Linear Functional Analysis Notes

Yuling Shi

Contents

1 Preliminaries

Theorem 1.1: Suppose that (M, d) is a metric space and $A \subset M$. Then:

1. if A is complete then it is closed;
2. if M is complete then A is complete if and only if it is closed;
3. if A is compact then it is closed and bounded;
4. (Bolzano-Weierstrass theorem) every closed, bounded subset of $\mathbb{F}^k$ is compact.

Def 1.1 (counting measure): We define counting measure

$$\mu(A) = \begin{cases} n & \text{if} A \text{ has exactly } n \text{ elements} \\ \infty & \text{otherwise} \end{cases}$$

Def 1.2 (essential supremum): Suppose that f is a measurable function and there exists a number b such that $f(x) \leq b a.e.$ Then we can define the **essential supremum** of f to be

$$ess\ sup\ f = \inf\{b : f(x) \leq b\ a.e.\}$$

Def 1.3: We define an equivalence relation $\equiv$ on $\mathcal{L}^1(X)$ by

$$f \equiv g \Longleftrightarrow f(x) = g(x) \text{ for a.e. } x \in X$$

This relation partitions the set $\mathcal{L}^1(X)$ into a space of equivalence classes, which we will denote by $L^1(X)$.

Def 1.4: Define the spaces

$$\begin{array}{l} \mathcal{L}^p(X) = \left\{ f : f \text{ is measurable and } \left(\int_X |f|^p d\mu \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty \\ \mathcal{L}^\infty(X) = \{ f : f \text{ is measurable and ess sup } |f| < \infty \} \end{array}$$

We also define the corresponding sets $L^P(X)$ by identifying functions in $\mathcal{L}^p(X)$ which are $a.e.$ equal. In practice, we simply refer to such $\mathcal{L}$ rather than classes themselves.

Theorem 1.2 (Two useful inequalities): Minkowski's inequality (for $1 \leq p < \infty$)

$$\begin{array}{c} \left(\int_X |f+g|^p d\mu \right)^{1/p} \leq \left(\int_X |f|^p d\mu \right)^{1/p} + \left(\int_X |g|^p d\mu \right)^{1/p} \\ ess\ \sup |f+g| \leq ess\ \sup |f| + ess\ \sup |g| \end{array}$$

Hölder's inequality (for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$):

$$\begin{array}{l} \int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu \right)^{1/p} \left(\int_X |g|^q d\mu \right)^{1/q} \\ \int_X |fg| d\mu \leq ess\ \sup |f| \int_X |g| d\mu \end{array}$$

Theorem 1.3: Suppose that $1 \leq p \leq \infty$. Then the metric space $L^P(X)$ is complete. In particular, the sequence space ℓ^P is complete.

Theorem 1.4: Suppose that $[a, b]$ is a bounded interval and $1 \leq p < \infty$. Then the set $C[a, b]$ is dense in $L^p[a, b]$

2 Normed Spaces

2.1 Examples of Normed Spaces

Def 2.1 (Norm): Let X be a vector space over $\mathbb{F}$. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y, \in X$ and $\alpha \in \mathbb{F}$

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = \|\alpha\|\|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

A vector space X on which there is a norm is called a normed vector space or just a normed space. If X is a normed space, a unit vector in X is a vector x such that $\|x\| = 1$

Lemma 2.1: Let X be a vector space with norm $\|\cdot\|$. If $d : X \times X \to \mathbb{R}$ is defined by $d(x, y) = \|x - y\|$ then (X, d) is a metric space.

Def 2.2 (Metric): If X is a vector space with norm $\|\cdot\|$ and d is the metric defined by $d(x, y) = \|x - y\|$ then d is called the metric associated with $\|\cdot\|\cdot$.

Def 2.3 (Convergence): We are trying to define convergence in our normed space. We will have strong convergence as $\lim_{n\to\infty} \|x_n - x\| = 0$ which is also written as $\lim_{n\to\infty} x_n = x$. And we also have weak convergence as $\lim_{n\to\infty} \|x_n\| = \|x\|$.

Theorem 2.2: Let X be a vector space over $\mathbb{F}$ with norm $\|\cdot\|$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X which converge to x, y in X respectively and let $\{\alpha_n\}$ be a sequence in $\mathbb{F}$ which converges to α in $\mathbb{F}$. Then:

1. $|\|x\| - \|y\|| \leq \|x - y\|$
2. $\lim_{n\to\infty} \|x_n\| = \|x\|$
3. $\lim_{n\to\infty} (x_n + y_n) = x + y$
4. $\lim_{n\to\infty} \alpha_n x_n = \alpha x$

2.2 Finite-dimensional Normed Spaces

Def 2.4 (Equivalence): Let X be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. The norm $\|\cdot\|_2$ is equivalent to the norm $\|\cdot\|_1$ if there exists $M, m > 0$ such that for all $x \in X$

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

Lemma 2.3: Let X be a vector space and let $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_3$ be three norms on X. Let $\|\cdot\|_2$ be equivalent to $\|\cdot\|_1$ and let $\|\cdot\|_3$ be equivalent to $\|\cdot\|_2$

1. $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ (exchanging position)
2. $\|\cdot\|_3$ is equivalent to $\|\cdot\|_1$ (passing)

Lemma 2.4: Let X be a vector space and let $\|\cdot\|$ and $\|\cdot\|_1$ be norms on X. Let d and d_1 be the metrics defined by $d(x, y) = \|x - y\|$ and $d_1(x, y) = \|x - y\|_1$. Suppose that there exists $K > 0$ such that $\|x\| \leq K\|x\|_1$ for all $x \in X$. Let $\{x_n\}$ be a sequence in X.

1. If $\{x_n\}$ converges to x in the metric space (X, d_1) then $\{x_n\}$ converges to x in the metric space (X, d)
2. If $\{x_n\}$ is Cauchy in the metric space (X, d_1) then $\{x_n\}$ is Cauchy in the metric space (X, d)

Corollary 2.1: Let X be a vector space and let $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on X. Let d and d_1 be the metrics defined by $d(x, y) = \|x - y\|$ and $d_1(x, y) = \|x - y\|_1$ Let $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ converges to x in the metric space $(X, d) \iff \{x_n\}$ converges to x in the metric space (X, d_1).
2. $\{x_n\}$ is Cauchy in the metric space $(X, d) \iff \{x_n\}$ is Cauchy in the metric space (X, d_1).
3. (X, d) is complete $\iff (X, d_1)$ is complete.

Theorem 2.5: Let X be a finite-dimensional vector space with norm $\|\cdot\|$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Another norm on X was defined by

$$\left\|\sum_{j=1}^{n} \lambda_j e_j\right\|_1 = \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^{\frac{1}{2}}$$

The norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

Corollary 2.2: If $\|\cdot\|$ and $\|\cdot\|_2$ are any two norms on a finite-dimensional vector space X then they are equivalent.

Lemma 2.6: Let X be a finite-dimensional vector space over $\mathbb{F}$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. If $\|\cdot\|_1 : X \to \mathbb{R}$ is the norm on X defined by $\left\|\sum_{j=1}^{n} \lambda_j e_j\right\|_1 = \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^{\frac{1}{2}}$ then X is a complete metric space.

Corollary 2.3: If $\|\cdot\|$ is any norm on a finite-dimensional space X then X is a complete metric space.

Corollary 2.4: If Y is a finite-dimensional subspace of a normed vector space X, then Y is closed.

2.3 Banach Spaces

In infinite-dimensional vector space X, many methods we discussed in the last section will expire.

Lemma 2.7: If X is a normed vector space and S is a linear subspace of X then $\bar{S}$ is a linear subspace of X.

Def 2.5: Let X be a normed vector space and let E be any non-empty subset of X. The closed linear span of E, denoted by $\overline{Sp}E$, is the intersection of all the closed linear subspaces of X which contain E.

Lemma 2.8: Let X be a normed space and let E be any non-empty subset of X.

1. $\overline{\mathrm{Sp}}E$ is a closed linear subspace of X which contains E.
2. $\overline{\mathrm{Sp}}E = \overline{\mathrm{Sp}\,E}$, that is, $\overline{\mathrm{Sp}}E$ is the closure of $\mathrm{Sp}\,E$

Theorem 2.9 (Riesz' Lemma): Suppose that X is a normed vector space, Y is a closed linear subspace of X such that $Y \neq X$ and α is a real number such that $0 < \alpha < 1$. Then there exists $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ for all $y \in Y$

Theorem 2.10: If X is an infinite-dimensional normed vector space then neither $D = \{x \in X : \|x\| \leq 1\}$ nor $K = \{x \in X : \|x\| = 1\}$ is compact.

Def 2.6 (Banach Space): A Banach space is a normed vector space which is complete under the metric associated with the norm.