Vectors and matrices

Ex 1.

two vectors: $x = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $y = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$

$$x + y = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 7 \end{pmatrix}$$

$3x - 2y$:

$$3x = 3 \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} \qquad 2y = 2 \cdot \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 8 \end{pmatrix}$$

$$3x - 2y = \begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$$

$x^T y$:

$$x^T y = 2 \cdot 5 + (-1) \cdot 0 + (3 \cdot 4) = 10 + 0 + 12 = 22$$

$x y^T$:

$$x y^T = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 & 2 \cdot 0 & 2 \cdot 4 \\ -1 \cdot 5 & -1 \cdot 0 & -1 \cdot 4 \\ 3 \cdot 5 & 3 \cdot 0 & 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 8 \\ -5 & 0 & -4 \\ 15 & 0 & 12 \end{pmatrix}$$

Ex 2

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 4 & -1 & 0 & 9 \end{bmatrix}$$

1. Dimensions of A and B:

- A has 3 rows and 2 columns, so its dimension is 3×2
- B has 2 rows and 4 columns, so its dimension is 2×4

2. Compute AB:

$$AB = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -2 & 5 \\ 4 & -1 & 0 & 9 \end{bmatrix}$$

1st row:
$$(2\cdot 1) + (-1)\cdot 4 = 2-4 = -2$$
$$(2\cdot 3) + (-1)\cdot(-1) = 6+1 = 7$$
$$(2\cdot(-2)) + (-1\cdot 0) = 4+0 = 4$$
$$(2\cdot 5) + (-1\cdot 9) = 10-9 = 1$$

2nd row:
$$(0\cdot 1) + (4\cdot 4) = 0+16 = 16$$
$$(0\cdot 3) + 4\cdot(-1) = -4$$
$$(0\cdot(-2)) + 4\cdot(0) = 0$$
$$(0\cdot 5) + (4\cdot 9) = 36$$

3rd row:
$$(3\cdot 1) + (5\cdot 4) = 3+20 = 23$$
$$(3\cdot 3) + (5\cdot(-1)) = 9-5 = 4$$
$$(3\cdot(-2)) + (5\cdot 0) = -6$$
$$(3\cdot 5) + (5\cdot 9) = 60$$

$$AB = \begin{bmatrix} -2 & 7 & 4 & 1 \\ 16 & -4 & 0 & 36 \\ 23 & 4 & -6 & 60 \end{bmatrix}$$

3. BA possible?

B has (2 × 4)

A has (3 × 2)

For multiplication BA to be defined, the number of columns of B (4) must match the number of rows of A(3)

Since $4 \neq 3$, the multiplication BA is not possible

Ex 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Compute AB

$$AB = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 1 & 1 \cdot 1 + 2 \cdot 0 \\ 3 \cdot 0 + 4 \cdot 1 & 3 \cdot 1 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

Compute BA

$$BA = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 3 & 0 \cdot 2 + 1 \cdot 4 \\ 1 \cdot 1 + 0 \cdot 3 & 1 \cdot 2 + 0 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

3. AB = BA?

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$AB \neq BA$.

Ex 4

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 5 & 1 \end{bmatrix}$$

Step 1: Check if A is invertible

$$\det(A) = 2(4\cdot 1 - 2\cdot 5) - 1(1\cdot 1 - 3\cdot 2) + 3(1\cdot 5 - 4\cdot 3) =$$

$$= 2(-6) - 1(-5) + 3(-7) = -12 + 5 - 21 = -28$$

$\det(A) \neq 0$, the matrix A is invertible

Step 2: Find the inverse of A

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$$

1. $C_{11} = (-1)^{1+1} \cdot \det\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} = 1\cdot(4\cdot 1 - 2\cdot 5) = 1\cdot(-6) = -6$

2. $C_{12} = (-1)^{1+2} \cdot \det\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = -1(1-6) = 5$

3. $C_{13} = (-1)^{1+3} \det\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} = 1(5-12) = -7$

4. $C_{21} = (-1)^{2+1} \cdot \det\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} = -1(1-15) = 14$

5. $C_{22} = (-1)^{2+2} \det\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = 1(2-9) = -7$

6. $C_{23} = (-1)^{2+3} \det\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = -1(10-3) = -7$

7. $C_{31} = (-1)^{3+1} \det \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = 1 \cdot (2-12) = -10$

8. $C_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = -1(4-3) = -1$

9. $C_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = 1(2 \cdot 4 - 1) = 7$

$$C = \begin{bmatrix} -6 & 5 & -7 \\ 14 & -7 & -7 \\ -10 & -1 & 7 \end{bmatrix}$$

$\operatorname{adj}(A) = C^T = \begin{bmatrix} -6 & 14 & -10 \\ 5 & -7 & -1 \\ -7 & -7 & 7 \end{bmatrix}$ adjugate matrix

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$A^{-1} = \frac{1}{-28} \begin{bmatrix} -6 & 14 & -10 \\ 5 & -7 & -1 \\ -7 & -7 & 7 \end{bmatrix} = \begin{bmatrix} \frac{-6}{-28} & \frac{14}{-28} & \frac{-10}{-28} \\ \frac{5}{-28} & \frac{-7}{-28} & \frac{-1}{-28} \\ \frac{-7}{-28} & \frac{-7}{-28} & \frac{7}{-28} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{3}{14} & -\frac{1}{2} & \frac{5}{14} \\ -\frac{5}{28} & \frac{1}{4} & \frac{1}{28} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Step 3: Verify $A \cdot A^{-1} = I$

Multiply A by A^{-1} and confirm that the result is the idendity matrix I.

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex 5

Linear regression, $Xw = y$:

$$w = (X^T X)^{-1} X^T y$$

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad y = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$X^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$X^T \cdot X = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 + 9 + 25 & 2 + 12 + 30 \\ 2 + 12 + 30 & 4 + 16 + 36 \end{bmatrix} = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$$

$$\det(X^T X) = 35 \cdot 56 - 44 \cdot 44 = 1960 - 1936 = 24$$

$$(X^T X) = \frac{1}{24} \begin{bmatrix} 56 & -44 \\ -44 & 35 \end{bmatrix} =$$

$$X^T \cdot y = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 + 24 + 45 \\ 14 + 32 + 54 \end{bmatrix} = \begin{bmatrix} 76 \\ 100 \end{bmatrix}$$

$$W = \frac{1}{24}\begin{bmatrix} 56 & -44 \\ -44 & 35 \end{bmatrix}\begin{bmatrix} 76 \\ 100 \end{bmatrix} =$$

$$= \frac{1}{24}\begin{bmatrix} 56 \cdot 76 + (-44 \cdot 100) \\ -44 \cdot 76 + (35 \cdot 100) \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 4256 - 4400 \\ -3344 + 3500 \end{bmatrix} =$$

$$= \frac{1}{24}\begin{bmatrix} -144 \\ 156 \end{bmatrix} = \begin{bmatrix} -6 \\ 6{,}5 \end{bmatrix}$$

Answer: $W = \begin{bmatrix} -6 \\ 6{,}5 \end{bmatrix}$