13.01.25

Part 1: SLR

Simple Linear Regression: $y = \alpha + \beta x + \epsilon$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

Step 1: Formulating the OLS problem

$$S(\alpha, \beta) = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

Step 2: Deriving the Estimator for $\hat{\alpha}$

$$0: \frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i)(-1) = -2 \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)$$

$$\alpha: \sum_{i=1}^{n} y_i - n\alpha - \beta \sum_{i=1}^{n} x_i = 0$$

$$n: n\alpha = \sum_{i=1}^{n} y_i - \beta \sum_{i=1}^{n} x_i$$

$$n: \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ are the simple means.

Step 3: Deriving the Estimator for $\hat{\beta}$

$$\frac{\partial S}{\partial \beta} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i)(-x_i) = -2 \sum_{i=1}^{n} x_i(y_i - \alpha - \beta x_i)$$

Setting this to zero:

$$\sum_{i=1}^{n} x_i y_i - \alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 = 0$$

Substituting $\alpha = \bar{y} - \beta \bar{x}$:

$$\sum_{i=1}^{n} x_i \cdot y_i - (\bar{y} - \beta \bar{x}) \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} x_i y_i - \bar{y} \sum_{i=1}^{n} x_i + \beta \bar{x} \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 = 0$$

Rearrange for β:

$$\beta\left(\sum_{i=1}^{n} x_i^2 - \bar{x} \sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} x_i y_i - \bar{y} \sum_{i=1}^{n} x_i$$

Using the identites:

$$\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and

$$\hat{\alpha} = \bar{y} - \hat{\beta} \bar{x}$$

Part 2: MLR

Multiple Linear Regression : $y = \beta x + \epsilon$

1. The $p \times 1$ vector of regression coefficients
2. The $n \times p$ design matrix containing independent variables
3. X is $n \times p$

   β is $p \times 1$
4. Proof of $\hat{\beta} = (X^T X)^{-1} X^T y$:

Solve OLS: $X^T y = X^T X \beta$

If $X^T X$ is invertible: $\hat{\beta} = (X^T X)^{-1} X^T y$

5. Dimensions of each element in $\hat{\beta}$:

- $\hat{\beta}$ is $p \times 1$
- $X^T X$ is $p \times p$
- $X^T y$ is $p \times 1$

6. Conditions for $X^T X$ to be invertible:

- X must be have full column rank
- $n \geq p$ (more observations than predictors)
- No linear dependence among columns of X.

Part 3: Calculate simple Linear Regression Coefficients

x	y
1	3
2	5
3	4
4	7
5	6.

1. Calculate the Intercept ($\hat{\alpha}$) and Slope ($\hat{\beta}$)

$$\hat{\beta} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

1) Compute means

$$\bar{x} = \frac{1+2+3+4+5}{5} = 3$$

$$\bar{y} = \frac{3+5+4+7+6}{5} = 5$$

2) Compute $\hat{\beta}$

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = (1-3)(3-5) + (2-3)(5-5) + (3-3)(4-5) + (4-3)(7-5) + (5-3)(6-5) =$$

$$= (-2\cdot(-2)) + (-1\cdot 0) + (0\cdot(-1)) + (1\cdot 2) + 2\cdot 1 = 8$$

$$\sum (x_i - \bar{x})^2 = (-2)^2 + (-1)^2 + (0)^2 + (1)^2 + 2^2 =$$

$$= 4 + 1 + 0 + 1 + 4 = 10.$$

$$\hat{\beta} = \frac{8}{10} = 0{,}8$$

3) Co

$\hat{\alpha} = 5$

$\hat{y} =$

3. Ca

Res

x
1
2
3
4
5

4. C

SS

=

5.

2) Compute $\hat{a}$

$\hat{a} = 5 - (0,8 \cdot 3) = 5 - 2,4 = 2,6$

$\hat{y} = 2.6 + 0,8x$

3. Calculate Residuals:

$Residual = y_i - \hat{y}_i$

x	y	$\hat{y}$	Residual $(y - \hat{y})$
1	3	$2,6 + 0,8 \cdot 1 = 3,4$	$3 - 3,4 = -0,4$
2	5	$2,6 + 0,8 \cdot 2 = 4,2$	$5 - 4,2 = 0,8$
3	4	$2,6 + 0,8 \cdot 3 = 5$	$4 - 5 = -1$
4	7	$2,6 + 0,8 \cdot 4 = 5,8$	$7 - 5,8 = 1,2$
5	6.	$2,6 + 0,8 \cdot 5 = 6,6$	$6 - 6,6 = -0,6$

4. Calculate SSR

$SSR = \Sigma (y_i - \hat{y}_i)^2 = (-0,4)^2 + (0,8)^2 + (-1,0)^2 + (1,2)^2 + (-0,6)^2 =$

$= 0,16 + 0,64 + 1 + 1,44 + 0,36 = 3,6$

5. Calculate R^2:

$$R^2 = 1 - \frac{SSR}{SST}$$

$SST = \Sigma (y_i - \bar{y})^2 = (3-5)^2 + (5-5)^2 + (4-5)^2 + (7-5)^2 + (6-5)^2 =$

$= (-2)^2 + 0 + (-1)^2 + 2^2 + 1 = 10$

$$R^2 = 1 - \frac{3,6}{10} = 1 - 0,36 = 0,64$$