \overrightarrow{OC}_x\right) = a\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)_x =$$

$$= a \cdot 0 = 0$$

VII. ВЕКТОРНОЕ И СМЕШАННОЕ ПРОИЗВЕДЕНИЕ

3.1 (1)

$a(3,-1,2)$

$b(2,-3,-5)$

$[a;b]$ – ?

$$[a;b]=\begin{vmatrix} e_1 & e_2 & e_3 \\ 3 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix} = e_1(-5+6)+ -e_2(15-4)+ + e_3(-9+2) = e_1-11e_2-7e_3 = (1,-11,-7)$$

3.7 (2) 2.35

Хотим проверить, правая ли тройка a, b, c

$(0;\frac{1}{\sqrt2},\frac{1}{\sqrt2}) = \frac{1}{\sqrt2}(0,1,1)$ $\quad(\frac{5\sqrt2}{14},-\frac{3\sqrt2}{14};-\frac{4\sqrt2}{7}) = \frac{\sqrt2}{14}(5,-3,-8)$

$(\vec a,\vec b,\vec c) = ([\vec a,\vec b],\vec c) > 0 \iff \vec a,\vec b,\vec c$ – правая тройка

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 5+5=10>0$$

правый

$$\begin{vmatrix} 1 & -1 & 1 \\ 5 & 1 & 1 \\ 5 & -3 & -8 \end{vmatrix} = -8+3-40-5-15-5 = -70<0$$

левый

Ответ: $\vec c(0,\frac{1}{\sqrt2},\frac{1}{\sqrt2})$

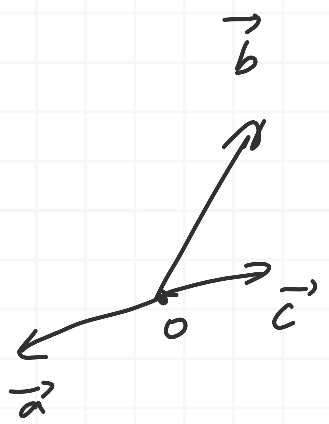
3.12

$[a,b]=[b,c]=[c,a] \iff a+b+c=0$

(неколлинеарные)

„$\Rightarrow$" пусть $\vec p=[\vec a,\vec b]=[\vec b,\vec c]=[\vec c,\vec a]$

$\vec p \perp \vec a,\vec b,\vec c$ ($\rightsquigarrow$ a, b, c – компланарны)



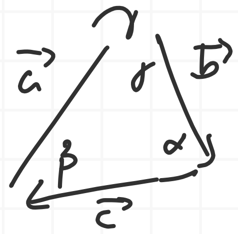
$$|a||b|\sin\angle(a,b) = |b||c|\sin\angle(b,c) = |a||c|\sin\angle(a,c)$$

$$1=\frac{|a|\sin\angle(a,b)}{|c|\sin\angle(b,c)} = \frac{|a|\sin\angle(a,c)}{|b|\sin\angle(b,c)}$$

$$\frac{\sin\angle(a,b)}{|c|} = \frac{\sin\angle(b,c)}{|a|}$$ ← т. синусов для тр. со сторонами a, b, c

Значит, a, b, c – образ. треугольник. Т.е. $\vec a+\vec b+\vec c=\vec 0$

„$\Leftarrow$" $\vec a + \vec b + \vec c = 0$, гл. $\vec a, \vec b, \vec c$ – компланарны



$$\frac{\sin\alpha}{|\vec a|} = \frac{\sin\beta}{|\vec b|} = \frac{\sin\gamma}{|\vec c|}$$

$$|\vec b|\sin\alpha = |\vec a|\sin\beta \qquad \Big|\cdot|\vec c|$$

$$[\vec b, \vec c] = [\vec c, \vec a]$$

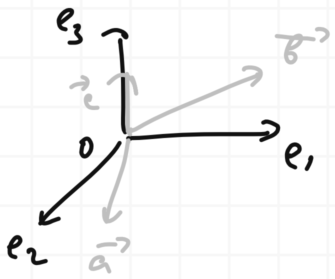
ан-но $[\vec b, \vec c] = [\vec a, \vec b]$

q.e.d.

3.13 (1,2)

(1) $$|[a,b]|^2 = \Big||a||b|\sin\angle(a,b)\Big|^2 = |a|^2|b|^2(1-\cos^2\angle(a,b)) = (a,a)(b,b) -$$
$$- (|a||b|\cos\angle(a,b))^2 = (a,a)(b,b) - (a,b)^2 = \begin{vmatrix} (a,a) & (a,b) \\ (a,b) & (b,b) \end{vmatrix}$$

(2) $[a,[b,c]] =$



Зафиксируем (·)O и онб $\vec e\,(e_1, e_2, e_3)$, т.ч. $c \uparrow\uparrow e_3$, $b \perp e_2$

$$a\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_e \quad b\begin{pmatrix} b_1 \\ 0 \\ b_3 \end{pmatrix} \quad c\begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix}$$

$$[b,c] = \begin{pmatrix} e_1 & e_2 & e_3 \\ b_1 & 0 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} = -e_2(b_1c_3)$$

$$[a,[b,c]] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ 0 & -b_1c_3 & 0 \end{pmatrix} = e_1(a_3b_1c_3) + e_3(-a_1b_1c_3)$$

$$b(a,c) = (b_1e_1 + b_3e_3)(a_3c_3) = e_1(a_3b_1c_3) + e_3(a_3b_3c_3)$$

$$c(a,b) = (c_3e_3)(a_1b_1 + a_3b_3) = e_3(a_1b_1c_3 + a_3b_3c_3)$$

$$b(a,c) - c(ab) = e_1(a_3b_1c_3) - e_3(a_1b_1c_3)$$

q.e.d.

3.16

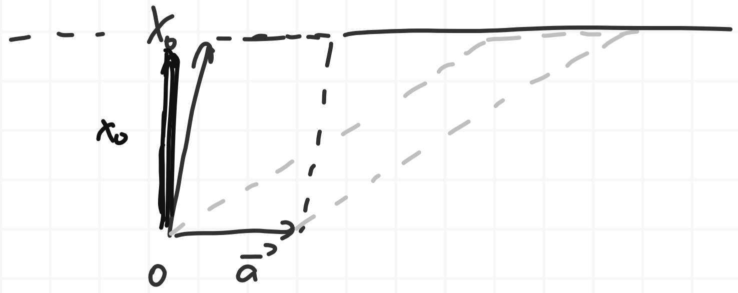
$\| \vec{a} \times \vec{b}$

$[x, a] = b$

$|x||a|\sin\angle(x, a) = S_{x} = |b|$

Пусть X — конец вектора x, O — нач. коорд. — fixed.

Хотим найти ГМТ X : $[\vec{OX}, a] = b$ — такие $\vec{OX}$, что параллелограмм, построенный на $\vec{OX}$ и $\vec{a}$ имеет площадь $|\vec{b}|$ и $\vec{OX}$ направл. в нужную полуплоск. отн. $\vec{a}$ (напр. $\vec{b}$)



Если X_0 — точка, кот. подходит. То ГМТ X — это прямая $\| \vec{a}$, проходящая через X_0

Проекц. точки O на прямую — это ортонормированный конец вектора $x_0 \perp a$, т.ч. $[x_0, a] = b$, $x_0 \perp b$

$x_0, a = b \qquad |x_0||a| = |b|$

$x_0 = \frac{[a, b]}{|a|^2} = \frac{|a||b|}{|a|^2}$ ← модули

3.20 (1)

компланарны ли?

$a(2,3,5)$
$b(7,1,-1)$
$c(3,-5,-11)$

если да, то $(a, b, c) = 0$

$$[a, b] = \begin{pmatrix} e_1 & e_2 & e_3 \\ 2 & 3 & 5 \\ 7 & 1 & -1 \end{pmatrix} = e_1(-3-5) - e_2(-2-35) + e_3(2-21) = -8e_1 + 37e_2 - 19e_3$$

$$([a, b], c) = -8 \cdot 3 + 37 \cdot (-5) + 19 \cdot 11 = -24 - 185 + 209 = 0$$

Ответ: да, компланарны

3.23

$A(2,1,-1)$
$B(3,0,2)$
$C(5,1,1)$
$D(0,-1,3)$

$\vec{DA}\ (2, 2, -4)$
$\vec{DB}\ (3, 1, -1)$
$\vec{DC}\ (5, 2, -2)$

$$// \ [DA, DB] = \begin{pmatrix} e_1 & e_2 & e_3 \\ 2 & 2 & -4 \\ 3 & 1 & -1 \end{pmatrix} = e_1(-2+4) - e_2(-2+12) + e_3(2-6) = 2e_1 - 10e_2 - 4e_3$$

$$|(\vec{DA}, \vec{DB}, \vec{DC})| = |2 \cdot 5 - 10 \cdot 2 + 2 \cdot 4| = |10 - 20 + 8| = |-2| = 2$$

(1) $V = \frac{1}{6}|(\vec{DA}, \vec{DB}, \vec{DC})| = \frac{1}{6} \cdot 2 = \frac{1}{3}$

$V = \frac{1}{3} h_c S_{ABD} = \frac{1}{3} h_c \cdot \frac{1}{2}\left|[\vec{AB}, \vec{AD}]\right|$

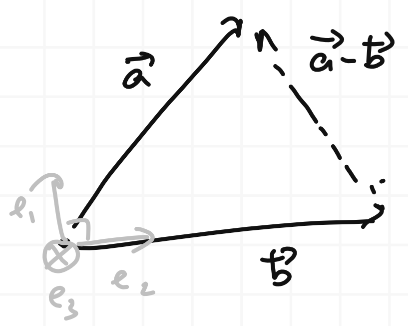
// $AB = (1,-1,3)$ $\quad \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix} = e_1(-4+6) - e_2(4+6) + e_3(-2-2) =$
$AD = (-2,-2,4)$ $\quad = 2e_1 - 10e_2 - 4e_3$

$|[AB, AD]| = \sqrt{4+100+16} = \sqrt{120} = 2\sqrt{30}$ //

(1) $\frac{1}{6} h_c \cdot 2\sqrt{30} = \frac{\sqrt{30}}{3} h_c$

(2) $\underline{h_c} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\sqrt{30}} = \underline{\frac{1}{\sqrt{30}}}$

<u>3.33</u>



$S = \frac{1}{2}|[\vec{a}, \vec{b}]|$

$S_M = \frac{1}{2}\left|\left[\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b},\ \frac{1}{2}\vec{a} + \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}\right]\right| =$

$= \frac{1}{2}\left|\left[\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b};\ \vec{a} - \frac{1}{2}\vec{b}\right]\right| =$

$= \frac{1}{4}\left|\left[\vec{a}+\vec{b},\ 2\vec{a}-\vec{b}\right]\right|$

Заведём базис, т.ч. $a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}_e \quad b = \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix}_e$

$[\vec{a}, \vec{b}] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & 0 \\ 0 & b_2 & 0 \end{pmatrix} = e_3(a_1 b_2)$

$\vec{a}+\vec{b} = \begin{pmatrix} a_1 \\ a_2 + b_2 \\ 0 \end{pmatrix}_e \quad 2\vec{a}-\vec{b} = \begin{pmatrix} 2a_1 \\ 2a_2 - b_2 \\ 0 \end{pmatrix}_e$

$[\vec{a}+\vec{b}, 2\vec{a}-\vec{b}] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2+b_2 & 0 \\ 2a_1 & 2a_2-b_2 & 0 \end{pmatrix} = e_3(\cancel{2a_1a_2} - a_1b_2 - \cancel{2a_1a_2} - 2a_1b_2) = e_3(-3a_1b_2)$

$S = \frac{1}{2}|[\vec{a}, \vec{b}]| = |a_1 b_2|$

$S_M = \frac{1}{4}|[\vec{a}+\vec{b}, 2\vec{a}-\vec{b}]| = \frac{3}{4}|a_1 b_2|$

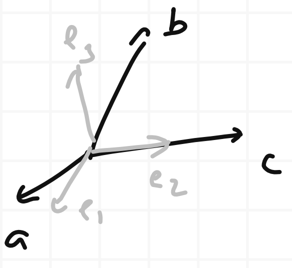
$\frac{S_M}{S} = \frac{3}{4}$ q.e.d.

<u>3.28</u>

$[a,b]$ $[b,c]$ $[c,a]$ - компл. $\iff ([a,b],[b,c],[c,a]) = 0$

a, b, c $\overset{?}{-}$ компл. т.е. д-ть, что $(a,b,c) = 0$

Введём ОНБ т.ч.:



$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_e \quad b = \begin{pmatrix} 0 \\ b_2 \\ b_3 \end{pmatrix}_e \quad c = \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix}$

$$[a,b] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \end{pmatrix}_e = e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3) + e_3(a_1b_2)$$

$$[b,c] = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix} = e_1(-b_3c_2)$$

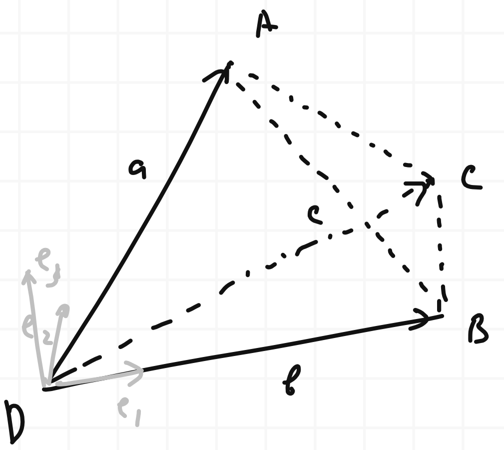
$$[c,a] = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & c_2 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} = e_1(a_3c_2) + e_3(-a_1c_2)$$

$$\big[[a,b],[b,c]\big] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_2b_3 - a_3b_2 & -a_1b_3 & a_1b_2 \\ -b_3c_2 & 0 & 0 \end{pmatrix} = +e_2(+a_1b_2b_3c_2) + e_3(-a_1b_3b_3c_2)$$

$$\Big(\big[[a,b],[b,c]\big],[c,a]\Big) = 0 + 0 + a_1a_1b_3b_3c_2c_2 = 0\ ; \quad (a_1b_3c_2)^2 = 0\ ; \quad \underline{a_1b_3c_2 = 0}$$

$$([a,b],c) = -a_1b_3c_2 = 0 \qquad \text{q.e.d.}$$

<u>T.2</u>



$$\frac{1}{2}[a,b] + \frac{1}{2}[b,c] + \frac{1}{2}[c,a] + \frac{1}{2}[\vec{BA},\vec{BC}] \overset{?}{=} 0$$

$$S = [a,b] + [b,c] + [c,a] + [a-b,\ c-b] = 0$$

Зададим ОНБ так, что:

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_e \quad b = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}_e \quad c = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}_e$$

$$[a,b] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & 0 & 0 \end{pmatrix} = -e_2(-a_3b_1) + e_3(-a_2b_1) = e_2(a_3b_1) + e_3(-a_2b_1)$$

$$[b,c] = \begin{pmatrix} e_1 & e_2 & e_3 \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix} = e_3(b_1c_2)$$

$$[c,a] = \begin{pmatrix} e_1 & e_2 & e_3 \\ c_1 & c_2 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} = e_1(a_3c_2) - e_2(a_3c_1) + e_3(a_2c_1 - a_1c_2)$$

$$a - b = \begin{pmatrix} a_1 - b_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad c - b = \begin{pmatrix} c_1 - b_1 \\ c_2 \\ 0 \end{pmatrix}$$

$$[a-b,c-b] = \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 - b_1 & a_2 & a_3 \\ c_1 - b_1 & c_2 & 0 \end{pmatrix} = e_1(-a_3c_2) - e_2(a_3b_1 - a_3c_1) +$$
$$+ e_3(a_1c_2 - b_1c_2 - a_2c_1 + a_2b_1)$$

$$S = e_1(\cancel{a_3c_2} - \cancel{a_3c_2}) + e_2(\cancel{a_3b_1} - \cancel{a_3c_1} - \cancel{a_3b_1} + \cancel{a_3c_1}) +$$
$$+ e_3(\cancel{b_1c_2} + \cancel{a_2c_1} - \cancel{a_1c_2} + \cancel{a_1c_2} - \cancel{b_1c_2} - \cancel{a_2c_1} + \cancel{a_2b_1} - \cancel{a_2b_1}) = 0 \qquad \text{q.e.d.}$$

3.29

(1) a_1, a_2, a_3 – не компл. $\iff \exists\, b_1, b_2, b_3 : \begin{cases} (a_i, b_j) = 0 & i \neq j \\ (a_i, b_j) = 1 & i = j \end{cases}$

"$\Rightarrow$" условие взаимности $(a_i, b_j) = 0$ $i \neq j$ однозначно задает направл. b_i при известных напр. a_i с точн. до знака

$$\begin{cases} (b_1, a_2) = 0 \\ (b_1, a_3) = 0 \end{cases} \iff \begin{cases} b_1 \perp a_2 \\ b_1 \perp a_3 \end{cases} \iff b_1 \parallel [a_2, a_3]$$

аналогично:

$$b_2 \parallel [a_1, a_3] \qquad b_3 \parallel [a_1, a_2]$$

а условие $(a_i, b_i) = \underbrace{|a_i||b_i|}_{\text{fixed}} \underbrace{\cos\angle(a_i, b_i)}_{(\pm 1)\cdot\text{fixed}}$ однозначно

выбирает знак напр. и задает длину

Так строится взаимная тройка векторов для некомпл. тр. в.

"$\Leftarrow$"

от противного. $\exists$ вз. тр. векторов $\exists$ и a_1, a_2, a_3 – компл.

$$b_3 \parallel [a_1, a_2] \qquad b_2 \parallel [a_1, a_3] \qquad b_1 \parallel [a_2, a_3]$$

пусть $\vec{n}$ – перп. к пл-ти векторов a_1, a_2, a_3

$$b_1 \parallel b_2 \parallel b_3 \parallel n$$

Тогда $(a_i, b_i) = 0$. противоречие.

(2) $b_1 = \alpha_1 [a_2, a_3]$ $\qquad (a_1, b_1) = \alpha_1 (a_1, [a_2, a_3]) = \alpha_1 (a_1, a_2, a_3) = 1$

$b_2 = \alpha_2 [a_3, a_1]$ $\qquad \alpha_1 = \dfrac{1}{(a_1, a_2, a_3)} = \alpha$

$b_3 = \alpha_3 [a_1, a_2]$

аналогично

$$\alpha_2 = \frac{1}{(a_2, a_3, a_1)} = \alpha \qquad \alpha_3 = \frac{1}{(a_3, a_1, a_2)} = \alpha$$

Ответ: $b_1 = \dfrac{[a_2, a_3]}{(a_1, a_2, a_3)}$; $b_2 = \dfrac{[a_3, a_1]}{(a_1, a_2, a_3)}$; $b_3 = \dfrac{[a_1, a_2]}{(a_1, a_2, a_3)}$

(3) $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) \overset{?}{>} 0$

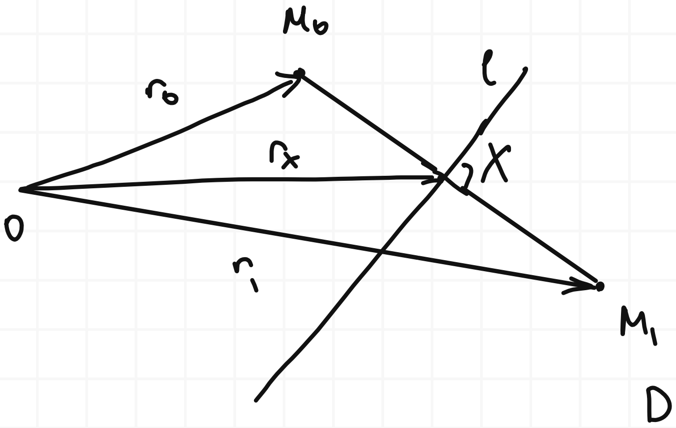
$$(b_1, b_2, b_3) = \left(\frac{[a_2, a_3]}{(a_1, a_2, a_3)}, \frac{[a_3, a_1]}{(a_1, a_2, a_3)}, \frac{[a_1, a_3]}{(a_1, a_2, a_3)} \right) = \frac{1}{(a_1, a_2, a_3)^3} ([a_2, a_3], [a_3, a_1], [a_1, a_3])$$

$$= \frac{(a_1, a_2, a_3)}{(a_1, a_2, a_3)^3} = \frac{1}{(a_1, a_2, a_3)^2}$$

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = \frac{1}{(a_1, a_2, a_3)}$$

VIII. ВЕКТОРНЫЕ УРАВНЕНИЯ

5.4(2)



$l: (r, n) = D$

$\frac{1}{2}(r_0 + r_1) = r_x$

$(r_0 - r_x, n) = 0; (r_0, n) = (r_x, n) = D$

$\overrightarrow{M_0 M_1} = 2\overrightarrow{M_0 X} = 2\alpha n = r_1 - r_0$

$\overrightarrow{M_0 X} = r_x - r_0$

$\mapsto \begin{array}{l} r_1 = r_0 + 2\alpha n \\ r_x = r_0 + \alpha n \end{array}$

$$D = (r_x, n) = (r_0 + \alpha n, n) = (r_0, n) + \alpha(n, n)$$

$$\alpha = \frac{D - (r_0, n)}{n^2}$$

(1) $r_x = r_0 + \alpha n = r_0 + \dfrac{D - (r_0, n)}{n^2} n$

(2) $r_1 = r_0 + 2\alpha n = r_0 + \dfrac{2(D - (r_0, n))}{n^2} n$

6.1 (1, 3, 4)

(1) $r = r_0 + au + bv \qquad n = [a; b]$

$(r - r_0, n) = 0$; $(r, n) = (r_0, n) = (r_0, [a, b]) = D$

$\underline{(r, [a, b]) = (r_0, [a, b])} \qquad a \nparallel b$

(2) $r = r_0 + at$,

$[r, a] = [r_0 + at, a] = [r_0, a] + \cancel{t\,[a, a]}^{\,=0} = [r_0, a] = b$

$\underline{[r, a] = [r_0, a]}$; $b = [r_0, a]$

(3) $[r, a] = b$

$[r_1, a] = b$
$[r_2, a] = b \qquad [r_2 - r_1, a] = 0$

$[r_0, a] = b$

Можем выбрать удобный r_0. Возьмем $r_0 \parallel [a, b]$: $r_0 = \alpha [a, b]$

$\alpha [[a, b], a] = b$

$\vec{b} = -\alpha [a, [a, b]] = -\alpha (a(a, b) - b(a, a)) = -\vec{a}\underbrace{\cancel{(a, b)}^{\,=0}}_{a \perp b} + \alpha a^2 \vec{b} = \alpha |a|^2 b$

$1 = |a|^2 \alpha$; $\alpha = \dfrac{1}{|a|^2}$

$r_0 = \alpha [a, b] = \dfrac{[a, b]}{|a|^2}$

$\underline{r = \dfrac{[a, b]}{|a|^2} + at}$

(4) $(r, n_1) = D_1$
$(r, n_2) = D_2 \qquad \rightsquigarrow [r, a] = b$

$a = [n_1, n_2]$

$b = [r, [n_1, n_2]] = n_1 (r, n_2) - n_2 (r, n_1) = n_1 D_2 - n_2 D_1$

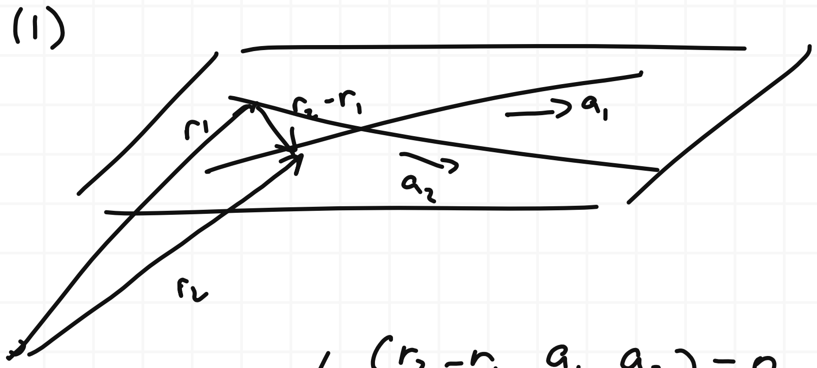
$\underline{[r, [n_1, n_2]] = n_1 D_2 - n_2 D_1}$

6.3

$r = r_1 + a_1 t$

$r = r_2 + a_2 t$

(1)



$(r_2 - r_1, a_1, a_2) = 0$ — условие, что прямые лежат в 1 пл-ти

$a_1 \nparallel a_2 \iff \forall \lambda \in \mathbb{R} \quad a \neq \lambda a_2 \iff [a_1, a_2] \neq 0$

$$\begin{cases} (r_2 - r_1, a_1, a_2) = 0 \\ [a_1, a_2] \neq 0 \end{cases}$$

(2) $(r_2 - r_1, a_1, a_2) \neq 0$

(3) $\begin{cases} [a_1, a_2] = 0 \\ [r_2 - r_1, a_1] \neq 0 \end{cases}$

(4) $\begin{cases} [a_1, a_2] = 0 \\ [r_2 - r_1, a_1] = 0 \end{cases}$

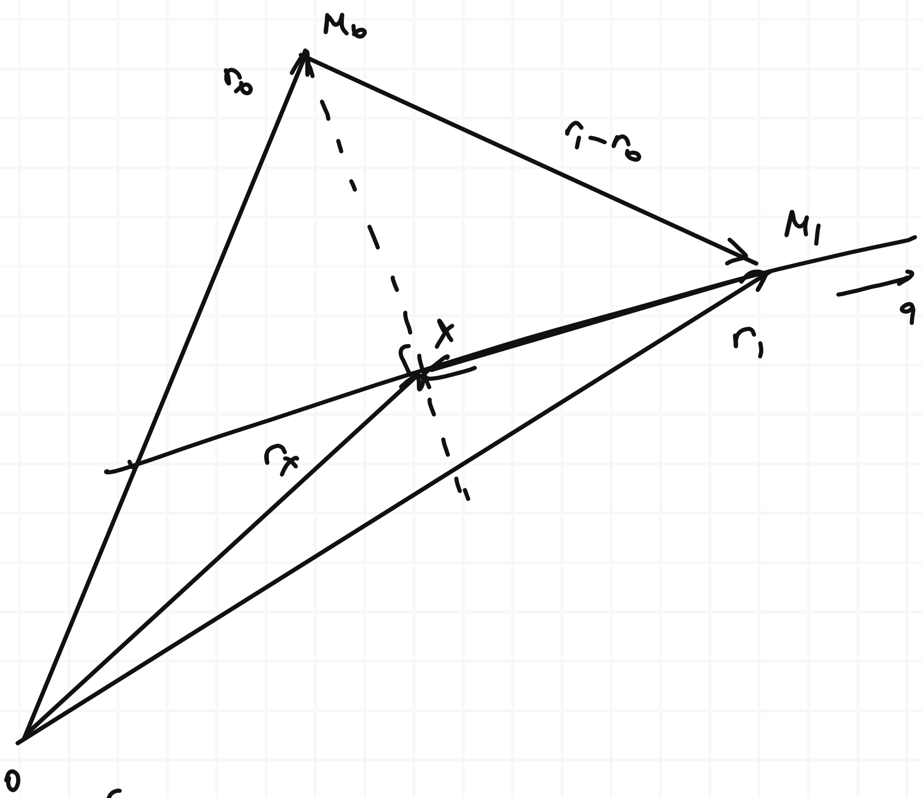
6.9(1)

$M_0(r_0) \qquad r = r_1 + at \quad (a \neq 0)$

$r_x = r_0 + \overrightarrow{M_0 X}$

$\overrightarrow{M_0 X} = r_1 - r_0 + \overrightarrow{M_1 X}$

$\overrightarrow{M_1 X} = \dfrac{(r_0 - r_1, a)}{|a|^2} a$



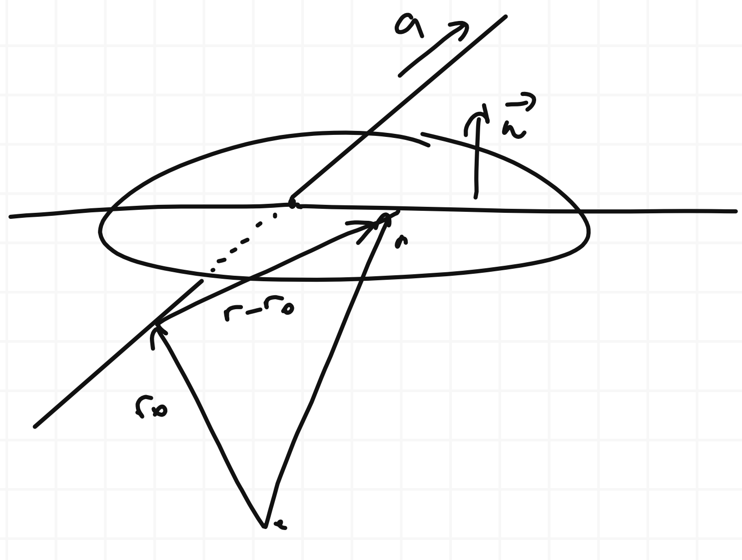
$$r_x = \cancel{r_0} + r_1 - \cancel{r_0} + \frac{(r_0 - r_1, a)}{|a|^2} a = r_1 + \frac{(r_0 - r_1, a)}{|a|^2} a$$

6.10(1,3,4)

(1) $r = r_0 + at$

$$\begin{cases} (r, n) = D \\ (r - r_0, a, n) = 0 \end{cases}$$

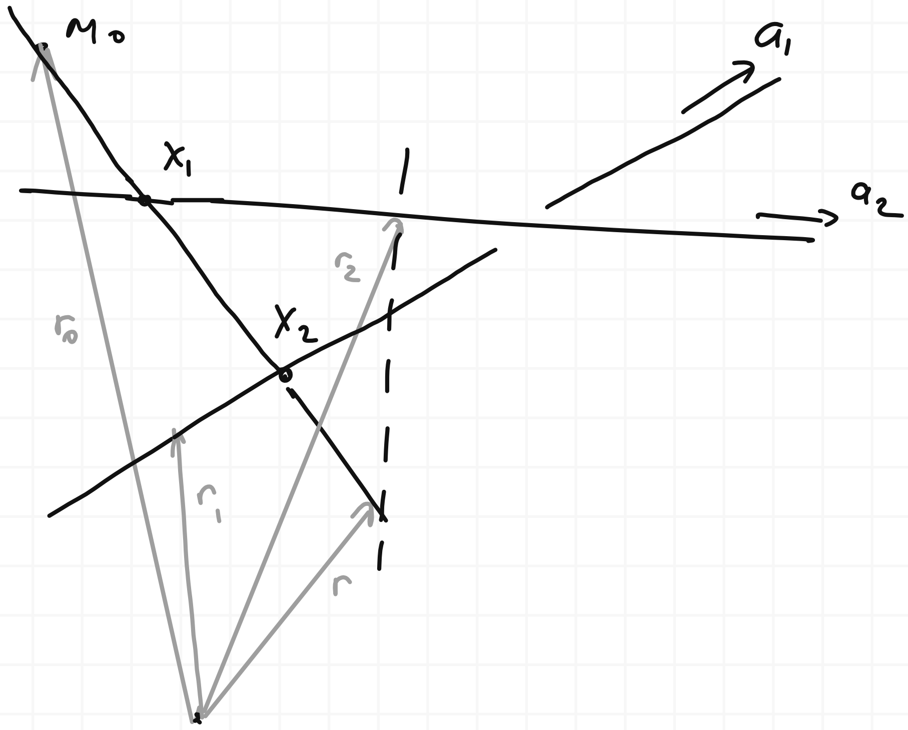
ответ



(3) $r = r_1 + a_1 t$

$r = r_2 + a_2 t$

$M_0(r_0)$

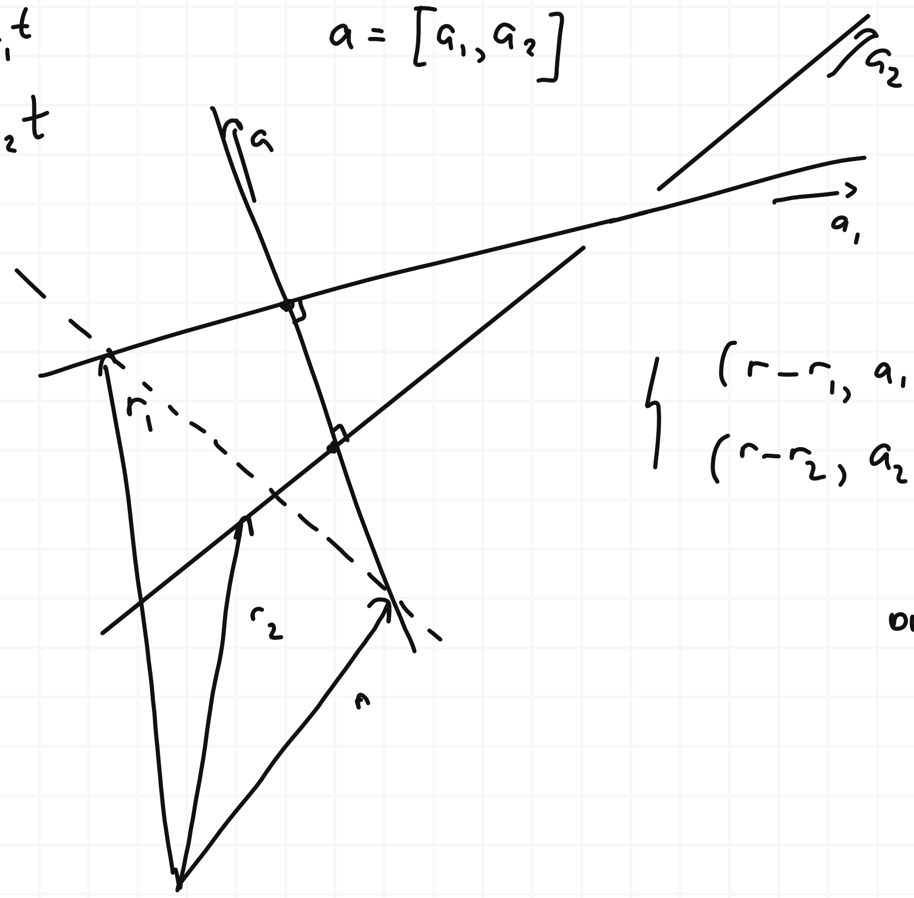


$$\begin{cases} (r - r_0, r_1, r - r_1) = 0 \\ (r - r_0, r_2, r - r_2) = 0 \end{cases}$$

ответ

(4) $r = r_1 + a_1 t$ $\qquad a = [a_1, a_2]$

$r = r_2 + a_2 t$



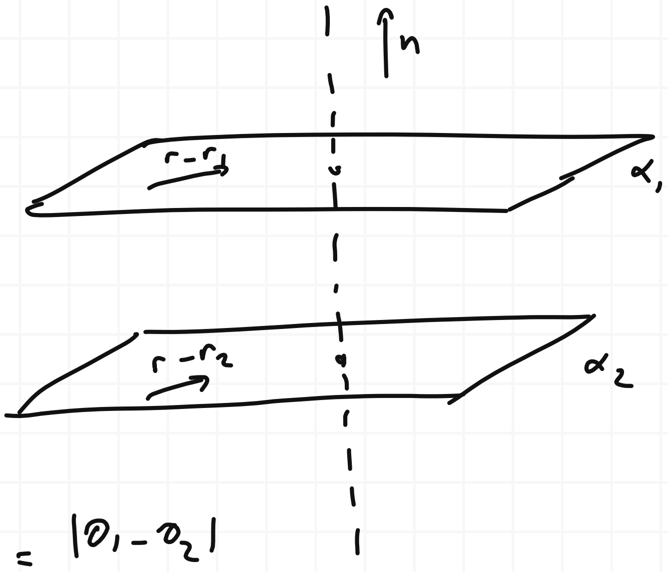
$$\begin{cases} (r - r_1, a_1, [a_1, a_2]) \\ (r - r_2, a_2, [a_1, a_2]) \end{cases}$$

ответ

6.11 (3, 4, 8)

(3) $(r, n) = D_1 = (r_1, n)$

$(r, n) = D_2 = (r_2, n)$

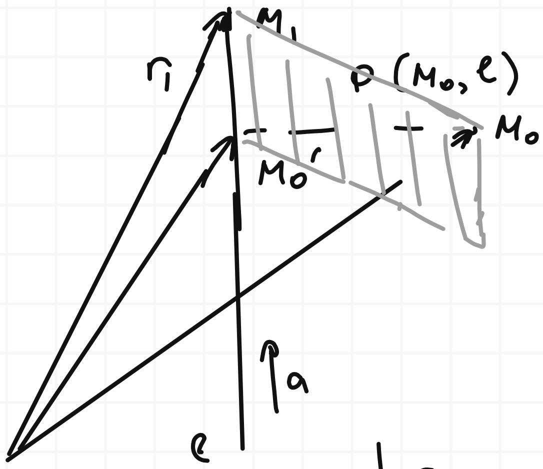


$$\rho(0; \alpha_i) = \frac{(r_i, n)}{|n|} = \frac{D_i}{|n|}$$

$$\underline{\rho(\alpha_1, \alpha_2)} = |\rho(0; \alpha_1) - \rho(0; \alpha_2)| = \underline{\frac{|D_1 - D_2|}{|n|}}$$

(7) $M_0(r_0)$

$r = r_1 + at$



$\rho(M_0,\ell) = |MM_0'|$

$$\vec{MM_0'} = \vec{M_0M_1} + \vec{M_1M_0'} =$$

$$= r_1 - r_0 - a\left(\frac{(r_1,a)}{|a|} - \frac{(r_0,a)}{|a|}\right) =$$

$$= r_1 - r_0 - \frac{(r_1-r_0,a)}{|a|}a$$

$$\left|r_1 - r_0 - \frac{(r_1-r_0,a)}{|a|}a\right|$$

II способ.

$$S_{///} = |M_0'M_1|\,\overset{\rho}{|M_0M_0'|} = \left|\left[\vec{M_0'M_1},\ r_1-r_0\right]\right| = \alpha\left[a,\ r_1-r_0\right]$$

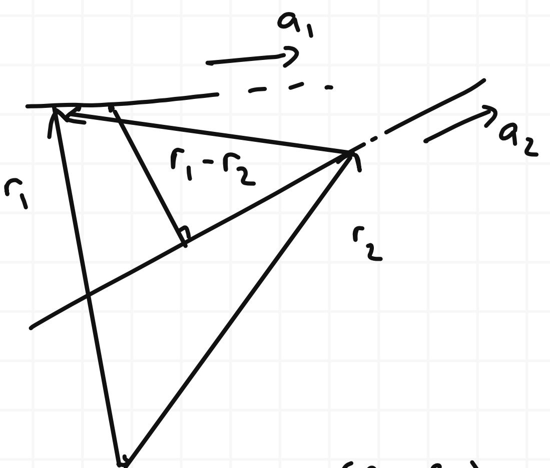
$$\rho = \frac{\cancel{\alpha}\,|[a, r_1-r_0]|}{\cancel{\alpha}\,|a|} = \frac{|[a, r_1-r_0]|}{|a|}$$

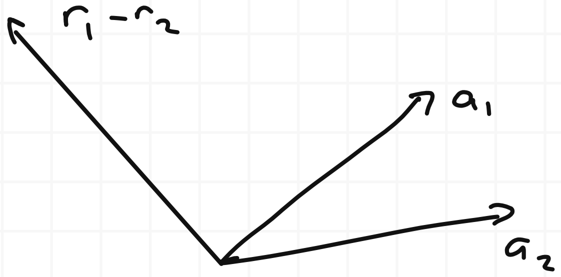
кажется, это одно и то же

(8) $r = r_1 + a_1 t$
$r = r_2 + a_2 t$

$n = [a_1, a_2]$

$r_1 - r_2$ лежит в пл-ти $\parallel$ напр. и обеим прямым





$V = |(r_1 - r_2, a_1, a_2)|$

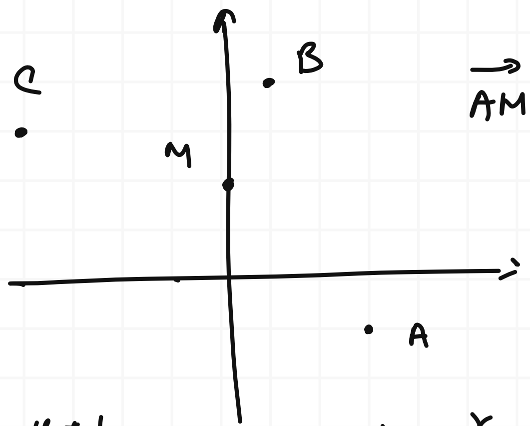
$V = \rho(\ell_1, \ell_2)\cdot|[a_1, a_2]|$

$$\rho(\ell_1,\ell_2) = \frac{|(r_1-r_2, a_1, a_2)|}{|[a_1,a_2]|}$$

IX. АФФИННЫЕ ЗАДАЧИ

5.15

$A(3,-1)$
$B(1;4)$
$M(0,2)$



$C(x,y)$

$$\vec{AM} = \frac{1}{3}\vec{AB} + \frac{1}{3}\vec{AC}\ ;\quad 3\begin{pmatrix}-3\\3\end{pmatrix} = \begin{pmatrix}-2\\5\end{pmatrix} + \begin{pmatrix}x-3\\y+1\end{pmatrix}$$

$$C: \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3-9+2\\-1+9-5\end{pmatrix} = \begin{pmatrix}-4\\3\end{pmatrix}$$

$\ell(AB):\ \frac{x-3}{1-3} = \frac{y+1}{4+1}$; $\quad 5x + 2y - 13 = 0$

$\ell(BC):\ \frac{x-1}{-4-1} = \frac{y-4}{3-4}$; $\quad x - 5y + 19 = 0$

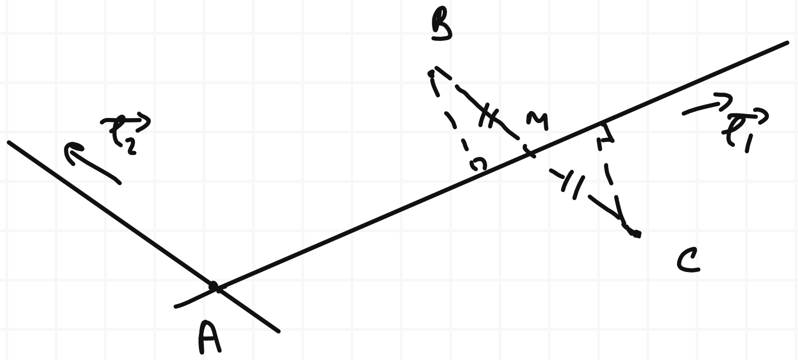
$\ell(AC):\ \frac{x+4}{3+4} = \frac{y-3}{-1-3}$; $\quad 4x + 7y - 5 = 0$

5.19

A(−1,5)
B(3;7)
C(1;−1)

$A(x-x_0)+B(y-y_0)=0$ — ур-е прямой прох. через точку (x_0,y_0) с $\vec{n}(A,B)$

$a_i(x+1)+b_i(y-5)=0$.



$\vec{l_1} \parallel \vec{AB}+\vec{AC} = \begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}2\\-6\end{pmatrix}=\begin{pmatrix}6\\-4\end{pmatrix} \parallel \begin{pmatrix}3\\-2\end{pmatrix}$

$\vec{l_2} \parallel \vec{CB} = \begin{pmatrix}2\\8\end{pmatrix} \parallel \begin{pmatrix}1\\4\end{pmatrix}$

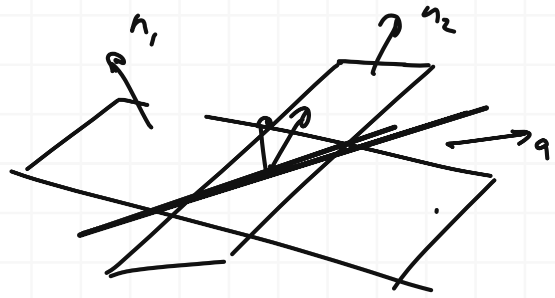
$\vec{l_1}\begin{pmatrix}3\\-2\end{pmatrix}$ $\vec{l_2}\begin{pmatrix}1\\4\end{pmatrix}$

$\vec{n_1}\begin{pmatrix}2\\3\end{pmatrix}$ $\vec{n_2}\begin{pmatrix}4\\-1\end{pmatrix}$

$$\left.\begin{array}{ll} l_1: \; 2(x+1)+3(y-5)=0 & 2x+3y-13=0 \\ l_2: \; 4(x+1)-(y-5)=0 & 4x-y+9=0 \end{array}\right\} \text{ответ}$$

6.16(2)

$$l: \begin{cases} x-y+2z+4=0 \\ -2x+y+z+3=0 \end{cases} \qquad n_1(1,-1,2) \quad n_2(-2,1,1)$$



$\vec{a}$ — напр. вектор l

$$\vec{a} = -[\vec{n_1},\vec{n_2}] = -\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix} = -(e_1(-1-2)-e_2(1+4)+e_3(1-2)) = -(-3e_1-5e_2-e_3) \; ; \quad \vec{a}\begin{pmatrix}3\\5\\1\end{pmatrix}$$

найдем $(x_0,y_0,z_0)\in l$.

$-x+3z+7=0$; $x=3z+7$; $y=x+4+2z = 3z+7+4+2z = 5z+11$

$\exists\, z=0 \quad x_0=7 \quad y_0=11 \quad z_0=0$

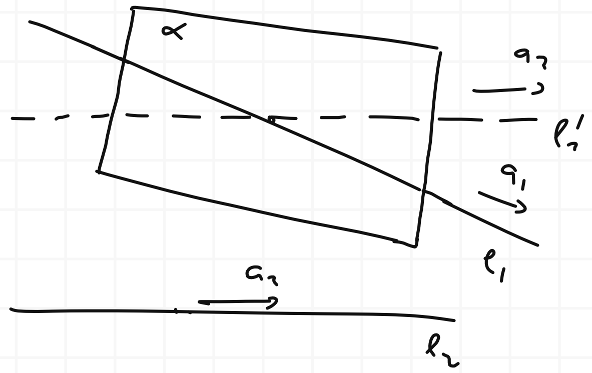
$$l: \begin{cases} x-x_0=a_1t \\ y-y_0=a_2t \\ z-z_0=a_3t \end{cases} \qquad \begin{cases} x-7=3t \\ y-11=5t \\ z=t \end{cases}$$ ← ответ (в парам. форме)

$[r-r_0, a]=0 \qquad \left[r-\begin{pmatrix}7\\11\\0\end{pmatrix} ; \begin{pmatrix}3\\5\\1\end{pmatrix}\right]=0$ ← ответ (в канонич. форме)

6.29(2)

$$l_1: \begin{cases} x=3+t \\ y=2+5t \\ z=-1+3t \end{cases} \qquad l_2: \begin{cases} x=4-2t \\ y=-8+t \\ z=5+2t \end{cases}$$

$a_1=(1,5,3) \qquad a_2(-2,1,2)$



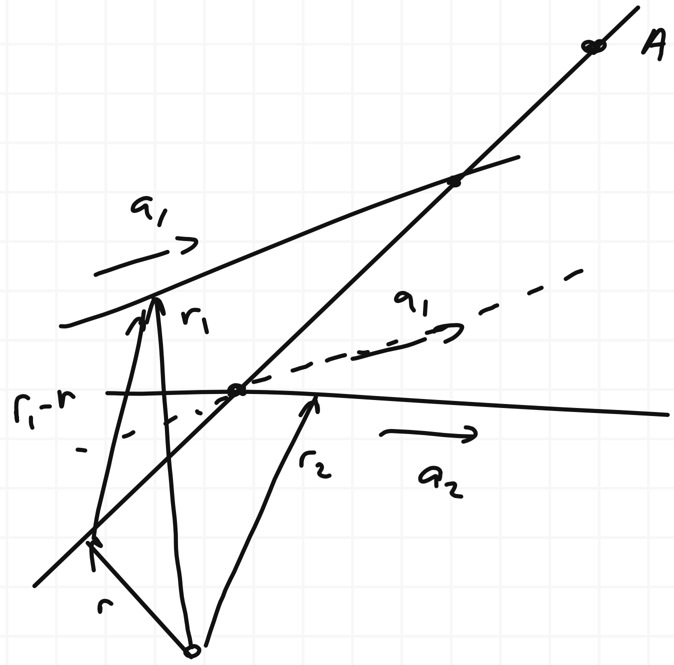
∀ точка $l_1 \in \alpha$. ∃ $\vec{r_0}\begin{pmatrix}3\\2\\-1\end{pmatrix}$

$\vec{r} - \vec{r_0} = \vec{a_1} t_1 + \vec{a_2} t_2$ ← ответ

$\vec{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \begin{pmatrix}1\\5\\3\end{pmatrix} t_1 + \begin{pmatrix}-2\\1\\2\end{pmatrix} t_2$

<u>6.38 (2)</u>

$A(-1, 1, -1)$

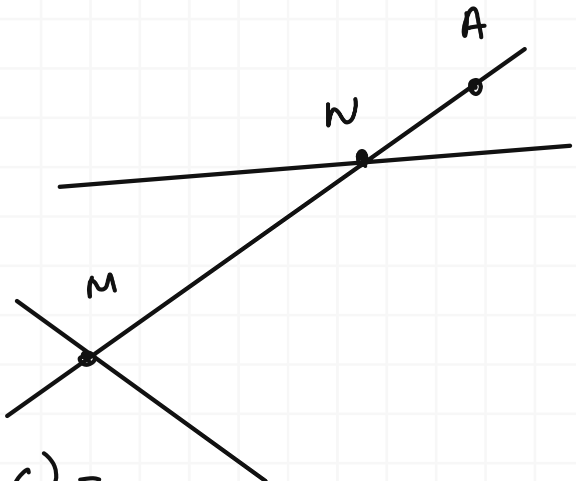


$l_1: \dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z}{-1}$ $\quad a_1\begin{pmatrix}2\\3\\-1\end{pmatrix}$ $\quad M_1\begin{pmatrix}1\\2\\0\end{pmatrix}$

$l_2: \dfrac{x}{4} = \dfrac{y+5}{-5} = \dfrac{z-3}{2}$ $\quad a_2\begin{pmatrix}4\\-5\\2\end{pmatrix}$ $\quad M_2\begin{pmatrix}0\\-5\\3\end{pmatrix}$

ответ — пересечение пл-тей, содерж. l_i и (·) A

Запишем их ур-я



$l_i \in \Pi_i$, $\parallel a_i$; $b_i = \overrightarrow{AM_i}$; $n_i \perp \Pi_i$

$b_1\begin{pmatrix}2\\1\\1\end{pmatrix}$ $\quad b_2\begin{pmatrix}1\\-6\\4\end{pmatrix}$ $\quad n_i = [a_i, b_i]$

$$n_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{vmatrix} = e_1(3+1) - e_2(2+2) + e_3(2-6) = = 4e_1 - 4e_2 - 3e_3 \qquad (4, -4, -4) \qquad n_1\begin{pmatrix}4\\-4\\-4\end{pmatrix}$$

$$n_2 = -\begin{vmatrix} e_1 & e_2 & e_3 \\ 4 & -5 & 2 \\ 1 & -6 & 4 \end{vmatrix} = -(e_1(-20+12) - e_2(16-2) + e_3(-24+5)) = = +8e_1 + 14e_2 + 19e_3 \qquad n_2\begin{pmatrix}8\\14\\19\end{pmatrix}$$

$\Pi_1: \quad 4x - 4y - 4z + d_1 = 0 \qquad -4 \cancel{-4} \cancel{+4} + d_1 = 0 \; ; \; d_1 = 4$

$\Pi_2: \quad 8x + 14y + 19z + d_2 = 0 \qquad -8 + 14 - 19 + d_2 = 0 \; ; \; d_2 = 13$

$A \in \Pi_1, \Pi_2$

Ответ: $\begin{cases} x - y - z + 1 = 0 \\ 8x + 14y + 19z + 13 = 0 \end{cases}$

X. МЕТРИЧЕСКИЕ ЗАДАЧИ

5.30

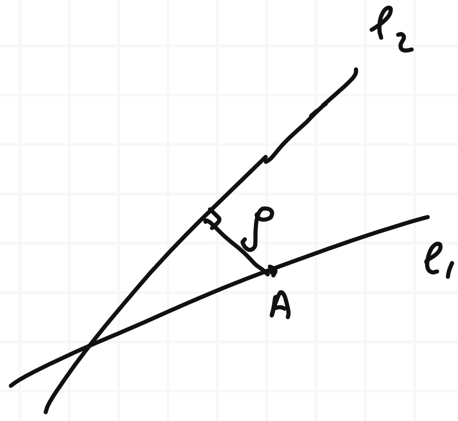
$A \in \ell_1$

$\ell_1: \ 2x-3y+4=0$

$\rho(A,\ell_2)=2$

$\ell_2: \ 4x-3y=0$

$A - ?$



$\exists A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$2x_0-3y_0+4=0; \quad 3y_0=2x_0+4$

$$\rho(A,\ell_2)=\frac{|4x_0-3y_0|}{\sqrt{16+9}}=\frac{|4x_0-2x_0-4|}{5}$$

$10=|2x_0-4| \qquad 5=|x_0-2|$

$$\left[\begin{array}{ll} x_0=7 & y_0=6 \\ x_0=-3 & y_0=-\frac{2}{3} \end{array}\right.$$

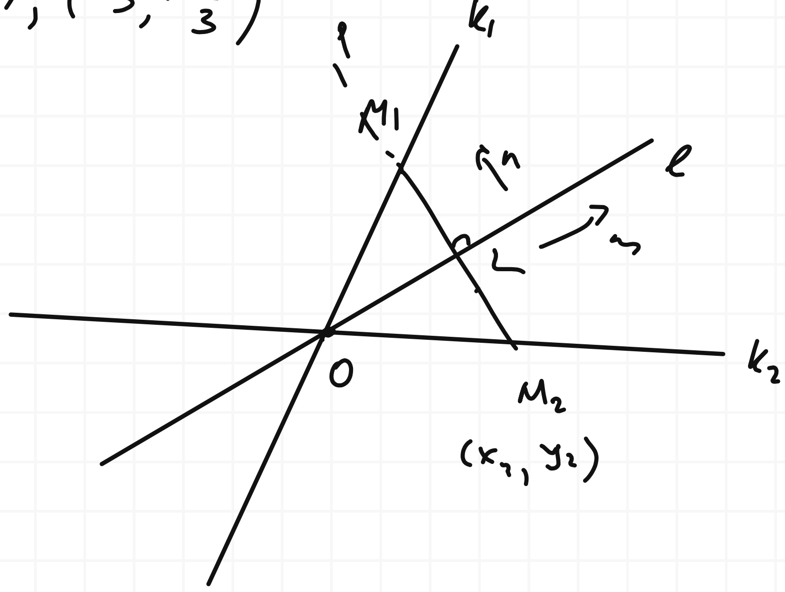
Ответ: $(7,6), \left(-3,-\frac{2}{3}\right)$

5.35

$k_1: \ 3x-y+5=0$

$\ell: \ x+y-1=0$

$n_\ell\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$\exists O(x_0,y_0)$

$4x_0+4=0; \quad x_0=-1$

$y_0=3x_0+5=-3+5=2$

$O(-1;2)$

$M_1\in k_1 \qquad M_1(x_1,y_1)$

$3x_1-y_1+5=0 \qquad \exists x_1=0; \ y_1=5$

$M(0;5)$

$m\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad p: \ -(x-0)+(y-5)=0; \quad -x+y-5=0; \quad x-y+5=0$

$L(x_L,y_L)=p\cap\ell$

$x_L-y_L+5=0$

$x_L+y_L-1=0$

$2x_L=-4; \quad x_L=-2; \quad y_L=3$

$L(-2;3)$

$LM_1\begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad LM_2=\begin{pmatrix} -2 \\ -2 \end{pmatrix}=\begin{pmatrix} x_2+2 \\ y_2-3 \end{pmatrix}; \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$k_2=(OM_2)$

$$\frac{x+1}{-4+1}=\frac{y-2}{1-2}; \quad \frac{x+1}{+3}=\frac{y-2}{+1}; \quad x+1=3y-6; \quad x-3y+7=0$$

Ответ: $x-3y+7=0$

6.45(2) Π – искомая пл-ть

Π_0: $x+3y-z+2=0$ $n_0(1,3,-1)$; $n_0 \parallel \Pi$

$l_2: \begin{cases} 2x-y+z=0 & \leftarrow n_1 \\ x+2y+z-3=0 & \leftarrow n_2 \end{cases}$ $a \parallel [n_1, n_2]$

$$[n_1,n_2] = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = e_1(-1-2) - e_2(2-1) + e_3(4+1) =$$
$$= -3e_1 - e_2 + 5e_1 \qquad (-3,-1,5)$$

$a\ (3,1,-5)$

$n_0, a \parallel \Pi$

$$\Pi: \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ 3 & 1 & -5 \\ 1 & 3 & -1 \end{vmatrix} = (x-x_0)(-1+15) - (y-y_0)(-3+5) + (z-z_0)(9-1) =$$
$$= 14(x-x_0) - 2(y-y_0) + 8(z-z_0) = 0$$

$7(x-x_0) - (y-y_0) + 4(z-z_0) = 0$

Π: $7x - y + 4z + D = 0$; $D = -7x_0 + y_0 - 4z_0$

найдем произв. точку $M \in \Pi$. $\exists M \in l_2 \in \Pi$. $\exists y_0 = 0$

$-\begin{cases} 2x_0 + z_0 = 0 \\ x_0 + z_0 - 3 = 0 \end{cases}$

$x_0 + 3 = 0$; $x_0 = -3$ $M(-3, 0, 6)$

$z_0 = -2x_0 = 6$

$D = +7\cdot 3 + 0 - 4\cdot 6 = 21 - 24 = -3$

Ответ: $7x - y + 4z - 3 = 0$

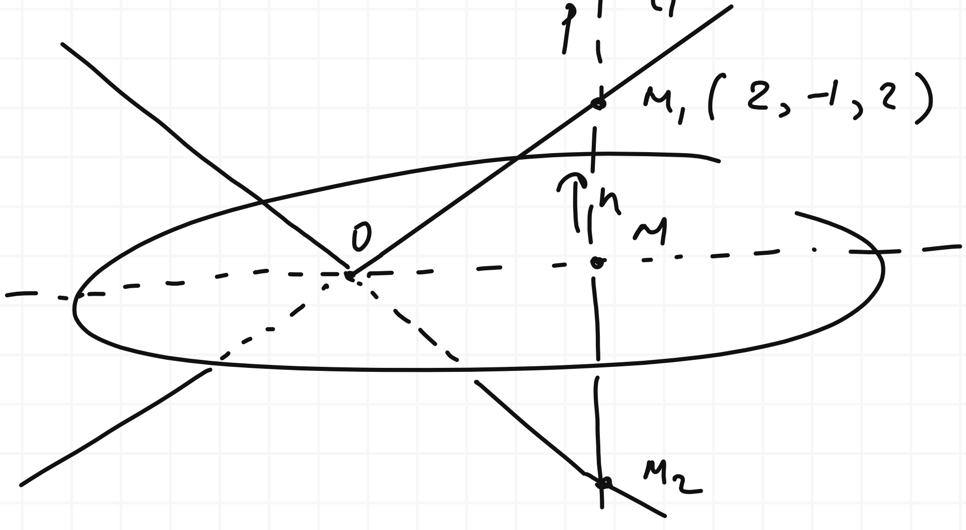
6.60

$l_1: \dfrac{x-2}{3} = \dfrac{y+1}{1} = \dfrac{z-2}{4}$

Π: $5x - y + z - 4 = 0$

$n(5,-1,1)$

Π' – плоскость $\perp \Pi$, через l_1



$O - ?$

$$\begin{cases} 5x + z - 4 = y \\ \dfrac{x-2}{3} = \dfrac{y+1}{1} = \dfrac{z-2}{4} \end{cases} \quad \begin{cases} x - 2 = 3(5x + z - 3) \\ 4(5x + z - 3) = z - 2 \end{cases} \quad \begin{cases} \cancel{x} - \cancel{2} = \cancel{15}x + 3z - \cancel{9} \quad (+14x \;\; +3z - 7) \\ 20x + \cancel{4}z - \cancel{12} = \cancel{z} - \cancel{2} \quad (+3z - 10 \;\; 0) \end{cases}$$

$$-\begin{cases} 14x + 3z - 7 = 0 \\ 20x + 3z - 10 = 0 \end{cases}$$

$6x - 3 = 0$

$x = \frac{1}{2} \quad z = 0 \quad y = \frac{5}{2} - \frac{8}{2} = -\frac{3}{2} \qquad O\left(\frac{1}{2}; -\frac{3}{2}; 0\right)$

$OM_1 \begin{pmatrix} 2 - \frac{1}{2} \\ -1 + \frac{3}{2} \\ 2 - 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 2 \end{pmatrix} \qquad n(5, -1, 1)$

$$MM_1 = \frac{(OM_1, n)}{|n|^2} n = \frac{\cancel{\frac{15}{2}} - \cancel{\frac{1}{2}} + 2}{27} n = \frac{1}{3}\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ -1/3 \\ 1/3 \end{pmatrix}$$

$+7$

$-M_2M_1 = -2MM_1 \; ; \; \begin{pmatrix} x_2 - 2 \\ y_2 + 1 \\ z_2 - 2 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} \\ 2/3 \\ -2/3 \end{pmatrix} ; \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \end{pmatrix} \; ; \; M_2 \begin{pmatrix} -\frac{4}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \end{pmatrix}$

$a_2 \parallel OM_2 \begin{pmatrix} -\frac{4}{3} - \frac{1}{2} \\ -\frac{1}{3} + \frac{3}{2} \\ \frac{4}{3} - 0 \end{pmatrix} = \begin{pmatrix} -\frac{11}{6} \\ \frac{7}{6} \\ \frac{4}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -11 \\ 7 \\ 8 \end{pmatrix}$

$a_2 \; (-11, 7, 8)$

$l_2: \dfrac{x + \frac{4}{3}}{-11} = \dfrac{y + \frac{1}{3}}{7} = \dfrac{z - \frac{1}{3}}{8} \leftarrow$ ответ

6.61(2)

$\Pi: \; x + 5y - z - 25 = 0 \qquad n(1, 5, -1)$

$l: \begin{cases} x - y + 2z - 1 = 0 \\ 3x - y + 2z + 2 = 0 \end{cases} \qquad \begin{matrix} n_1(1, -1, 2) \\ n_2(3, -1, 2) \end{matrix} \qquad a \parallel [n_1, n_2]$

$$[n_1, n_2] = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 3 & -1 & 2 \end{vmatrix} = e_1(-2 + 2) - e_2(2 - 6) + e_3(-1 + 3) =$$
$$= 4e_2 + 2e_3 \qquad (0; 4; 2)$$

$a(0; 2; 1)$

$$n_1 \parallel a, n : \; n_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 5 & -1 \\ 0 & 2 & 1 \end{vmatrix} = e_1(5 + 2) - e_2(1) + e_3(2)$$

$n_1(7, -1, 2)$

$n_\perp: \quad 7x - y + 2z + D = 0$

$O = n \cap \ell:$

$$\begin{cases} x + 5y - z - 25 = 0 \\ x - y + 2z - 1 = 0 \\ 3x - y + 2z + 2 = 0 \end{cases} \rightarrow$$

$-3 \qquad +2x$

$\cancel{y-1} = \cancel{3}x + \cancel{2} \qquad x = -\frac{3}{2}$

$y = x + 2z - 1 = 2z - \frac{5}{2}$

$z = x + 5y - 25 = -\frac{3}{2} + 10z - \frac{25}{2} - 25 =$

$= +10z - 14 - 25 =$

$= 10z - 39$

$\overset{13}{\cancel{39}} = \overset{3}{\cancel{9}}z \quad ; \quad z = \frac{13}{3}$

$y = \frac{26}{3} - \frac{5}{2} = \frac{52-15}{6} = \frac{37}{6}$

$O\left(-\frac{3}{2}; \frac{37}{6}; \frac{13}{3}\right)$

$D = -7x_0 + y_0 - 2z_0 = \frac{21}{2} + \frac{37}{6} - \frac{52}{6} = \frac{63}{6} - \frac{15}{6} = 8$

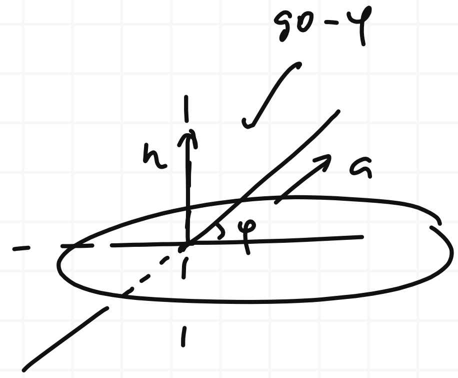
Ответ: $\begin{cases} 7x - y + 2z + 8 = 0 \\ x + 5y - z - 25 = 0 \end{cases}$

6.64(1)

$4x + 4y - 7z + 1 = 0 \qquad n(4, 4, -7)$

$\begin{cases} x + y + z + 1 = 0 \\ 2x + y + 3z + 2 = 0 \end{cases} \qquad \begin{matrix} n_1(1, 1, 1) \\ n_2(2, 1, 3) \end{matrix}$

$a \parallel [n_1, n_2]$



$[n_1, n_2] = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = e_1(3-1) - e_2(3-2) + e_3(1-2) = 2e_1 - e_2 - e_3$

$(2, -1, -1)$

$a(2, -1, -1)$

$(a, n) = |a||n|\cos(90 - \varphi) \quad ; \quad \sin\varphi = \frac{(a, n)}{|a||n|} = \frac{8 - 4 + 7}{\sqrt{4+1+1}\sqrt{16+16+49}} =$

$= \frac{11}{9\sqrt{6}} = \frac{11\sqrt{6}}{9 \cdot 6} = \frac{11\sqrt{6}}{54}$

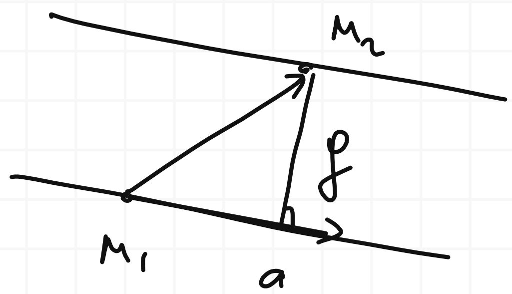
$\varphi = \arcsin\left(\frac{11\sqrt{6}}{54}\right)$

6.72(1)

$\ell_1: \frac{x-4}{3} = \frac{y+1}{6} = \frac{z-1}{-2} \qquad \begin{matrix} M_1(4, -1, 1) \\ a_1(3, 6, -2) \end{matrix}$

$M_1M_2(1, 1, -1) \rightarrow a_1 \parallel a_2$, т.е.

$\ell_2: \frac{x-5}{-6} = \frac{y}{-12} = \frac{z}{4} \qquad \begin{matrix} M_2(5; 0; 0) \\ a_2(-6, -12, 4) \parallel (-3, -6, 2) \end{matrix} \rightarrow \ell_1 \parallel \ell_2$



$$\rho = \frac{|[M_1M_2, a_1]|}{|a_1|} = \frac{\sqrt{16+1+9}}{\sqrt{9+36+4}} = \frac{\sqrt{26}}{7}$$

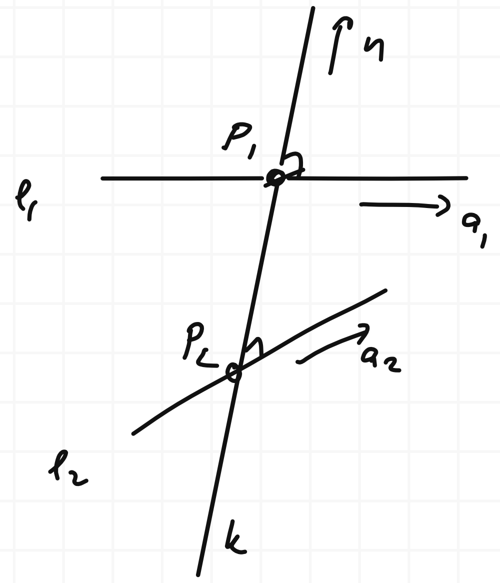
$$\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & -1 \\ 3 & 6 & -2 \end{vmatrix} = e_1(-2+6) - e_2(-2+3) + e_3(6-3) =$$
$$= 4e_1 - e_2 + 3e_3 \qquad (4, -1, 3)$$

Ответ: $\frac{\sqrt{26}}{7}$

6.73(3)

$l_1: \ \frac{x-6}{1} = \frac{y-1}{2} = \frac{z-10}{-1}$ $\qquad a_1(1,2,-1)$, $M_1(6,1,10)$

$l_2: \ \frac{x+4}{-7} = \frac{y-3}{2} = \frac{z-4}{3}$ $\qquad a_2(-7,2,3)$, $M_2(-4,3,4)$



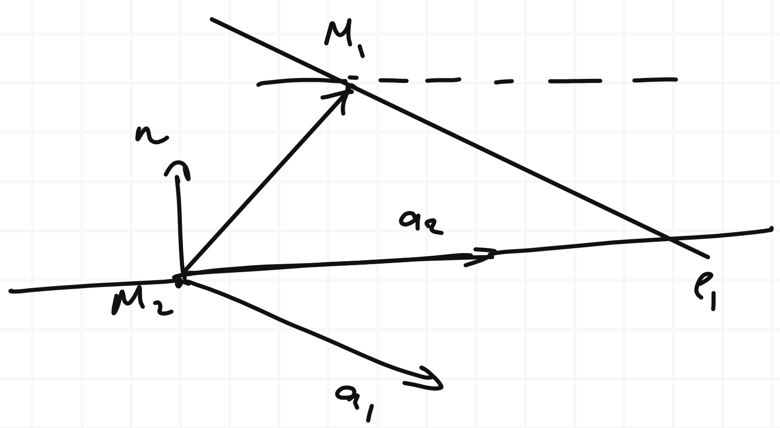
$n \parallel [a_1, a_2]$

$$[a_1, a_2] = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ -7 & 2 & 3 \end{vmatrix} = e_1(6+2) - e_2(3-7) + e_3(2+14) =$$
$$= 8e_1 + 4e_2 + 16e_3$$

$$(8,4,16) \parallel (2,1,4)$$

$n(2,1,4)$

$$M_2M_1 = \begin{pmatrix} 6+4 \\ 1-3 \\ 10-4 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} \quad ; M_2M_1(10,-2,6)$$



(c) $\rho(l_1, l_2) = \frac{|(M_2M_1, a_1, a_2)|}{|[a_1, a_2]|} =$

$$= \frac{\left|\left(\begin{pmatrix} 8 \\ 4 \\ 16 \end{pmatrix}, \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix}\right)\right|}{4\sqrt{4+1+16}} = \frac{80 \overset{+88}{\cancel{-8+96}}}{4\sqrt{21}} = \frac{\overset{42}{\cancel{168}}}{\cancel{4}\sqrt{21}} = \frac{\overset{2}{\cancel{42}}\sqrt{21}}{\cancel{21}} = 2\sqrt{21}$$

$$P_2P_1 = n\frac{\rho(l_1,l_2)}{|n|} = \frac{2\sqrt{21}}{\sqrt{21}} n = 2n = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} \quad ; P_2P_1(4,2,8)$$

k – общий $\perp$. $\qquad k = \Pi_1 \cap \Pi_2$, где

$$\Pi_1: \quad n_{\Pi_1} \parallel [a_1, n] = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ 2 & 1 & 4 \end{vmatrix} = e_1(8+1) - e_2(4+2) + e_3(1-4) =$$
$$= 9e_1 - 6e_2 - 3e_3 \qquad (9,-6,-3) \parallel (3,-2,-1)$$

$3x - 2y - z + d_1 = 0 \qquad M_1(6;1;10)$

$d_1 = -18 + 2 + 10 = -6$

$n_2: \quad n_{n_2} \parallel [a_2, n] = \begin{vmatrix} e_1 & e_2 & e_3 \\ -7 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = e_1(8-3) + e_2(+28+6) + e_3(-7-4) =$

$= 5e_1 + 34e_2 - 11e_3$

$5x + 34y - 11z + d_2 = 0 \qquad M_2(-4, 3, 4)$

$-d_2 = -20 + 102 - 44 = 38 \;;\; d_2 = -38$

(a) $k: \begin{cases} 3x - 2y - z - 6 = 0 \\ 5x + 34y - 11z - 38 = 0 \end{cases}$

$l_1: \begin{cases} 2x - 12 = y - 1 \\ 1 - y = 2z - 20 \end{cases}$

$\begin{cases} 2x - y - 11 = 0 \\ y + 2z - 21 = 0 \end{cases} \qquad y = 2x - 11 \qquad 2x - 11 = 21 - 2z \;; \qquad 2z = 21 + 11 - 2x = -2x + 32 \;;$

$z = -x + 16$

$(x, 2x - 11, -x + 16)$

$\cancel{3x} - \cancel{4x} + \cancel{22} \cancel{+x} - \cancel{16} \cancel{-6} = 0 \qquad$ true

$5x + 68x - 374 + 11x - 176 - 38 = 0$

$84x = 586$

$x = \frac{294}{42} = \frac{147}{21} = \frac{21}{3} = 7$

$P_1(7, 3, 9)$

$\begin{array}{r} 68 \\ 11 \\ 5 \\ \hline 84 \end{array} \qquad \begin{array}{r} 374 \\ 176 \\ 38 \\ \hline 588 \end{array}$

$l_2: \begin{cases} 2x + 8 = -7y + 21 \\ 3y - 9 = 2z - 8 \end{cases} \qquad \begin{cases} 2x + 7y - 13 = 0 \\ 3y - 2z - 1 = 0 \end{cases} \qquad \begin{array}{l} x = -\frac{7}{2}y + \frac{13}{2} \\ z = \frac{3}{2}y - \frac{1}{2} \end{array}$

$\left(-\frac{7}{2}y + \frac{13}{2} ; y ; \frac{3}{2}y - \frac{1}{2}\right)$

$\cancel{-\frac{21}{2}y} + \cancel{\frac{39}{2}} \cancel{-2y} - \cancel{\frac{3}{2}y} \cancel{+\frac{1}{2}} \cancel{-6} = 0 \qquad y = 1$

$\cancel{-12y} \; \cancel{120} \; -14y \qquad +14$

$P_2\left(-\frac{7}{2} + \frac{13}{2} ; 1 ; \frac{3}{2} - \frac{1}{2}\right) = (3; 1; 1)$

(b) $P_1(7; 3, 9)$

$P_2(3; 1; 1)$

6.80

$n_1: x - z - 5 = 0 \qquad n_1(1, 0, -1)$

$n_2: 3x + 5y + 4z = 0 \qquad n_2(3, 5, 4)$

$n \parallel \frac{1}{2}\left(\frac{n_1}{|n_1|} + \frac{n_2}{|n_2|}\right) = \frac{1}{2}\left(\frac{n_1}{\sqrt{2}} + \frac{n_2}{\sqrt{9+25+16}}\right) = \frac{1}{2}\left(\frac{5n_1}{5\sqrt{2}} + \frac{n_2}{5\sqrt{2}}\right) = \frac{1}{10\sqrt{2}}(5n_1 + n_2) =$

$$= \frac{1}{10\sqrt{2}} \begin{pmatrix} 5+3 \\ 0+5 \\ -5+4 \end{pmatrix} = \frac{1}{10\sqrt{2}} \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix} \qquad n(8,5,-1)$$

$$n:\ 8x+5y-z+d=0$$

$$\begin{cases} x-z-5=0 \\ 3x+5y+4z=0 \end{cases} \qquad \exists y=0 \qquad x=z+5$$

$$3z+15+4z=0;\ 7z=-15;\ z=-\frac{15}{7}$$

$$x=\frac{-15+35}{7}=\frac{20}{7};$$

$$\left(\frac{20}{7}; 0; -\frac{15}{7}\right)$$

$$\frac{160}{7}+0+\frac{15}{7}+d=0 \qquad d=-\frac{175}{7}=-25$$

Ответ: $8x+5y-z-25=0$