SIS 2

Exercise 1: Gauss - Markov Assumptions

Consider the multi-linear model:

$y_i = X\beta + \epsilon_i$

- y_i is an $(n \times 1)$ vector of observations,
- X is an $(n \times k)$ matrix of independent variables,
- β is a $(k \times 1)$ vector of parameters,
- ϵ_i is an $(n \times 1)$ vector of error terms

And the OLS estimator $\hat{\beta} = (X'X)^{-1}X'y$

1. What does the Gauss-Markov theorem state?

The Gauss - Markov theorem states that under the classical linear regression assumptions, the Ordinary Least Squares (OLS) estimator $\hat{\beta}$ is the Best Linear Unbiased Estimator (BLUE)

This means:

- Best: It has the smallest variance among all linear unbiased estimators.
- Linear: It is a linear function of the observed data.
- Unbiased: Its expected value is equal to

the true parameter value, i.e, $E[\hat{\beta}] = \beta$.

Thus, OLS is most efficient estimator under the given assumptions.

2. Which assumptions do you need in order to show that $\hat{\beta}$ is identifed?

For $\hat{\beta}$ to be identifed, we need the assumption of full column rank of X:

$$rank(X) = k.$$

This ensures that $X'X$ is invertible

Proof of identification:

$\hat{\beta} = (X'X)^{-1} X'y$.

For $\hat{\beta}$ to be uniquely determined, $(X'X)^{-1}$ must exist. this requires that $X'X$ is a full-rank matrix, meaning the columns of X are linearly independent. If this condition is met, $\hat{\beta}$ is identifed.

3. Show that $\hat{\beta}$ is unbiased, i.e. $E[\hat{\beta}] = \beta$

OLS estimator:

$$\hat{\beta} = (X'X)^{-1} X'y$$

Substituting $y = X\beta + \epsilon$:

$$\hat{\beta} = (X'X)^{-1} X'(X\beta + \epsilon). \Rightarrow$$

$$\Rightarrow \hat{\beta} = (X'X)^{-1} X'X\beta + (X'X)^{-1} X'\epsilon$$

Since $(X'X)^{-1} X'X = I_k$ (idendity matrix), we get:

$\hat{\beta} = \beta + (X'X)^{-1} X'\epsilon.$

Taking expectations:

$$E[\hat{\beta}] = E[\beta + (X'X)^{-1} X'\epsilon]$$

Since $E[\epsilon] = 0$ by assumption, we get:

$$E[\hat{\beta}] = \beta$$

Thus, $\hat{\beta}$ is an unbiased estimator of β

4. Prove that $Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

$\hat{\beta} = \beta + (X'X)^{-1} X'\epsilon.$

Computing variance:

$$Var(\hat{\beta}) = Var((X'X)^{-1} X'\epsilon).$$

Since $Var(\epsilon) = \sigma^2 I_n$, we get

$Var(\hat{\beta}) = (X'X)^{-1} X' \, Var(\epsilon) \, X (X'X)^{-1}$

Substitung $Var(\epsilon) = \sigma^2 I_n$:

$Var(\hat{\beta}) = (X'X)^{-1} X' (\sigma^2 I_n) X (X'X)^{-1}$

Since $X' I_n X = X'X$, we obtain:

$Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

5. Efficienty of $\hat{\beta}$ and the Gauss-Markov theorem

Let $\hat{\beta}$ be any other linear and unbiased estimator:

$$\hat{\beta} = Ay$$

For $\hat{\beta}$ to be unbiased:

$E[\hat{\beta}] = AE[y] = AX\beta = \beta$

This implies:

$$AX = I_k$$

Using Cochran's theorem, we can show that:

$Var(\hat{\beta}) - Var(\hat{\beta})$ is positive semi-definite.

This means that $\hat{\beta}$ has the smallest variance among all linear unbiased estimators, proving that it is the Best Linear Unbiased Estimator (BLUE)

6. Consistency of β

To show consistency, we prove:

$$\hat{\beta}_n \xrightarrow{p} \beta \quad \text{as} \quad n \to \infty$$

Rewriting the OLS estimator:

$$\hat{\beta} = \beta + (X'X)^{-1} X'\epsilon.$$

Since $E[\epsilon] = 0$, we analyze the term $(X'X)^{-1}X'\epsilon$.

Using the Law of Large Numbers (LLN):

$$\frac{1}{n} X'X \xrightarrow{p} Q, \quad \frac{1}{n} X'\epsilon \xrightarrow{p} 0.$$

If Q is invertible, then:

$$(X'X)^{-1} X'e \xrightarrow{p} 0.$$

Thus: $\hat{\beta} \xrightarrow{p} \beta.$

This proves that $\hat{\beta}$ is consistent under standard assumptions.