# Второе задание.

## I. СРАВНЕНИЕ ФУНКЦИЙ

### 9.50

1) $x^2 = o(x)$ а) $x \to 0$ б) $x \to \infty$

$\lim\limits_{x\to 0} \frac{x^2}{x} = 0$ $\lim\limits_{x\to\infty} \frac{x^2}{x} = \infty$

а) true б) false

2) $x = o(x^2)$ а) $x \to 0$ б) $x \to \infty$

$\lim\limits_{x\to 0} \frac{x}{x^2} = \infty$ $\lim \frac{x}{x^2} = 0$

а) false б) true

### 9.51(1)

$x \to 0$ $n \in \mathbb{N}$ $k \in \mathbb{N}$ $n \geq k$

$o(x^n) + o(x^k) = o(x^k)$

$\forall f_n = o(x^n), \forall f_k = o(x^k)$

$\frac{f_n}{x^n} \to 0$; $\frac{f_k}{x^k} \to 0$

$\frac{f_n}{x^n} + \frac{f_k}{x^k} \geq \frac{f_n}{x^k} + \frac{f_k}{x^k} = \frac{f_n + f_k}{x^k} \to 0$

### T.1

$x \to 0$ $f(x) = o(g(x))$ и $g(x) \sim h(x) \overset{?}{\Rightarrow} f(x) = o(h(x))$ $x \to 0$

$\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = 0$ $\lim\limits_{x\to 0} \frac{g(x)}{h(x)} = 1$ $\exists \lim \frac{f(x)}{h(x)} \overset{?}{=} 0$

$\underset{\downarrow \atop 0}{\frac{f(x)}{\cancel{g(x)}}} \cdot \underset{\downarrow \atop 1}{\frac{\cancel{g(x)}}{h(x)}} \to 0$ q.e.d.

T.2

$x_0 \in \mathbb{R} - ?$

$x^2-4x+4 = o(x^2-3x+2)$ $\quad x \to x_0$ $\quad \begin{cases} x_0 \neq 1 \\ x_0 \neq 2 \end{cases}$

$$\lim_{x\to x_0} \frac{x^2-4x+4}{x^2-3x+2} = \frac{x^2-3x-x+2+2}{x^2-3x+2} = 1 + \frac{-x+2}{x^2-3x+2} = 0$$

$$\lim \frac{x-2}{x^2-3x+2} = \lim \frac{\cancel{x-2}}{\cancel{(x-2)}(x-1)} = \lim \frac{1}{x-1} = 1$$

$x_0 = 2$

ответ: 2

## II. Дифференцируемость. Дифференциалы

13.197 (2,3)

(2) $y = 2x - \frac{1}{2}\cos x$ $\qquad y_0 = -\frac{1}{2}$

$$\frac{d f^{-1}(y_0)}{dy} = \frac{1}{\frac{df(x_0)}{dx}} =$$

$y_0 = f(x_0)$; $\quad f'(x) = 2 + \frac{1}{2}\sin\frac{x}{2}$

$f(0) = -\frac{1}{2}$ ; $x_0 = 0$ подходит

$$= \frac{1}{2+\frac{1}{2}\sin\frac{x}{2}} = \frac{1}{2}$$

(3) $y = 0{,}1x + e^{0{,}1x}$ ; $y_0 = 1$

$1 = 0{,}1x_0 + e^{0{,}1x_0}$

$x_0 = 0$: $\quad 1 = 0 + e^0 = 0+1$ — подх.

$$\frac{d f^{-1}(y_0)}{dy} = \frac{1}{\frac{df(x_0)}{dx}} = \frac{1}{0{,}1+0{,}1e^{0{,}1x_0}} = \frac{1}{0{,}1+0{,}1} = 5$$

13. 201 (3, 7)

(3) $x(t) = a\cos t \quad y(t) = b\sin t$

$$y'_x = \frac{y'_t}{x'_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\,\mathrm{ctg}\,t$$

$$\frac{x}{y} = \frac{a}{b}\,\mathrm{ctg}\,t \;;\; \mathrm{ctg}\,t = \frac{b}{a}\cdot\frac{x}{y}$$

// $\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$

$a^2 y\,dy + b^2 x\,dx = 0$ //

$$1 + \mathrm{tg}^2 t = \frac{1}{\cos^2 t} = \frac{1}{\left(\frac{x(t)}{a}\right)^2} = \frac{a^2}{x^2(t)}$$

$$\mathrm{tg}^2 t = \frac{a^2 - x^2}{x^2} \;;\; \mathrm{ctg}^2 t = \frac{x^2}{a^2 - x^2}$$

$\forall t \in (0; \frac{\pi}{2}) \quad y(t) > 0$

$$\mathrm{ctg}\,t = \frac{x}{\sqrt{a^2 - x^2}}$$

$$y'_x = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}$$

(7) $x = a(t - \sin t) \quad x'_t = a - a\cos t$

$y = a(1 - \cos t) \quad y'_t = a\sin t$

$$y'_x = \frac{y'_t}{x'_t} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t} = \mathrm{ctg}\frac{t}{2} = \frac{1}{\mathrm{tg}\frac{t}{2}}$$

// пусть $\mathrm{tg}\frac{t}{2} = z$

$$\sin t = \frac{2z}{1+z^2} \quad \cos t = \frac{1-z^2}{1+z^2}$$

$$\frac{x}{a} = t - \frac{2z}{1+z^2}$$

$$1 - \frac{y}{a} = \cos t = \frac{1-z^2}{1+z^2} = \frac{1+z^2-2z^2}{1+z^2} = 1 - \frac{2z^2}{1+z^2} \;;\; \frac{y}{a} = \frac{2z^2}{1+z^2}$$

$$y + yz^2 = 2az^2 \;;\; y = (2a-y)z^2 \;;\; z^2 = \frac{y}{2a-y} \;;\; \frac{1}{z^2} = \frac{2a-y}{y} = \frac{2a}{y} - 1$$ //

13.213(1)

$$d(e^{-x}+\ln x) = d(e^{-x}) + d(\ln x) = \left(-e^{-x}+\frac{1}{x}\right)dx$$

13.173

$$y = \begin{cases} |x|^\alpha \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(1) при $\alpha > 0$: $|x|^\alpha$ — беск. малая, $\sin\left(\frac{1}{x}\right) \in [-1;1]$ — огранич.

$\Rightarrow |x|^\alpha \sin\left(\frac{1}{x}\right)$ — беск. малая

$$\lim_{x\to 0} |x|^\alpha \sin\left(\frac{1}{x}\right) = 0 = f(0)$$ — непрерывна

при $\alpha = 0$: $\sin\left(\frac{1}{x}\right)$

$\sin\left(\frac{\pi}{2}+2\pi n\right) = 1$

$\sin\left(-\frac{\pi}{2}+2\pi m\right) = -1$

$\forall \varepsilon > 0$ начиная с какого-то $n_0$ и $m_0$ $x = \frac{1}{\frac{\pi}{2}+2\pi n}$ и $x = \frac{1}{-\frac{\pi}{2}+2\pi m}$ начнёт попадать в $U_\varepsilon(0)$. начиная с $\max(n_0, m_0)$, одновременно.

Выберем послед. Гейне

$\{x_n\}$: $x_n = \frac{1}{\frac{\pi}{2}+2\pi n} \wedge n \geq n_0$ $\quad f(\{x_n\}) \to 1$

$\{x_m\}$: $x_m = \frac{1}{-\frac{\pi}{2}+2\pi m} \wedge m \geq m_0$ $\quad f(\{x_m\}) \to -1$

а хотим 0
противоре-е

$\alpha < 0$: аналогичные $\{x_n\}$ и $\{x_m\}$

$|x|^\alpha \sin\left(\frac{1}{x}\right)$

$f(\{x_n\}) \to +\infty$

$f(\{x_m\}) \to -\infty$

итак, $f$ непр. $\forall \alpha > 0$.

(2) исследуем $f$ на наличие производной

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$ — производная $f(x)$ в $x_0$

$$\lim_{x\to 0}\frac{f(x)-f(0)^{=0}}{x}=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{|x|^{\alpha}\sin\left(\frac{1}{x}\right)}{x}=$$

$$\underbrace{=\lim_{x\to 0}\underline{\operatorname{sign}(x)\,|x|^{\alpha-1}\sin\left(\tfrac{1}{x}\right)}}_{\text{хотим понять, когда } \exists} \leftarrow g(x) \quad \ominus$$

$$g(-x)=g(x)=|x|^{\alpha-1}\sin\left(\frac{1}{|x|}\right)$$

$$\ominus\lim_{x\to 0}|x|^{\alpha-1}\sin\left(\frac{1}{|x|}\right)=\lim_{x\to 0+}x^{\alpha-1}\sin\frac{1}{x}$$

if $\alpha-1>0$ $\exists$

else $\neg\exists$

(см. пункт 1)

итак, $f$ имеет произв $\forall\alpha>1$

(3) когда эта производная непрерывна?

$x>0$

$f(x)=x^{\alpha}\sin\frac{1}{x}$;

$f'(x)=\alpha x^{\alpha-1}\sin\frac{1}{x}-x^{\alpha-2}\cos\frac{1}{x}$

$x<0$

$f(x)=(-x)^{\alpha}\sin\frac{1}{x}$

$f'(x)=\alpha(-x)^{\alpha-1}\sin\frac{1}{x}-(-x)^{\alpha-2}\cos\frac{1}{x}$

таким образом,

$$f'(x\neq 0)=\alpha|x|^{\alpha-1}\sin\left(\frac{1}{x}\right)-|x|^{\alpha-2}\cos\frac{1}{x}$$

— непрерывна во всех точках, кроме, возможно, нуля

$\lim_{x\to 0}f'(x\neq 0)=0$ при $\alpha>2$

итак, производн. непр. $\forall\alpha>2$

13.179(1)

$$y=|x^3(x+1)^2(x+2)|=x^2(x+1)^2|x(x+2)|=f(x)$$

$$f'(x)=x^2(x+1)^2\left(|x(x+2)|\right)'+2x\cdot 2(x+1)=$$

$$=x^2(x+1)^2\left(|x(x+2)|\right)'+4x(x+1)$$

проблемы с дифф-стью могут возникнуть только в точках смены знака внутри модуля.

$$\left[\begin{array}{l}x=0\\x=-2\end{array}\right.$$

во всех остальных случаях дифф-ть сохр: если до отражения относит. оси абсцисс была касат. с конечным накл. коэф, то и после тоже будет $(k \to -k,\ k, -k \in \mathbb{R})$

1) $x = 0$

$$? \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0) = 0$$

$$? \lim_{x \to 0+} \frac{f(x)}{x} = \lim_{x \to 0-} \frac{f(x)}{x}$$

проверим

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^3(x+1)^2(x+2) = 0$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \left(-x^3(x+1)^2(x-2)\right) = 0$$

$$\lim_{x \to 0+} \frac{f(x)}{x} = \lim x^2(x+1)^2(x-2) = 0$$

$$\lim_{x \to 0-} \frac{f(x)}{x} = \lim \frac{|x| \cdot x^2 (x+1)^2 (x-2)}{\cancel{x}} = 0$$

все усл-я выполняются, $f(x)$ дифф в $x_0 = 0$

2) $x = -2$; $t = x + 2$; $x = t - 2$

$$f(t) = \left|(t-2)^3 (t-1)^2 t\right|$$

$$\lim_{t \to 0\pm} \frac{f(t) - 0}{t - 0} = \lim_{t \to 0\pm} \frac{f(t)}{t} = \operatorname{sign}(t)(t-1)^2(t-2)^2|t-2| =$$

при $t \to 0+$: $= (-1)^2(-2)^2|-2| = 1 \cdot 4 \cdot 2 = 8$

при $t \to 0-$: $= -8$

$\neq$ а значит, $f(t)$ не дифф-ма в $t = 0$

$f(x)$ не дифф в $x = -2$

Ответ: $f(x)$ дифф в $\forall x_0 \in (-\infty; -2) \cup (-2; +\infty)$

14.10(3)

$y^4 - 4x^4 - 6xy = 0$ $\qquad$ $M(1;2)$

1) $\exists\, y = y(x)$

$y^4(x) - 4x^4 + 6xy(x) = 0$

$4y^3(x)\underline{y'(x)} - 16x^3 + 6x\underline{y'(x)} + 6y(x) = 0$

$y'(x)\left(4y^3(x) + 6x\right) = 16x^3 - 6y(x)$

$y'(x) = \dfrac{16x^3 - 6y(x)}{4y^3(x) + 6x} = f'(x)$

ур-е касательной в (.)$x_0$: $y = f'(x_0)(x - x_0) + f(x_0)$

Для (.)$M(1,2)$: $y = \dfrac{16 \cdot 1 + 12}{32 - 6}(x - 1) + 2 = \dfrac{14}{13}x + \dfrac{12}{13}$

$13y - 14x - 12 = 0$

Нормаль к кривой в (.)M:

$\ell: ay + bx + c$

$\vec{m} = (13, -14)$, $\vec{m} \perp \vec{\ell}$; $\vec{\ell}(14, 13)$

$M \in \ell \Rightarrow 14 \cdot 2 + 13 \cdot 1 + c = 0 \Rightarrow c = -28 - 13 = -41$

таким образом, $\ell: 14y + 13x - 41 = 0$

Ответ: 1) $13y - 14x - 12 = 0$
2) $14y + 13x - 41 = 0$

# III. ПРОИЗВОДНЫЕ И ДИФФЕРЕНЦИАЛЫ ВЫСШИХ ПОРЯДКОВ

15.9(4)

$$y = x^x \qquad (x^x)' = (e^{x\ln x})' = e^{x\ln x}\cdot(\ln x+1) = x^x(\ln x+1)$$

$$dy = d(x^x) = x^x(\ln x+1)\,dx$$

$$d^2y = d(x^x(\ln x+1)dx) = d(x^x(\ln x+1))\,dx^2 \ominus$$

$$(x^x(\ln x+1))' = x^x\cdot\frac{1}{x} + x^x(\ln+1)(\ln x+1) = x^x\left(\frac{1}{x}+(\ln x+1)^2\right)$$

$$\ominus\ x^x\left(\frac{1}{x}+(\ln x+1)^2\right)dx^2$$

15.13(2)

$$y = u\ln v$$

$$dy = u\,d(\ln v) + \ln v\,du = \frac{u}{v}dv + \ln v\,du$$

$$d^2y = d\left(\frac{u}{v}dv\right) + d(\ln v\,du) = d\left(\frac{u}{v}\right)dv + \frac{u}{v}d^2v + d(\ln v)du +$$

$$+\ln v\,d^2u = \frac{u}{v}d^2v + \ln v\,d^2u + \frac{u\,dv - v\,du}{v^2}dv + \frac{1}{v}dv\,du =$$

$$= \frac{u}{v}d^2v + \ln v\,d^2u + \frac{u}{v^2}dv^2 - \cancel{\frac{1}{v}dv\,du} + \cancel{\frac{1}{v}dv\,du} = \underline{\underline{\frac{u}{v}d^2v + \ln v\,d^2u + \frac{u}{v^2}dv^2}}$$

15.14(4)

$$x = a\cos t$$
$$y = b\sin t \quad ; \quad \sin t = \frac{y}{b}$$

$$\frac{dx}{dt} = -a\sin t;\quad \frac{dy}{dt} = b\cos t;\quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{b\cos t}{a\sin t} = -\frac{b}{a}\operatorname{ctg} t$$

$$\frac{d^2y}{dx^2} = +\frac{b}{a}\left(+\frac{1}{\sin^2 t}\right)\cdot\frac{1}{-a\sin t} = -\frac{b}{a^2}\cdot\frac{1}{\sin^3 t} = -\frac{b}{a^2}\cdot\frac{b^3}{y^3} = -\frac{b^4}{a^2}\cdot\frac{1}{y^3}$$

15.22(3)

$$x^3y + \arcsin(y-x) = 1 \qquad (1;1)$$

$$d(x^3y + \arcsin(y-x) - 1) = 0$$

$$d(x^3y) = y\cdot d(x^3) + x^3dy = y\cdot 3x^2dx + x^3dy$$

$$d(\arcsin(y-x)) = \frac{d(y-x)}{\sqrt{1-(y-x)^2}} = \frac{dy-dx}{\sqrt{1-(y-x)^2}}$$

$$3x^2y\,dx + x^3dy + \frac{dy-dx}{\sqrt{1-(y-x)^2}} = 0$$

$$(1;1):\ \cancel{3}^{2}dx + dy + dy - \cancel{dx} = 0\ ;\quad dx = -dy$$

$$6xy\,dx^2 + 3x^2dx\,dy + 3x^2dx\,dy + x^3d^2y + \frac{(d^2y - d^2x)\sqrt{1-(y-x)^2}}{1-(y-x)^2} -$$

$$+\ \frac{(dy-dx)^2\,2(x-y)}{2\,(1-(y-x)^2)^{3/2}} = 0$$

$$(1;1):\ \cancel{6dx^2} - \cancel{6dx^2} - 3d^2x = 0$$

$$d^2x = 0$$

$$\underline{\underline{d^2y = 0}}$$

15.24 (13, 14)

13) $y = \dfrac{x}{x^2-4x-12} = \dfrac{x}{(x-2)(x+6)}$ // $\frac{D}{2} = 4 + 12 = 16$

$-2 \pm 4$ $\quad \begin{matrix}+2\\-6\end{matrix}$ //

$$\frac{a}{x-2} + \frac{b}{x+6} = \frac{(a+b)x + 6a - 2b}{(\ \ )(\ \ )}$$

$a + b = 1$

$3\cancel{6}a = \cancel{2}b$ $\qquad a + 3a = 1\ ;\ a = \frac{1}{4}\ ;\ b = 1 - a = \frac{3}{4}$

$$y = \underbrace{\frac{1}{4(x-2)}}_{f(x)} + \underbrace{\frac{3}{4(x+6)}}_{g(x)}$$

$$f'(x) = \frac{1}{4}\cdot(-1)\frac{1}{(x-2)^2}\ ;\quad f''(x) = \frac{1}{4}(-1)(-2)\frac{1}{(x-2)^3}$$

$$f^{(n)} = \frac{1}{4}(-1)^n\frac{n!}{(x-2)^n} \qquad g^{(n)} = \frac{3}{4}(-1)^n\frac{n!}{(x+6)^n}$$

$$y^{(n)} = \frac{1}{4}(-1)^n\,n!\left(\frac{1}{(x-2)^n} + 3\frac{1}{(x+6)^n}\right)$$

(14) $y=\frac{1+x^2}{1-x^2} = \frac{1-x^2+2x^2}{1-x^2} = 1+\frac{2x^2}{1-x^2}$

при $n>1$ $\quad y^{(n)} = \left(\frac{2x^2}{1-x^2}\right)^{(n)}$

$$2x^2\cdot\frac{1}{1-x^2} = 2x^2\left(\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)$$

$$\left(2x^2\cdot\frac{1}{1-x^2}\right)^{(n)} = \sum_{k=0}^{n} C_n^k (2x^2)^{(n-k)}\left(\frac{1}{1-x^2}\right)^{k}$$

$(2x^2)^{(n)}=0$ при $n>2$

$$\left(2x^2\cdot\frac{1}{1-x^2}\right)^{(n)} = C_n^{n-2}\cdot 4\cdot\left(\frac{1}{1-x^2}\right)^{(n-2)} + C_n^{n-1}(4x)\left(\frac{1}{1-x^2}\right)^{(n-1)} + C_n^n(2x^2)\left(\frac{1}{1-x^2}\right)^{(n)}$$

$$\left(\frac{1}{1-x^2}\right)^{(n)} = \frac12\left(\frac{1}{1-x}\right)^{(n)}+\frac12\left(\frac{1}{1+x}\right)^{(n)} = \frac12(-1)^n(-1)(-2)\dots(-1-n+1)(1-x)^{-1-n} + \frac12(-1)(-2)\dots(-1-n+1)(1+x)^{-1-n} = \frac12(-1)\ n!\left((-1)^n\frac{1}{(1-x)^{n+1}}+\frac{1}{(1+x)^{n+1}}\right)$$

$$\left(2x^2\frac{1}{1-x^2}\right)^{(n)} = n!(-1)^{n-2}\left((-1)^{n-2}\frac{1}{(1-x)^{n-1}}+\frac{1}{(1+x)^{n-1}}\right) + \frac{n!}{(n-1)!}\cdot4x\cdot\frac12(-1)^{n-1}(n-1)!\left((-1)^{n-1}\frac{1}{(1-x)^n}+\frac{1}{(1+x)^n}\right) + 2x^2\cdot\frac12(-1)^n n!\left((-1)^n\left(\frac{1}{(1-x)^{n+1}}+\frac{1}{(1+x)^{n+1}}\right) =\right.$$

$$= n!(-1)^{n-2}\left(\frac{(-1)^{n-2}}{(1-x)^{n-1}}+\frac{1}{(1+x)^{n-1}} - 2x\left(\frac{(-1)^{n-1}}{(1-x)^n}+\frac{1}{(x+1)^n}\right)+ x^2\left(\frac{(-1)^n}{(1-x)^{n+1}}+\frac{1}{(1+x)^{n+1}}\right)\right) = n!\Big((1-x)^{-n+1}+(-1)^{n-2}(1+x)^{-n+1} + 2x(1-x)^{-n} - 2x(-1)^{n-2}(x+1)^{-n} + x^2(1-x)^{-n-1}+(-1)^{n-2}x^2(1+x)^{n-1}\Big) =$$

$$= n!\Big((1-x)^{-n-1}\big((1-x)^2+2x-\cancel{2x^2}+\cancel{x^2}\big)+(-1)^n(1+x)^{-n-1}\big((1+x)^2-2x(x+1)+x^2\big)\Big) = n!\left((1-x)^{-n-1}+(-1)^n(1+x)^{-n-1}\right)$$

Ответ: $n!\left((1-x)^{-n-1}+(-1)^n(1+x)^{-n-1}\right)$

$(e^{2-3x})' = -3e^{2-3x}$

$(e^{2-3x})^{(n)} = (-3)^n e^{2-3x}$

15.25 (3, 6, 9)

3) $y = (3-2x)^2 e^{2-3x}$

$$y^{(n)} = \sum_{k=0}^{n} C_n^k \left((3-2x)^2\right)^{(k)} \left(e^{2-3x}\right)^{(n-k)} =$$

$$= (3-2x)^2 (-3)^n e^{2-3x} + \overset{n}{\vee} \underbrace{2(3-2x)(-2)}_{=(-1)4(3-2x)}(-3)^{n-1} e^{2-3x} +$$

$$+ \overset{\frac{1}{2}n(n-1)}{\vee}(-1)4(-2)(-3)^{n-2} e^{2-3x} =$$

$$= (-1)^n 3^n (3-2x)^2 e^{2-3x} + n(-1)^n 3^{n-1} 4(3-2x) e^{2-3x} + \overset{\frac{1}{2}n(n-1)}{\vee}(-1)^n 3^{n-2}\, 8 e^{2-3x} =$$

$$= (-1)^n 3^{n-2} e^{2-3x} \left(9(3-2x)^2 + 12n(3-2x) + 4n(n-1)\right)$$

6) $y = x \ln\frac{3+x}{3-x}$

$$y^{(n)} = \sum_{k=0}^{n} C_n^k x^{(k)} \left(\ln\frac{3+x}{3-x}\right)^{(n-k)} = x \ln\frac{3+x}{3-x}^{(n)} + n \ln\frac{3+x}{3-x}^{(n-1)} \;(=)$$

$$\left(\ln\frac{3+x}{3-x}\right)' = \frac{\cancel{3-x}}{3+x} \cdot \frac{1(3-x)-(-1)(3+x)}{(3-x)^{\cancel{2}}} = \frac{3\cancel{-x}+3\cancel{+x}}{(3+x)(3-x)} = \frac{6}{9-x^2}$$

// $\frac{6}{(3+x)(3-x)} = \frac{1}{3+x} + \frac{1}{3-x} = \frac{3\cancel{-x}+3\cancel{+x}}{(\ \ )(\ \ )} = \frac{6}{(\ \ )(\ \ )}$, ✓ //

$$\left(\ln\frac{3+x}{3-x}\right)' = \frac{1}{3-x} + \frac{1}{3+x} = \frac{1}{x+3} - \frac{1}{x-3}$$

$$\left(\frac{1}{x\pm3}\right)'' = \left(-\frac{1}{(x\pm3)^2}\right)' = (-1)^2\, 2\, \frac{1}{(x\pm3)^3} \;; \quad \left(\frac{1}{x\pm3}\right)^{(n)} = (-1)^n \frac{n!}{(x\pm3)^{n+1}}$$

$$\left(\ln\frac{3+x}{3-x}\right)^{(n+1)} = \left(\frac{1}{x+3} - \frac{1}{x-3}\right)^{(n)} = (-1)^n n! \left(\frac{1}{(x+3)^{n+1}} - \frac{1}{(x-3)^{n+1}}\right)$$

$$(n) \quad = (-1)^{n-1}(n-1)! \left(\frac{1}{(x+3)^n} - \frac{1}{(x-3)^n}\right)$$

если $n-2>1$

$$(=)\; x(-1)^{n-1}(n-1)!\left(\frac{1}{(x+3)^n} - \frac{1}{(x-3)^n}\right) + n(-1)^{n-2}(n-2)!\left(\frac{1}{(x+3)^{n-1}} - \frac{1}{(x-3)^{n-1}}\right)$$

9) $y = 2x\cos^2\left(\frac{x}{3}\right)$

$$y^{(n)} = \sum_{k=0}^{n} C_n^k (2x)^{(k)} \left(\cos^2\frac{x}{3}\right)^{(n-k)} = 2x\left(\cos^2\frac{x}{3}\right)^{(n)} + 2n\left(\cos^2\frac{x}{3}\right)^{(n-1)} \ominus$$

$$\left(\cos^2\left(\frac{x}{3}\right)\right)' = 2\cos\frac{x}{3}\cdot\frac{1}{3} = \frac{2}{3}\cos\frac{x}{3}$$

$$\left(\cos^2\left(\frac{x}{3}\right)\right)'' = \frac{2}{3}\cdot\frac{1}{3}(-1)\sin\frac{x}{3} = \cos\left(\frac{x}{3}+\frac{\pi}{2}\right)$$

$$^{(3)} = 2\cdot\frac{1}{3^3}(-1)\cos\frac{x}{3} = \frac{2}{3^3}\cos\left(\frac{x}{3}+\frac{\pi\cdot 2}{2}\right)$$

$$^{(4)} = 2\cdot\frac{1}{3^4}(-1)^2\sin\left(\frac{x}{3}\right) = \frac{2}{3^4}\cos\left(\frac{x}{3}+\frac{\pi\cdot 3}{2}\right)$$

если $n-1>0$

$$\ominus\ 2x\cdot\frac{2}{3^n}\cos\left(\frac{x}{3}+\frac{\pi(n-1)}{2}\right) + 2\cdot\frac{2}{3^{n-1}}n\cos\left(\frac{x}{3}+\frac{\pi(n-2)}{2}\right) =$$

$$= \frac{4}{3^n}x\cos\left(\frac{x}{3}+\frac{\pi}{2}(n-1)\right) + \frac{4}{3^{n-1}}n\cos\left(\frac{x}{3}+\frac{\pi}{2}(n-2)\right)$$

15.26(2)

$$y = \frac{x^2}{\sqrt{1-2x}} = x^2\cdot\frac{1}{\sqrt{1-2x}}$$

$$\left(\frac{1}{\sqrt{1-2x}}\right)' = \left((1-2x)^{-\frac{1}{2}}\right)' = -\frac{1}{\cancel{2}}(1+2x)^{-\frac{3}{2}}\cdot\cancel{2} = (1+2x)^{-\frac{3}{2}}$$

$$'' = -\frac{3}{\cancel{2}}(1+2x)^{-\frac{5}{2}}\cdot\cancel{2} = -3(1+2x)^{-\frac{5}{2}}$$

$$''' = (-1)^2\, 3\cdot 5\,(1+2x)^{-\frac{7}{2}} \qquad -\left(3+\frac{1}{2}\right) = \frac{2\cdot 3+1}{2}$$

$$^{(n)} = (-1)^{(n-1)}(2n-1)!!\,(1+2x)^{-\frac{2n+1}{2}}$$

$$y^{(n)} = \sum C_n^k (x^2)^k\left(\frac{1}{\sqrt{1-2x}}\right)^{(n-k)} = x^2(-1)^{n-1}(2n-1)!!\,(1+2x)^{-\frac{2n+1}{2}} +$$

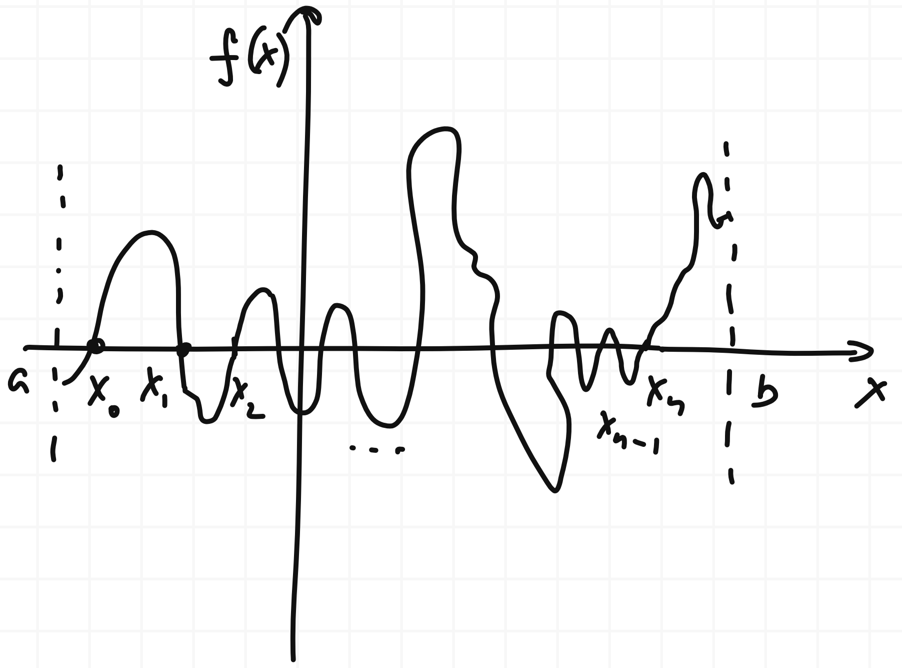
$$+ n\cdot 2x(-1)^{n-2}(2n-3)!!\,(1+2x)^{-\frac{2n-1}{2}} + n(n-1)(-1)^{n-3}(2n-5)!!\,(1+2x)^{-\frac{2n-3}{2}}$$

# IV. ТЕОРЕМЫ О СРЕДНЕМ

## 16.5

$f$ дифф. $n$ раз на $[a,b]$ и обращ. на нем в нуль в $n+1$ точках

? $\exists\, \xi \in (a;b): f^{(n)}(\xi)=0$

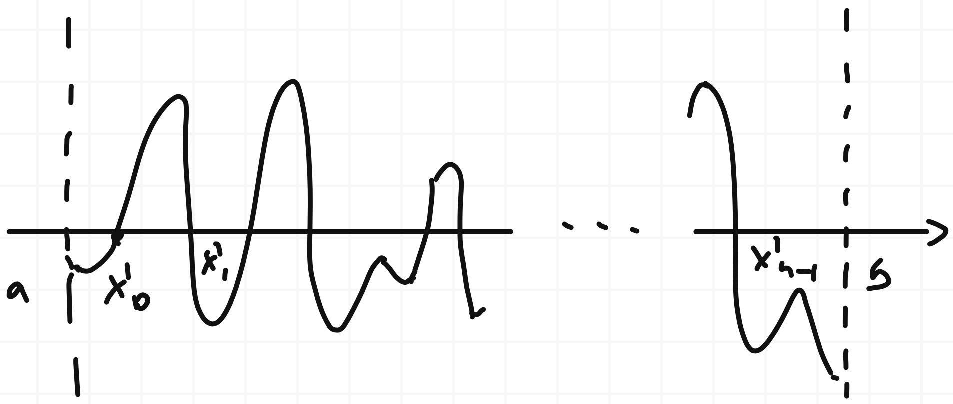



по т. Ролля $\forall k = 0, 1, \ldots, n-1$

$\exists\, \xi \in (x_k, x_{k+1}) \hookrightarrow f'(\xi)=0$

для интервалов от $(x_0, x_1)$, $(x_1, x_2), \ldots, (x_{n-1}, x_n)$ получаем $n$ строго различных точек в которых $f'=0$

дальше




$x_0', \ldots, x_{n-1}' : \quad f'(x_i')=0$

ан-но применим т. Ролля

и т.д. до $n$-й производной

...

$\exists f^{(n)}(\xi)=0$ q.e.d.

## 16.15 (3)

$e^x \geq 1+x \;;\; e^x - x - 1 \geq 0$

$f(x) = e^x - x - 1$

$f(0) = 1 - 0 - 1 = 0$

по т. Лагранжа $\forall x \in (0;+\infty)\ \exists\, \xi : f'(\xi) = \dfrac{f(x)-f(0)}{x-0} = \dfrac{f(x)}{x}$

т.е. $f(x) = x f'(\xi) = \underset{\substack{\vee\\0}}{x}(\underset{\substack{\vee\\1}}{e^{\xi}}-1) > 0$

$\forall x \in (0;+\infty) \hookrightarrow e^x \geq 1+x$

по т. Лагранжа $\forall x \in (-\infty; 0)$ $\exists \xi: f'(\xi) = \frac{f(0)-f(x)}{0-x} = \frac{f(x)}{x}$

т.е. $f(x) = xf'(\xi) = \underset{\wedge\atop 0}{x}\underbrace{(e^{\xi}-1)}_{\wedge\atop 0} > 0$

т.е. нер-во вып.

q.e.d.

<u>16.15(2)</u>

$$\underset{(1)}{\frac{x}{1+x} < \ln(1+x)} \underset{(2)}{<} x \qquad x > 0$$

(1) $\frac{x}{1+x} < \ln(1+x)$; $f(x) = \ln(1+x) - \frac{x}{1+x} \overset{?}{>} 0$; $f(0) = \ln 1 - 0 = 0$

по т. Лагранжа $\forall x \in (0; +\infty)$ $\exists \xi \in (0; +\infty): f'(\xi) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$

т.е. $f(x) = x\left(\cancel{\frac{1}{1+x}} - \cancel{\frac{1}{1+x}} + \frac{x}{(1+x)^2}\right) = \left(\frac{x}{1+x}\right)^2 > 0$

q.e.d.

(2) $f(x) = x - \ln(1+x) \overset{?}{>} 0$ ; $f(0) = 0$

— " — $f(x) = \left(1 - \frac{1}{1+x}\right)x = \left(\frac{\cancel{1+x-1}}{1+x}\right)x = \frac{x^2}{1+x} > 0$ q.e.d.

<u>16.19</u>

$f$ дифф. и неогр. на $(a;b)$ ; $a, b \in \mathbb{R}$ // далее $A = \lim\limits_{x\to a+0} f(x)$ $B = \lim\limits_{x \to b-0} f(b)$ //

? $f'$ неогр. на $(a,b)$

$f$ неогр.: $\forall M' \in \mathbb{R} \hookrightarrow \exists x \in (a;b): |f(x)| > M$

по т. Лагранжа

$\forall x \in (a;x)$ $\exists \xi_1 \in (a;x)$: $f'(\xi_1) = \frac{f(x)-A}{x-a} \geq \frac{1}{|a-b|}(f(x)-A)$

$\forall x \in (x;b)$ $\exists \xi_2 \in (x;b)$: $f'(\xi_2) = \frac{B-f(x)}{b-x} \geq \underbrace{\frac{1}{|a-b|}}_{c}(B-f(x))$

$0 \leq$ знаменатель $\leq |a-b|$

$\forall M \in \mathbb{R} \hookrightarrow \exists x \in (a;b)$

$f(x) > M$

$\exists \xi_1: f'(\xi_1) \geq c(f(x) - f(a)) \geq$
$\geq c(M-A)$

$-f(x) > M$

$\exists \xi_2: f'(\xi_2) \geq c(f(b) - f(x)) \geq$
$\geq c(B-M)$

// $M' = c(M-A) = cM - cA$ // $M' = c(B-M)$

$M = \frac{M'+cA}{c} = A + \frac{M'}{c}$ // $M = B - \frac{M'}{c}$ //

q.e.d.

16.30

f дифф. на $[1;2]$

? $\exists \xi \in (1;2)$ $f(2) - f(1) = \frac{\xi^2}{2} f'(\xi)$

по т. Коши $\exists \xi \in (1;2)$: $\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(\xi)}{g'(\xi)}$, если $g$ дифф на $(a;b)$, $g'(t) \neq 0$ $\forall t$

$$\frac{f(2)-f(1)}{f'(\xi)} = \frac{g(2)-g(1)}{g'(\xi)} = \frac{\xi^2}{2}$$

$g(t) = -\frac{2}{t}$ подходит

q.e.d.

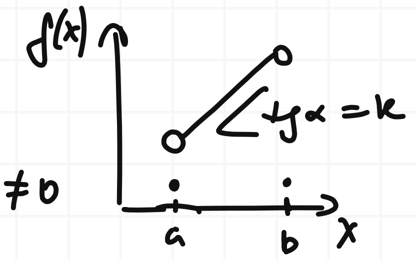
16.33

Всегда ли $\exists \xi \in (a;b)$ $f'(\xi) = 0$, если

1) имеет произв. на $(a;b)$ — нет
$f(a) = f(b)$

контрпример:

$\forall x \in (a;b)$ $\exists f'(x) = k \neq 0$

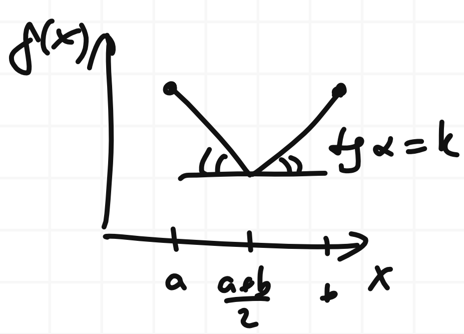



2) непр. на $[a,b]$ — нет
$f(a) = f(b)$

контрпример:

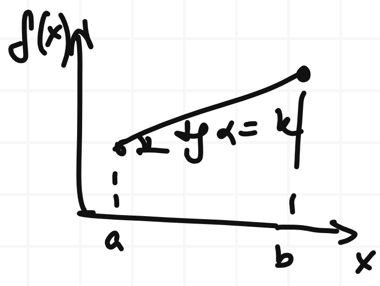
$\forall x \in (a; \frac{a+b}{2})$ $\exists f'(x) = k = 0$

$\forall x \in (\frac{a+b}{2}; b)$ $\exists f'(x) = k \neq 0$




3) непр. на $[a;b]$ — нет
им. пр. на $(a;b)$

$\forall x \in (a;b)$ $\exists f'(x) = k \neq 0$



16.20

$\exists \lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to b-0} f(x) \in \overline{\mathbb{R}}$ $\overset{?}{\Longrightarrow}$ $\exists \xi \in (a; b): f'(\xi) = 0$

$f$ дифф на $(a; b)$

## V. ФОРМУЛЫ МАКЛОРЕНА и ТЕЙЛОРА

**7.3**

$$f(x) = (x+x^2-2x^3+x^4)^3 =$$

$$= (x^2 + x^2 - 2x^4 + \cancel{x^5} + x^3 + x^4 - \cancel{2x^5} + x^6 - 2x^4 - 2x^5 + 4x^6 - 2x^7 +$$

$$+ \cancel{x^5} + x^6 - 2x^7 + x^8)(x+x^2-2x^3+x^4) =$$

$$= (x^2 + 2x^3 - 3x^4 - 2x^5 + \underbrace{6x^6 - 4x^7 + x^8}_{o(x^5)})(x+x^2-2x^3+x^4) =$$

$$= x^3 + 2x^4 - 3x^5 + x^4 + 2x^3 - 2x^5 + o(x^5) = 3x^3 + 3x^4 - 5x^5 + o(x^5)$$

$$f(x) = 3x^3 + 3x^4 - 5x^5 + o(x^5)$$

**18.2(9)**

$$\ln(2+x-x^2) = \ln(-(x-2)(x+1)) = \ln(2-x) + \ln(x+1) \circleddash$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n)$$

$$\ln(2-x) = \ln 2 + \ln\left(1-\frac{x}{2}\right) = \ln 2 - \sum_{k=1}^{n}\frac{\left(\frac{x}{2}\right)^k}{k} + o(x^n)$$

$$\circleddash \ln 2 + \sum_{k=1}^{n}\left((-1)^{k-1}\frac{x^k}{k} - \frac{\left(\frac{x}{2}\right)^k}{k}\right) + o(x^n) =$$

$$= \ln 2 + \sum_{k=1}^{n}\left(\frac{x^k}{k}\left((-1)^{k-1} - \frac{1}{2^k}\right)\right) + o(x^n)$$

**18.3(5,9)**

5) $$\frac{x}{\sqrt[3]{9-6x+x^2}} = \frac{x}{\sqrt[3]{(3-x)^2}} = x(3-x)^{-\frac{2}{3}} = 3^{-\frac{2}{3}} x \left(1-\frac{x}{3}\right)^{-\frac{2}{3}} =$$

$$= 3^{-\frac{2}{3}} x \left(\sum_{k=0}^{n-1} C^k_{-\frac{2}{3}}\left(-\frac{x}{3}\right)^k + o(x^{n-1})\right) = 3^{-\frac{2}{3}} \sum_{k=0}^{n-1} \frac{(-1)^k}{3^k} C^k_{-\frac{2}{3}} x^{k+1} + o(x^n) \circleddash$$

$$// C^k_\alpha = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-(k-1))}{k!} = \left[\alpha = -\frac{2}{3}\right] = \frac{-\frac{2}{3}\left(-\frac{2}{3}-1\right)\cdot\ldots\left(-\frac{2}{3}-(k-1)\right)}{k!} =$$

$$= \frac{(-1)^k}{3^k k!}\cdot 2\cdot(2+3)(2+6)\ldots(2+3k-3) = \frac{(-1)^k 2^k}{k!\,3^k}\cdot 1\cdot\left(1+\frac{3}{2}\right)\left(1+\frac{6}{2}\right)\ldots\left(1+\frac{3(k-1)}{2}\right) //$$

$$\circleddash 3^{-\frac{2}{3}} \sum_{k=0}^{n-1}\left(\frac{(-1)^k}{3^k}\cdot\frac{(-1)^k}{k!}\cdot\frac{2^k}{3^k}\cdot\prod_{i=0}^{k-1}\left(1+\frac{3i}{2}\right)\cdot x^{k+1}\right) + o(x^n) =$$

$$= 3^{-\frac{2}{3}} \sum_{k=0}^{n-1} \left( \frac{2^k}{3^{2k}} \prod_{i=0}^{k-1} \left(1+\frac{3i}{2}\right) \frac{x^{k+1}}{k!} \right) + o(x^n)$$

(9) $(1-x)\ln(1+5x+6x^2) = (1-x)\ln\left(6\left(x+\frac{1}{2}\right)\left(x+\frac{1}{3}\right)\right) \overset{x\to 0}{=}$

// $D = 25-4\cdot 6 = 1$

$\frac{-5\pm 1}{2\cdot 6} = \left[ \begin{array}{l} \frac{-6}{2\cdot 6} = -\frac{1}{2} \\ \frac{-4}{12} = -\frac{1}{3} \end{array} \right.$

$6\left(x+\frac{1}{2}\right)\left(x+\frac{1}{3}\right)$ //

$$= (1-x)\ln((1+2x)(1+3x)) =$$

$$= (1-x)(\ln(1+2x) + \ln(1+3x)) =$$

$$= (1-x)\left((2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \dots + \frac{(-1)^{n-1}(2x)^n}{n} + \right.$$

$$\left. + (3x) - \frac{(3x)^2}{2} + \dots + \frac{(-1)^{n-1}(3x)^n}{n}\right) + o(x^n) =$$

$$= (1-x) \sum_{k=1}^{n} \left( \frac{(-1)^{k-1}(2x)^k}{k} + \frac{(-1)^{k-1}(3x)^k}{k} \right) + o(x^n) = (1-x)\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(2^k+3^k)x^k + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(2^k+3^k)x^k - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(2^k+3^k)x^{k+1} =$$

$k+1 = \tilde{k}$

$k = \tilde{k}-1$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(2^k+3^k)x^k - \sum_{k=2}^{n+1} \frac{(-1)^k}{k-1}(2^{k-1}+3^{k-1})x^k =$$

$$= 5x - \underbrace{\frac{(-1)^{n+1}}{n}(2^n+3^n)x^{n+1}}_{o(x^n)} + \sum_{k=2}^{n} \left( \frac{(-1)^{k-1}}{k}(2^k+3^k) + \frac{(-1)^{k-1}}{k-1}(2^{k-1}+3^{k-1}) \right) x^k + o(x^n) =$$

$$= 5x + \sum_{k=2}^{n} (-1)^{k-1} \left( \frac{2^k+3^k}{k} + \frac{2^{k-1}+3^{k-1}}{k-1} \right) x^k + o(x^n)$$

18.4(7)

$$\frac{x^2+4x-1}{x^2+2x-3} = \frac{x^2+2x+2x\cancel{-1}-3\cancel{+3}+2}{(x-1)(x+3)} = 1 + \frac{2x+2}{(x-1)(x+3)} = 1 + \frac{1}{x-1} + \frac{1}{x+3} =$$

// $x^2+2x-3$

$\frac{D}{4} = 1+3 = 4$

$-1\pm 2 = \begin{array}{l} 1 \\ -3 \end{array}$ //

$$= 1 - \frac{1}{1-x} + \frac{1}{3}\cdot\frac{1}{1+\frac{x}{3}} =$$

$$= 1 - \sum_{k=0}^{n} x^k + o(x^n) + \frac{1}{3}\sum_{k=0}^{n} \frac{x^k}{3^k} + o(x^n) =$$

$$= 1 + \sum_{k=0}^{n} \left( \frac{1}{3^{k+1}} - 1 \right) x^k + o(x^n)$$

**18.5(3)**  $o(x^{2n})$

$x\sin^2 2x =$

// $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots =$

$= \sum_{k=0}(-1)^k\frac{x^{2k}}{(2k)!} + o(x^{2n+1})$ //

// $\sin^2 2x = \frac12(1-\cos 2x)$ //

$= \frac12 x(1-\cos 2x) = \frac12 x - \frac12 x\left(\sum_{k=0}^{n-1}(-1)^k\frac{x^{2k}}{(2k)!} + o(x^{2n-1})\right) =$

$= \frac12 x - \frac12\cdot\sum_{k=0}^{n-1}(-1)^k\frac{x^{2k+1}}{(2k)!} + o(x^{2n}) = \frac12\left(\cancel{x}-\cancel{x}+\frac{x^3}{2}-\dots\right)+o(x^{2n}) =$

$= \frac12\sum_{k=1}^{n-1}(-1)^k\frac{x^{2k+1}}{(2k)!} + o(x^{2n})$

**18.14(3)**

$x\ln(2-3x+x^2)$ ; $x_0=-2$

// $x^2-3x+2=(x-1)(x-2)$ //

$x\ln((1-x)(2-x)) = \begin{bmatrix} t=x-x_0= \\ =x+2 \\ x=t-2 \end{bmatrix} = (t-2)\ln\left((\overset{+3}{\cancel{1}}-t\cancel{+2})(\overset{+4}{\cancel{2}}-t\cancel{+2})\right) =$

$= (t-2)\ln\left(12(1-\frac t3)(1-\frac t4)\right) = (t-2)\left(+\ln 12+\ln(1-\frac t3)+\ln(1+\frac t4)\right) =$

$= (t-2)\left(+\ln 12-\sum_{k=1}^{n}\frac{t^k}{3^k k}-\sum_{k=1}^{n}\frac{t^k}{4^k k}\right) \overset{+o(t^n)}{=} (2-t)\left(\ln 12+\sum_{k=1}^{n}\left((\frac1{3^k}+\frac1{4^k})\frac{t^k}{k}\right)\right)+o(t^n) =$

$= -2\ln 12 + t\ln 12 + 2\sum_{k=1}^{n}(\frac1{3^k}+\frac1{4^k})\frac{t^k}{k} - \sum_{k=1}^{n-1}(\frac1{3^k}+\frac1{4^k})\frac{t^{k+1}}{k} + o(t^n) = \begin{bmatrix}\tilde k = k+1 \\ k=\tilde k-1\end{bmatrix}$

$= -\ln 144 + (\ln 12)t + 2\sum_{k=1}^{n}(\frac1{3^k}+\frac1{4^k})\frac{t^k}{k} - \sum_{k=2}^{n}(\frac1{3^{k-1}}+\frac1{4^{k-1}})\frac{t^k}{k-1} + o(t^n) =$

$= -\ln 144 + (\ln 12)t + 2\cdot\overset{\frac{7}{12}}{(\frac13+\frac14)}t + \sum_{k=2}^{n}\left(\frac2k(1+\frac1{4^k})-\frac1{k-1}(1+\frac1{4^{k-1}})\right)t^k =$

$= -\ln 144 + (\frac76+\ln 12)(x-x_0) + \sum_{k=2}^{n}\left[\left(\frac2k(\frac1{3^k}+\frac1{4^k})-\frac1{k-1}(\frac1{3^{k-1}}+\frac1{4^{k-1}})\right)(x-x_0)^k\right] + o((x-x_0)^n)$

**18.20(6)**

$x(x-2)2^{x^2-2x-1}$, $x_0=1$   $t=x-x_0=x-1$ ; $x=t+1$

$(t+1)(t-1)2^{t^2\cancel{+2t}\cancel{+1}\cancel{-2t}-2\cancel{-1}} = (t+1)(t-1)2^{t^2-2}$

$f(t)=(t^2-1)2^{t^2-2} = \frac14\underbrace{2^{t^2}}_{g(t)}\underbrace{(t^2-1)}_{h(t)}$

$$g(t) = 2^{t^2} = e^{\underbrace{t^2 \ln 2}} \rightsquigarrow w(t)$$

$$h(t) = t^2 - 1$$

$$w(t) = t^2 \ln 2$$

$$g(w(t)) = e^{w(t)} = \sum_{k=0}^{n} \frac{w^2}{k!} + o(w^n) = \sum_{k=0}^{n} \frac{(\ln 2)^k t^{2k}}{k!} + o(t^{2n})$$

$$f(t) = \frac{1}{4} h(t) g(t) = \frac{1}{4}(t^2 - 1)\left(\sum \frac{(\ln 2)^k t^{2k}}{k!} + o(t^{2n})\right) =$$

$$= \frac{1}{4} \sum^{n} \frac{(\ln 2)^k t^{2k+2}}{k!} - \frac{1}{4} \sum_{k=0}^{n} \frac{(\ln 2)^k t^{2k}}{k!} + o(t^{2n+1}) =$$

$$= \frac{1}{4} \sum_{k=0}^{n-1} \frac{(\ln 2)^k t^{2k+2}}{k!} - \frac{1}{4} \sum_{k=0}^{n} \frac{(\ln 2)^k t^{2k}}{k!} + o(t^{2n+1}) =$$

$$= -\frac{1}{4} + \frac{1}{4} \sum_{k=1}^{n} \left( \frac{(\ln 2)^{k-1}}{(k-1)!} - \frac{(\ln 2)^k}{k!} \right) t^{2k} + o(t^{2n+1}) =$$

$$= -\frac{1}{4} + \sum_{k=1}^{n} \frac{(\ln 2)^{k-1}}{k!} (k - \ln 2)(x-1)^{2k} + o((x-1)^{2n+1})$$

<u>18.39(5)</u> $\quad o(x^5)$

$$\ln \frac{\sin x}{x} = \ln\left(\frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)\right) = \ln\left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + o(x^5)\right)\right) =$$

$$// \ \ln(1-t) = -\sum_{k=1}^{n} \frac{t^k}{k} = (-1)\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots\right) //$$

$$= (-1)\left( \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{1}{2}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + o(x^5)\right)^2 + o(x^5)\right) =$$

$$= (-1)\left(\frac{x^2}{6} - \frac{x^4}{120} + \frac{x^4}{72} + o(x^5)\right) = (-1)\left(\frac{x^2}{6} + \frac{x^4}{180}\right) + o(x^5) \ominus$$

$$// \quad -\frac{1}{120}^{\backslash 3} + \frac{1}{72}^{\backslash 5} = -\frac{3}{360} + \frac{5}{360} = \frac{2}{360} = \frac{1}{180} //$$

$$\ominus \ -\frac{x^2}{6} - \frac{x^4}{180} + o(x^5)$$

T.4

а) $y = \arcsin x = f(x)$

$$f(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k + o(x^n)$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad f'(0) = 1$$

$$f''(x) = \left((1-x^2)^{-\frac{1}{2}}\right)' = +\frac{\cancel{1}}{\cancel{2}}(1-x^2)^{-\frac{3}{2}}(+\cancel{2}x) = x(1-x^2)^{-\frac{3}{2}} \qquad f''(0) = 0$$

$$f'''(x) = (1-x^2)^{-\frac{3}{2}} + \frac{3}{\cancel{2}} x(1-x^2)^{-\frac{5}{2}}\cancel{(-2x)} = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}} \qquad f'''(0) = 1$$

$$f^{(4)}(x) = +\frac{3}{\cancel{2}}(+\cancel{2}x)(1-x^2)^{-\frac{5}{2}} + 3x^2\left(+\frac{5}{\cancel{2}}\right)(+\cancel{2}x)(1-x^2)^{-\frac{7}{2}} + 6x(1-x^2)^{-\frac{5}{2}}; \ f^{(4)}(0) = 0$$

$$\underline{\underline{\arcsin x}} = 0 + x + \frac{x^3}{6} + o(x^4) = \underline{\underline{x + \frac{x^3}{6} + o(x^4)}}$$

б) $\operatorname{th} x = f(x)$

// лирическое отступление

$\operatorname{sh} x = \frac{1}{2}(e^x - e^{-x})$ ; $\operatorname{sh}(0) = 0$

$\operatorname{ch} x = \frac{1}{2}(e^x + e^{-x})$ ; $\operatorname{ch}(0) = \frac{1}{2}(1+1) = 1$

$\operatorname{th}' x = \left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)' = \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x} = 1 - \operatorname{th}^2 x$

//

$f'(x) = \frac{1}{\operatorname{ch}^2(x)}$ ; $f'(0) = \frac{1}{1^2} = 1$

$f''(x) = -2\operatorname{th} x(1 - \operatorname{th}^2 x)$ ; $f''(0) = 0$

$$f'''(x) = -2\left((1-\operatorname{th}^2 x)^2 + \operatorname{th} x\left(-2\operatorname{th} x(1-\operatorname{th}^2 x)\right)\right) =$$

$$= -2(1-\operatorname{th}^2 x)^2 + 4\operatorname{th}^2 x(1-\operatorname{th}^2 x); \ f'''(0) = -2$$

$$f^{(4)}(x) = \frac{-4(1-\operatorname{th}^2 x)(-2\operatorname{th} x)}{\operatorname{ch}^2 x} + 4\left(\frac{2\operatorname{th} x}{\operatorname{ch}^2 x}(1-\operatorname{th}^2 x) + \operatorname{th}^2 x\left(-\frac{2\operatorname{th} x}{\operatorname{ch}^2 x}\right)\right); \ f^{(4)}(0) = 0$$

$$\underline{\underline{\operatorname{th} x}} = 0 + \frac{1}{1}x + \frac{-2}{6}x^3 + o(x^4) = \underline{\underline{x - \frac{x^3}{3} + o(x^4)}}$$

T.5

а) $y = \operatorname{tg} x = \frac{\sin x}{\cos x}$

$\operatorname{tg} x$ – нечетная ф-я, значит, в разложении будут присутствовать только нечетные степени

$\operatorname{tg} x = ax + bx^3 + cx^5 + o(x^6)$

$$\operatorname{tg} x = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6)}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + o(x^6)}$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6) = (ax + bx^3 + cx^5 + o(x^6))(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + o(x^6)) =$$

$$= ax - \frac{1}{2}ax^3 + \frac{1}{24}ax^5 + bx^3 - \frac{1}{2}bx^5 + cx^5 + o(x^6) =$$

$$= ax + (-\frac{1}{2}a + b)x^3 + (\frac{1}{24}a - \frac{1}{2}b + c)x^5 + o(x^6)$$

$$\begin{cases} a = 1 \\ -\frac{1}{2}a + b = -\frac{1}{6} \\ \frac{1}{24}a - \frac{1}{2}b + c = \frac{1}{120} \end{cases} \qquad \begin{cases} b = \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{1}{3} \\ \frac{1}{24} - \frac{1}{2}b + c = \frac{1}{120} \end{cases} \qquad \frac{1}{24} - \frac{4}{24} + c = \frac{1}{120}$$

$$c = \frac{1}{120} + \frac{3}{24} = \frac{\cancel{16}}{\cancel{120}} = \frac{2}{15}$$

$a = 1$; $b = \frac{1}{3}$; $c = \frac{2}{15}$

$$\operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6)$$

б) $y = \operatorname{arctg} x$

$y' = \frac{1}{1+x^2} = (1+x^2)^{-1}$ ; $y'(0) = 1$

$$y'' = (-1)(1+x^2)^{-2}2x = -2x(1+x^2)^{-2}$$

$$y''' = +2x(+2(1+x^2)^{-3}2x) - 2(1+x^2)^{-2} = \underbrace{8}_{2^2 \cdot 2!}x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$$

$$y^{(4)} = 8x^2(-3(1+x^2)^{-4}2x) + 16x(1+x^2)^{-3} - 2(-2(1+x^2)^{-3}2x) =$$

$$= -2^3 \cdot 3!\, x^3(1+x^2)^{-5} + 16x(1+x^2)^{-3} + 8x(1-x^2)^{-3}$$

$$y^{(5)} = -2^3 \cdot 3!\,(2x^2(1-x^2)^{-5} + x^3(-5)(1+x^2)^{-6}2x) + 16((1+x^2)^{-3} + x(-3)(1+x^2)^{-4}2x)$$

$$+ 8((1-x^2)^{-3} + x(-3(1-x^2)^{-4}2x))$$

$y'''(0) = -2$

$y^{(5)}(0) = 0+16(1+0)+8(1+0) = 16+8 = 24$

$\operatorname{arctg} x = x - \cancel{\frac{2}{6}}\tfrac{1}{3} x^3 + \cancel{\frac{24}{120}}\tfrac{1}{5} x^5 + o(x^6) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + o(x^6)$

**18.30(1)**

$x^3|x| + \cos^2 x = \operatorname{sign}(x)\cdot x^4 + \frac{1}{2}(1+\cos 2x) \ominus$

$// \; 1+\cos 2x = 1 + \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \; //$

$= \operatorname{sign}(x) x^4 + \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) =$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\frac{x^4}{24} + \underset{0 \text{ (в } x=0)}{\operatorname{sign}(x) x^4} + \frac{1}{2}\sum_{k=3}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) =$

$= 1 - \frac{x^2}{4} + \frac{x^4}{48} + \ldots = 1 + \frac{1}{2}\sum_{k=1}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$

## VI. Вычисление пределов с помощью правила Лопиталя

**17.39**

$\lim\limits_{x\to+0} \frac{\ln x}{\ln \sin x} = \lim\limits_{x\to+0} \frac{\frac{1}{x}}{\frac{1}{\sin x}\cdot\cos x} = \lim\limits_{x\to+0} \frac{\operatorname{tg} x}{x} = \lim\limits_{x\to+0} \frac{\frac{1}{\cos^2 x}}{1} =$

$= \lim\limits_{x\to+0} \frac{1}{\cos^2 x} = 1$

**17.49**

$\lim\limits_{x\to+0} \frac{3+\ln x}{2-3\ln \sin x} = \lim\limits_{x\to+0} \frac{\frac{1}{x}}{-3\frac{1}{\sin x}\cos x} = \lim\left(-\frac{1}{3}\right)\frac{\operatorname{tg} x}{x} \overset{17.39}{=\!=} -\frac{1}{3}$

**17.64**

$\lim\limits_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left(\exp\left(\frac{\frac{2}{\pi}\arccos x}{x}\right)\right) = e^{\frac{2}{\pi}}$

$// \; \lim\limits_{x\to 0}\frac{\frac{2}{\pi}\arccos x}{x} = \lim\limits_{x\to 0}\frac{2}{\pi}\cdot\frac{1}{1+x^2} = \frac{2}{\pi} \; //$

## 17.76(1)

$$\lim_{x\to\infty}\frac{(x+\cos x)'}{(x-\cos x)'}=\lim_{x\to\infty}\frac{1-\sin x}{1+\sin x}$$ — не имеет предела при $x\to\infty$

$$\lim_{x\to\infty}\frac{x+\cos x}{x-\cos x}=\lim_{x\to\infty}\frac{1+\frac{\cos x}{x}\to 0}{1-\frac{\cos x}{x}\to 0}=1$$

## 17.80

$$f(x)=\begin{cases} e^{-1/x^2} & ; x\neq 0\\ 0 & ; x=0\end{cases} \qquad \lim_{x\to+0}e^{-1/x^2}=\lim_{x\to-0}e^{-1/x^2}=0$$

(1) Д-м, что $f(x)$ беск. дифф. на $\mathbb{R}$, т.е. $\forall n\in\mathbb{N}\ \exists f^{(n)}(x)$ на $\mathbb{R}$ ) по индукции

покажем, что $\forall n\in\mathbb{N}$ $f^{(n)}(x)=\begin{cases}P_{m(n)}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}, & x\neq 0\\ 0, & x=0\end{cases}$

база: $n=1$

$x\neq 0$ $\quad f'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}=P_{m(1)}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}$

$x=0$ $\quad f'(0)=\lim\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{1}{h}e^{-\frac{1}{h^2}}=\left[t=\frac{1}{h}\right]=$

$$=\lim_{t\to\infty}\frac{t}{e^{t^2}}\overset{\text{(лопиталь)}}{=}\lim_{t\to\infty}\frac{1}{2te^{t^2}}=0$$

непрерывность $f'$:

$$\lim f'(x)=f'(0)=0=\lim_{x\to 0}\frac{2}{x^3}\frac{1}{e^{1/x^2}}=\lim_{t\to\infty}2t^3\frac{1}{e^{t^2}}=0$$

инд. предп: $]\ \exists f^{(k)}(x)$ на $\mathbb{R}$

переход. Д-м, что $\exists f^{(k+1)}(x)$ на $\mathbb{R}$

$x\neq 0$: $\quad f^{(k+1)}(x)=\left(\frac{dP_{m(k)}}{d\left(\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)+P_{m(k)}\left(\frac{1}{x}\right)\left(\frac{2}{x^3}\right)\right)e^{-\frac{1}{x^2}}=P_{m(k+1)}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}$

$x=0$: $\quad \lim\frac{1}{h}\left(P_{m(k)}\left(\frac{1}{h}\right)e^{-\frac{1}{h^2}}\right)=\lim\left(Q_{(m(k)+1)}\left(\frac{1}{h}\right)e^{-\frac{1}{h^2}}\right)=\lim_{t\to\infty}\frac{Q_{m(k)+1}(t)}{e^{t^2}}=0$

непр-ть: $\lim f^{(k+1)}(x)=f^{(k+1)}(0)=0=$

$$=\lim P_{m(k+1)}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}=\left[t=\frac{1}{x}\right]=\lim_{t\to\infty}\frac{P_{m(k+1)}(t)}{e^{t^2}}=0$$

$f^{(k)}(0)=0$ $\qquad$ q.e.d.

## VII. Вычисление пределов с помощью ф-лы Тейлора

### 19.7(2)

$$\lim_{x\to0}\frac{\sqrt{1+2\operatorname{tg}x}-e^x+x^2}{\arcsin x-\sin x}=$$

$$=\lim_{x\to0}\frac{\sqrt{1+2x+\frac{2}{3}x^3+\frac{4}{15}x^5+o(x^6)}-1-x-\frac{x^2}{2}-\frac{x^3}{6}+x^2+o(x^3)}{\cancel{x}+\underline{\underline{\frac{x^3}{6}}}+\frac{3x^5}{40}-\cancel{x}+\underline{\underline{\frac{x^3}{6}}}-\frac{x^5}{120}+o(x^6)}=$$

$$=\lim_{x\to0}\frac{\sqrt{1+2x+o(x^2)}-1-x+\frac{1}{2}x^2}{\frac{x^3}{3}+o(x^3)}=\lim_{x\to0}\frac{\cancel{1}+\cancel{\frac{2x}{2}}-\cancel{\frac{4x^2}{8}}+\frac{5}{6}x^3-\cancel{1}-\cancel{x}+\cancel{\frac{1}{2}x^2}-\frac{1}{6}x^3+o(x^3)}{\frac{x^3}{3}+o(x^3)}=$$

$$=\lim_{x\to0}\frac{\frac{2}{3}x^2+o(x^3)}{\frac{x^3}{3}+o(x^3)}=\underline{\underline{2}}$$

### 19.8(6)

$$\lim_{x\to0}\frac{\sqrt{2-e^{2x}}-\cos 2x+\ln(1+x)}{\sin x-\arcsin\operatorname{tg}x}=$$

$$=\lim_{x\to0}\frac{\sqrt{\cancel{2}^{\,1}-\cancel{1}-2x-\frac{\cancel{4}^{\,2}x^2}{\cancel{2}}-\frac{8x^3}{6}+o(x^3)}-1+\frac{\cancel{4}^{\,2}x^2}{\cancel{2}}-\frac{16x^4}{24}+\frac{256x^6}{720}+o(x^7)+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+o(x^4)}{x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^6)-\operatorname{tg}x-\frac{\operatorname{tg}^3x}{6}-\frac{3\operatorname{tg}^5x}{40}+o(x^6)}$$

$$=\lim_{x\to0}\frac{\sqrt{1-2x-2x^2-\frac{4}{3}x^3+o(x^3)}-1-2x^2+x+\frac{x^3}{3}+o(x^3)}{\cancel{x}-\frac{x^3}{6}+\frac{x^5}{120}-\cancel{x}-\frac{2x^3}{6}-\frac{2x^5}{15}-\frac{1}{6}\left(x+o(x^2)\right)^3+o(x^3)}=$$

$$=\lim_{x\to0}\frac{1-\left(x+x^2+\frac{2x^3}{3}\right)-\frac{1}{8}\left(-2x-2x^2+\frac{4x^3}{3}+o(x^3)\right)^2+\frac{1}{16}(-2x)^3+o(x^3)+\underbrace{\ldots}^{\cos,\ln}}{-\frac{3}{6}x^3-\frac{1}{6}x^3+o(x^3)}=$$

$$=\lim_{x\to0}\frac{\cancel{1}-\cancel{x}-\cancel{x^2}-\frac{2}{3}x^3-\cancel{\frac{1}{2}x^2}-2x^3-\frac{1}{2}x^3-\cancel{1}+\cancel{x}+\cancel{2x^2}-\cancel{\frac{x^2}{2}}+\frac{x^3}{3}+o(x)^?}{-\frac{2}{3}x^3+o(x^3)}=$$

$$=\lim\frac{-\frac{11}{6}x^3+o(x^3)}{-\frac{2}{3}x^3+o(x^3)}=\underline{\underline{\frac{11}{4}}}$$

19. 14(5)

$$\lim_{x\to 0} \frac{e^{\frac{x}{1-x}} - sh\,x - \cos x}{\sqrt[6]{1+x} + \sqrt[6]{1-x} - 2} \ \textcircled{=}$$

знам: $(1+x)^{\frac{1}{6}} + (1-x)^{\frac{1}{6}} - 2 =$

$= \cancel{1} + \cancel{\frac{1}{6}x} + \frac{\frac{1}{6}(\frac{1}{6}-1)}{2}x^2 + o(x^2) + \cancel{1} - \cancel{\frac{1}{6}x} + \frac{\frac{1}{6}(\frac{1}{6}-1)}{2}x^2 + o(x^2) \cancel{-2} =$

$= \frac{1}{6}(\frac{1}{6}-1)x^2 + o(x^2) = -\frac{1}{6}\cdot\frac{5}{6}x^2 + o(x^2) = -\frac{5}{36}x^2 + o(x^2)$

числ: $e^{\frac{x}{1-x}} - sh\,x - \cos x =$

$= \cancel{1} + \underbrace{\frac{x}{1-x} + \frac{1}{2}\left(\frac{x}{1-x}\right)^2}_{} + \frac{1}{6}\left(\frac{x}{1-x}\right)^3 + \frac{1}{24}\left(\frac{x}{1-x}\right)^4 \cancel{-1} \overset{-x}{\checkmark} - \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + o(x^4) \textcircled{=}$

$= \cancel{x} + x^2 + \frac{1}{2}(x + x^2 + o(x^2))^2 + \frac{1}{2}x^2 \overset{\cancel{-x}}{\checkmark} + o(x^2) = \frac{3}{2}x^3 + \frac{1}{2}x^2 + o(x^2) = 2x^2 + o(x^2)$

$$\textcircled{=} \frac{2x^2 + o(x^2)}{-\frac{5}{36}x^2 + o(x^2)} = \underline{\underline{-\frac{72}{5}}}$$

19. 21(4)

$$\lim_{x\to 0} \underbrace{\left(\frac{tg\,x}{arctg\,x}\right)}_{f(x)}^{\overset{g(x)}{\frac{1}{x^2}}} = e^{\lim_{x\to 0}(\ln f(x))\,g(x)}$$

$f$: $f = \frac{x + \frac{x^3}{3} + o(x^4)}{x - \frac{x^3}{3} + o(x^4)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3)$

$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3))\left(x - \frac{x^3}{3} + o(x^4)\right) = x + \frac{x^3}{3} + o(x^4)$

$a_0 x + a_1 x^2 + \left(-\frac{a_0}{3} + a_2\right)x^3 + o(x^3) = x + \frac{x^3}{3} + o(x^4)$

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 - \frac{a_0}{3} = \frac{1}{3} \end{cases} \qquad \begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = \frac{2}{3} \end{cases}$$

$\frac{tg\,x}{arctg\,x} = 1 + \frac{2}{3}x^2 + o(x^2)$

$$\ln f(x) = \ln(1 + \tfrac{2}{3}x^2 + o(x^2)) = \tfrac{2}{3}x^2 + o(x^2)$$

$$\lim_{x \to 0} (\ln(f(x))g(x)) = \lim_{x \to 0} \frac{\frac{2}{3}x^2 + o(x^2)}{x^2} = \frac{2}{3}$$

Ответ: $e^{\frac{2}{3}}$

15.30(4)

$$\lim_{x \to 0} \underbrace{\left( \frac{2-x}{2+x} + \sin \ln (1+x) \right)}_{f}^{\frac{1}{x^3} = g} = e^{\lim (g(x) \ln (f(x)))}$$

$$f: \ \frac{2-x}{2+x} = \frac{1 - \frac{x}{2}}{1 + \frac{x}{2}} = \left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + o(x^3)\right) =$$

$$= 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + o(x^3) = 1 - x + \frac{x^2}{2} - \frac{x^3}{4} + o(x^3)$$

$$] \ln(1+x) = w$$

$$\sin w = w - \frac{w^3}{6} + o(w^4)$$

$$w = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\sin \ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{6} + o(x^3) = x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$f(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{4} + x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) = 1 - \frac{x^3}{12} + o(x^3)$$

$$\ln f(x) = \ln\left(1 - \frac{x^3}{12} + o(x^3)\right) = -\frac{x^3}{12} + o(x^3)$$

$$\lim (\ln f(x)) g(x) = \frac{-\frac{x^3}{12} + o(x^3)}{x^3} = -\frac{1}{12}$$

Ответ: $\underline{\underline{e^{-\frac{1}{12}}}}$

VIII. ИССЛЕДОВАНИЕ Ф-Й

20.20(2)

$y = f(x) = \frac{\ln^2 x}{x}$;

$y' = (\ln^2 x \cdot x^{-1})' = 2\ln x \cdot \frac{1}{x} \cdot \frac{1}{x} + \ln^2 x \cdot (-1)\frac{1}{x^2} = \frac{2\ln x}{x^2} - \frac{\ln^2 x}{x^2} = \frac{\ln x}{x^2}(2 - \ln x)$

крит. точки:

$\left[\begin{array}{ll} \ln x = 0 & x = 1 \\ \ln x = 2 & x = e^2 \\ x = 0 & \end{array}\right.$

$y'' = \left(\frac{\ln x}{x^2}\right)'(2 - \ln x) + \frac{\ln x}{x^2}\left(-\frac{1}{x}\right) = \left(\frac{1}{x^3} + \ln x(-2)\frac{1}{x^3}\right)(2 - \ln x) - \frac{\ln x}{x^3} =$

$= \frac{(2 - \ln x)}{x^3}(1 - 2\ln x) - \frac{\ln x}{x^3}$

$y''(1) = \frac{2}{1}(1 - 0) - 0 = 2 > 0$ — local min

$y''(e^2) = \frac{(2-2)}{e^6}(1 - 4) - \frac{2}{e^6} = -\frac{2}{e^6} < 0$ — local max

Ответ: $x = 1$ — лок. мин, $x = e^2$ — лок. макс.

20.23(1)

$y = |x-5|(x-3)^3 = \operatorname{sgn}(x-5)(x-3)^3(x-5)$

$y' = \operatorname{sgn}(x-5)\left(3(x-3)^2(x-5) + (x-3)^3\right) = \operatorname{sgn}(x-5)(x-3)^2(\overset{4x}{\cancel{3x}} \overset{-18}{\cancel{-15}} + \cancel{x} \cancel{-3})$

$= 4\operatorname{sgn}(x-5)(x-3)^2\left(x - \frac{9}{2}\right)$

$y' = 0: \left[\begin{array}{l} x = 5 \\ x = 3 \\ x = \frac{9}{2} \end{array}\right.$

$y'' = 4\operatorname{sgn}(x-5)\left(2(x-3)\left(x - \frac{9}{2}\right) + (x-3)^2\right) = 4\operatorname{sgn}(x-5)(x-3)(\overset{3x}{\cancel{2x}} \overset{-12}{\cancel{-9}} + \cancel{x} \cancel{-3}) =$

$= 12\operatorname{sgn}(x-5)(x-3)(x-4)$

$y''(5) = 0$

$y''(3) = 0$

$y''\left(\frac{9}{2}\right) = -12 \cdot \frac{3}{2} \cdot \frac{1}{2} < 0$ — local max

$x=5$ local min

$+\infty$ — max

$-\infty$ — min

Ответ: $x=5$ — лок. мин, $x=\frac{9}{2}$ — лок. макс., $-\infty$ — мин, $+\infty$ — макс.

20.39(5)

$y=(x-3)e^{|x+1|}, \quad x\in[-2;4]$

$y'=\left((x-3)e^{\operatorname{sgn}(x+1)(x+1)}\right)'= e^{|x+1|}+(x-3)\operatorname{sgn}(x+1)\,e^{|x+1|}=$

$=e^{|x+1|}\left(1+\operatorname{sgn}(x+1)\cdot(x-3)\right)$

$y'=0; \quad \operatorname{sgn}(x+1)(x-3)+1=0; \quad \begin{cases} x-3+1=0 & \text{если } x+1\geq 0 \\ -x+3+1=0 & \text{если } x+1\leq 0\end{cases}$

$\begin{cases} x=2 & \text{если } x\geq -1 & \text{— подх.} \\ x=4 & \text{если } x\leq -1 & \text{— не подх.}\end{cases}$

$y''=\operatorname{sgn}(x+1)e^{|x+1|}+e^{|x+1|}\operatorname{sgn}(x+1)=2\operatorname{sgn}(x+1)e^{|x+1|}$

$y''(2)=2e^3>0$ — local min

$y(2)=-e^3$

$y(-2)=(-5)e=-5e$

$y(4)=1\cdot e^5=e^5$

$\max\limits_{[-2;4]} y=\max\left(y(-2),y(4)\right)=e^5 \qquad x=4$

$\min\limits_{[-2;4]} y=\min\left(y(-2),y(4);y(2)\right)=-e^3 \qquad x=2$

T.6

$e^{\frac{x+y}{2}} \overset{?}{\leq} \frac{e^x+e^y}{2}$

$\triangleleft\ f(x)=e^x \quad f''(x)=e^x$

$f''(x)>0 \ \forall x\in\mathbb{R} \Rightarrow f(x)$ строго выпукла вниз на $\mathbb{R}$, т.е.

$\forall x,y\in\mathbb{R}\ \forall\alpha_1,\alpha_2\geq 0: \alpha_1+\alpha_2=1 \Rightarrow f(\alpha_1x_1+\alpha_2x_2)\leq \alpha_1 f(x_1)+\alpha_2 f(x_2)$

В частности, для $\alpha_1=\alpha_2=\frac{1}{2}$:

$e^{\frac{x+y}{2}}\leq \frac{1}{2}e^x+\frac{1}{2}e^y \qquad$ q.e.d.

<u>20.3s</u>

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases} \qquad g(x)=\begin{cases} xe^{-\frac{1}{x^2}} & x\neq 0 \\ 0 & x=0 \end{cases}$$

(1)

IX. ПОСТРОЕНИЕ ГРАФИКОВ Ф-Й

21.5(2)

$$y = \frac{(x-1)^3}{(x-2)^2}$$

$D_f = \mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2; +\infty)$

корни: $x = 1$

асимптоты:

вертикальные: $\lim\limits_{x \to x_0} y = \infty$ $\qquad x = 2$

наклонная:

$$\lim_{x \to +\infty} \frac{y}{x} = \lim_{x \to +\infty} \frac{(x-1)^3}{x(x-2)^2} = \lim_{x \to +\infty} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - 4x^2 + 4x} = \lim_{x \to +\infty} \frac{1 + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 1$$

$$\lim_{x \to -\infty} \frac{y}{x} = 1$$

$\hookrightarrow k = 1$

$$\lim_{x \to +\infty} (y - kx) = \lim_{x \to +\infty} \left( \frac{(x-1)^3}{(x-2)^2} - x \right) = \lim_{x \to +\infty} \frac{\cancel{x^3} - \cancel{3x^2} + \cancel{3x} - 1 - \cancel{x^3} + \cancel{4x^2} - \cancel{4x}}{x^2 - 4x + 4} =$$

$+x^2$ $\quad -x$

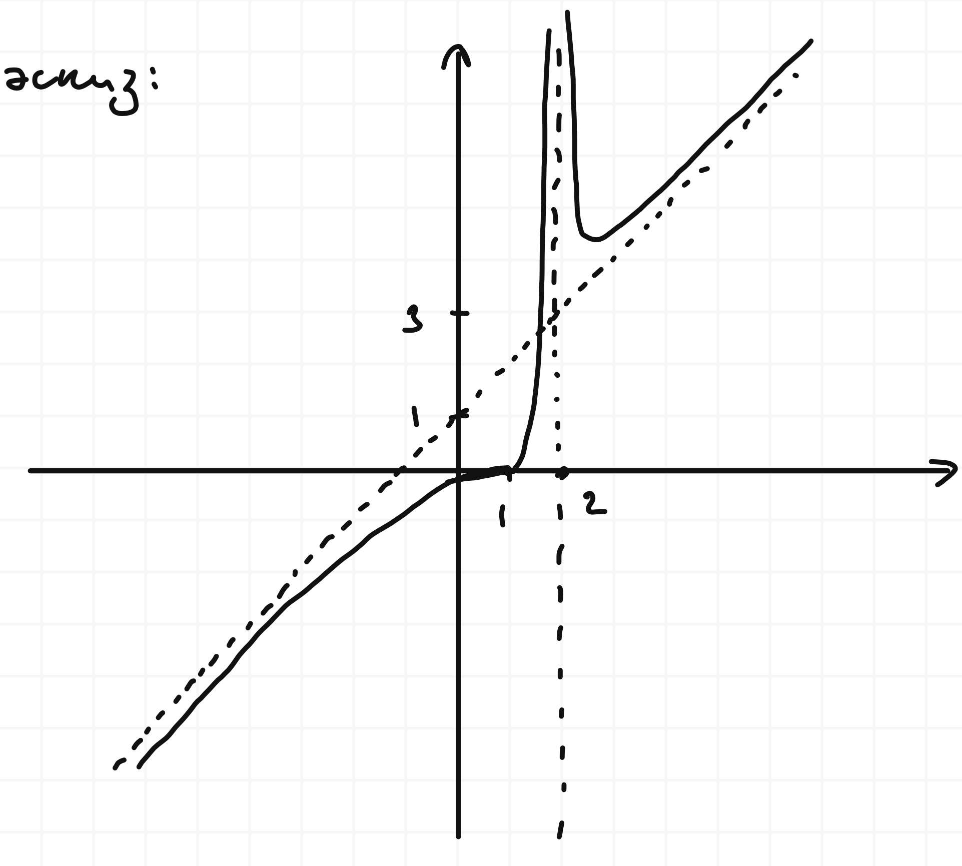
$$= \lim_{x \to +\infty} \frac{x^2 - x - 1}{x^2 - 4x + 4} = \lim_{x \to +\infty} \frac{1 - \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 1$$

$$\lim_{x \to -\infty} (y - kx) = 1$$

$\hookrightarrow b = 1$

$$y = x + 1$$

эскиз:

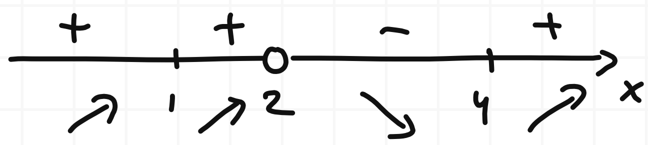


$$y = \frac{(x-1)^3}{(x-2)^2} = (x-1)^3 (x-2)^{-2}$$

$$y' = 3(x-1)^2(x-2)^{-2} + (x-1)^3(-2)(x-2)^{-3} = \frac{3(x-1)^2}{(x-2)^2} - \frac{2(x-1)^3}{(x-2)^3}$$

$$y' = 0; \quad \frac{(x-1)^2}{(x-2)^2}\left(3 - \frac{2(x-1)}{x-2}\right) = \frac{(x-1)^2}{(x-2)^3}\left(\overset{+x\ -4}{\cancel{3x}\cancel{-6} - \cancel{2x}\cancel{+2}}\right) = \frac{(x-1)^2(x-4)}{(x-2)^3}$$

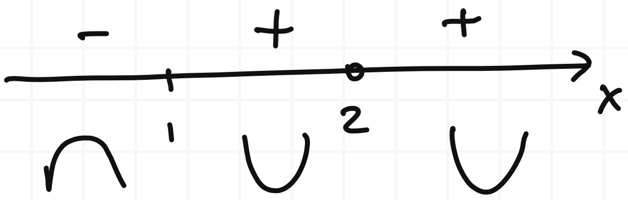
$$\left[\begin{array}{l} x=1 \\ \overset{x}{\cancel{3x}}\cancel{-6} = \cancel{2x}\overset{4}{\cancel{-2}} \end{array}\right. \qquad \left[\begin{array}{l} x=1 \\ x=4 \end{array}\right.$$




$$y'' = 3\left(2(x-1)(x-2)^{-2} + (x-1)^2(-2)(x-2)^{-3}\right) - 2\left(3(x-1)^2(x-2)^{-3} + (x-1)^3(-3)(x-2)^{-4}\right)$$

$$= 3\left(\frac{2(x-1)}{(x-2)^2} - \frac{2(x-1)^2}{(x-2)^3}\right) - 2\left(\frac{3(x-1)^2}{(x-2)^3} - \frac{3(x-1)^3}{(x-2)^4}\right) = \frac{6(x-1)}{(x-2)^2}\left(1 - \frac{x-1}{x-2}\right) -$$

$$- 6\frac{(x-1)^2}{(x-2)^3}\left(1 - \frac{x-1}{x-2}\right) = \frac{6(x-1)}{(x-2)^2}\left(\frac{\overset{-1}{\cancel{x}\cancel{-2}\cancel{-x}\cancel{+1}}}{x-2}\right)\left(1 - \frac{x-1}{x-2}\right) = \frac{6(x-1)}{(x-2)^4}$$




$y(1) = 0$ — точка перегиба

$y(2)$ не определено

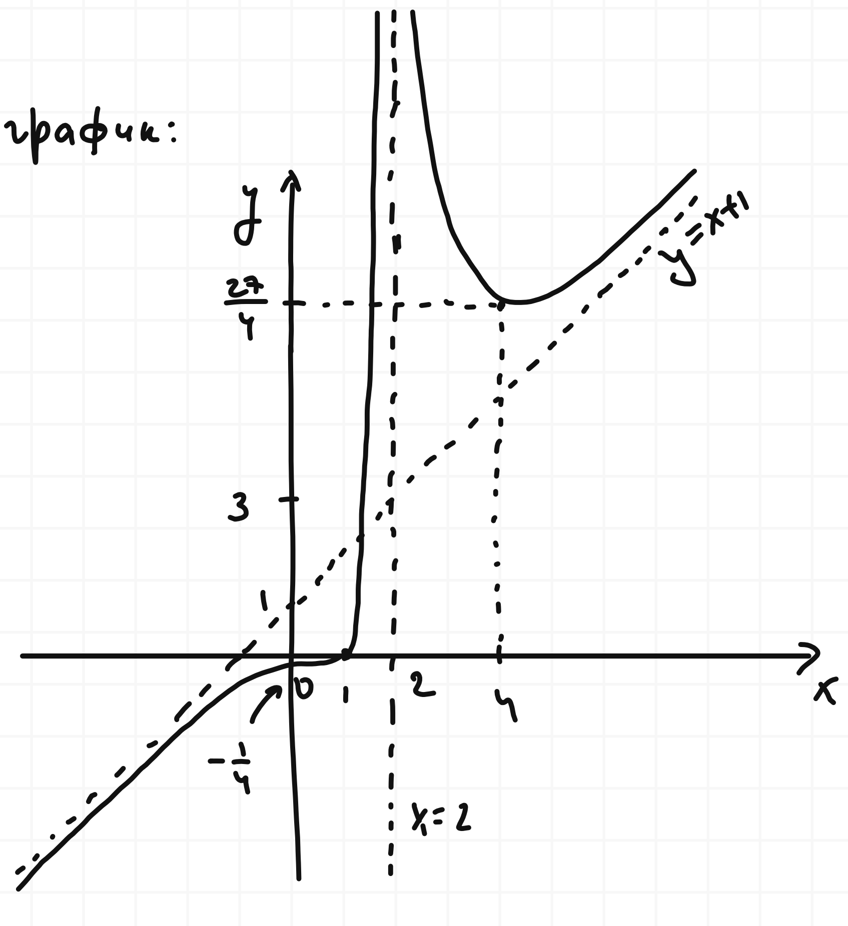
$y(4) = \frac{27}{4}$

$y'(1) = 0$

точка пересеч. с $y$:

$y(0) = \frac{-1}{4} = -\frac{1}{4}$

график:



21.5(3)

$$y=\frac{(x-5)^3}{(x-7)^2}$$

$D_f=(-\infty;7)\cup(7;+\infty)$

Точки пересеч. с осями:

корни: $x=7$ $\quad(7;0)$

с $y$: $y(0)=\frac{-125}{49}$ ; $\left(0;-\frac{125}{49}\right)$

асимптоты:

верт: $x=7$

накл:

$$\lim_{x\to+\infty}\frac{(x-5)^3}{x(x-7)^2}=\lim_{x\to+\infty}\frac{x^3-15x^2+75x+125}{x^3-14x^2+49x}=\lim_{x\to+\infty}\frac{1-\frac{15}{x}+\frac{75}{x^2}+\frac{125}{x^3}}{1-\frac{14}{x}+\frac{49}{x^2}}=1$$

$$\lim_{x\to-\infty}\frac{(x-5)^3}{x(x-7)^2}=1$$

$\hookrightarrow k=1$

$$\lim_{x\to+\infty}(y-kx)=\lim_{x\to+\infty}\left(\frac{(x-5)^3}{(x-7)^2}-x\right)=\lim_{x\to+\infty}\frac{\cancel{x^3}\overbrace{-\cancel{15x^2}}^{-x^2}+75x+125-\cancel{x^3}+\cancel{14x^2}-49x}{(x-7)^2}=$$

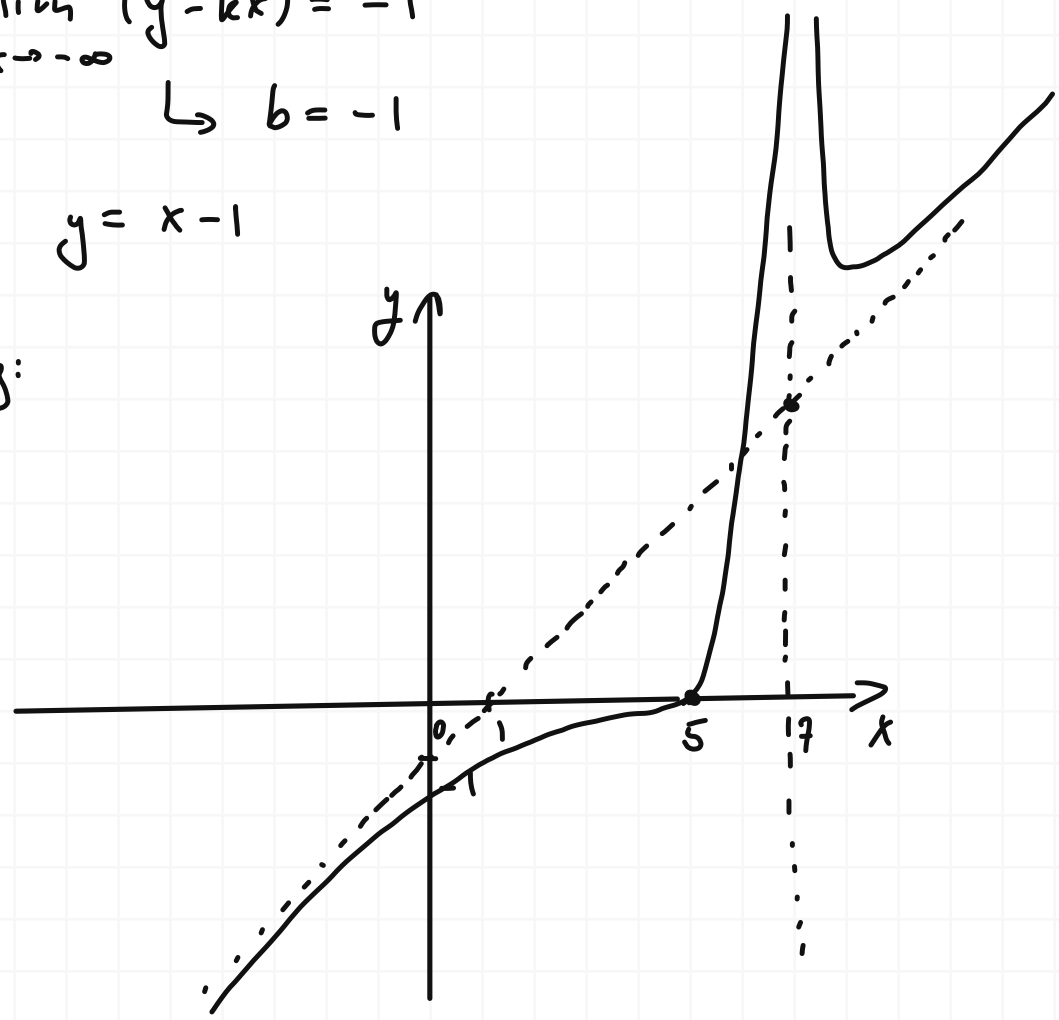
$$=\lim_{x\to+\infty}\frac{-x^2+\dots}{x^2+\dots}=-1$$

$$\lim_{x\to-\infty}(y-kx)=-1$$

$\hookrightarrow b=-1$

$y=x-1$

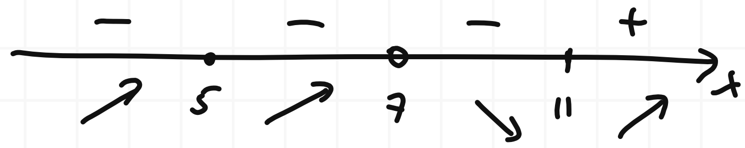
эскиз:



$$y = \frac{(x-5)^3}{(x-7)^2} = (x-5)^3 (x-7)^{-2}$$

$$y' = \frac{3(x-5)^2}{(x-7)^2} + \frac{(x-5)^3(-2)}{(x-7)^3} = \frac{(x-5)^2}{(x-7)^2}\left(3 - 2\,\frac{x-5}{x-7}\right) = \frac{(x-5)^2}{(x-7)^2}\left(\frac{\overset{x-11}{\cancel{3x-21-2x+10}}}{x-7}\right) =$$

$$= \frac{(x-5)^2(x-11)}{(x-7)^3}$$




$y(5) = 0$

$y(7)$ not defined

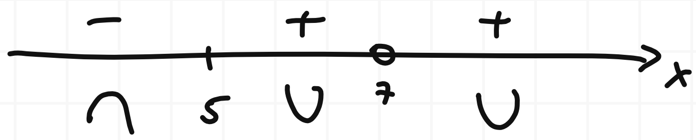
$$y(11) = \frac{6^3}{4^2} = \frac{\cancel{6}^3\cdot\cancel{6}^3\cdot\cancel{6}^3}{\cancel{2}\cancel{4}\cdot\cancel{4}_2} = \frac{27}{2}$$

$$y'' = \frac{\left((x-5)^2(x-11)\right)'}{(x-7)^3} + \frac{(x-5)^2(x-11)(-3)}{(x-7)^4} = \frac{\left(2(x-5)(x-11) + (x-5)^2\right)}{(x-7)^3} -$$

$$\frac{3(x-5)^2(x-11)}{(x-7)^4} = \frac{(x-5)}{(x-7)^3}\left(\left(\overset{3x-27}{\cancel{2x}-22+\cancel{x}-5}\right) - \frac{3(x-5)(x-11)}{(x-7)}\right) =$$

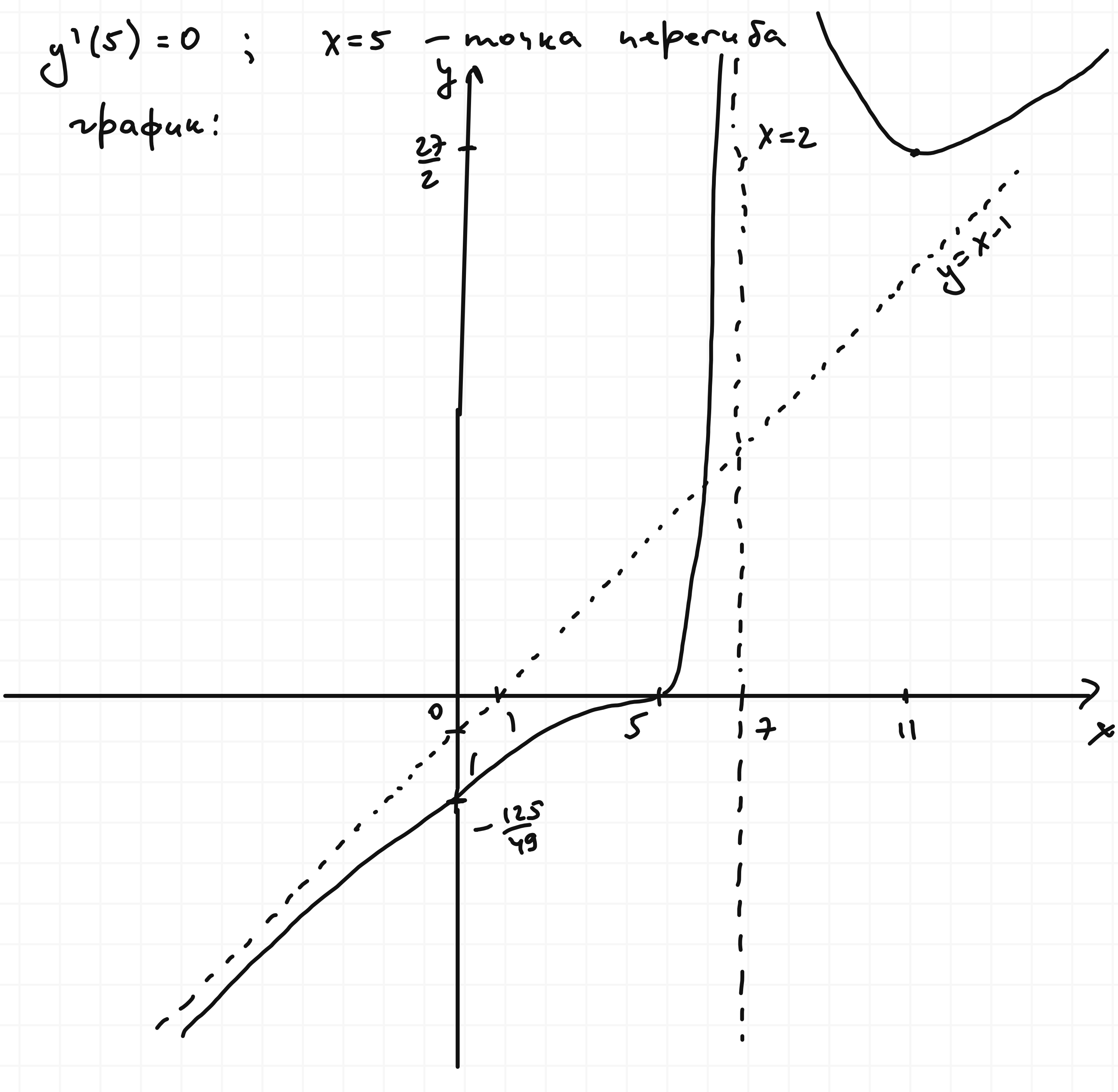
$$\frac{x-5}{(x-7)^4}\left((3x-27)(x-7) - 3(x^2-16x+55)\right) = \frac{x-5}{(x-7)^4}\left(\cancel{3x^2} \cancel{-27x} \cancel{-21x} + \underset{+24}{\cancel{189}} - \cancel{3x^2} + \cancel{48x} \cancel{-165}\right) =$$

$$= \frac{24(x-5)}{(x-7)^4}$$




$y'(5) = 0$ ; $x = 5$ − точка перегиба

график:



21.9(3)

$$y=\frac{8x}{\sqrt{x^2-4}}$$

$D_f$:

$x^2-4>0$ ; $(x-2)(x+2)>0$ $-2$ $2$ $x$

$D_f=(-\infty;-2)\cup(2;+\infty)$

Точки пересеч. с осями:

корень: $x=0$ $(0;0)$

$y$: $y(0)=0$ $(0;0)$ $\Big\}\ \notin D_f$

асимптоты:

верт: $x=\pm 2$

накл:

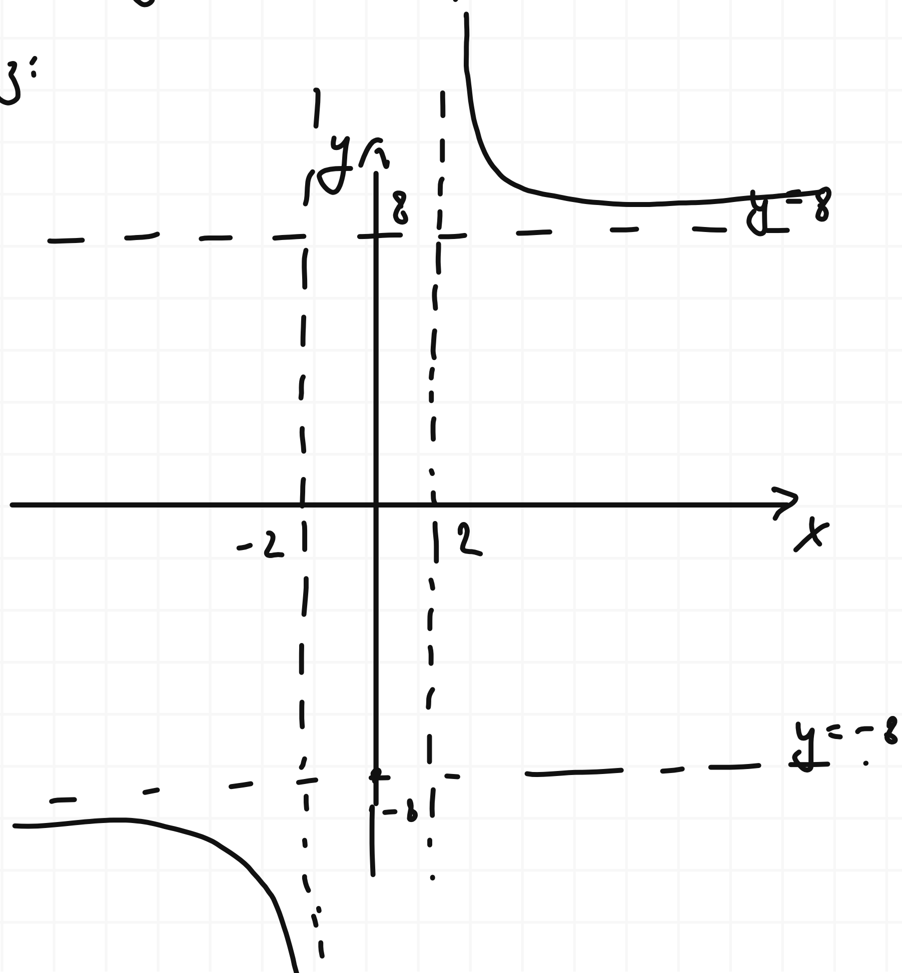
$$\lim_{x\to+\infty}\frac{8x}{x\sqrt{x^2-4}}=\lim_{x\to+\infty}\frac{8}{\sqrt{x^2-4}}=0\ ;\ \lim_{x\to-\infty}\frac{8x}{x\sqrt{x^2-4}}=0\ \to k=0$$

$$\lim_{x\to+\infty}\left(\frac{8x}{\sqrt{x^2-4}}-0\cdot x\right)=\lim_{x\to+\infty}\frac{8x}{x\sqrt{1-\frac{4}{x^2}}}=\lim_{x\to+\infty}\frac{8}{\sqrt{1-\underbrace{\frac{4}{x^2}}_{\to 0}}}=8$$

$$\lim_{x\to-\infty}\frac{8x}{\sqrt{x^2-4}}=\lim_{x\to-\infty}\frac{8x}{|x|\sqrt{1-\frac{4}{x^2}}}=\lim_{x\to-\infty}\frac{-8}{\sqrt{1-\frac{4}{x^2}}}=-8$$

$\hookrightarrow\ y=\pm 8\ \begin{pmatrix}+\infty\\-\infty\end{pmatrix}$

эскиз:




$$\lim_{x\to 2}\frac{8x}{\sqrt{x^2-4}}=+\infty$$

$$\lim_{x\to -2}\frac{8x}{\sqrt{x^2-4}}=-\infty$$

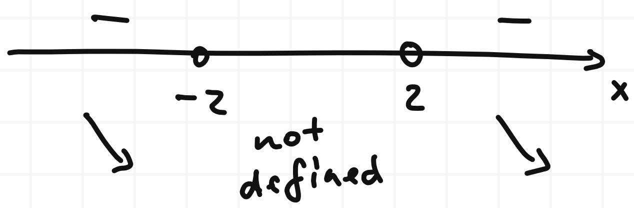
нечетная ф-я

$$y = \frac{8x}{\sqrt{x^2-4}} = 8x(x^2-4)^{-\frac{1}{2}}$$

$$y' = \frac{8}{\sqrt{x^2-4}} + \frac{8x(-\cancel{\frac{1}{2}})\cdot\cancel{2}x}{\sqrt{x^2-4}\,(x^2-4)} = \frac{8}{\sqrt{x^2-4}}\left(1-\frac{x^2}{x^2-4}\right) = \frac{8}{\sqrt{x^2-4}}\left(\frac{x^2-4-x^2}{x^2-4}\right) =$$

$$= -32(x^2-4)^{-\frac{3}{2}}$$

$y' = 0$; $x = \pm 2$




$$y'' = +32\left(+\frac{3}{\cancel{2}}(x^2-4)^{-\frac{5}{2}}\cdot\cancel{2}x\right) = 96x(x^2-4)^{-\frac{5}{2}}$$ ; $y''=0$ при $x=\pm 2$

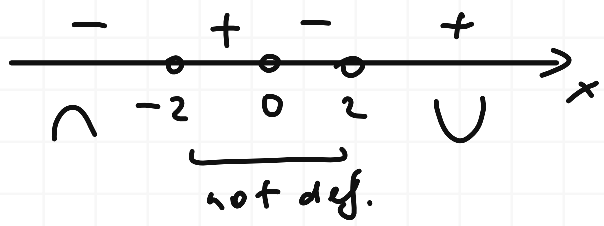
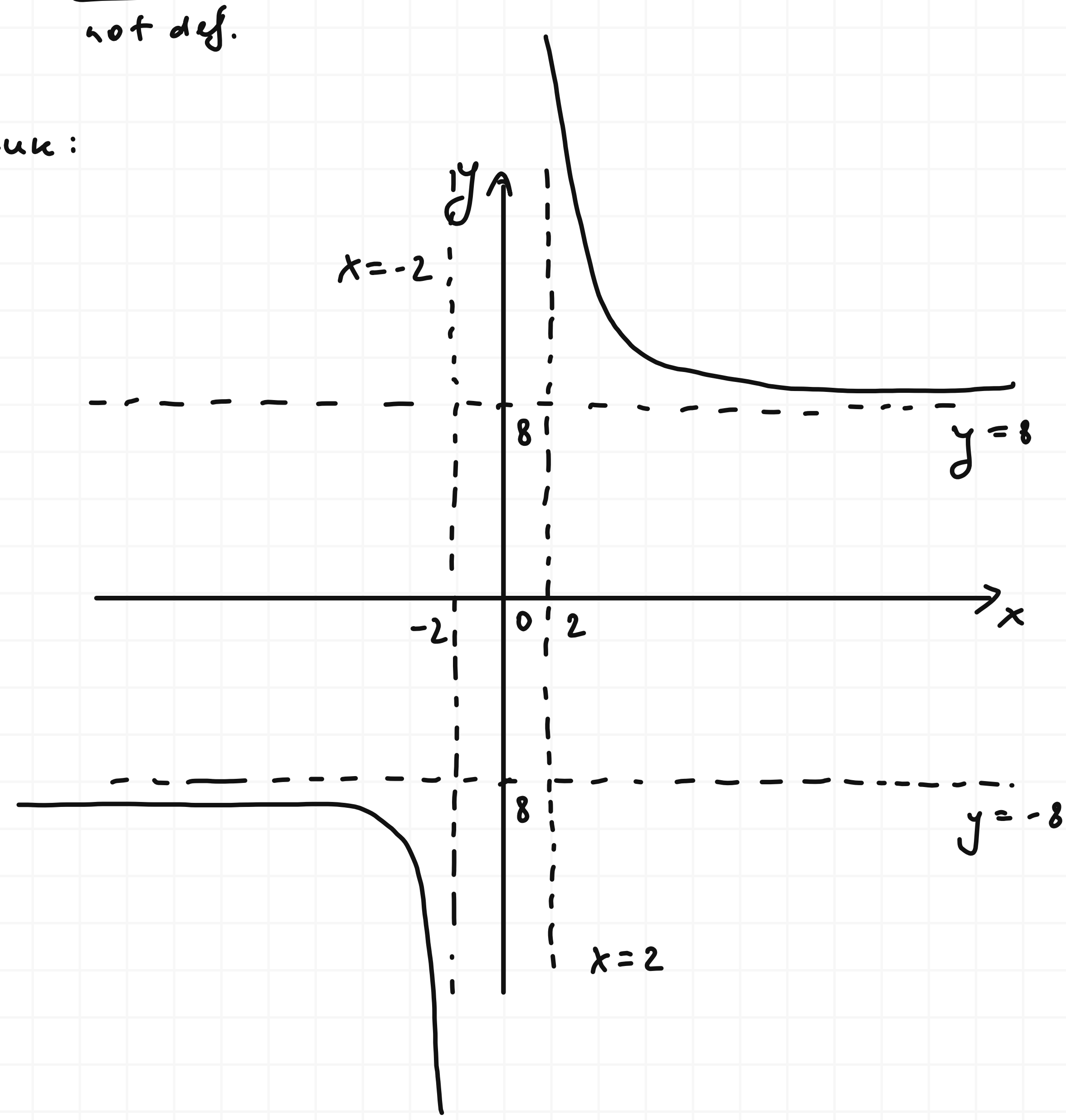



график:



21.10(г)

$y = \sqrt[3]{x^2(3-x)}$

область опр-я: $D_f = \mathbb{R}$

точки пересеч. с осями:

корни: $x = 0$ $(0;0)$
$x = 3$ $(3;0)$

с $y$: $y(0) = 0$ $(0,0)$

асимптоты:

верт: нет

накл:

$$\lim_{x\to+\infty} \frac{\sqrt[3]{x^2(3-x)}}{x} = \lim_{x\to+\infty} \frac{x\sqrt[3]{\frac{3}{x}-1}}{x} = \lim_{x\to+\infty} \sqrt[3]{\underbrace{\frac{3}{x}}_{\to 0}-1} = -1$$

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x^2(3-x)}}{x} = -1$$

$\to k = -1$

$$\lim_{x\to+\infty} (y-kx) = \lim_{x\to+\infty}\left(\sqrt[3]{x^2(3-x)} + x\right) = \lim_{x\to+\infty} \underbrace{x}_{\to+\infty}\underbrace{\left(1+\sqrt[3]{\frac{3}{x}-1}\right)}_{\to 0} =$$

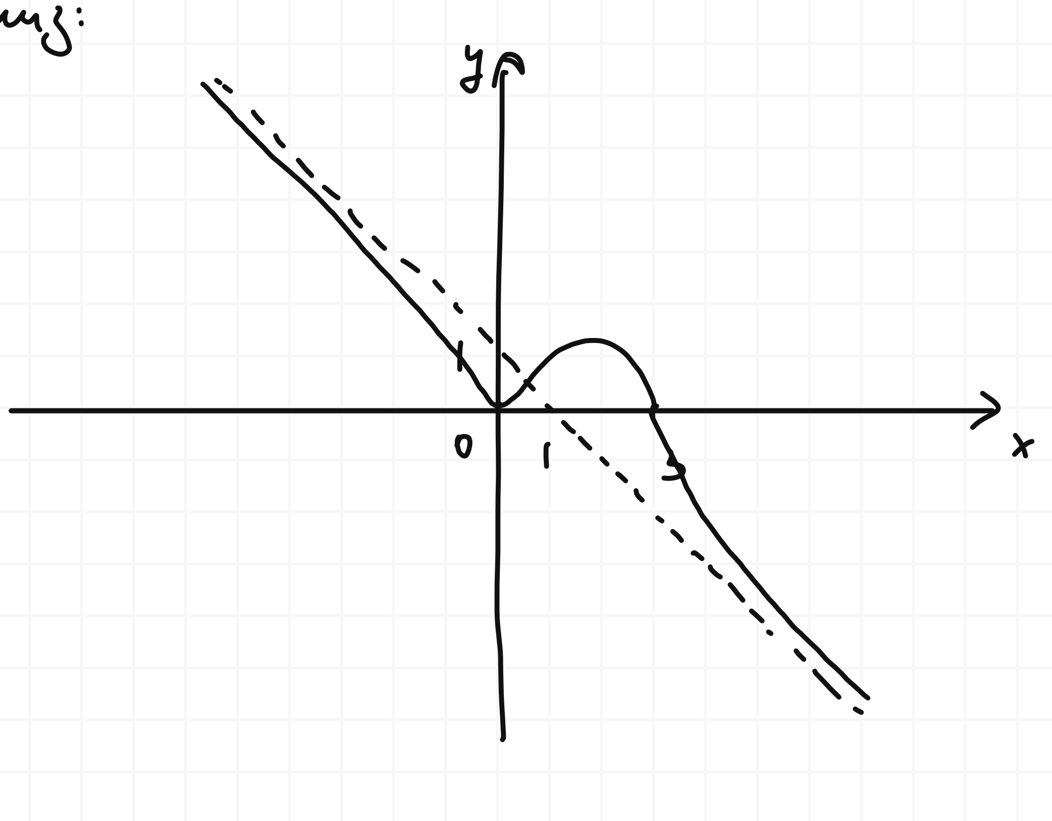
$$= \lim_{x\to+\infty} \frac{1+\sqrt[3]{\frac{3}{x}-1}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\cancel{\frac{1}{3}}\frac{1}{\sqrt[3]{\frac{3}{x}-1}^2}\cdot \cancel{\frac{3}{x^2}}}{\cancel{\frac{1}{x^2}}} = \lim_{x\to+\infty} \frac{1}{\sqrt[3]{\underbrace{\frac{3}{x}}_{\to 0}-1}^2} = 1$$

$$\lim_{x\to-\infty} (y-kx) = -1$$

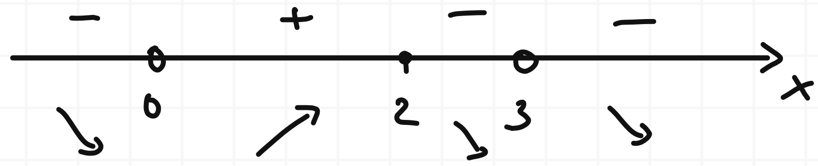
$\to b = 1$

$\hookrightarrow y = -x+1$

эскиз:



$y=\sqrt[3]{x^2(3-x)} = \sqrt[3]{-x^3+3x^2}$




$y' = \frac{1}{\cancel{3}}\,\frac{-\cancel{3}x^2+\cancel{6}^{\,2}x}{\sqrt[3]{(-x^3+3x^2)^2}} = \frac{x(2-x)}{\sqrt[3]{x^2(3-x)}^{\,2}}$

$y(0)=0$ $\quad y(2)=\sqrt[3]{4\cdot 1}=\sqrt[3]{4}$

$y(3)=0$

$y'=0$ при $x=2$

// я буду использовать дробные степени для удобства записи, подразумевая, что ОДЗ сохраняется //

$y' = x(2-x)\left(x^2(3-x)\right)^{-\frac{2}{3}}$

$y'' = (x(2-x))'\left(x^2(3-x)\right)^{-\frac{2}{3}} + x(2-x)\left(-\frac{2}{3}\right)\left(x^2(3-x)\right)^{-\frac{5}{3}}\left(2x(3-x)+x^2(-1)\right) =$

$= (2-x+x(-1))\left(x^2(3-x)\right)^{-\frac{2}{3}} - \frac{2}{3}x^2(2-x)\left(x^2(3-x)\right)^{-\frac{5}{3}}\underbrace{(6-2x-x)}_{=3(2-x)} =$

$= 2(1-x)\left(x^2(3-x)\right)^{-\frac{2}{3}} - 2x^2(2-x)^2\left(x^2(3-x)\right)^{-\frac{5}{3}} =$

$= 2\frac{1-x}{(x^2(3-x))^{\frac{2}{3}}} - 2\frac{x^2(2-x)^2}{(x^2(3-x))^{-\frac{5}{3}}}$

$y''=0;\quad \frac{1-x}{\cancel{(x^2(3-x))^{\frac{2}{3}}}} = \frac{x^2(2-x)^2}{\cancel{(x^2(3-x))^{\frac{2}{3}}}(x^2(3-x))};\quad \left[\begin{array}{l}(1-x)(\cancel{x^2}(3-x)) = \cancel{x^2}(2-x)^2;\\ x=0\\ x=3\end{array}\right.$

$\left[\begin{array}{l}3-\cancel{x}-\cancel{3x}+\cancel{x^2}=\cancel{x^2}-\cancel{4x}+4 \quad \text{не подх}\\ x=0\\ x=3\end{array}\right.$ $\qquad \left[\begin{array}{l}x=0\\ x=3\end{array}\right.$

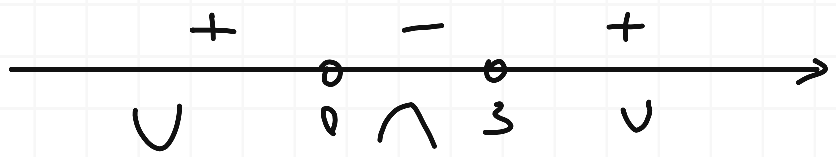
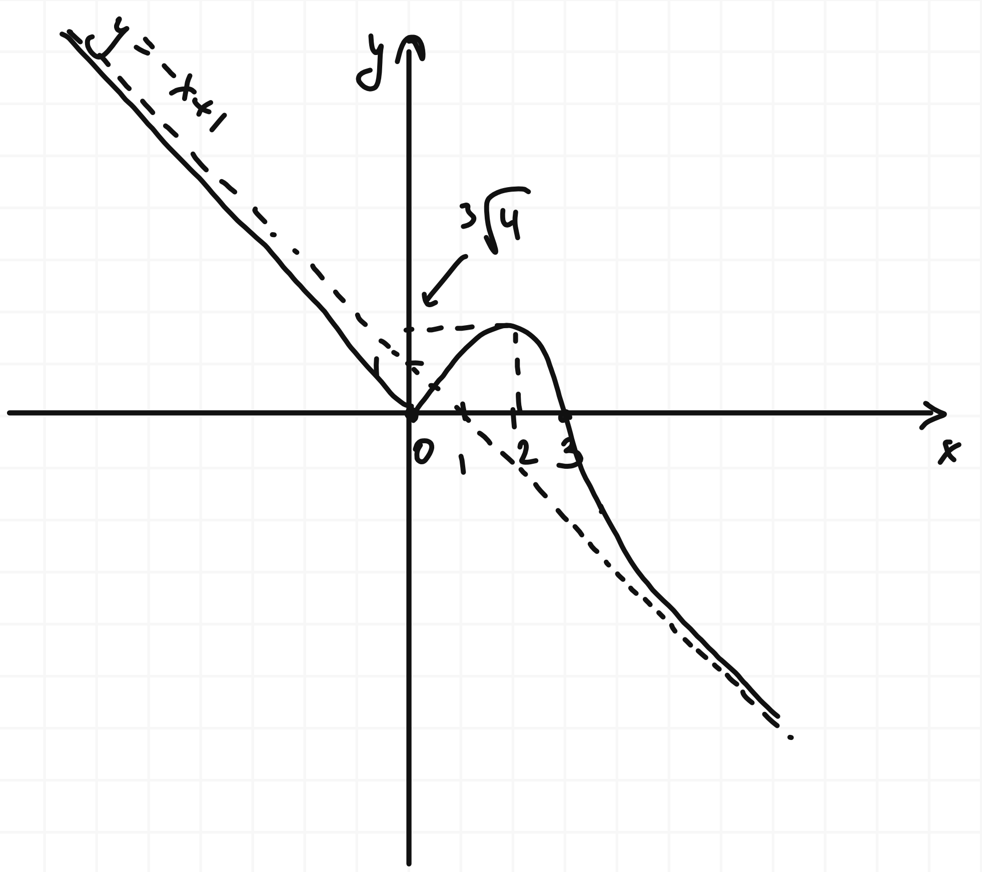



график:



## 21.12(1)

$y = |x|\sqrt{1-x^2}$ четная ф-я

обл. опр-я:

$1-x^2 \geq 0$ ; $x^2-1 \leq 0$ ; $(x-1)(x+1) \leq 0$

$D_f = [-1;1]$

точки пересеч. с осями: асимптоты: $x = \pm 1$

корни: $x \in \{0; \pm 1\}$

$y(0) = 0$

$$y = |x|\sqrt{1-x^2} = sgn(x)\, x\sqrt{1-x^2}$$

$$y' = sgn(x)\left(\sqrt{1-x^2} + x\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right) = sgn(x)\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$$

$$y' = 0; \quad \left[\begin{array}{l} sgn(x) = 0 \\ \dfrac{1-x^2-x^2}{\sqrt{1-x^2}} = 0 \end{array}\right. \quad \left[\begin{array}{l} x = 0 \\ x^2 - \frac{1}{2} = 0 \end{array}\right. \quad \left[\begin{array}{l} x = 0 \\ x = \pm\frac{1}{\sqrt{2}} \end{array}\right.$$

$y(0) = 0$

$y\left(\frac{1}{\sqrt{2}}\right) = y\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = \frac{1}{2}$

$$y'' = sgn(x)\left(\frac{\overset{(-1)}{\cancel{-2}x}}{\cancel{2}\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} - \frac{x^2(-\cancel{2}x)}{\cancel{(-2)}\sqrt{1-x^2}^3}\right) =$$

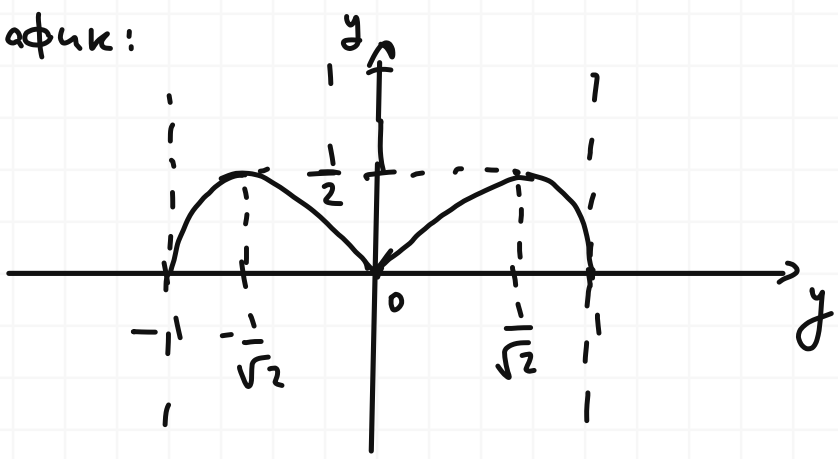
$$= sgn(x)\left(\frac{-3x}{\sqrt{1-x^2}} - \frac{x^3}{\sqrt{1-x^2}^3}\right) = \frac{sgn(x)\cdot x\cdot(-1)\left(3 + \frac{x^2}{1-x^2}\right)}{\sqrt{1-x^2}} =$$

$$= \frac{-sgn(x)\cdot x}{\sqrt{1-x^2}}\left(\frac{3 \overset{-2x^2}{-\cancel{3x^2}} \cancel{+x^2}}{1-x^2}\right) = \frac{2x\, sgn(x)\left(x^2 - \frac{3}{2}\right)}{(1-x^2)\sqrt{1-x^2}}$$

$$y'' = 0 \quad \left[\begin{array}{l} x = 0 \\ x = \pm\sqrt{\frac{3}{2}} \notin D_f \end{array}\right.$$

−1 ∩ 0 ∩ 1

График:



21.13(11)

$$y = \frac{e^{-x}}{1-x}$$

$D_f = (-\infty; 1) \cup (1; +\infty)$

Точки пересечения:

корней нет

$y(0) = \frac{1}{1} = 1 \qquad (0; 1)$

асимптоты:

верт: $x = 1$

накл:

$$\lim_{x\to+\infty} \frac{e^{-x}}{x(1-x)} = \lim_{x\to+\infty} \underset{\downarrow\, 0}{\frac{1}{x}} \cdot \underset{\downarrow\, 0}{\frac{1}{1-x}} \cdot \underset{\downarrow\, 0}{e^{-x}} = 0 \qquad k = 0$$

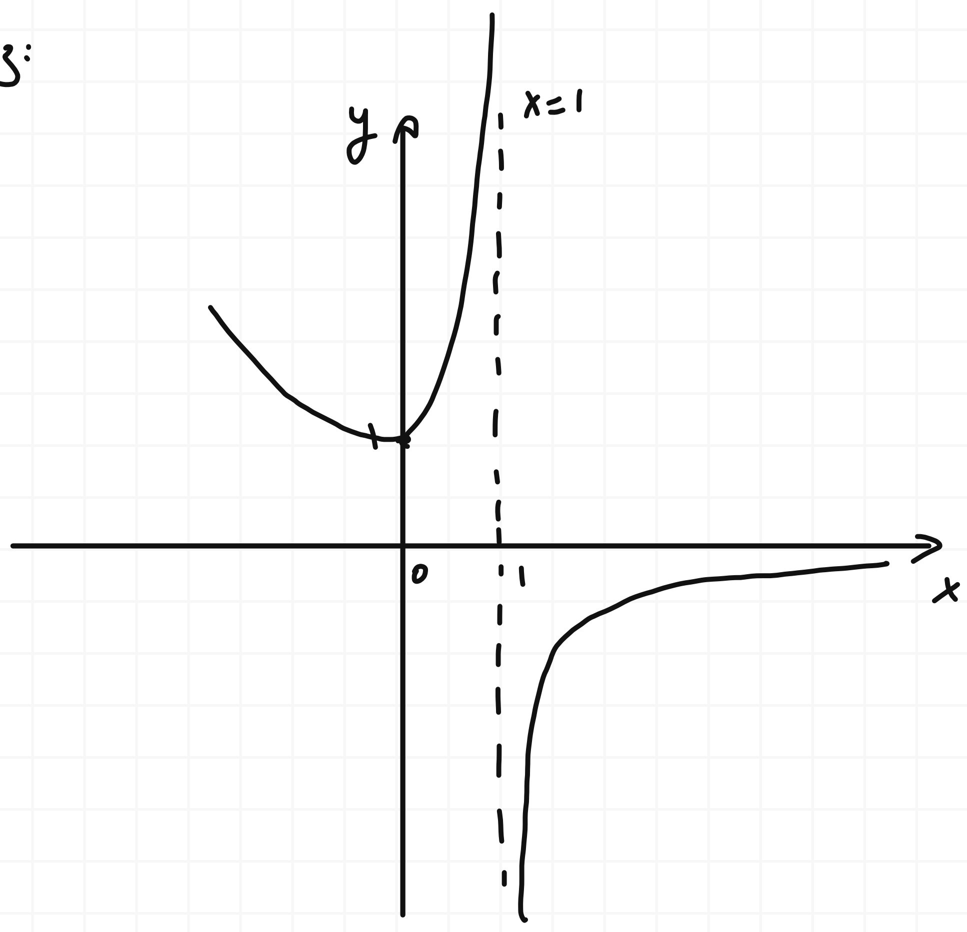
$$\lim_{x\to+\infty} \frac{e^{-x}}{1-x} = 0$$

$\hookrightarrow$ $y = 0$ при $x \to +\infty$

$$\lim_{x\to-\infty} \frac{e^{-x} \to +\infty}{x(1-x) \to \infty} = \lim_{x\to-\infty} \frac{-e^{-x}}{1-2x} = \lim_{x\to-\infty} \frac{e^{-x} \to \infty}{-2} = -\infty$$

это про верт. асимптоту, разобр. выше

эскиз:



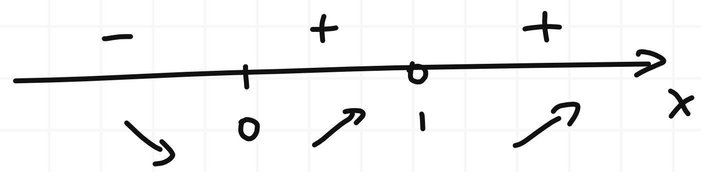

$$\lim_{x\to1+} \frac{e^{-x}}{1-x} = -\infty$$

$$\lim_{x\to1^-} \frac{e^{-x}}{1-x} = +\infty$$

$$y = \frac{e^{-x}}{1-x}$$

$$y' = \frac{-e^{-x}(1-x) + e^{-x}\cancel{(-1)}}{(1-x)^2} = \frac{e^{-x}(x \cancel{-1} \cancel{+1})}{(1-x)^2} = \frac{e^{-x}x}{(1-x)^2}$$

$y' = 0$ при $x = 0$



$$y'' = \frac{(e^{-x}x)'(1-x)^2 + e^{-x}x \cdot 2\cancel{(1-x)}(-1)}{(1-x)^{\cancel{4}3}} = \frac{(-e^{-x}x + e^{-x})(1-x) + 2xe^{-x}}{(1-x)^3} =$$

$$= \frac{e^{-x}(1-x)^2 + 2xe^{-x}}{(1-x)^3} = \frac{e^{-x}(1+x^2 \cancel{-2x} \cancel{+2x})}{(1-x)^3} = \frac{e^{-x}(1+x^2)}{(1-x)^3}$$

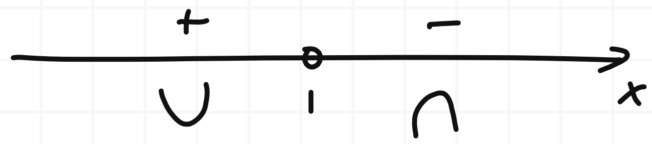
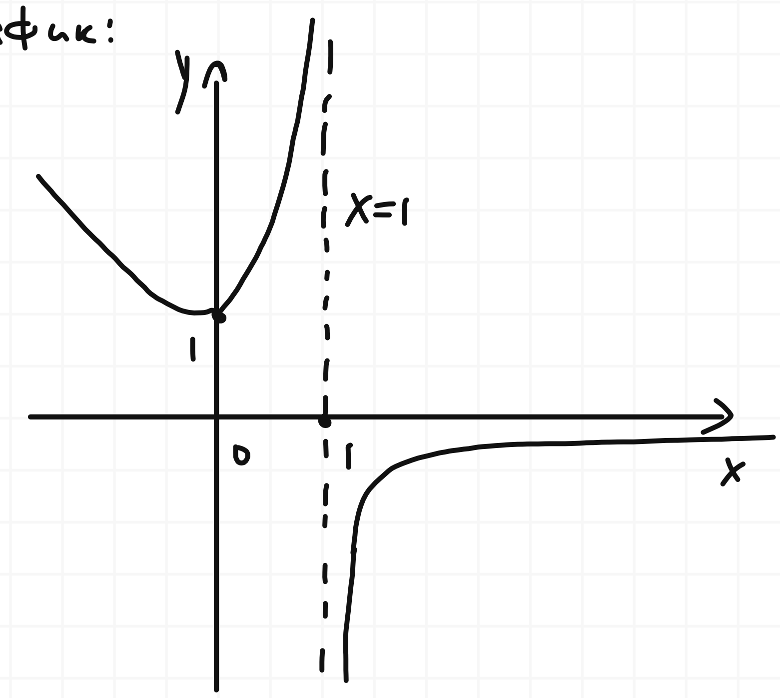


график:



21.15(7)

$y = \frac{\ln x}{x}$

$D_f = (0; +\infty)$

точки пер-я с осями:

с $y$ нет, т.к. $x = 0 \notin D_f$

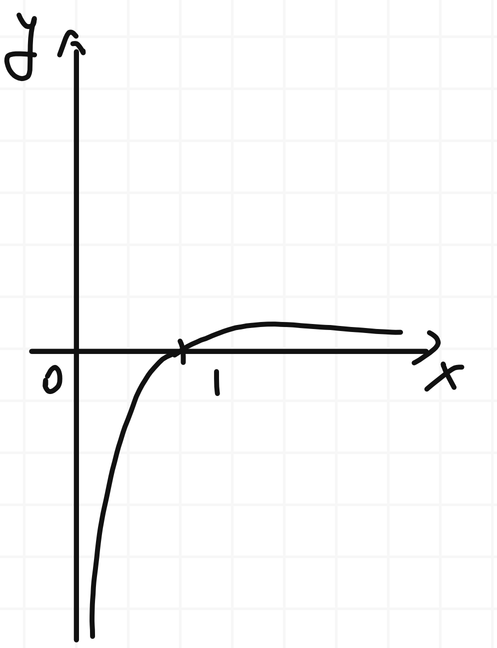
корни: $\ln x = 0$; $x = 1$

асимптоты:

верт: $x = 0$

эскиз:

$$\lim_{x \to 0+} \frac{\ln x}{x} = -\infty$$



накл:

$$\lim_{x\to+\infty}\frac{\ln x^{\to\infty}}{x^2_{\to\infty}}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{2x}=\lim_{x\to+\infty}\frac{1}{2x^2}=0 \hookrightarrow k=0$$

$$\lim_{x\to+\infty}\left(\frac{\ln x}{x}-0\right)=\lim_{x\to+\infty}\frac{\ln x}{x}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to+\infty}\frac{1}{x}=0 \hookrightarrow b=0$$

$\hookrightarrow y=0$

$$y=\frac{\ln x}{x}$$

$$y'=\frac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}\ ;\ y'=0 \text{ при } \ln x=1\ ;\ x=e$$

$$\begin{array}{ccccc} & + & & - & \\ 0 & \nearrow & e & \searrow & \end{array}$$

$$y(e)=\frac{\ln e}{e}=\frac{1}{e}$$

$$y''=\frac{-\frac{1}{\cancel{x}}\cdot x^{\cancel{2}}-(1-\ln x)2x}{x^4}=\frac{-\cancel{x}+2\cancel{x}(\ln x-1)}{x^{\cancel{4}3}}=\frac{-1+2\ln x-2}{x^3}=\frac{-3+2\ln x}{x^3}$$

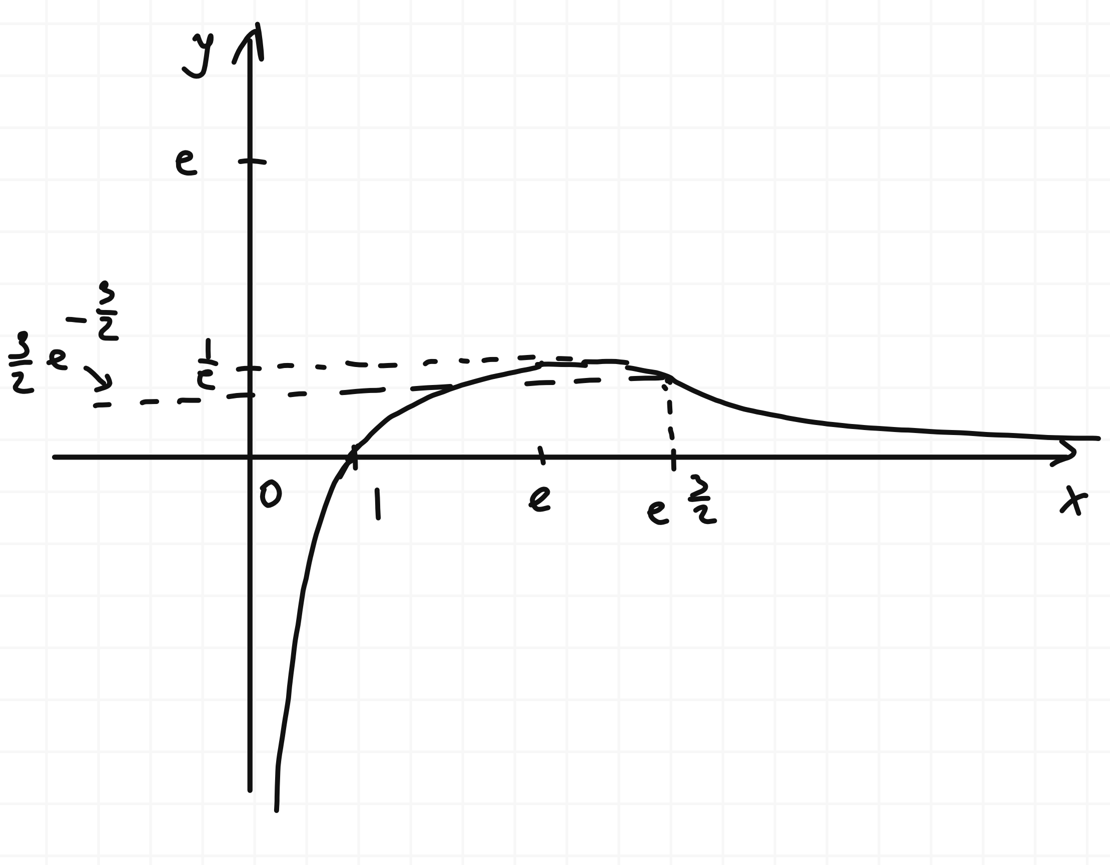
$$y''=0 \text{ при } \ln x=\frac{3}{2}\ ;\ x=e^{\frac{3}{2}}$$

$$y(e^{\frac{3}{2}})=\frac{3/2}{e^{3/2}}=\frac{3}{2}e^{-\frac{3}{2}}$$

$$y'(e^{\frac{3}{2}})=\frac{1-\frac{3}{2}}{e^3}=-\frac{1}{2}e^{-3}$$

$x=e^{\frac{3}{2}}$ – точка перегиба

График:



$$\left((3x-2)^{\frac{1}{2}}\right)^{(n)} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-n+1\right)(3x-2)^{\frac{1}{2}-n}\cdot 3^n$$

$$\left((3x-2)^{\frac{1}{2}}\right)' = \frac{1}{2}(3x-2)^{\frac{1}{2}-1}\cdot 3$$

$$'' = \frac{1}{2}\cdot\left(\frac{1}{2}-1\right)(3x-2)^{\frac{1}{2}-2}\cdot 3^2$$

$$\vdots$$

Др-рац. ф-я

$$\frac{1}{(ax+b)^p}$$

$$(\sin(ax+b))^{(n)} = \sin\left(ax+b+\frac{\pi n}{2}\right)\cdot a^n$$

---

$x_0 = -1$ до $o\left((x-x_0)^4\right)$

$$y(x) = \frac{x^3-3x}{x^3-3x^2+3x-1}$$

$x_0 = -1$

$x - x_0 = x+1 = t$ ; $x = t-1$

$$y(x) = \frac{x(x^2-3)}{(x-1)^3}$$

$$y(t) = \frac{(t-1)(t^2-2t\overset{-2}{\cancel{+1-3}})}{(t-2)^3}$$

$(x^2-3)$

Алгоритм

1. замена переменной, переход к ф-ле Маклорена
2. алгебраическая подготовка к почти табличной ф-и
3. представл. к почти табл. ф-и
4. приведение подобных слагаемых (замена индекса суммирования)
5. обратная замена